The theory behind T A B O O

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Release 1.0

October 9, 2003

Published by the Samizdat Press

Center for Wave Phenomena

Department of Geophysics

Colorado School of Mines

Golden, Colorado 80401

and

New England Research

76 Olcott Drive

White River Junction, Vermont 05001

Samizdat Press publications are available via FTP or WWW from

samizdat.mines.edu

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Preface

This is an attempt to collect in a single document the basic traits of the theory describing the deformations of the Earth under peculiar surface loads: the ice sheets. The book has a mainly pedagogical purpose. It is written in a simple way, and an effort is made to avoid the sentence *it can be shown that*. Almost all of the propositions given here are demonstrated step–by–step, even when they may appear obvious a priori. This is mainly done to facilitate the beginners in the 'art' of the postglacial rebound, but I also hope that this transparent style of writing could be useful for more experienced investigators. The book is written according to an austere minimalism: we only give the statements which are strictly needed to understand the basic concepts. For this reason, the chapter devoted to the mathematical background is largely biased towards the main tools, such has differential operators and spherical harmonics.

The theory illustrated here is implemented in the source code `taboo.f90` which is freely distributed by the Samizdat Press along with this document and the accompanying user guide. At the core of `TABOO` there is the assumption that the Earth is spherically layered. In the common language this means that the problems which can be solved by `TABOO` are 1D problems. Nowadays several research groups have developed more advanced codes, which account for the 2D or even for the 3D structure of the lithosphere and the mantle. However, these codes are not publically available to date, mainly for two reasons. First, their are not totally developed, and some work is still to be done. Second, differently from `TABOO` they are often based on numerical techniques developed with the aid of software packages that are not publically available. In a sense, `TABOO` has the aim of closing the chapter of the 1D problems giving the chance of obtaining a portable source code and a full account of the theory behind. It is hoped that this will encourage the developers of 2D and 3D models to to the same with their procedures in the future.

The reader should be warned that `TABOO` is *not* a sealevel equation [4] solver! The sealevel equation will be the subject of a separate review coming in the next months along with a freely available code (`SELEN`).

The theory behind `TABOO` has only the purpose of collecting formulas and results in an ordered structure. By no means the results presented here are the product of my own research work. Rather, they constitute a theoretical framework which has been constructed by a number of Authors in the course of the last decades. It is not possible to mention all of the contributors to this enormous (but sparse) work, and for this reason I must apologize for the very poor bibliography that I have written at the end of this document. The full set of original papers where the basic ideas have been first developed can be reconstructed on the basis of bibliographies of the manuscripts and books quoted here.

While I have done my best to present a complete account of the theory behind `TABOO` (and consequently a complete source code), some work is still to be done. In particular, the present version of `TABOO` does not explicitely compute relevant physical quantities such as the gravity anomalies, the stress field in the lithosphere and the rotational variations of the Earth. It is my intention to include these topics in the next versions of the code (which will also include figures).

A final note concerning notation. I do not like to write vectors by bold face letters, so that I use arrows throughout. I have been very pedantic in the demonstration of the various propositions given in this document, certainly too much for an experienced reader. This is admittedly boring, but I hope it helps the novices, who are indeed the main target of this booklet. Since the source code `TABOO` is totally accessible, I have not described in detail how and where the single propositions are numerically implemented.

I have been involved in the research on these topics for fifteen years, first as a student, and later as a teacher. Both need a place where a given formula can be easily found and demonstrated. After all, this is the main purpose of `TABOO`.

The future releases of this document (if any) will benefit from the feedback of the readers of this first edition. Please feel free to write to

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for questions, comments, and suggestions.

Urbino, October 10, 2003.

Acknowledgments

This booklet is particularly dedicated to my friend and colleague Carlo Giunchi. He has taught me that a book or a software should be written *per n* (he knows what I mean). Following his hint, I have decided to write the source code of `TABOO`, the accompanying user guide, and finally *The theory behind* `TABOO`. Sofia has made her best to help during the preparation of the manuscript, learning and teaching LaTeX.

I owe much to Enzo Boschi, Roberto Sabadini, Dave Yuen and Yanick Ricard who encouraged me to undertake the research in the field of global geodynamics. In the course of the years I have benefited from discussions and exchange of opinions with many scientists involved in the research on global geodynamics and postglacial rebound. I mention (the order is random) Jerry Mitrovica, Lapo Boschi, Ondrej Čadek and his group of the Charles University in Prague, Spina Cianetti, Benjamin Fong Chao, Maurizio Bonafede, Gabriella D'Agostino, Nicola Piana Agostinetti, Luce Fleitout, Claude Froidevaux, Detlef Wolf, Laura Alfonsi, Ilaire Legros, Bert Vermeersen, Giorgio Ranalli, Paolo Gasperini, Antonio Piersanti, Patrick Wu, Marianne Gregg–Lefftz, Paul Johnston, Paul Morin, and many others. I have benefited from the aid and the patience of my environmental sciences students Valter Brandi, Gabriele Galvani, Paolo Stocchi, and Francesco Frattallone, who have also helped in the preparation of the accompanying software `TABOO`. Finally, I owe much to my friends and colleagues Gianluca Maria Guidi and Renzo Lupini, who have convinced me that strange attractors and the foundations of mathematics are worth to be studied, at least as global geodynamics. I am particularly grateful to Roberto Casadio for the long oxygenating walks in the pinewood of Ravenna (the Dante *Inferno*) and for its continuous encouragement.

The preparation of this document, of the manual of the software `TABOO`, and the development of the source code have been possible thanks to the financial support of the Faculty of Environmental Sciences of the University of Urbino, Italy, with grants "Ex 60%", and that of MIUR (Ministero dell'Istruzione, dell'Universitá e della Ricerca) by a FIRB grant.

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Chapter 1

Mathematical background

This chapter introduces the basic differential operators (gradient, divergence, curl, and Laplacian) in spherical geometry, a number of conventions, and definitions concerning the spherical harmonic functions. The Complex (Surface) Spherical Harmonics (CSH), largely employed in quantum mechanics [8] are useful for their simple and compact algebra. For numerical implementations, it is more convenient to employ the Real (Surface) Spherical Harmonics (RSH). We also discuss the Fully Normalized (Surface) Spherical Harmonics (FNSH), which are sometimes useful for studies concerning the Earth gravity field. The last part of the chapter is devoted to the definition the ocean function, the time–dependent functions in general, the Laplace transform, and the convolution product.

1.1 Differential operators on the sphere

1.1.1 Spherical coordinates

Given a Cartesian reference frame Oxyz, the polar spherical coordinates of a given point in space will be conventionally denoted with r (radius, $0 \leq r \leq \infty$), θ (colatitude, $0 \leq \theta \leq \pi$), and λ (longitude, $0 \leq \lambda \leq 2\pi$).

The polar spherical coordinates are related to the Cartesian coordinates x, y, and z by the formulas:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} \sin\theta\cos\lambda \\ \sin\theta\sin\lambda \\ \cos\theta \end{bmatrix}, \tag{1.1}$$

with

$$x^2 + y^2 + z^2 = r^2. \tag{1.2}$$

Any vector $\vec{u}$ can be written as a combination of the unit, mutually orthogonal vectors $\hat{e}_x$, $\hat{e}_y$ and $\hat{e}_z$, which point along the x, y, and z axes of the Cartesian reference frame:

$$\vec{u} = u_x\hat{e}_x + u_y\hat{e}_y + u_z\hat{e}_z, \tag{1.3}$$

where the vector components u_x, u_y, and u_z are functions of x, y, and z, and

$$\hat{e}_x \times \hat{e}_y = \hat{e}_z, \quad \hat{e}_y \times \hat{e}_z = \hat{e}_x, \quad \hat{e}_z \times \hat{e}_x = \hat{e}_y, \tag{1.4}$$

where $\times$ is the vector product (in the following, the symbol $(\cdot)$ will be used to indicate the scalar product). In a similar manner, it is possible to write $\vec{u}$ as a combination of the unit, mutually orthogonal vectors $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_\lambda$, which point to the directions of increasing r, θ, λ:

$$\vec{u} = u_r\hat{e}_r + u_\theta\hat{e}_\theta + u_\lambda\hat{e}_\lambda, \tag{1.5}$$

where the vector components u_r, u_θ, and u_λ are functions of r, θ, and λ, and

$$\hat{e}_\theta \times \hat{e}_\lambda = \hat{e}_r, \quad \hat{e}_\lambda \times \hat{e}_r = \hat{e}_\theta, \quad \hat{e}_r \times \hat{e}_\theta = \hat{e}_\lambda. \tag{1.6}$$

The relationships between the Cartesian and spherical components of $\vec{u}$ are

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\lambda & \cos\theta\cos\lambda & -\sin\lambda \\ \sin\theta\sin\lambda & \cos\theta\sin\lambda & \cos\lambda \\ \cos\theta & -\sin\lambda & 0 \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \\ u_\lambda \end{bmatrix}, \tag{1.7}$$

and conversely

$$\begin{bmatrix} u_r \\ u_\theta \\ u_\lambda \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\lambda & \sin\theta\sin\lambda & \cos\theta \\ \cos\theta\cos\lambda & \cos\theta\sin\lambda & -\sin\theta \\ -\sin\lambda & \cos\lambda & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}. \tag{1.8}$$

1.1.2 Partial derivatives

We employ the following notation for the partial derivatives:

$$\partial_\xi \;=\; \frac{\partial}{\partial\xi} \tag{1.9}$$

where ξ is one among r, θ, λ.

1.1.3 Gradient

The three–dimensional *gradient* operator is defined as

$$\nabla = \hat{e}_r \partial_r + \frac{1}{r} \nabla_h, \tag{1.10}$$

where

$$\nabla_h = \hat{e}_\theta \partial_\theta + \hat{e}_\lambda \frac{1}{\sin\theta} \partial_\lambda \tag{1.11}$$

is the *surface gradient* operator.

1.1.4 Divergence

Given a vector $\vec{u}$ in the form (1.5), its *divergence* is

$$\nabla \cdot \vec{u} = \partial_r u_r + \frac{2}{r} u_r + \frac{1}{r} \partial_\theta u_\theta + \frac{\cot\theta}{r} u_\theta + \frac{1}{r \sin\theta} \partial_\lambda u_\lambda, \tag{1.12}$$

or:

$$\nabla \cdot \vec{u} = \left(\partial_r + \frac{2}{r} \right) u_r + \frac{1}{r} \nabla_h \cdot \vec{u}, \tag{1.13}$$

where the *surface divergence* of $\vec{u}$ is

$$\nabla_h \cdot \vec{u} = \partial_\theta u_\theta + \cot\theta u_\theta + \frac{1}{\sin\theta} \partial_\lambda u_\lambda. \tag{1.14}$$

1.1.5 Curl

Given a vector field $\vec{u}$ in the form (1.5), its *curl* is

$$\begin{aligned} \nabla \times \vec{u} \quad = \quad & \frac{\hat{e}_r}{r} \left(\partial_\theta u_\lambda + \cot\theta u_\lambda - \frac{1}{\sin\theta} \partial_\lambda u_\theta \right) + \\ & \frac{\hat{e}_\theta}{r} \left(\frac{1}{\sin\theta} \partial_\lambda u_r - r \partial_r u_\lambda - u_\lambda \right) + \\ & \frac{\hat{e}_\lambda}{r} \left(r \partial_r u_\theta + u_\theta - \partial_\theta u_r \right), \end{aligned} \tag{1.15}$$

while the *surface curl* operator is

$$\hat{e}_r \times \nabla_h = -\hat{e}_\theta \frac{1}{\sin\theta} \partial_\lambda + \hat{e}_\lambda \partial_\theta. \tag{1.16}$$

1.1.6 Laplacian

The *Laplacian* operator is defined as

$$\nabla^2 = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\nabla_h^2, \tag{1.17}$$

where the *surface Laplacian* is

$$\nabla_h^2 = \nabla_h \cdot \nabla_h = \partial_\theta^2 + \cot\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial_\lambda^2. \tag{1.18}$$

1.2 Complex Spherical Harmonics

The complex spherical harmonics (CSH) formalism is traditionally employed in quantum mechanics. For a summary on the CSH and their properties the reader is referred to the Appendices of the book of Messiah [8].

We first define the associated Legendre function of *degree* l ($l = 0, 1, 2, \ldots$) and *order* m ($m = 0, 1, 2, \ldots l$) as

$$P_{lm}(x) = (-)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x), \tag{1.19}$$

where $(-) \equiv (-1)$, $x = \cos\theta$, θ is colatitude, and the Legendre polynomials of degree l are defined by the Rodriguez formula

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l. \tag{1.20}$$

With the above definitions, the CSH are

$$Y_{lm}(\theta,\lambda) = \mu_{lm}P_{lm}(\cos\theta)\ \mathrm{e}^{\iota m\lambda}, \tag{1.21}$$

where

$$\iota = \sqrt{-1}. \tag{1.22}$$

The normalization constant

$$\mu_{lm} = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \tag{1.23}$$

ensures that the following *orthogonality relationship* holds

$$\int_\Omega Y_{l'm'}^*(\theta,\lambda)Y_{lm}(\theta,\lambda)d\Omega = \delta_{ll'}\delta_{mm'}, \tag{1.24}$$

where the asterisk denotes complex conjugation, δ_{ij} is the Kronecker delta[1], and

$$\int_{\Omega}(\cdot)d\Omega \equiv \int_0^{2\pi}\int_0^{\pi}(\cdot)\sin\theta d\theta d\lambda, \tag{1.25}$$

where $(\cdot)$ is any scalar function.

We finally observe that the CSH with negative order can be obtained from those with positive orders by the definition:

$$Y_{l-m}(\theta,\lambda) \equiv (-)^m Y^*_{lm}(\theta,\lambda). \tag{1.26}$$

1.3 Properties of Y_{lm}, P_{lm}, and P_l

A full account of the properties of the Y_{lm}, P_{lm}, and P_l functions is beyond our purposes. We only give a few identities useful for the ensuing discussion. The reader is referred to [1] and [8] for a more complete list of definitions, formulas, and identities.

1.3.1 Properties of Y_{lm}

Property 1. The spherical harmonic functions Y_{lm} (1.21) are eigenfunctions of $-\nabla_h^2$ with eigenvalue $l(l+1)$:

$$\nabla_h^2 Y_{lm} = -l(l+1)Y_{lm}, \tag{1.27}$$

where the surface Laplacian ∇_h^2 his given by (1.18).

Property 2 (*addition theorem*). Let (θ,λ) and (θ',λ') the polar spherical coordinates of two points on the surface of a sphere, and let Θ be the colatitude of the second relative to the first, such that

$$\cos\Theta = \frac{\vec{r}' \cdot \vec{r}}{rr'}, \tag{1.28}$$

with $r' = \|\vec{r}'\|$ and $r = \|\vec{r}\|$. The *addition theorem* states that

$$P_l(\cos\Theta) = \frac{4\pi}{2l+1}\sum_{m=-l}^{+l} Y^*_{lm}(\theta',\lambda')Y_{lm}(\theta,\lambda). \tag{1.29}$$

[1]The Kronecker delta is $\delta_{ij}=1$ if $i=j$, and $\delta_{ij}=0$ if $i \neq j$.

A proof of the addition theorem can be found in [11].

degree l	order m	$Y_{lm}(\theta,\lambda) =$ $= k$	$\cdot f(\theta,\lambda)$
0	0	$(1/4\pi)^{1/2}$	$\cdot 1$
1	0	$(1/2)(3/\pi)^{1/2}$	$\cdot \cos\theta$
1	1	$-(1/2)(3/2\pi)^{1/2}$	$\cdot \sin\theta e^{\iota\lambda}$
2	0	$(1/4)(5/4\pi)^{1/2}$	$\cdot(3\cos^2\theta - 1)$
2	1	$-(1/2)(15/2\pi)^{1/2}$	$\cdot \sin\theta\cos\theta e^{\iota\lambda}$
2	2	$(1/4)(15/4\pi)^{1/2}$	$\cdot \sin^2\theta e^{2\iota\lambda}$
3	0	$(1/4)(7/\pi)^{1/2}$	$\cdot(5\cos^3\theta - 3\cos\theta)$
3	1	$-(1/8)(21/\pi)^{1/2}$	$\cdot \sin\theta(5\cos^2\theta - 1)e^{\iota\lambda}$
3	2	$(1/4)(105/2\pi)^{1/2}$	$\cdot \sin^2\theta\cos\theta e^{2\iota\lambda}$
3	3	$-(1/8)(35/2\pi)^{1/2}$	$\cdot \sin^3\theta e^{3\iota\lambda}$

Table 1.1: Complex spherical harmonics
Table of $Y_{lm}(\theta,\lambda)$ for degree $0 \le l \le 3$ and order $m \ge 0$. The harmonics are factorized as $Y_{lm}(\theta,\lambda) = k \cdot f(\theta,\lambda)$. Harmonics with negative orders can be obtained by (1.26).

1.3.2 Properties of P_{lm}

Property 1.

$$P_{l0}(\cos\theta) = P_l(\cos\theta). \tag{1.30}$$

Proof. This is a straightforward consequence of (1.19).

Property 2 (*an integral property*).

$$\int_\Omega P_{lm}^2(\cos\theta)\left\{\begin{array}{c}\cos^2 m\lambda\\ \sin^2 m\lambda\end{array}\right\}d\Omega = \frac{2\pi}{2l+1}\frac{(l+m)!}{(l-m)!} \quad (m \neq 0). \tag{1.31}$$

Proof. According to (1.25), the lefthand side of (1.31) is equivalent to

$$\int_0^\pi P_{lm}^2(\cos\theta)\sin\theta d\theta \cdot \int_0^{2\pi}\left\{\begin{array}{c}\cos^2 m\lambda\\ \sin^2 m\lambda\end{array}\right\}d\lambda, \tag{1.32}$$

where it is easy to verify that:

$$\int_0^\pi \cos^2 m\lambda d\lambda = \int_0^\pi \sin^2 m\lambda d\lambda = \pi \quad (m \neq 0). \tag{1.33}$$

From the orthogonality relationship (1.24) written for $l = l'$ we obtain:

$$\begin{aligned}
\delta_{mm'} &= \int_\Omega Y_{lm}(\theta,\lambda) Y^*_{lm'}(\theta,\lambda) d\Omega \\
\delta_{mm'} &= \text{(1.21)} = \mu_{lm}\mu_{lm'} \int_0^{2\pi} \mathrm{e}^{\iota(m-m')\lambda} d\lambda \,\cdot \\
&\quad \cdot \int_0^\pi P_{lm}(\cos\theta) P_{lm'}(\cos\theta) \sin\theta d\theta \\
\delta_{mm'} &= 2\pi \mu_{lm}\mu_{lm'}\delta_{mm'} \int_0^\pi P_{lm}(\cos\theta) P_{lm'}(\cos\theta) \sin\theta d\theta \\
1 &= 2\pi\mu_{lm}^2 \int_0^\pi P_{lm}^2(\cos\theta) \sin\theta d\theta \\
\frac{1}{2\pi\mu_{lm}^2} &= \int_0^\pi P_{lm}^2(\cos\theta) \sin\theta d\theta \\
\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!} &= \text{(1.23)} = \int_0^\pi P_{lm}^2(\cos\theta) \sin\theta d\theta.
\end{aligned} \tag{1.34}$$

Once (1.34) and (1.33) are inserted into (1.32), (1.31) is proved •

1.3.3 Properties of P_l

Property 1 (*orthogonality*). The Legendre polynomials are mutually orthogonal in the interval $[0, \pi]$:

$$\int_0^\pi P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta d\theta \equiv \int_{-1}^{+1} P_l(x) P_{l'}(x) dx = \frac{2\delta_{ll'}}{2l+1}. \tag{1.35}$$

Proof. The statement (1.35) is a consequence of the CSH orthogonality relationship (1.24) written for $m = m' = 0$:

$$\begin{aligned}
\delta_{ll'}\delta_{mm'} &= \int_\Omega Y^*_{l'm'}(\theta,\lambda) Y_{lm}(\theta,\lambda) d\Omega \\
\delta_{ll'} &= \text{(1.21)} = \int_\Omega \mu_{l0}\mu_{l'0} P_{l0}(\cos\theta) P_{l'0}(\cos\theta) d\Omega \\
&= \text{(1.23)} = 2\pi \frac{\sqrt{(2l+1)}\sqrt{(2l'+1)}}{4\pi} \int_{-1}^{+1} P_l(x) P_{l'}(x) dx,
\end{aligned} \tag{1.36}$$

l	m	$P_{lm}(\cos\theta)$
0	0	1
1	0	$\cos\theta$
1	1	$-\sin\theta$
2	0	$(1/2)(3\cos^2\theta - 1)$
2	1	$-3\sin\theta\cos\theta$
2	2	$3\sin^2\theta$
3	0	$(1/2)(5\cos^3\theta - 3\cos\theta)$
3	1	$(-3/2)\sin\theta(5\cos^2\theta - 1)$
3	2	$15\sin^2\theta\cos\theta$
3	3	$-15\sin^3\theta$

Table 1.2: Associated Legendre functions
Table of the associated Legendre functions $P_{lm}(\cos\theta)$ for degrees $0 \le l \le 3$.

l	$P_l(x)$
0	1
1	x
2	$(1/2)(3x^2 - 1)$
3	$(1/2)(5x^3 - 3x)$
4	$(1/8)(35x^4 - 30x^2 + 3)$
5	$(1/8)(63x^5 - 70x^3 + 15x)$

Table 1.3: Legendre polynomials
Table of the Legendre polynomials $P_l(x)$ for degrees $0 \le l \le 5$, with $x = \cos\theta$.

so that:

$$\int_{-1}^{+1} P_l(x)P_{l'}(x)dx = \frac{2\delta_{ll'}}{\sqrt{(2l+1)}\sqrt{(2l'+1)}} = \frac{2\delta_{ll'}}{2l+1} \quad \bullet \tag{1.37}$$

Property 2 (*link with the Chebichev polynomials*). A useful integral property of the Legendre polynomials is

$$\int_{z}^{1} \frac{P_n(x)dx}{\sqrt{x-z}} = \frac{T_n(z) - T_{n+1}(z)}{(n+\frac{1}{2})\sqrt{1-z}}, \tag{1.38}$$

where

$$T_n(z) \equiv \cos(nz), \quad (n = 0, 1, 2, \ldots) \tag{1.39}$$

are the Chebichev polynomials of 2^{nd} kind [1].

Property 3 (*shifted derivatives*).

$$P_l(x) = \frac{P'_{l+1}(x) - P'_{l-1}(x)}{2l+1}, \quad (l \geq 1), \tag{1.40}$$

where $P'_l(x) \equiv \dfrac{dP_l(x)}{dx}$.

Property 4 (*values for $x = \pm 1$*).

$$\begin{aligned} P_l(1) &= 1 \\ P_l(-1) &= (-)^l. \end{aligned} \tag{1.41}$$

Property 5 (*a Legendre sum*).

$$\sum_{l=0}^{\infty} P_l(\cos\theta) = \frac{1}{2\sin\frac{\theta}{2}}. \tag{1.42}$$

Property 6 (*Legendre polynomials generating function*).

$$\sum_{l=0}^{\infty} z^l P_l(\cos\theta) = \frac{1}{\sqrt{1 - 2z\cos\theta + z^2}}, \quad |z| \leq 1. \tag{1.43}$$

Property 7 (*a useful integral*).

$$I_o(\theta_1,\theta_2) \equiv \int_{\cos\theta_1}^{\cos\theta_2} P_l(x)dx \tag{1.44}$$

$$= {}_{(1.40)} = \frac{P_{l+1}(\cos\theta_2) - P_{l-1}(\cos\theta_2)}{2l+1} + - \frac{P_{l+1}(\cos\theta_1) - P_{l-1}(\cos\theta_1)}{2l+1}. \tag{1.45}$$

In particular:

$$I_o(\pi,\alpha) = {}_{(1.44)} = \int_{-1}^{\cos\alpha} P_l(x)dx$$

$$= {}_{(1.41,\ 1.45)} = \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1}. \tag{1.46}$$

1.4 Spherical harmonics expansions of scalar fields

Here we illustrate the various forms of the spherical harmonics expansion valid for a scalar field, and the relationships between them. A generic scalar function of colatitude and longitude can be expanded in series of complex spherical harmonics (CSH), real spherical harmonics (RSH) or fully normalized spherical harmonics (FNSH). For the functions which only depend on colatitude (axis–symmetrical functions), a LEG expansion suffices.

1.4.1 CSH expansion

We denote with $F(\theta,\lambda)$ a scalar field. It is often necessary to expand F on the basis of the CSH, i.e., to determine the (complex) coefficients f_{lm} such that

$$F(\theta,\lambda) = \Sigma_{lm} f_{lm} Y_{lm}(\theta,\lambda), \tag{1.47}$$

where[2] we conventionally write:

$$\Sigma_{lm} \equiv \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} . \tag{1.48}$$

[2]We are not concerned here on the conditions which ensure the convergence of (1.47). The reader is referred to [11] and to [2] for these issues.

In the following, we will refer to f_{lm} as to the *CSH coefficients* of the scalar function F.

The *degree variance* of $F(\theta, \lambda)$ is:

$$S_l = \sqrt{\frac{1}{2l+1} \sum_{m=-l}^{+l} |f_{lm}|^2}. \tag{1.49}$$

Proposition 1 *The coefficients of the CSH expansion (1.47) are*

$$f_{lm} = \int_\Omega Y^*_{lm}(\theta, \lambda) F(\theta, \lambda) d\Omega. \tag{1.50}$$

Proof.

$$\begin{aligned}
F &= \text{(1.47)} = \quad \Sigma_{l'm'} f_{l'm'} Y_{l'm'} \\
FY^*_{lm} &= \Sigma_{l'm'} f_{l'm'} Y_{l'm'} Y^*_{lm} \\
\int_\Omega Y^*_{lm} F d\Omega &= \Sigma_{l'm'} f_{l'm'} \int_\Omega Y_{l'm'} Y^*_{lm} d\Omega \\
\int_\Omega Y^*_{lm} F d\Omega &= \text{(1.24)} = \quad \Sigma_{l'm'} f_{l'm'} \delta_{ll'} \delta_{mm'} \\
\int_\Omega Y^*_{lm} F d\Omega &= f_{lm} \quad \bullet
\end{aligned} \tag{1.51}$$

Proposition 2 *If the field F is real, i.e., if*

$$F = F^*, \tag{1.52}$$

the CSH coefficients satisfy the relationship:

$$f_{l-m} = (-)^m f^*_{lm}. \tag{1.53}$$

Proof.

$$\begin{aligned}
f_{l-m} &= \text{(1.50)} = \quad \int_\Omega Y^*_{l-m} F d\Omega \\
&= \text{(1.26)} = \quad \int_\Omega (-)^m Y_{lm} F d\Omega \\
&= (-)^m \left(\int_\Omega Y^*_{lm} F^* d\Omega \right)^* \\
&= \text{(1.52)} = \quad (-)^m \left(\int_\Omega Y^*_{lm} F d\Omega \right)^* \\
&= \text{(1.50)} = \quad (-)^m f^*_{lm} \quad \bullet
\end{aligned} \tag{1.54}$$

Proposition 3 *If the field F is real (see 1.52), the degree 0 CSH coefficient of the expansion (1.47) is real.*

Proof. The fact that f_{00} is real follows immediately from (1.53). The explicit expression of f_{00} is

$$\begin{aligned} f_{00} &= {\scriptstyle (1.50)} = \int_\Omega Y_{00}^* F d\Omega \\ &= {\scriptstyle (\text{table } 1.1)} = \frac{1}{\sqrt{4\pi}} \int_\Omega F d\Omega \quad \bullet \end{aligned} \tag{1.55}$$

Proposition 4 *The mean value of a scalar field F on the sphere, defined as*

$$\langle F \rangle_\Omega \equiv \frac{\int_\Omega F d\Omega}{\int_\Omega d\Omega}, \tag{1.56}$$

is proportional to the degree 0 CSH coefficient of (1.47):

$$\langle F \rangle_\Omega = \frac{f_{00}}{\sqrt{4\pi}} \tag{1.57}$$

Proof.

$$\begin{aligned} \langle F \rangle_\Omega &\equiv {\scriptstyle (1.56)} = \frac{\int_\Omega F d\Omega}{\int_\Omega d\Omega} \\ &= {\scriptstyle (1.55)} = \frac{\sqrt{4\pi} f_{00}}{4\pi} \\ &= \frac{f_{00}}{\sqrt{4\pi}} \quad \bullet \end{aligned} \tag{1.58}$$

1.4.2 RSH expansion

Due to their simple algebra, the CSH are convenient for theoretical purposes. However, for computational purposes, it is by far more practical to employ a real representation of the spherical harmonics. The Real Spherical Harmonic (RSH) function of degree l and order m has the form

$$S_{lm}(\theta, \lambda) = (c_{lm} \cos m\lambda + s_{lm} \sin m\lambda) P_{lm}(\cos\theta), \tag{1.59}$$

where c_{lm} and s_{lm} ($l = 0, 1, 2 \ldots; m = 0, \ldots, l$) are referred as to cosine and sine coefficients of the RSH, and $P_{lm}(\cos\theta)$ is given by (1.19).

A RSH can be of one of three types. If $m = 0$, the RSH is a *zonal* RSH, which is only function of colatitude. For $0 < m < l$, the RSH is a *tesseral* RSH, and finally, for $m = l$, the RSH is called *sectorial.* The geometrical features of the three families are well illustrated in e.g. [7].

Below we give some recipes showing how to convert a CSH expansion into a RSH expansion.

Proposition 5 *The CSH expansion of a scalar function*

$$F(\theta, \lambda) = \Sigma_{lm} f_{lm} Y_{lm}(\theta, \lambda), \tag{1.60}$$

can be equivalently written as a RSH expansion:

$$F(\theta, \lambda) = \Sigma'_{lm}(c_{lm} \cos m\lambda + s_{lm} \sin m\lambda) P_{lm}(\cos\theta), \tag{1.61}$$

where the prime indicates that the sum is restricted to $m \geq 0$:

$$\Sigma'_{lm} \equiv \sum_{l=0}^{\infty} \sum_{m=0}^{+l}, \tag{1.62}$$

and the cosine and sine coefficients (or, more simply, the RSH coefficients) of $F(\theta, \lambda)$ are

$$\left\{ \begin{array}{c} c_{lm} \\ s_{lm} \end{array} \right\} = (2 - \delta_{0m}) \mu_{lm} \left\{ \begin{array}{cc} & \mathrm{Re}(f_{lm}) \\ - & \mathrm{Im}(f_{lm}) \end{array} \right\}, \quad (l \geq 0, 0 \leq m \leq l), \tag{1.63}$$

where μ_{lm} is given by (1.23), and $\mathrm{Re}(f_{lm})$ and $\mathrm{Im}(f_{lm})$ are the real and imaginary parts of f_{lm}, respectively.

Proof. It suffices to observe that

$$\begin{aligned}
F(\theta, \lambda) &= {\scriptstyle (1.47)} = \quad \Sigma_{lm} f_{lm} Y_{lm} \\
&= \Sigma_l (\Sigma_{m<0} f_{lm} Y_{lm} \quad + f_{l0} Y_{l0} \quad + \Sigma_{m>0} f_{lm} Y_{lm}) \\
&= \Sigma_l (\Sigma_{p>0} f_{l-p} Y_{l-p} \quad + f_{l0} Y_{l0} \quad + \Sigma_{m>0} f_{lm} Y_{lm}) = {\scriptstyle (1.26, 1.53)} = \\
&= \Sigma_l (\Sigma_{p>0} (-1)^p f^*_{lp} (-1)^p Y^*_{lp} \quad + f_{l0} Y_{l0} \quad + \Sigma_{m>0} f_{lm} Y_{lm}) \\
&= \Sigma_l (\Sigma_{m>0} f^*_{lm} Y^*_{lm} \quad + f_{l0} Y_{l0} \quad + \Sigma_{m>0} f_{lm} Y_{lm}) \\
&= \Sigma_l [2\mathrm{Re}(\Sigma_{m>0} f_{lm} Y_{lm}) \quad + f_{l0} Y_{l0}] \\
&= \Sigma_l (2 - \delta_{0m}) \mathrm{Re}(\Sigma_{m\geq 0} f_{lm} Y_{lm}) = {\scriptstyle (1.21)} = \\
&= \Sigma'_{lm} (2 - \delta_{0m}) \mathrm{Re}[f_{lm} \mu_{lm} P_{lm}(\cos\theta) \mathrm{e}^{\iota m\lambda}] \\
&= \Sigma'_{lm} (2 - \delta_{0m}) \mu_{lm} P_{lm} \mathrm{Re}(f_{lm} \mathrm{e}^{\iota m\lambda}) \\
&= \Sigma'_{lm} (2 - \delta_{0m}) \mu_{lm} [\mathrm{Re}(f_{lm}) \cos m\lambda - \mathrm{Im}(f_{lm}) \sin m\lambda)] P_{lm} \\
&= \Sigma'_{lm} (c_{lm} \cos m\lambda + s_{lm} \sin m\lambda) P_{lm}(\cos\theta),
\end{aligned} \tag{1.64}$$

where c_{lm} and s_{lm} are given by (1.63) •

1.4.3 FNSH expansion

The Fully Normalized Spherical Harmonics (FNSH) differ from the RSH for their normalization. Given a real scalar function $F(\theta, \lambda)$, its FNSH expansion is

$$F(\theta, \lambda) = \Sigma'_{lm}(\bar{c}_{lm} \cos m\lambda + \bar{s}_{lm} \sin m\lambda)\bar{P}_{lm}(\cos\theta), \quad (1.65)$$

where Σ'_{lm} is defined by (1.62), and the fully normalized associated Legendre polynomials $\bar{P}_{lm}(\cos\theta)$ are such that

$$\int_{\Omega} \bar{P}^2_{lm}(\cos\theta) \left\{ \begin{array}{c} \cos^2 m\lambda \\ \sin^2 m\lambda \end{array} \right\} d\Omega = 4\pi, \quad (m \neq 0). \quad (1.66)$$

By comparison of (1.31) with (1.66) we obtain:

$$\bar{P}_{lm}(\cos\theta) = \sqrt{2(2l+1)\frac{(l-m)!}{(l+m)!}} P_{lm}(\cos\theta), \quad (m \neq 0), \quad (1.67)$$

which can be extended to the case $m = 0$ requiring that $\bar{P}_{00}(\cos\theta) = 1$:

$$\bar{P}_{lm}(\cos\theta) = \sqrt{(2-\delta_{0m})(2l+1)\frac{(l-m)!}{(l+m)!}} P_{lm}(\cos\theta), \quad (m \geq 0). \quad (1.68)$$

Proposition 6 *Given the RSH expansion*

$$F(\theta, \lambda) = \Sigma'_{lm}(c_{lm} \cos m\lambda + s_{lm} \sin m\lambda) P_{lm}(\cos\theta), \quad (1.69)$$

the sine and cosine coefficients of the FNSH expansion

$$F(\theta, \lambda) = \Sigma'_{lm}(\bar{c}_{lm} \cos m\lambda + \bar{s}_{lm} \sin m\lambda)\bar{P}_{lm}(\cos\theta) \quad (1.70)$$

are

$$\left\{ \begin{array}{c} \bar{c}_{lm} \\ \bar{s}_{lm} \end{array} \right\} = h_{lm} \left\{ \begin{array}{c} c_{lm} \\ s_{lm} \end{array} \right\}, \quad (1.71)$$

where

$$h_{lm} = \sqrt{\frac{1}{(2-\delta_{0m})(2l+1)} \frac{(l+m)!}{(l-m)!}}. \quad (1.72)$$

Proof. It suffices to use (1.68) into (1.70) and to compare the result with (1.69) •

1.4.4 LEG expansion

When a scalar function $g(\theta)$ only depends on colatitude, the RSH and CSH expansions reduce to a sum on the Legendre polynomials (LEG expansion). Candidates to a LEG expansion are those functions which show an axial symmetry with respect the z–axis of the Cartesian reference frame. For this reason, we will refer to them as to *axis–symmetrical functions*. In the RSH expansion (1.61) of an axis–symmetrical function $g(\theta)$ only the zonal terms appear:

$$g(\theta) = \sum_{l=0}^{\infty} g_l P_l(\cos\theta), \tag{1.73}$$

where g_l is the *LEG coefficient* of degree l of $g(\theta)$. The main results for the LEG expansions are given in the following three propositions.

Proposition 7 *The LEG coefficients of the expansion:*

$$g(\theta) = \sum_{l=0}^{\infty} g_l P_l(\cos\theta) \tag{1.74}$$

are

$$g_l = \frac{2l+1}{2} \int_0^{\pi} g(\theta) P_l(\cos\theta) \sin\theta d\theta, \tag{1.75}$$

or, equivalently:

$$g_l = \frac{2l+1}{2} \int_{-1}^{+1} g(x) P_l(x) dx, \quad (x \equiv \cos\theta). \tag{1.76}$$

Proof.

$$\begin{aligned}
g(\theta) &= \text{(1.74)} = \sum_{l=0}^{\infty} g_l P_l(\cos\theta) \\
g(\theta) P_{l'}(\cos\theta) &= \sum_{l=0}^{\infty} g_l P_l(\cos\theta) P_{l'}(\cos\theta) \\
\int_0^{\pi} g(\theta) P_{l'}(\cos\theta) \sin\theta d\theta &= \sum_{l=0}^{\infty} g_l \int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta d\theta \\
\int_0^{\pi} g(\theta) P_{l'}(\cos\theta) \sin\theta d\theta &= \text{(1.35)} = \sum_{l=0}^{\infty} g_l \frac{2\delta_{ll'}}{2l'+1}
\end{aligned}$$

$$\begin{aligned}\int_0^{\pi} g(\theta)P_l(\cos\theta)\sin\theta d\theta &= g_l\frac{2}{2l+1}\\ g_l &= \frac{2l+1}{2}\int_0^{\pi} g(\theta)P_l(\cos\theta)\sin\theta d\theta\\ &= \frac{2l+1}{2}\int_{-1}^{+1} g(x)P_l(x)dx \quad \bullet \end{aligned} \tag{1.77}$$

Proposition 8 *Given the axis–symmetrical function $g(\theta)$, its LEG expansion (1.74) is equivalent to a RSH expansion with coefficients $c_{lm} = g_l\delta_{m0}$ and $s_{lm} = 0$.*

Proof.

$$\begin{aligned} g(\theta) &= \text{(1.74)} = \sum_{l=0}^{\infty} g_l P_l(\cos\theta)\\ &= \text{(1.30)} = \Sigma'_{lm} g_l \delta_{m0} P_{lm}(\cos\theta)\\ &= \Sigma'_{lm} g_l\delta_{m0}\cos m\lambda P_{lm}(\cos\theta)\\ &= \Sigma'_{lm}(c_{lm}\cos m\lambda + s_{lm}\sin m\lambda)P_{lm}(\cos\theta), \end{aligned} \tag{1.78}$$

with $c_{lm} = g_l\delta_{m0}$ and $s_{lm} = 0$ $\bullet$

Proposition 9 *Given the axis–symmetrical function $g(\theta)$, its LEG expansion (1.74) is equivalent to a CSH expansion with coefficients $f_{lm} = \frac{1}{\mu_{lm}}\delta_{m0}g_l$.*

Proof.

$$\begin{aligned} g(\theta) &= \text{(1.74)} = \sum_{l=0}^{\infty} g_l P_l(\cos\theta)\\ &= \text{(1.30)} = \Sigma_{lm} g_l \frac{1}{\mu_{lm}}\delta_{m0}\mu_{lm}P_{lm}(\cos\theta)e^{im\lambda}\\ &= \text{(1.21)} = \Sigma_{lm} g_l \frac{1}{\mu_{lm}}\delta_{m0} Y_{lm}(\theta,\lambda)\\ &= \Sigma_{lm} f_{lm} Y_{lm}(\theta,\lambda), \end{aligned} \tag{1.79}$$

with $f_{lm} = \frac{1}{\mu_{lm}}\delta_{m0}g_l$ and where μ_{lm} is given by (1.23) $\bullet$

1.4.5 Summary conversion Tables

Here we provide summary tables showing how to convert a given spherical harmonics expansion into another. Some of the formulas displayed are deduced explicitely in the previous sections, some others can be obtained by simple algebra from those that have been demonstrated. The first and the second columns show the types of the original (old) and of the final (new) expansions, respectively. The third gives the harmonic coefficients of the old form, and the equation giving the expansion in the old form is referenced in the fourth column. The fifth column shows the relationship between the old and the new coefficients, and the sixth provides a reference equation to the new expansion.

from	to	old coeff.	see eq.	new coeff.		see eq.
CSH	RSH	f_{lm}	(1.47)	$c_{lm} =$	$\mathrm{Re}(f_{lm})\mu_{lm}(2-\delta_{0m})$	(1.61)
				$s_{lm} =$	$-\mathrm{Im}(f_{lm})\mu_{lm}(2-\delta_{0m})$	
CSH	FNSH	f_{lm}	(1.47)	$\bar{c}_{lm} =$	$\mathrm{Re}(f_{lm})(2-\delta_{0m})/4\pi)^{1/2}$	(1.65)
				$\bar{s}_{lm} =$	$-\mathrm{Im}(f_{lm})(2-\delta_{0m})/4\pi)^{1/2}$	

Table 1.4: CSH conversion table

Conversion table for CSH to RSH and to FNSH. The coefficient μ_{lm} is given by (1.23). $\mathrm{Re}(f_{lm})$ and $\mathrm{Im}(f_{lm})$ are the real and imaginary part of the CSH coefficient f_{lm}, respectively.

from	to	old coeff.	see eq.	new coeff.		see eq.
RSH	CSH	c_{lm} s_{lm}	(1.61)	$f_{lm} =$	$(c_{lm} - \iota s_{lm})/(2-\delta_{0m})\mu_{lm}$	(1.47)
RSH	FNSH	c_{lm} s_{lm}	(1.61)	$\bar{c}_{lm} =$ $\bar{s}_{lm} =$	$h_{lm}c_{lm}$ $h_{lm}s_{lm}$	(1.65)

Table 1.5: RSH conversion table

Conversion table for RSH to CSH and for RSH to FNSH. The coefficients h_{lm} and μ_{lm} are defined by (1.72) and by (1.23), respectively. The symbol ι denotes $\sqrt{-1}$.

from	to	old coeff.	see eq.	new coeff.		see eq.
FNSH	CSH	$\bar{c}_{lm}$ $\bar{s}_{lm}$	(1.65)	$\mathrm{Re}(f_{lm}) =$ $\mathrm{Im}(f_{lm}) =$	$\bar{c}_{lm}[4\pi/(2-\delta_{0m})]^{1/2}$ $-\bar{s}_{lm}[4\pi/(2-\delta_{0m})]^{1/2}$	(1.47)
FNSH	RSH	$\bar{c}_{lm}$ $\bar{s}_{lm}$	(1.65)	$c_{lm} =$ $s_{lm} =$	$\bar{c}_{lm}/h_{lm}$ $\bar{s}_{lm}/h_{lm}$	(1.61)

Table 1.6: FNSH conversion table

Conversion table for FNSH to CSH and to RSH. The coefficients μ_{lm} and h_{lm} are given by (1.23) and (1.72). $\mathrm{Re}(f_{lm})$ and $\mathrm{Im}(f_{lm})$ denote the real and imaginary part of the coefficient f_{lm}, respectively.

from	to	old coeff.	see eq.	new coeff.		see eq.
LEG	CSH	g_l	(1.74)	$f_{lm} =$	$g_l\delta_{m0}/\mu_{lm}$	(1.47)
LEG	RSH	g_l	(1.74)	$c_{lm} =$ $s_{lm} =$	$g_l\delta_{m0}$ 0	(1.61)
LEG	FNSH	g_l	(1.74)	$\bar{c}_{lm} =$ $\bar{s}_{lm} =$	$g_l h_{lm}\delta_{m0}$ 0	(1.65)

Table 1.7: LEG conversion table

Conversion table for LEG to CSH, LEG to RSH, and LEG to FNSH. The coefficients h_{lm} and μ_{lm} are defined by (1.72) and by (1.23), respectively.

1.5 Ocean function

The *ocean function* is defined as follows:

$$\mathcal{O}(\theta,\lambda) = \begin{cases} 1 & \text{if} \quad (\theta,\lambda) \in \text{ Oceans} \\ 0 & \text{if} \quad (\theta,\lambda) \in \text{ Land,} \end{cases} \tag{1.80}$$

where θ and λ colatitude and longitude, respectively.

Proposition 10 *The coefficients of the CSH ocean function expansion*

$$\mathcal{O}(\theta,\lambda) = \sum_{lm} o_{lm} Y_{lm}(\theta,\lambda) \tag{1.81}$$

are:

$$o_{lm} = \int_{\Omega \in \text{Oceans}} Y^*_{lm} d\Omega. \tag{1.82}$$

Proof.

$$\begin{aligned} o_{lm} &\equiv \text{(1.50)} = \int_{\Omega} \mathcal{O} Y^*_{lm} d\Omega \\ &= \text{(1.80)} = \int_{\Omega \in \text{Oceans}} Y^*_{lm} d\Omega \quad \bullet \end{aligned} \tag{1.83}$$

Proposition 11 *The RSH coefficients of the ocean function are:*

$$\left\{ \begin{array}{c} c^{\mathcal{O}}_{lm} \\ s^{\mathcal{O}}_{lm} \end{array} \right\} = (2 - \delta_{0m}) \mu_{lm} \left\{ \begin{array}{cc} & \mathrm{Re}(o_{lm}) \\ - & \mathrm{Im}(o_{lm}) \end{array} \right\}. \tag{1.84}$$

Proof. This is a direct consequence of (1.63) •

Proposition 12 *The area of the surface of the oceans is*

$$A_{oc} = \sqrt{4\pi} a^2 o_{00} = 4\pi a^2 c^{\mathcal{O}}_{00}, \tag{1.85}$$

where a is the radius of the Earth.

Proof. The area of the surface of the oceans is

$$A_{oc} = \int_{\Omega \in \text{Oceans}} dA, \tag{1.86}$$

where

$$dA = a^2 d\Omega \tag{1.87}$$

is the element of area, with

$$d\Omega = \sin\theta d\theta d\lambda \tag{1.88}$$

(see also 1.25). Hence:

$$\begin{aligned} A_{oc} &= a^2 \int_{\Omega \in \text{Oceans}} d\Omega \\ &= \sqrt{4\pi} a^2 \int_{\Omega \in \text{Oceans}} \frac{1}{\sqrt{4\pi}} d\Omega \\ &= \text{(table 1.1)} = \sqrt{4\pi} a^2 \int_{\Omega \in \text{Oceans}} Y^*_{00} d\Omega \\ &= \text{(1.82)} = \sqrt{4\pi} a^2 o_{00}. \end{aligned} \tag{1.89}$$

The right equality in (1.85) follows from the first of (1.84) •

1.5.1 Ocean function low-degree RSH coefficients

Below we give the double precision numerical value of some low-degree RSH coefficients of the ocean function defined by (1.80). An expansion to harmonic degree 128 is contained in the file oceano.128 in the `TABOO` package.

l	m	$c^{\mathcal{O}}_{lm}$	E	$s^{\mathcal{O}}_{lm}$	E
0	0	0.71311686	0	0.00000000	0
1	0	-.19514428	0	0.00000000	0
1	1	0.18556270	0	0.10244521	0
2	0	-.12529769	0	0.00000000	0
2	1	0.51594067	-1	0.79778124	-1
2	2	0.24431601	-1	-.62126902	-3
3	0	0.14005141	0	0.00000000	0
3	1	-.48729831	-1	0.43897959	-1
3	2	0.21701498	-1	-.30707776	-1
3	3	0.15241890	-2	0.11474542	-1
4	0	-.83171371	-1	0.00000000	0
4	1	-.35013115	-1	-.22466979	-1
4	2	0.19243888	-1	-.53146845	-2
4	3	0.29394751	-2	-.30391754	-3
4	4	0.34639739	-3	-.21057688	-2
5	0	0.34944621	0	0.00000000	0
5	1	0.26141714	-2	-.10522861	-1
5	2	0.77676002	-2	0.42565974	-2
5	3	0.10048663	-2	0.39066894	-3
5	4	-.72146556	-3	0.23722730	-3
5	5	0.10315201	-5	0.11962016	-3

Table 1.8: Ocean function
Table of the RSH coefficients $c^{\mathcal{O}}_{lm}$ and $s^{\mathcal{O}}_{lm}$ of the ocean function $\mathcal{O}(\theta, \lambda)$ for degrees and orders ≤ 5. The coefficients are in the form $a \cdot 10^E$.

1.6 Time and Laplace domains

We introduce three basic time–dependent functions (the Dirac delta, the Heaviside function, and the (multi)exponential function) and subsequently we review the basic properties of the Laplace transforms.

1.6.1 Time histories

The Dirac delta

The Dirac delta $\delta(t)$ is actually a distribution, defined by its integral property

$$f(t') = \int_{-\infty}^{+\infty} \delta(t - t') f(t) dt, \tag{1.90}$$

where $f(t)$ is any continuous function of time.

The Heaviside function

This is also known as *step function*:

$$H(t) = \begin{cases} 1 & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0, \end{cases} \tag{1.91}$$

with derivative

$$\delta(t) = \frac{dH(t)}{dt}, \tag{1.92}$$

where $\delta(t)$ is the Dirac delta (see 1.90).

The multi–exponential function

We define the *multi–exponential* function as

$$\mathrm{mexp}(t) = \delta(t) f_o + \sum_{i=1}^{N} e^{s_i t} f_i, \tag{1.93}$$

where f_o, f_i, and s_i are real constants ($s_i < 0$), and N is an integer. The response of a viscoelastic, incompressible, self–gravitating, spherically symmetric Earth model to a δ–like forcing is of multi–exponential type, as shown in §4.2.2.

1.6.2 Laplace transforms

Laplace transforms are useful to compute the response of a viscoelastic Earth to applied loads, as explained in §4.1.1. We only recall the basic facts.

Definition

Given a function of time $f(t)$ defined for $t \geq 0$, its Laplace transform (LT) is

$$f(s) \equiv \int_0^\infty e^{st} f(t)dt, \tag{1.94}$$

where the complex variable s is the *Laplace variable*. It is assumed that the integral (1.94) exists so that $f(s)$ is well defined. We will also use the notation

$$f(s) = \mathrm{LT}[f(t)] \tag{1.95}$$

to indicate that $f(s)$ is the LT of $f(t)$ and

$$f(t) = \mathrm{LT}^{-1}[f(s)] \tag{1.96}$$

to say that $f(t)$ is the *inverse* LT of $f(s)$.

LT of a derivative

Using the definition (1.94) and integrating by parts it is straightforward to show that:

$$\mathrm{LT}[f'(t)] = s\mathrm{LT}[f(t)] - f(0). \tag{1.97}$$

where $f'(t) = \dfrac{df(t)}{dt}$.

LT transforms of simple functions

From (1.94) we can simply obtain the LT transforms of the elementary functions introduced above:

1.6.3 Time convolution

Definition

Given two functions of time $f(t)$ and $g(t)$, their *convolution product* is:

$$c(t) \equiv \int_{-\infty}^{t} f(t-t')g(t')dt', \tag{1.98}$$

which we also denote by:

$$c(t) = f(t) \otimes g(t). \tag{1.99}$$

$f(t)$	$\mathrm{LT}[f(t)]$
$\delta(t)$	1
$H(t)$	$\frac{1}{s}$
$e^{\alpha t}, \alpha < 0$	$\frac{1}{s-\alpha}$
$\mathrm{mexp}(t)$	$f_0 + \sum_{i=1}^{N} \frac{f_i}{s-s_i}$

Table 1.9: Elementary Laplace transforms.

A property of the convolution product

Given two functions $f(s) = \mathrm{LT}[f(t)]$ and $g(s) = \mathrm{LT}[g(t)]$, *"it can be shown that"*:

$$\mathrm{LT}^{-1}[f(s)g(s)] = f(t) \otimes g(t), \tag{1.100}$$

i.e., the inverse Laplace transform of the product $f(s)g(s)$ is the convolution product of the original functions (see e. g. [11]).

Chapter 2

Displacement and Gravity

This Chapter is devoted to the study of the two relevant geophysical quantities, i.e., the displacement field and the variations of the gravity potential resulting from forces which perturb the equilibrium of the Earth. The reader is referred to the literature for a broader and self–contained discussion.

2.1 Toroidal–Poloidal decomposition of displacement

The displacement field is defined as

$$\vec{u} = \vec{r}(t) - \vec{r}_o, \tag{2.1}$$

where $\vec{r}(t)$ is the position of a particle of continuum at time t, and $\vec{r}_o$ is its position in a given reference state.

For most applications, we can assume that the Earth is a perfectly incompressible body[1]. If the Earth equilibrium is perturbed in some way, the resulting displacement field may be thus regarded as a *solenoidal* (i. e., divergence–free) field:

$$\nabla \cdot \vec{u} = 0. \tag{2.2}$$

[1]The current version of `TABOO` (1.0) is fully based on this assumption.

Proposition 13 *If the vector field $\vec{u}$ is solenoidal, there are unique scalar fields $T(r,\theta,\lambda)$ and $P(r,\theta,\lambda)$ with zero average on the surface of the sphere such that*

$$\vec{u} = \vec{u}^t + \vec{u}^p = \nabla \times \hat{e}_r T + \nabla \times (\nabla \times \hat{e}_r P), \tag{2.3}$$

where $T = T(\vec{r})$ and $P(\vec{r})$ are the toroidal and poloidal scalars, and $\vec{u}^t$ and $\vec{u}^p$ are the toroidal and poloidal parts of $\vec{u}$, respectively. The expression (2.3) is known as "Mie representation" of the solenoidal vector $\vec{u}$ [2].

Proof. The Mie representation of a solenoidal vector field (2.3) derives from the Helmholtz representation of a tangent vector. Details are given in [2] •

2.1.1 CSH expansion of the displacement field

Our purpose in this section is to write the general expansion of the (solenoidal) displacement field on the CSH basis. The starting point is the expansion of the toroidal and the poloidal scalars:

$$\left\{ \begin{array}{c} T \\ P \end{array} \right\} (\vec{r}) = \Sigma_{lm} \left\{ \begin{array}{c} t_{lm}(r) \\ p_{lm}(r) \end{array} \right\} Y_{lm}(\theta,\lambda), \tag{2.4}$$

where, according to proposition 13 and (1.57):

$$t_{00}(r) = p_{00}(r) = 0. \tag{2.5}$$

Proposition 14 *The components of the (solenoidal) displacement field $\vec{u}$ can be expanded as follows:*

$$\left\{ \begin{array}{rcl} u_r(\vec{r}) & = & \Sigma_{lm} u^{(1)}_{lm}(r) Y_{lm} \\ u_\theta(\vec{r}) & = & \Sigma_{lm} \left[+ u^{(2)}_{lm}(r) \partial_\theta Y_{lm} + \dfrac{v^{(1)}_{lm}(r)}{\sin\theta} \partial_\lambda Y_{lm} \right] \\ u_\lambda(\vec{r}) & = & \Sigma_{lm} \left[- v^{(1)}_{lm}(r) \partial_\theta Y_{lm} + \dfrac{u^{(2)}_{lm}(r)}{\sin\theta} \partial_\lambda Y_{lm} \right] \end{array} \right. \tag{2.6}$$

with:

$$u^{(1)}_{lm}(r) = \frac{l(l+1)}{r^2} p_{lm}, \quad u^{(2)}_{lm}(r) = \frac{1}{r}\frac{dp_{lm}}{dr}, \quad v^{(1)}_{lm}(r) = \frac{t_{lm}}{r}, \tag{2.7}$$

where t_{lm} and p_{lm} are the CSH coefficients of the toroidal and poloidal scalars, respectively (see 2.4). We observe that, due to (2.5), the degree 0 coefficients of (2.6) vanish identically. This result, which is valid for solenoidal fields, is also demonstrated in [10].

Proof. We use the definition of curl (1.15) with (2.3) and simple algebra:

$$u_r^t = 0, \tag{2.8}$$

$$u_\theta^t = = \frac{1}{r\sin\theta}\partial_\lambda T = \frac{1}{r\sin\theta}\Sigma_{lm} t_{lm}\partial_\lambda Y_{lm}, \tag{2.9}$$

$$u_\lambda^t = -\frac{1}{r}\partial_\theta T = -\frac{1}{r}\Sigma_{lm} t_{lm}\partial_\theta Y_{lm}, \tag{2.10}$$

$$u_r^p = -\frac{1}{r^2}\nabla_h^2 P = -\frac{1}{r^2}\Sigma_{lm} p_{lm}\nabla_h^2 Y_{lm} = \frac{1}{r^2}\Sigma_{lm} l(l+1) p_{lm} Y_{lm}, \tag{2.11}$$

$$u_\theta^p = \frac{1}{r}\partial_{r\theta}^2 P = \frac{1}{r}\Sigma_{lm}\frac{dp_{lm}}{dr}\partial_\theta Y_{lm}, \tag{2.12}$$

$$u_\lambda^p = \frac{1}{r\sin\theta}\partial_{r\lambda}^2 P = \frac{1}{r\sin\theta}\Sigma_{lm}\frac{dp_{lm}}{dr}\partial_\lambda Y_{lm}, \tag{2.13}$$

which can be summarized as follows:

$$u_r = u_r^t + u_r^p = \Sigma_{lm}\frac{l(l+1)}{r^2} p_{lm} Y_{lm} \tag{2.14}$$

$$u_\theta = u_\theta^t + u_\theta^p = \frac{1}{\sin\theta}\Sigma_{lm}\frac{t_{lm}}{r}\partial_\lambda Y_{lm} + \Sigma_{lm}\frac{1}{r}\frac{dp_{lm}}{dr}\partial_\theta Y_{lm} \tag{2.15}$$

$$u_\lambda = u_\lambda^t + u_\lambda^p = -\Sigma_{lm}\frac{t_{lm}}{r}\partial_\theta Y_{lm} + \frac{1}{\sin\theta}\Sigma_{lm}\frac{1}{r}\frac{dp_{lm}}{dr}\partial_\lambda Y_{lm}. \tag{2.16}$$

With the definitions (2.7) the expansions (2.6) are thus demonstrated. The radial functions $u_{lm}^{(1)}(r)$ and $u_{lm}^{(2)}(r)$, related to p_{lm}, have a poloidal nature, whereas $v_{lm}^{(1)}$ has a toroidal character being related to t_{lm} (see 2.7) •

2.2 The gravity field

In this section we first consider the gravity field of a non–rotating body, without making assumptions on its internal density distribution. The Stokes coefficients are defined in §2.2.4. The inertia tensor and the position of the center of mass of the body, introduced in §2.2.2 and 2.2.3 are related with the low–degree Stokes coefficients of the gravity field, as shown in §2.2.5. In §2.2.6, 2.2.7 and 2.2.8 the general concepts previously outlined are used to describe the gravity field within the assumptions of `TABOO`. In particular, we define the potential perturbation and the geoid height change in response to perturbing forces acting at the Earth surface, and we show how these quantities are related to variations in the Stokes coefficients and of the inertia tensor.

2.2.1 Gravity and gravity potential

We consider an arbitrarily shaped body B of finite extent and we denote by $\vec{r'}$ the position of a mass element dm of B in a Cartesian reference frame Oxyz. By Newton's Law of gravitation, the gravity potential at an external point P with position $\vec{r}$ is:

$$dU(\vec{r}) = G\frac{dm}{\|\vec{r'} - \vec{r}\|}, \tag{2.17}$$

with

$$dm = \rho(\vec{r'})dV, \tag{2.18}$$

where $\rho(\vec{r'})$ is the density at $\vec{r'}$ and

$$dV = r'^2 dr' \sin\theta' d\theta' d\lambda' \tag{2.19}$$

is the volume element in spherical geometry.

The potential of the gravity field due to the whole mass distribution can be obtained by integration of (2.17) over B:

$$U(\vec{r}) = G\int_B \frac{dm}{\|\vec{r'} - \vec{r}\|}, \tag{2.20}$$

and is related to the gravity field by

$$\vec{g}(\vec{r}) = \overrightarrow{\nabla U} = \hat{e}_r \partial_r U + \frac{1}{r}\overrightarrow{\nabla_h U} \tag{2.21}$$

where the surface gradient operator ∇_h is given by (1.11).

An *equipotential (EP) surface* is a surface on which U takes a constant value:

$$U(\vec{r}) = c, \tag{2.22}$$

and that particular EP surface corresponding to the free surface of the oceans in the absence of winds and currents is called *geoid*.

In the special case of a body characterized by a radial density distribution:

$$\rho(\vec{r}') = \rho(r'), \quad \text{(radial density distribution)} \tag{2.23}$$

the gravity potential can be obtained explicitely from (2.20):

$$U(r) = \frac{GM}{r}, \tag{2.24}$$

where r is the distance from the center of the body. The result (2.24) shows that, for a body with radial density distribution, the gravity potential is the same as if the whole mass of the body were concentrated in its center (see 2.17). The EP surfaces have a spherical shape, and from (2.21) the external gravity field is directed along the radial direction:

$$\vec{g}(r) = -\hat{e}_r \frac{GM}{r^2}. \tag{2.25}$$

2.2.2 Inertia tensor

The elements of the symmetric inertia tensor are defined as follows:

$$i_{xx} = \int_B (y'^2 + z'^2) dm \tag{2.26}$$

$$i_{yy} = \int_B (z'^2 + x'^2) dm \tag{2.27}$$

$$i_{zz} = \int_B (x'^2 + y'^2) dm \tag{2.28}$$

$$i_{xy} \equiv i_{yx} = -\int_B x'y' dm \tag{2.29}$$

$$i_{xz} \equiv i_{zx} = -\int_B x'z' dm \tag{2.30}$$

$$i_{yz} \equiv i_{zy} = -\int_B y'z' dm, \tag{2.31}$$

where the integrals are over the volume of the body, and dm is the mass element with coordinates (x', y', z') in a Cartesian reference frame Oxyz. The trace of the inertia tensor is:

$$\begin{aligned} \mathrm{Tr}(I) &\equiv i_{xx} + i_{yy} + i_{zz} \\ &= {\scriptstyle (2.26-2.28)} = \quad 2 \int_B (x'^2 + y'^2 + z'^2) dm \\ &= {\scriptstyle (1.2)} = 2 \int_B r'^2 dm, \end{aligned} \tag{2.32}$$

2.2.3 Center of mass

The center of mass (CM) is defined as:

$$\vec{R}_{cm} = \frac{\int_B \vec{r}' dm}{\int_B dm}, \tag{2.33}$$

where dm is the mass element with position vector $\vec{r}'$. The Cartesian components of $\vec{R}_{cm}$ are:

$$\begin{aligned} \begin{bmatrix} x_{cm} \\ y_{cm} \\ z_{cm} \end{bmatrix} &= \frac{1}{M} \int_B \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} dm \\ &= {\scriptstyle (1.1)} = \frac{1}{M} \int_B \begin{bmatrix} r' \sin\theta' \cos\lambda' \\ r' \sin\theta' \sin\lambda' \\ r' \cos\theta' \end{bmatrix} dm, \end{aligned} \tag{2.34}$$

where

$$M = \int_B dm \tag{2.35}$$

is the mass of the body B.

2.2.4 Stokes coefficients

Proposition 15 *The gravity potential external to a body B can be expressed by a multipole expansion:*

$$U(\vec{r}) = \sum_{l=0}^{\infty} \frac{G}{r} \int_B \left(\frac{r'}{r}\right)^l P_l(\cos\beta) dm, \tag{2.36}$$

where β is the angle between $\vec{r}$ and $\vec{r}'$, such that

$$\cos\beta = \frac{\vec{r} \cdot \vec{r}'}{rr'}, \tag{2.37}$$

where $r = \|\vec{r}\|$ and $r' = \|\vec{r}'\|$.

Proof. By the law of cosines:

$$\|\vec{r}' - \vec{r}\| = r\sqrt{1 - 2\left(\frac{r'}{r}\right)\cos\beta + \left(\frac{r'}{r}\right)^2}, \tag{2.38}$$

hence

$$\begin{aligned} U(\vec{r}) &= (2.20) = \frac{G}{r}\int_B \frac{dm}{\sqrt{1-2\left(\frac{r'}{r}\right)\cos\beta+\left(\frac{r'}{r}\right)^2}}, \\ &= (1.43) = \sum_{l=0}^{\infty}\frac{G}{r}\int_B\left(\frac{r'}{r}\right)^l P_l(\cos\beta)dm \quad \bullet \end{aligned} \tag{2.39}$$

Proposition 16 *The potential of the gravity field external to a body B can be expanded in series of RSH as follows:*

$$U(\vec{r}) = \frac{GM}{r}\Sigma'_{lm}\left(\frac{a}{r}\right)^l (c_{lm}\cos m\lambda + s_{lm}\sin m\lambda)P_{lm}(\cos\theta), \tag{2.40}$$

where

$$\left\{\begin{array}{c} c_{lm} \\ s_{lm} \end{array}\right\} = (2-\delta_{0m})\frac{(l-m)!}{(l+m)!}\frac{1}{M}\int_B\left(\frac{r'}{a}\right)^l P_{lm}(\cos\theta')\left\{\begin{array}{c}\cos m\lambda' \\ \sin m\lambda'\end{array}\right\}dm, \tag{2.41}$$

are referred as to cosine and sine Stokes coefficients, respectively, "a" is a conventionally chosen reference radius (the average radius is a suitable choice for quasi–spherical bodies), and M is the mass of the body.

Proof.

$$\begin{aligned} U(\vec{r}) &= (2.36) = \sum_{l=0}^{\infty}\frac{G}{r}\int_B\left(\frac{r'}{r}\right)^l P_l(\cos\beta)dm \\ &= (1.29) = \sum_{l=0}^{\infty}\frac{G}{r}\int_B\left(\frac{r'}{r}\right)^l\frac{4\pi}{2l+1}\sum_{m=-l}^{+l}Y^*_{lm}(\theta',\lambda')Y_{lm}(\theta,\lambda)dm \\ &= \frac{GM}{r}\Sigma_{lm}\left(\frac{a}{r}\right)^l\Lambda_{lm}Y_{lm}(\theta,\lambda), \end{aligned} \tag{2.42}$$

with coefficients

$$\Lambda_{lm} = \frac{1}{M}\frac{4\pi}{2l+1}\int_B\left(\frac{r'}{a}\right)^l Y^*_{lm}(\theta',\lambda')dm. \tag{2.43}$$

According to (1.63), (2.42) can be converted into the equivalent RSH form

$$U(\vec{r}) = \frac{GM}{r}\Sigma'_{lm}\left(\frac{a}{r}\right)^l (c_{lm}\cos m\lambda + s_{lm}\sin m\lambda)P_{lm}(\cos\theta), \tag{2.44}$$

with

$$\left\{\begin{array}{c} c_{lm} \\ s_{lm} \end{array}\right\} = (2-\delta_{0m})\mu_{lm}\left\{\begin{array}{c} \mathrm{Re}(\Lambda_{lm}) \\ -\ \mathrm{Im}(\Lambda_{lm}) \end{array}\right\}$$

$$= (2-\delta_{0m})\mu_{lm}^2\frac{1}{M}\frac{4\pi}{2l+1}\int_B\left(\frac{r'}{a}\right)^l\left\{\begin{array}{c} \cos m\lambda' \\ \sin m\lambda' \end{array}\right\}P_{lm}(\cos\theta)dm$$

$$= (2-\delta_{0m})\frac{(l-m)!}{(l+m)!}\frac{1}{M}\int_B\left(\frac{r'}{a}\right)^l P_{lm}(\cos\theta')\left\{\begin{array}{c} \cos m\lambda' \\ \sin m\lambda' \end{array}\right\}dm \bullet \tag{2.45}$$

Proposition 17 *For a body with arbitrary density distribution, the zonal Stokes sine coefficients vanish identically:*

$$s_{l0} = 0. \tag{2.46}$$

Proof. This result derives from (2.41) observing that $\sin m\lambda' = 0$ for $m = 0$.

Proposition 18 *For a body with arbitrary density distribution, the degree zero cosine Stokes coefficient is*

$$c_{00} = 1. \tag{2.47}$$

Proof.

$$c_{00} = \text{(2.41)} = \frac{1}{M}\int_B P_{00}(\cos\theta')dm$$

$$= \text{(table 1.2)} = \frac{1}{M}\int_B dm$$

$$= \text{(2.35)} = 1 \bullet \tag{2.48}$$

Proposition 19 *The mass of a spherical body of radius "a" characterized by a radial density distribution (see 2.23) is:*

$$M = 4\pi\int_0^a dr'\rho(r')r'^2, \tag{2.49}$$

and the only non vanishing Stokes coefficient of the body is

$$c_{00} = 1. \tag{2.50}$$

Proof.

$$
\begin{aligned}
M &= \text{(2.35, 2.23, 2.18)} = \int_B \rho(r')dV \\
&= \text{(2.19)} = \int_0^a \int_0^{2\pi} \int_0^{\pi} \rho(r')r'^2 dr' \sin\theta' d\theta' d\lambda' = \\
&= 4\pi \int_0^a dr' \rho(r') r'^2 \quad \bullet
\end{aligned}
\tag{2.51}
$$

To prove the second part of the proposition, it suffices to recall from (2.24) that in the case of radial density distribution:

$$
U(r) = \frac{GM}{r}, \tag{2.52}
$$

which can be cast in the RSH form (2.40), with $c_{00} = 1$, $s_{00} = 0$, and $c_{lm} = s_{lm} = 0$ for $l \geq 1$ $\bullet$

2.2.5 Low–degree Stokes coefficients

The low–degree Stokes coefficients ($l = 0, 1, 2$) have a special physical meaning, which is discussed in the following.

Proposition 20 *For a body with arbitrary density distribution, the degree* 0 *Stokes coefficients are:*

$$
\begin{cases} c_{00} = 1 \\ s_{00} = 0. \end{cases} \tag{2.53}
$$

Proof. The first is a repetition of propositions 18, while the second is obtained from proposition 17 with $l = 0$ $\bullet$

Proposition 21 *For a body with arbitrary density distribution, the degree 1 Stokes coefficients are:*

$$
\begin{Bmatrix} c_{11} \\ s_{11} \\ c_{10} \end{Bmatrix} = \frac{1}{a} \begin{Bmatrix} x_{cm} \\ y_{cm} \\ z_{cm} \end{Bmatrix}, \tag{2.54}
$$

where x_{cm}, y_{cm}, and z_{cm} are the Cartesian coordinates of the CM (2.34). A direct consequence is that the degree 1 *Stokes coefficients vanish identically if the origin of the Cartesian reference frame coincides with the CM of the body.*

Proof. Once again, the relationships (2.54) are a direct consequence of the definition (2.41). In detail, we have:

$$\begin{aligned} c_{11} &= \text{(2.41)} = \frac{1}{M}\int_B \frac{r'}{a} P_{11}(\cos\theta')\cos\lambda' dm \\ &= \text{(table 1.2)} = \frac{1}{a}\frac{1}{M}\int_B r'\sin\theta'\cos\lambda' dm \\ &= \text{(1.1)} = \frac{1}{a}\frac{1}{M}\int_B x' dm \\ &= \text{(2.34)} = \frac{x_{cm}}{a} \quad \bullet \end{aligned} \tag{2.55}$$

$$\begin{aligned} s_{11} &= \text{(2.41)} = \frac{1}{M}\int_B \frac{r'}{a} P_{11}(\cos\theta')\sin\lambda' dm \\ &= \text{(table 1.2)} = \frac{1}{a}\frac{1}{M}\int_B r'\sin\theta'\sin\lambda' dm \\ &= \text{(1.1)} = \frac{1}{a}\frac{1}{M}\int_B y' dm \\ &= \text{(2.34)} = \frac{y_{cm}}{a} \quad \bullet \end{aligned} \tag{2.56}$$

$$\begin{aligned} c_{10} &= \text{(2.41)} = \frac{1}{M}\int_B \frac{r'}{a} P_{10}(\cos\theta') dm \\ &= \text{(table 1.2)} = \frac{1}{a}\frac{1}{M}\int_B r'\cos\theta' dm \\ &= \text{(1.1)} = \frac{1}{a}\frac{1}{M}\int_B z' dm \\ &= \text{(2.34)} = \frac{z_{cm}}{a} \quad \bullet \end{aligned} \tag{2.57}$$

Proposition 22 *For a body with arbitrary density distribution, the Stokes coefficients of harmonic degree 2 can be expressed as linear combinations of the elements of the inertia tensor:*

$$\left\{\begin{array}{c} c_{20} \\ c_{21} \\ c_{22} \end{array}\right\} = \frac{1}{Ma^2}\left\{\begin{array}{c} -[i_{zz} - (i_{xx} + i_{yy})/2] \\ i_{xz} \\ -(i_{xx} - i_{yy})/4 \end{array}\right\}, \tag{2.58}$$

$$\left\{\begin{array}{c} s_{20} \\ s_{21} \\ s_{22} \end{array}\right\} = \frac{1}{Ma^2}\left\{\begin{array}{c} 0 \\ i_{yz} \\ -i_{xy}/2 \end{array}\right\}, \tag{2.59}$$

where i_{kl} *is given by (2.26-2.31).*

Proof. The demonstration of (2.58) and (2.59) is straightforward but somewhat cumbersome. The details are given in the following.

c_{20} **Stokes coefficient.**

$$
\begin{aligned}
c_{20} &= \text{(2.41)} = \frac{1}{M}\int_B \left(\frac{r'}{a}\right)^2 P_{20}(\cos\theta')dm \\
&= \text{(table 1.2)} = \frac{1}{2Ma^2}\int_B r'^2(3\cos^2\theta' - 1)dm \\
&= \text{(1.1)} = \frac{1}{2Ma^2}\int_B (3z'^2 - r'^2)dm. \qquad (2.60)
\end{aligned}
$$

The last integral can be transformed observing that

$$
\begin{aligned}
i_{xx} + i_{yy} &= \text{(2.26, 2.27)} = \int_B (x'^2 + y'^2 + 2z'^2)dm \\
&= \text{(2.28)} = i_{zz} + 2\int_B z'^2 dm, \qquad (2.61)
\end{aligned}
$$

hence

$$
\int_B z'^2 dm = \frac{i_{xx} + i_{yy} - i_{zz}}{2}, \qquad (2.62)
$$

and from (2.32):

$$
\int_B r'^2 dm = \frac{i_{xx} + i_{yy} + i_{zz}}{2}. \qquad (2.63)
$$

We therefore obtain

$$
\begin{aligned}
c_{20} &= \text{(2.60)} = \frac{1}{2Ma^2}\left[\frac{3}{2}(i_{xx} + i_{yy} - i_{zz}) - \frac{1}{2}(i_{xx} + i_{yy} + i_{zz})\right] \\
&= \frac{1}{Ma^2}\left[i_{zz} - \frac{i_{xx} + i_{yy}}{2}\right] \quad \bullet \qquad (2.64)
\end{aligned}
$$

c_{21} **Stokes coefficient.**

$$
\begin{aligned}
c_{21} &= \text{(2.41)} = \frac{1}{3}\frac{1}{M}\int_B \left(\frac{r'}{a}\right)^2 P_{21}(\cos\theta')\cos\lambda' dm \\
&= \text{(table 1.2)} = \frac{1}{3}\frac{1}{Ma^2}\int_B r'^2(-3\cos\theta'\sin\theta')\cos\lambda' dm \\
&= -\frac{1}{Ma^2}\int_B (r'\sin\theta'\cos\lambda')(r'\cos\theta')dm \\
&= \text{(1.1)} = -\frac{1}{Ma^2}\int_B x'z'dm \\
&= \text{(2.30)} = \frac{i_{xz}}{Ma^2} \quad \bullet
\end{aligned}
\tag{2.65}
$$

c_{22} **Stokes coefficient.**

$$
\begin{aligned}
c_{22} &= \text{(2.41)} = \frac{1}{12}\frac{1}{M}\int_B \left(\frac{r'}{a}\right)^2 P_{22}(\cos\theta')\cos 2\lambda' dm \\
&= \text{(table 1.2)} = \frac{1}{12}\frac{1}{Ma^2}\int_B r'^2(3\sin^2\theta')(\cos^2\lambda' - \sin^2\lambda')dm \\
&= \frac{1}{4Ma^2}\int_B (r'^2\sin^2\theta'\cos^2\lambda' - r'^2\sin^2\theta'\sin^2\lambda')dm \\
&= \text{(1.1)} = \frac{1}{4Ma^2}\int_B (x'^2 - y'^2)dm \\
&= \text{(2.26, 2.27)} = -\frac{i_{xx} - i_{yy}}{4Ma^2} \quad \bullet
\end{aligned}
\tag{2.66}
$$

s_{21} **Stokes coefficient.**

$$
\begin{aligned}
s_{21} &= \text{(2.41)} = \frac{1}{3}\frac{1}{M}\int_B \left(\frac{r'}{a}\right)^2 P_{21}(\cos\theta')\sin\lambda' dm \\
&= \text{(table 1.2)} = \frac{1}{3}\frac{1}{Ma^2}\int_B r'^2(-3\cos\theta'\sin\theta')\sin\lambda' dm \\
&= -\frac{1}{Ma^2}\int_B (r'\sin\theta'\sin\lambda')(r'\cos\theta')dm \\
&= \text{(1.1)} = -\frac{1}{Ma^2}\int_B y'z'dm \\
&= \text{(2.31)} = \frac{i_{yz}}{Ma^2} \quad \bullet
\end{aligned}
\tag{2.67}
$$

s_{22} Stokes coefficient.

$$\begin{aligned}
s_{22} &= \text{(2.41)} = \frac{1}{12}\frac{1}{M}\int_B \left(\frac{r'}{a}\right)^2 P_{22}(\cos\theta')\sin 2\lambda' dm \\
&= \text{(table 1.2)} = \frac{1}{12}\frac{1}{Ma^2}\int_B r'^2(3\sin^2\theta')(2\sin\lambda'\cos\lambda')dm \\
&= \frac{1}{2Ma^2}\int_B (r'\sin\theta'\cos\lambda')(r'\sin\theta'\sin\lambda')dm \\
&= \text{(1.1)} = \frac{1}{2Ma^2}\int_B x'y'dm \\
&= \text{(2.29)} = \frac{i_{xy}}{2Ma^2} \quad \bullet \qquad (2.68)
\end{aligned}$$

2.2.6 Potential perturbation and geoid height

In the following, we consider two distinct states of the Earth: a *reference* and a *perturbed* state. In the reference state (hereafter referred as to *ref* state), the gravity field is that of a non–rotating, spherically symmetrical body: the solid surface of the Earth has a spherical shape, and the relevant geophysical parameters (density, rigidity, and viscosity) only depend on radius. The *perturbed* state results from the action of forces applied at the surface of the Earth. The spherical symmetry of the *ref* state is lost, but it is assumed that the mass of the Earth is unchanged. Since we are only concerned with surface forces, no lateral density variations at depth are produced in addition to those due to the deformation of the internal boundaries. The perturbing forces may be arranged so that to describe the load due to ice sheets or even the exchange of mass between ice and fresh water reservoirs.[2] Since these *surface loads* correspond to specific imposed force systems that mimic mass conservation, they never imply a creation or the disruption of actual mass on the Earth surface. The Earth mass is always constant to its value in the *ref* state.

[2]However, this procedure *is not* self–consistent, since the oceans water distribution, and hence the ocean loads, should be naturally determined by the changes of the gravity field of the Earth due to deformation, and not imposed as it is done here. The only way to escape to this difficulty is to solve the sealevel equation, according to the theory illustrated by Farrell and Clark [4]. This is done by the numerical code `SELEN`, which will be soon made available by my postglacial rebound group. In `SELEN` we have implemented the finite elements approach to the sealevel equation described by [5].

Since in the *ref* state it is assumed a radial density distribution (2.23), we represent the potential of the Earth gravity field as

$$U^{ref}(r) = \frac{Gm_e}{r}, \tag{2.69}$$

where r is the radius measured with respect to the CM of the Earth, m_e is the mass of the Earth, and G is the Newton constant (see 2.24). Since in the *ref* state any spherical surface is an EP surface, the unperturbed solid surface of the Earth is a particular EP surface before deformation.

According to (2.21), the gravity field in the *ref* state can be expressed as

$$\vec{g}^{ref}(r) = \nabla U^{ref}(r) = -\hat{e}_r \frac{Gm_e}{r^2} \equiv -\hat{e}_r \gamma_o(r) \tag{2.70}$$

where

$$\gamma_o(r) = \frac{Gm_e}{r^2} \tag{2.71}$$

is the modulus of the unperturbed gravity field at a distance r from the CM. The *ref* gravity computed at the unperturbed Earth radius $r = a$ is

$$\gamma_o \equiv \gamma_o(a) = \frac{Gm_e}{a^2}. \tag{2.72}$$

In the perturbed state we assume that the gravity potential and the gravity acceleration slightly depart from their values in the *ref* state. Accordingly, we write:

$$U(t, \vec{r}) = U^{ref}(r) + \Phi(t, \vec{r}) \tag{2.73}$$

and

$$\vec{g}(t, \vec{r}) = \vec{g}^{ref}(r) + \vec{g}'(t, \vec{r}), \tag{2.74}$$

where $\Phi(t, \vec{r})$ and $\vec{g}'(t, \vec{r})$ are the *potential perturbation* and the *gravity perturbation*, respectively, with

$$|\Phi(t, \vec{r})| \ll |U^{ref}(r)| \tag{2.75}$$

$$\|\vec{g}'(t, \vec{r})\| \ll \|\vec{g}^{old}(r)\|, \tag{2.76}$$

and

$$\vec{g}'(t, \vec{r}) = \nabla\Phi(t, \vec{r}). \tag{2.77}$$

Since the incremental potential is not totally negligible in front of the perturbed potential, the body is a *self–gravitating* body. A simply *gravitating* body is one for which $\Phi = 0$, and consequently $\vec{g} = \vec{g}^{ref}$. In our following discussion, we will always deal with self–gravitating bodies.

The potential perturbation $\Phi(t, \vec{r})$ accounts for:

1. forces acting on the solid surface of the Earth, which mimic the effect of continental ice sheets,

2. forces acting on the oceans bottom, which describe modifications of the water load due to the accretion or ablation of the ice sheets of point 1. above,

3. the further change in the gravity potential due to the distortions of the solid Earth under the effect of ice and the water loads.

Accordingly, we write

$$\Phi(t, \vec{r}) = \Phi^r(t, \vec{r}) + \Phi^{def}(t, \vec{r}), \tag{2.78}$$

where Φ^r describes the effects 1. and 2. above[3], and Φ^{def} describes the effect 3. The decomposition (2.78) will be reconsidered in §4.2.1 from another point of view.

It is convenient to transform the potential perturbation into a physical quantity with the same dimensions of a displacement, the *geoid height*:[4]

$$N(t, \theta, \lambda) \equiv \frac{\Phi(t, a, \theta, \lambda)}{\gamma_o}, \tag{2.79}$$

where $\Phi(t, a, \theta, \lambda) = \Phi(t, \vec{r})|_{r=a}$, a is the reference radius of the Earth, and γ_o is the reference gravity (2.72). Since Φ is a small quantity if compared to U^{ref} (see 2.75), we can assume that N is small quantity as well, if compared to the *ref* radius of the Earth:

$$N(t, \theta, \lambda) \ll a. \tag{2.80}$$

[3]The upperscript r stands for *rigid*, since Φ^r can be computed as the Earth was rigid given that this term is only dependent on the load.

[4]The use of the term *geoid height* is conventional. As stated in the text, the geoid is in fact that particular equipotential surface corresponding to the free surface of the oceans. The lack of gravitationally self–consistent oceans in `TABOO` makes impossible a rigorous implementation of the definition of geoid.

As any scalar field, the potential perturbation evaluated at the undeformed surface of the solid Earth can be expanded in series of CSH functions (see 1.47):

$$\Phi(t, a, \theta, \lambda) = \Sigma_{lm} \Phi_{lm}(t, a) Y_{lm}(\theta, \lambda), \tag{2.81}$$

so that the CSH expansion of the geoid height is

$$N(t, \theta, \lambda) = \Sigma_{lm} n_{lm}(t) Y_{lm}(\theta, \lambda), \tag{2.82}$$

with harmonic coefficients

$$n_{lm}(t) = \frac{\Phi_{lm}(t, a)}{\gamma_o}. \tag{2.83}$$

Proposition 23 *The geoid height N is such that*

$$U(a + N) = U^{ref}(a), \tag{2.84}$$

where $U(a + N) = U(t, \vec{r})|_{r=a+N}$. In words, the perturbed gravity potential computed on the surface $r = a + N$ equals the old potential computed on the unperturbed surface $r = a$. Since $r = a$ is an EP surface

$$U^{ref}(a) = c, \tag{2.85}$$

(see 2.69), it follows from (2.84) that $r = a + N$ is also an EP surface, corresponding to the same constant c. Notice that while $r = a$ is a solid surface, $r = a + N$ generally is not a solid surface.

Proof. To first order in N, we have:

$$U(a + N) \simeq U(a) + N \frac{\partial U}{\partial r}\bigg|_{r=a}, \tag{2.86}$$

where

$$\begin{aligned} N \frac{\partial U}{\partial r}\bigg|_{r=a} &= \text{(2.21)} = \quad N \hat{e}_r \cdot \vec{g}(t, a, \theta, \lambda) \\ &= \text{(2.74)} = \quad N \hat{e}_r \cdot (\vec{g}^{ref}(a) + \vec{g}'(t, a, \theta, \lambda)) \\ &\simeq \text{(2.70)} \simeq \quad -N \gamma_o, \end{aligned} \tag{2.87}$$

where we have neglected the product of small quantities $N\hat{e}_r \cdot \vec{g}'$ (see 2.80 and 2.76). From (2.86), (2.87), and (2.73) we obtain

$$\begin{aligned} U(a+N) &= U(a) - \gamma_o N \\ &= \Phi(t,a,\theta,\lambda) + U^{ref}(a) - \gamma_o N \\ &= \Phi(t,a,\theta,\lambda) + U^{ref}(a) - \Phi(t,a,\theta,\lambda) \\ &= U^{ref}(a), \end{aligned} \tag{2.88}$$

so that from (2.85) we conclude that the surface $r = a + N$ is an EP in the perturbed state, with $U(a+N) = c$ •

2.2.7 Stokes coefficients variations

Here we compute the variations of the Stokes coefficients and of the inertia tensor with respect to the *ref* state defined in the previous section.

Proposition 24 *In the perturbed state, the geoid height is:*

$$N(t,\theta,\lambda) = a\ \Sigma'_{lm}(\delta c_{lm}(t)\cos m\lambda + \delta s_{lm}(t)\sin m\lambda)P_{lm}(\cos\theta), \tag{2.89}$$

where a is the Earth radius in the ref state, and

$$\left\{ \begin{array}{l} \delta c_{lm}(t) = c_{lm}(t) - c_{lm}^{ref} \\ \delta s_{lm}(t) = s_{lm}(t) - s_{lm}^{ref} \end{array} \right. \tag{2.90}$$

are the variations of the Stokes coefficients with respect to the ref state. Since the mass of the Earth is constant, and it is assumed that the origin of the reference frame coincides with the CM of the Earth both in the ref and in the perturbed state, the only non–vanishing terms in the RSH expansion (2.89) are those with degree $l \geq 2$.

Proof. The potential of the gravity field in the spherically symmetric *ref* state is

$$U^{ref}(r) = \frac{Gm_e}{r}, \tag{2.91}$$

where m_e is the Earth mass in the *ref* state, and we assume that the origin of the reference system coincides with the CM (see 2.69). The reference potential (2.91) can be formally expanded as

$$U^{ref}(r) = \frac{Gm_e}{r}\Sigma'_{lm}\left(\frac{a}{r}\right)^l (c_{lm}^{ref}\cos m\lambda + s_{lm}^{ref}\sin m\lambda)P_{lm}(\cos\theta), \tag{2.92}$$

where a is the *ref* Earth radius, and the only non–vanishing term is that of degree zero:

$$\begin{cases} c_{00}^{ref} &= 1 \\ c_{1m}^{ref} &= s_{1m}^{ref} = 0, \qquad (m = 0, 1) \\ c_{lm}^{ref} &= s_{lm}^{ref} = 0, \qquad (l \geq 2). \end{cases} \tag{2.93}$$

According to the general result (2.40) the gravity potential can be expanded as follows in the perturbed state:

$$U(t, \vec{r}) = \frac{Gm_e}{r} \Sigma'_{lm} \left(\frac{a}{r}\right)^l (c_{lm}(t) \cos m\lambda + s_{lm}(t) \sin m\lambda) P_{lm}(\cos\theta), \tag{2.94}$$

where $(c_{lm}(t),\, s_{lm}(t))$ and m_e are the Stokes coefficients and the Earth mass in the perturbed state, respectively, and it is assumed that the origin of the reference frame still coincides with the CM. Due to (2.47) and (2.54):

$$\begin{cases} c_{00} &= 1 \\ c_{1m} &= s_{1m} = 0, \qquad (m = 0, 1) \\ c_{lm} &\neq s_{lm} \neq 0, \qquad (l \geq 2). \end{cases} \tag{2.95}$$

From (2.73), the potential perturbation is the difference between the potential of the gravity field in the perturbed and in the reference state:

$$\Phi(t, \vec{r}) = U(t, \vec{r}) - U^{ref}(r), \tag{2.96}$$

which according to (2.94) and (2.92) can be written as:

$$\Phi(t, \vec{r}) = \frac{Gm_e}{r} \Sigma'_{lm} \left(\frac{a}{r}\right)^l (\delta c_{lm}(t) \cos m\lambda + \delta s_{lm}(t) \sin m\lambda) P_{lm}(\cos\theta), \tag{2.97}$$

where $\delta c_{lm}(t)$ and $\delta s_{lm}(t)$ are the variations of the Stokes coefficients:

$$\begin{cases} \delta c_{lm}(t) = c_{lm}(t) - c_{lm}^{ref} \\ \delta s_{lm}(t) = s_{lm}(t) - s_{lm}^{ref}, \end{cases} \tag{2.98}$$

with

$$\text{if} \quad (l = 0, 1) : \begin{cases} \delta c_{lm}(t) &= 0 \\ \delta s_{lm}(t) &= 0 \end{cases} \tag{2.99}$$

and

$$\text{if} \quad (l \geq 2) : \begin{cases} \delta c_{lm}(t) &= c_{lm} \\ \delta s_{lm}(t) &= s_{lm}, \end{cases} \tag{2.100}$$

where we have used (2.95) and (2.93). Using (2.97) and the definition of geoid height (2.79), we finally obtain:

$$\begin{aligned} N(t,\theta,\lambda) &= \frac{\Phi(t,a,\theta,\lambda)}{\gamma_o} \\ &= a\ \Sigma'_{lm}(\delta c_{lm}(t)\cos m\lambda + \delta s_{lm}(t)\sin m\lambda)P_{lm}(\cos\theta), \end{aligned} \qquad (2.101)$$

where due to (2.99) and (2.100) the lowest degree of the RSH expansion is $l = 2$. We also observe from (2.100) that for $l \geq 2$ the variations of the Stokes coefficients are *the* Stokes coefficients in the perturbed state •

2.2.8 Inertia tensor variations

According to (2.58) and (2.59), the variations of the Stokes coefficients of harmonic degree 2 are related to the variations of the elements of the inertia tensor by

$$\left\{ \begin{array}{c} \delta c_{20} \\ \delta c_{21} \\ \delta c_{22} \end{array} \right\}(t) \;=\; \frac{1}{m_e a^2} \left\{ \begin{array}{c} -[\delta i_{zz} - (\delta i_{xx} + \delta i_{yy})/2] \\ \delta i_{xz} \\ -(\delta i_{xx} - \delta i_{yy})/4 \end{array} \right\}(t), \qquad (2.102)$$

and

$$\left\{ \begin{array}{c} \delta s_{20} \\ \delta s_{21} \\ \delta s_{22} \end{array} \right\}(t) \;=\; \frac{1}{m_e a^2} \left\{ \begin{array}{c} 0 \\ \delta i_{yz} \\ -\delta i_{xy}/2 \end{array} \right\}(t), \qquad (2.103)$$

where m_e and a are the mass and the reference radius of the Earth, respectively. The relationships above cannot be unequivocally inverted in order to obtain the inertia tensor variations from the Stokes coefficients. A further condition must be provided, as stated in the following proposition.

Proposition 25 *As shown in [10], the trace of the inertia tensor does not vary provided that the displacement field induced by the perturbing forces is solenoidal (this statement corresponds to the so–called Darwin's Theorem). Since we have assumed that the Earth is incompressible, the displacement field is solenoidal (2.2). Using the constraint:*

$$\delta i_{xx} + \delta i_{yy} + \delta i_{zz} = 0, \qquad (2.104)$$

in (2.102) and (2.103), we obtain:

$$\left\{ \begin{array}{c} \delta\bar{i}_{xx} \\ \delta\bar{i}_{yy} \\ \delta\bar{i}_{zz} \\ \delta\bar{i}_{xz} \\ \delta\bar{i}_{yz} \\ \delta\bar{i}_{xy} \end{array} \right\}(t) \quad = \quad \left\{ \begin{array}{c} \delta c_{20}/3 - 2\delta c_{22} \\ \delta c_{20}/3 + 2\delta c_{22} \\ -2\delta c_{20}/3 \\ \delta c_{21} \\ \delta s_{21} \\ -2\delta s_{22} \end{array} \right\}(t), \tag{2.105}$$

where we have introduced the normalized inertia tensor:

$$\bar{i}(t) \equiv \frac{i(t)}{m_e a^2}. \tag{2.106}$$

Chapter 3

Surface loads

In the previous Chapter we have associated the displacement field (2.6) and the geoid height (2.89) to generic perturbing forces causing deformation of the solid Earth. The forces of concern in `TABOO` are indeed particular, in that they are related to the accretion or ablation of ice loads placed on the Earth surface and to the consequent variations of ocean mass. Here we describe these *surface loads* in mathematical terms; their relationship with the displacements and the geoid height is discussed in the next Chapter.

3.1 General properties

3.1.1 Definition

In our discussion we are only concerned with geophysical processes which can be modeled in terms of normal forces acting on the Earth surface. We define the *surface load* as

$$L(t, \theta, \lambda) = -\frac{1}{\gamma_0} \frac{df_n}{dA}(t, \theta, \lambda), \tag{3.1}$$

where df_n is the normal force on the surface element of area dA of the Earth surface at time t, and γ_o is the reference gravity acceleration at the surface of the Earth (2.72). The surface load has dimensions of a mass per unit surface.

In this discussion we will limit our attention to the special case of surface loads which can be factorized as follows:

$$L(t, \theta, \lambda) = f(t)\sigma(\theta, \lambda), \tag{3.2}$$

where the function $\sigma(\theta, \lambda)$, called *load function*, defines the spatial features of the surface load, and the non–dimensional function $f(t)$ is the *load time–history*, which describes its time–evolution.

All of the surface loads considered in this booklet are characterized by load functions of the type

$$\sigma(\theta, \lambda) = \begin{cases} H_{\mathcal{D}}(\theta, \lambda) & \text{if} \quad (\theta, \lambda) \in \mathcal{D} \\ c & \text{if} \quad (\theta, \lambda) \notin \mathcal{D}, \end{cases} \tag{3.3}$$

where $\mathcal{D} \subseteq \Omega$ is the *load function domain*, Ω is the surface of the sphere, $H_{\mathcal{D}}$ is a function defined on $\mathcal{D}$, and c is a constant. The particular load functions and time–histories available in `TABOO` will be described in §3.2, 3.3, and 7.1.

3.1.2 Dynamic and static load mass

Once $L(t, \theta, \lambda)$ is defined by (3.2) and (3.3), it is useful to introduce the *dynamic mass of the load*:

$$\mu(t) \equiv \int_{\Omega} L(t, \theta, \lambda) dA \tag{3.4}$$

$$= \text{(3.2)} = \quad f(t) \int_{\Omega} \sigma(\theta, \lambda) dA, \tag{3.5}$$

where

$$dA = a^2 d\Omega \tag{3.6}$$

is the element of area on the surface of the sphere of radius a, with $d\Omega = \sin\theta d\theta d\lambda$ (see also 1.25).

It is also useful to introduce the *static load mass* m_s as

$$m_s = \int_{\Omega} \sigma(\theta, \lambda) dA, \tag{3.7}$$

so that:

$$\mu(t) = f(t) m_s. \tag{3.8}$$

The dynamic mass $\mu(t)$ gives information of the spatially averaged surface load at time t, and as such it can be characterized by negative, null, or positive values:

$$
\begin{aligned}
\mu(t) &= \text{(3.4)} = \int_\Omega L(t,\theta,\lambda)dA \\
&= A_e \frac{\int_\Omega L(t,\theta,\lambda)dA}{A_e} \\
&= \text{(3.6)} = \quad A_e \frac{\int_\Omega L(t,\theta,\lambda)a^2 d\Omega}{\int_\Omega a^2 d\Omega} \\
&= \text{(1.56)} = \quad A_e \langle L(t,\theta,\lambda)\rangle_\Omega \\
&= \text{(3.2)} = \quad A_e f(t)\langle \sigma(\theta,\lambda)\rangle_\Omega,
\end{aligned}
\tag{3.9}
$$

where A_e is the area of the surface of the Earth and $\langle ... \rangle_\Omega$ indicates the average on Ω (see 1.56).

3.1.3 Balanced loads

In the special case of a surface load with vanishing dynamic mass at any time t, we are dealing with a *balanced load*. Due to (3.8), a sufficient condition for $\mu(t) = 0$ is

$$m_s = 0 \quad \text{(balanced load)}. \tag{3.10}$$

Since the balanced loads never create nor destroy a net static mass on the Earth surface, they are useful to mimic the principle of mass conservation. It should be remarked that the surface loads, being them balanced or not, always correspond to distributed forces applied at the Earth surface, which do not imply any alteration of the effective mass of the Earth.

Suppose that a non–balanced surface load with dynamic mass $\mu^1(t)$ is assigned on a domain $\mathcal{D}^1$ and that we want to balance that load introducing an appropriate *compensating surface load* with dynamic mass $\mu^2(t)$ on a domain $\mathcal{D}^2$ (see 3.3). The two surface loads will be also referred as to *primary* and *secondary* surface loads, respectively. In the special case $\mathcal{D}^1 \cup \mathcal{D}^2 = \Omega$, the secondary load is *complementary* to the primary.

The load resulting from the juxtaposition of the two unbalanced surface loads is balanced if

$$\mu^{(1+2)}(t) = \mu^1(t) + \mu^2(t) = f^1(t)m_s^1 + f^2(t)m_s^2 \equiv 0, \tag{3.11}$$

where $f^1(t)$ and $f^2(t)$ are the time–histories of the two loads, and m_s^1 and m_s^2 are their static masses, respectively. The condition (3.11) is satisfied if

$f^1(t) = f^2(t)$ and $m_s^1 = -m_s^2$, i.e., if the two loads have the same time–history but opposite static masses. This is the strategy that we will employ in the following in order to build the compensating secondary surface load once the primary is assigned. In general, the static mass of the primary load is ≥ 0, since the primary surface load is associated with an *excess* of mass on the Earth surface (e.g., an ice dome, or other). As a consequence, a static mass ≤ 0 is assigned to the secondary load, which is usually associated with a mass deficiency (e.g., the sealevel drop due to ice accumulation within the ice caps).

A special case is that of *self–balanced* loads, for which the dynamic mass vanishes without the need of introducing a secondary compensating load. An example of self–balanced load is given in §3.3.6.

3.1.4 AX and NAX surface loads

For our following discussion, it is important to classify the load functions $\sigma(\theta, \lambda)$ into two families. The first contains those load functions which possess an *axis of symmetry*, and the corresponding surface loads are called *AX loads*. The second family contains the non–axissimmetric loads, referred as to *NAX loads*. Of course, any AX load can be viewed as a particular NAX load.

Suppose that, once the load function $\sigma(\theta, \lambda)$ and the domain $\mathcal{D}$ in (3.3) are specified in the geographical reference frame (GRF), it is possible to determine a new frame in which σ is only function of the new colatitude Θ. The z–axis of the new frame, that we call *load reference frame* (LRF) is the axis of symmetry of the load. The *pole* of the (AX) load is that point where the axis of symmetry pierces the sphere, so that the pole has colatitude $\Theta = 0$ in the LRF. The existence of the LRF depends on the geometrical features of both $\sigma(\theta, \lambda)$ and $\mathcal{D}$. The x and y axes of the LRF can be assigned arbitrarily on the plane perpendicular to z due to the symmetry of the load.

If Θ is the colatitude of point P in the LRF, the cosines theorem of spherical geometry ensures that

$$\cos\Theta = \cos\theta\cos\theta_c + \sin\theta\sin\theta_c\cos(\lambda - \lambda_c), \tag{3.12}$$

where (θ, λ) and (θ_c, λ_c) are the spherical coordinates of P and of the pole of the load in the GRF, respectively.

3.1.5 Expansion of the NAX load function

Proposition 26 *Given a generic NAX surface load, its load function can be expanded as*

$$\sigma(\theta,\lambda) = \Sigma_{lm}\sigma_{lm}Y_{lm}(\theta,\lambda), \tag{3.13}$$

where the CSH coefficients are:

$$\sigma_{lm} = \int_{\Omega}\sigma(\theta,\lambda)Y^{*}_{lm}(\theta,\lambda)d\Omega. \tag{3.14}$$

Proof. The formula (3.13) is just a consequence of the general CSH expansion theorem (1.47), and (3.14) follows from (1.50).

Proposition 27 *Given the CSH expansion of the load function associated with a generic NAX load (3.13), the coefficients of the RSH equivalent expansion*

$$\sigma(\theta,\lambda) = \Sigma'_{lm}(c^{\sigma}_{lm}\cos m\lambda + s^{\sigma}_{lm}\sin m\lambda)P_{lm}(\cos\theta) \tag{3.15}$$

are

$$\left\{\begin{array}{c} c^{\sigma}_{lm} \\ s^{\sigma}_{lm} \end{array}\right\} = (2-\delta_{0m})\mu_{lm}\left\{\begin{array}{cc} & \mathrm{Re}(\sigma_{lm}) \\ - & \mathrm{Im}(\sigma_{lm}) \end{array}\right\}, \tag{3.16}$$

with

$$\Sigma'_{lm} \equiv \sum_{l=0}^{\infty}\sum_{m=0}^{+l}. \tag{3.17}$$

Proof. The formula (3.15) is just a a consequence of (1.63) •

Proposition 28 *The static mass of a generic NAX surface load is*

$$m_s = a^2\sqrt{4\pi}\sigma_{00} = 4\pi a^2 c^{\sigma}_{00}, \tag{3.18}$$

where a is the reference radius of the Earth.

Proof. We prove the left equality first:

$$
\begin{aligned}
m_s &= \text{(3.7)} = a^2 \int_\Omega \sigma(\theta,\lambda) d\Omega \\
&= \text{(3.13)} = a^2 \int_\Omega \Sigma_{lm}\sigma_{lm} Y_{lm} d\Omega \\
&= a^2 \Sigma_{lm}\sigma_{lm} \int_\Omega Y_{lm} d\Omega \\
&= a^2 \Sigma_{lm}\sigma_{lm}\sqrt{4\pi} \int_\Omega \frac{1}{\sqrt{4\pi}} Y_{lm} d\Omega \\
&= \text{(table 1.1)} = a^2 \Sigma_{lm}\sigma_{lm}\sqrt{4\pi} \int_\Omega Y_{00} Y_{lm} d\Omega \\
&= a^2 \Sigma_{lm}\sigma_{lm}\sqrt{4\pi} \int_\Omega Y^*_{00} Y_{lm} d\Omega \\
&= \text{(1.24)} = a^2 \Sigma_{lm}\sigma_{lm}\sqrt{4\pi}\delta_{l0}\delta_{m0} \\
&= a^2\sqrt{4\pi}\sigma_{00} \quad \bullet
\end{aligned} \tag{3.19}
$$

The right equality in (3.18) can be recognized as true observing that from the first of (3.16) we have $c^\sigma_{00} = \frac{1}{\sqrt{4\pi}}\text{Re}(\sigma_{00})$ (see also 1.23). But since $\sigma(\theta,\lambda)$ and σ_{00} are real (1.55), we have $\sigma_{00} = \sqrt{4\pi}c^\sigma_{00}$, which proves (3.18) •

3.2 Two useful NAX loads

We will consider two specific kinds of NAX loads: the rectangular and the ocean surface loads.

3.2.1 Rectangular load

The *rectangular surface load* is defined as

$$L(t,\theta,\lambda) = f(t)\sigma^r(\theta,\lambda), \tag{3.20}$$

where $f(t)$ is the load time–history, and the load function is:

$$\sigma^r(\theta,\lambda) = \rho_i \begin{cases} h & \text{if} \quad (\theta_1 \le \theta \le \theta_2) \quad \text{and} \quad (\lambda_1 \le \lambda \le \lambda_2) \\ 0 & \text{elsewere,} \end{cases} \tag{3.21}$$

where the parameter h is called *load thickness*, ρ_i is the density of the material which constitutes the load[1], and (λ_1, λ_2) and (θ_1, θ_2) are the longitudes of

[1]the label i in ρ_i stands for ice, since generally (but not necessarily) the disc load is used to model an ice cap.

the two meridians and the colatitudes of the two parallels which bound the rectangular load, respectively. As we will show below, the product $\rho_i h$ is proportional to the static mass of the rectangular load m_s^r.

Proposition 29 *The CSH coefficients of the rectangular load function expansion:*

$$\sigma^r(\theta, \lambda) = \Sigma_{lm} \sigma_{lm}^r Y_{lm}(\theta, \lambda) \tag{3.22}$$

are

$$\sigma_{lm}^r = \rho_i h \; \mu_{lm} \; \gamma_{lm} \; (\alpha_m + \iota \beta_m), \tag{3.23}$$

where μ_{lm} is given by (1.23), and

$$\left\{ \begin{array}{c} \alpha_0 \\ \beta_0 \end{array} \right\} = \left\{ \begin{array}{c} \lambda_2 - \lambda_1 \\ 0 \end{array} \right\}, \quad (m = 0) \tag{3.24}$$

$$\left\{ \begin{array}{c} \alpha_m \\ \beta_m \end{array} \right\} = \frac{1}{m} \left\{ \begin{array}{c} \sin m\lambda_2 - \sin m\lambda_1 \\ \cos m\lambda_2 - \cos m\lambda_1 \end{array} \right\}, \quad (m \neq 0), \tag{3.25}$$

$$\gamma_{lm} \equiv - \int_{\cos\theta_1}^{\cos\theta_2} P_{lm}(x) dx. \tag{3.26}$$

Proof.

$$\begin{aligned} \sigma_{lm}^r &= \text{(3.14)} = \int_\Omega \sigma^r Y_{lm}^* d\Omega \\ &= \text{(3.21)} = \rho_i h \mu_{lm} \int_{\lambda_1}^{\lambda_2} d\lambda e^{-im\lambda} \int_{\theta_1}^{\theta_2} P_{lm}(\cos\theta) \sin\theta d\theta \\ &= -\rho_i h \mu_{lm} \int_{\lambda_1}^{\lambda_2} d\lambda e^{-im\lambda} \int_{\cos\theta_1}^{\cos\theta_2} P_{lm}(x) dx \\ &= \text{(3.26)} = \rho_i h \mu_{lm} \gamma_{lm} \int_{\lambda_1}^{\lambda_2} d\lambda e^{-im\lambda}. \end{aligned} \tag{3.27}$$

For $m = 0$:

$$\begin{aligned} \int_{\lambda_1}^{\lambda_2} d\lambda e^{-im\lambda} &= \int_{\lambda_1}^{\lambda_2} d\lambda \\ &= \lambda_2 - \lambda_1 \\ &= \alpha_0 + \iota\beta_0, \end{aligned} \tag{3.28}$$

where α_0 and β_0 are given by (3.24).

For $m \neq 0$:

$$\begin{aligned}
\int_{\lambda_1}^{\lambda_2} d\lambda e^{-im\lambda} &= -\frac{1}{\iota m}\left[e^{-im\lambda}\right]_{\lambda_1}^{\lambda_2} \\
&= \frac{\iota}{m}(e^{-im\lambda_2} - e^{-im\lambda_1}) \\
&= \frac{\iota}{m}(\cos\lambda_2 - \iota\sin\lambda_2 - \cos\lambda_1 + \iota\sin\lambda_1) \\
&= \frac{\iota}{m}[(\cos\lambda_2 - \cos\lambda_1) - \iota(\sin\lambda_2 - \sin\lambda_1)] \\
&= \frac{1}{m}(\sin\lambda_2 - \sin\lambda_1) + \iota\frac{1}{m}(\cos\lambda_2 - \cos\lambda_1) \\
&= \alpha_m + \iota\beta_m, \qquad (3.29)
\end{aligned}$$

where α_m and β_m are given by (3.25) •

Proposition 30 *The static mass of a rectangular load of thickness h and density ρ_i is*

$$m_s^r = \rho_i h a^2(\lambda_2 - \lambda_1)(\cos\theta_1 - \cos\theta_2). \qquad (3.30)$$

Proof. From (3.18), the static mass of the load is $m_s^r = \sqrt{4\pi}a^2\sigma_{00}^r$, where σ_{00}^r is the degree 0 CSH coefficient of the load function expansion. But from (3.23), (1.23), and (3.26), we also have $\sigma_{00}^r = \frac{\rho_i h}{\sqrt{4\pi}}\,(\lambda_2 - \lambda_1)\,(\cos\theta_1 - \cos\theta_2)$, so that for a rectangular load (3.30) holds •

3.2.2 Ocean surface load

The *ocean surface load* is defined by

$$L(t, \theta, \lambda) = f(t)\sigma^{oc}(\theta, \lambda), \qquad (3.31)$$

where $f(t)$ is the load time–history, and the *ocean load function* is

$$\sigma^{oc}(\theta, \lambda) = \frac{m_s^{oc}}{A_{oc}}\mathcal{O}(\theta, \lambda), \qquad (3.32)$$

where $\mathcal{O}(\theta, \lambda)$ is given by (1.80), m_s^{oc} is the static mass of the load, and A_{oc} is the area of the surface of the oceans (1.85). By definition of static mass (see 3.7):

$$m_s^{oc} = a^2\int_\Omega \sigma^{oc} d\Omega. \qquad (3.33)$$

3.3 AX loads

As seen in §3.1.4, a generic AX load can be expressed in the LRF by a load function $\sigma^{AX}(\Theta)$, where Θ is the colatitude measured with respect to the axis of symmetry of the load. When the LRF is not coincident with the GRF, it is by far more economical to take advantage of the load symmetry and to expand the load function in Legendre polynomials in the LRF instead of writing a CSH expansion in the GRF. We thus write:

$$\sigma^{AX}(\Theta) = \sum_{l=0}^{\infty} \sigma_l^{AX} P_l(\cos\Theta), \tag{3.34}$$

where, according to (1.75), the LEG coefficients of the expansion are:

$$\sigma_l^{AX} = \frac{2l+1}{2} \int_0^{\pi} \sigma^{AX}(\Theta) P_l(\cos\Theta) \sin\Theta d\Theta, \tag{3.35}$$

or

$$\sigma_l^{AX} = \frac{2l+1}{2} \int_{-1}^{+1} \sigma^{AX}(x) P_l(x) dx, \quad x \equiv \cos\theta. \tag{3.36}$$

Proposition 31 *The static mass of an AX load is*

$$m_s^{AX} = 2\pi a^2 \int_0^{\pi} \sigma^{AX}(\Theta) \sin\Theta d\Theta. \tag{3.37}$$

Proof. This result can be obtained from the general formula (3.7) written in the LRF:

$$\begin{aligned} m_s^{AX} &= a^2 \int_{\Omega} \sigma^{AX}(\Theta) \sin\Theta d\Theta d\Lambda \\ &= {\scriptstyle (1.25)} = a^2 \int_0^{2\pi} \int_0^{\pi} \sigma^{AX}(\Theta) \sin\Theta d\Theta d\Lambda \\ &= 2\pi a^2 \int_0^{\pi} \sigma^{AX}(\Theta) \sin\Theta d\Theta, \end{aligned} \tag{3.38}$$

where Λ and Θ the longitude and the colatitude in the LRF, and we have taken advantage from the load symmetry •

Proposition 32 *The explicit expression for the static mass of an AX load is:*

$$m_s^{AX} = 4\pi a^2 \sigma_0^{AX}, \tag{3.39}$$

where σ_0^{AX} is the degree 0 LEG coefficient of the expansion of the load function in the LRF.

Proof.

$$\begin{aligned}
m_s^{AX} &\equiv \text{(3.37)} = 2\pi a^2 \int_0^\pi \sigma^{AX}(\Theta) \sin\Theta d\Theta \\
&= \text{(3.34)} = 2\pi a^2 \int_0^\pi \sum_{l=0}^{\infty} \sigma_l^{AX} P_l(\cos\Theta) \sin\Theta d\Theta \\
&= \text{(table 1.3)} = 2\pi a^2 \sum_{l=0}^{\infty} \sigma_l^{AX} \int_0^\pi P_0(\cos\Theta) P_l(\cos\Theta) \sin\Theta d\Theta \\
&= \text{(1.35)} = 2\pi a^2 \sum_{l=0}^{\infty} \sigma_l^{AX} \frac{2\delta_{l0}}{2l+1} \\
&= 4\pi a^2 \sigma_0^{AX} \quad \bullet
\end{aligned} \tag{3.40}$$

Proposition 33 *Let us consider an AX load and its LEG expansion in the LRF:*

$$\sigma^{AX}(\Theta) = \sum_{l=0}^{\infty} \sigma_l^{AX} P_l(\cos\Theta), \tag{3.41}$$

where Θ is the colatitude of an arbitrary point on the Earth surface. This same point has coordinates (θ, λ) in the GRF. The load function $\sigma^{AX}(\Theta)$ will depend on θ and λ through the dependence of Θ from θ and λ (see 3.12):

$$\sigma^{AX}(\Theta) = \sigma^{ax}(\theta, \lambda), \tag{3.42}$$

where the lower–case upperscript 'ax' indicates that the corresponding quantity is written in the GRF. Our purpose here is to show that the coefficients of the CSH expansion

$$\sigma^{ax}(\theta, \lambda) = \Sigma_{lm} \sigma_{lm}^{ax} Y_{lm}(\theta, \lambda), \tag{3.43}$$

are

$$\sigma_{lm}^{ax} = \frac{4\pi Y_{lm}^*(\theta_c, \lambda_c)}{2l+1} \sigma_l^{AX}, \tag{3.44}$$

where (θ_c, λ_c) are the coordinates of the pole of the load in the GRF.

Proof. The proof is a direct consequence of the addition theorem:

$$\begin{aligned}
\sigma^{AX}(\Theta) &= \text{(3.34)} = \sum_{l=0}^{\infty} \sigma_l^{AX} P_l(\cos\Theta) \\
\sigma^{ax}(\theta,\lambda) &= \text{(1.29)} = \sum_{l=0}^{\infty} \sigma_l^{AX} \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} Y^*_{lm}(\theta_c,\lambda_c) Y_{lm}(\theta,\lambda) \\
&= \Sigma_{lm} \sigma^{ax}_{lm} Y_{lm}(\theta,\lambda), \qquad (3.45)
\end{aligned}$$

with

$$\sigma^{ax}_{lm} = \frac{4\pi Y^*_{lm}(\theta_c,\lambda_c)}{2l+1} \sigma_l^{AX} \quad \bullet \qquad (3.46)$$

3.3.1 Unit load

The response of the Earth to a unit surface load allows to construct the response to any other load. For this reason, it is important in our discussion. The unit surface load is defined as:

$$L(t,\Theta) = f(t)\sigma^\delta(\Theta), \qquad (3.47)$$

with load function

$$\sigma^\delta(\Theta) = \frac{m_s^\delta}{2\pi a^2}\delta(1-\cos\Theta), \qquad (3.48)$$

where $\delta(x)$ is the Dirac delta and a is the reference Earth radius. According to (3.37) the static mass m_s^δ of the unit load is

$$m_s^\delta = 2\pi a^2 \int_\Omega \sigma^\delta(\Theta) \sin\Theta d\Theta, \qquad (3.49)$$

where Θ is the colatitude in the LRF.

For later purposes it is convenient to introduce the potential perturbation ϕ^p associated with a point source with static mass m_s^δ placed on the surface of the Earth in the *ref* state (§2.2.6). By Newton's Law of gravitation:

$$\phi^p = \frac{G m_s^\delta}{d}, \qquad (3.50)$$

where d is the distance between the observer and the source. By the cosines theorem:

$$\phi^p(a,\Theta) = \frac{G m_s^\delta}{\sqrt{2a^2(1-\cos\Theta)}}, \qquad (3.51)$$

where a is the reference radius of the Earth, and Θ is the colatitude of the observer with respect to the point source. Since $1 - \cos\Theta = 2\sin^2(\Theta/2)$, we obtain:

$$\phi^p(a, \Theta) = \frac{Gm_s^\delta}{2a \sin \frac{\Theta}{2}}, \tag{3.52}$$

which can be transformed recalling the Legendre sum (1.42):

$$\begin{aligned} \phi^p(a, \Theta) &= \frac{Gm_s^\delta}{a} \sum_{l=0}^{\infty} P_l(\cos\Theta) \\ &= \sum_{l=0}^{\infty} \phi_l^p(a) P_l(\cos\Theta), \end{aligned} \tag{3.53}$$

where the LEG coefficients of the expansion are

$$\phi_l^p(a) = \frac{Gm_s^\delta}{a}. \tag{3.54}$$

Proposition 34 *The coefficients of the LEG expansion of the unit load function*

$$\sigma^\delta(\Theta) = \sum_{l=0}^{\infty} \sigma_l^\delta P_l(\cos\Theta) \tag{3.55}$$

are

$$\sigma_l^\delta = m_s^\delta \left(\frac{2l+1}{4\pi a^2} \right). \tag{3.56}$$

Proof.

$$\begin{aligned} \sigma_l^\delta &= {\scriptstyle (3.36)} = \frac{2l+1}{2} \int_{-1}^{+1} \sigma^\delta(x) P_l(x) dx \\ &= {\scriptstyle (3.48)} = \frac{2l+1}{2} \frac{m_s^\delta}{2\pi a^2} \int_{-1}^{+1} \delta(1-x) P_l(x) dx \\ &= {\scriptstyle (1.90)} = m_s^\delta \left(\frac{2l+1}{4\pi a^2} \right) P_l(1) \\ &= {\scriptstyle (1.41)} = m_s^\delta \left(\frac{2l+1}{4\pi a^2} \right). \end{aligned} \tag{3.57}$$

From (3.56) we observe that

$$m_s^\delta = 4\pi a^2 \sigma_0^\delta, \tag{3.58}$$

in agreement with the general relationship (3.39) •

3.3.2 Disc load

The disc load is the particular AX surface load defined as

$$L(t,\Theta) = f(t)\sigma^d(\Theta), \tag{3.59}$$

where $f(t)$ is the load time–history and the load function in the LRF is

$$\sigma^d(\Theta) = \begin{cases} \rho_i h & \text{if} \quad 0 \le \Theta \le \alpha \\ 0 & \text{if} \quad \alpha < \Theta \le \pi, \end{cases} \tag{3.60}$$

where α is the *half–amplitude* of the disc load ($0 \le \alpha \le \pi$), h is the *thickness* of the load, and ρ_i is the ice density (see the footnote of page 50).

Proposition 35 *The static mass of the disc load is*

$$m_s^d = 2\pi a^2 \rho_i h(1 - \cos\alpha), \tag{3.61}$$

so that an alternative form of the disc load function is:

$$\sigma^d(\Theta) = \begin{cases} \dfrac{m_s^d}{2\pi a^2(1-\cos\alpha)} & \text{if} \quad 0 \le \Theta \le \alpha \\ 0 & \text{if} \quad \alpha < \Theta \le \pi. \end{cases} \tag{3.62}$$

Proof.

$$\begin{aligned} m_s^d &= {\scriptstyle (3.37)} = 2\pi a^2 \int_0^\pi \sigma^d(\Theta)\sin\Theta d\Theta \\ &= {\scriptstyle (3.60)} = 2\pi a^2 \rho_i h \int_0^\alpha \sin\Theta d\Theta \\ &= 2\pi a^2 \rho_i h(1-\cos\alpha) \quad \bullet \end{aligned} \tag{3.63}$$

Proposition 36 *The coefficients of the LEG expansion of the disc load function in the LRF*

$$\sigma^d(\Theta) = \sum_{l=0}^{\infty} \sigma_l^d P_l(\cos\Theta) \tag{3.64}$$

are

$$\sigma_l^d = \frac{\rho_i h}{2} \begin{cases} 1 - \cos\alpha & \text{if} \quad l = 0 \\ [-P_{l+1}(\cos\alpha) + P_{l-1}(\cos\alpha)] & \text{if} \quad l \ge 1. \end{cases} \tag{3.65}$$

Proof. We apply to the disc load the general result (3.35) and we consider separately the cases $l = 0$ and $l \geq 1$.

1. For $l = 0$:

$$\begin{aligned}
\sigma_0^d &= \text{(3.35)} = \frac{1}{2}\int_0^\pi \sigma^d(\Theta)P_0(\cos\Theta)\sin\Theta d\Theta \\
&= \text{(3.60)} = \frac{\rho_i h}{2}\int_0^\alpha P_0(\cos\Theta)\sin\Theta d\Theta \\
&= \text{(table 1.3)} = \frac{\rho_i h}{2}\int_0^\alpha \sin\Theta d\Theta \\
&= \frac{\rho_i h}{2}(1-\cos\alpha). \qquad (3.66)
\end{aligned}$$

2. For $l \geq 1$:

$$\begin{aligned}
\sigma_l^d &= \text{(3.35)} = \frac{2l+1}{2}\int_0^\pi \sigma^d(\Theta)P_l(\cos\Theta)\sin\Theta d\Theta \\
&= \text{(3.60)} = \frac{2l+1}{2}\rho_i h\int_0^\alpha P_l(\cos\Theta)\sin\Theta d\Theta \qquad (3.67)\\
&= \frac{2l+1}{2}\rho_i h\int_{\cos\alpha}^1 P_l(x)dx \\
&= \text{(1.40)} = \frac{2l+1}{2}\rho_i h\int_{\cos\alpha}^1 \frac{P'_{l+1}(x)-P'_{l-1}(x)}{2l+1}dx \\
&= \frac{\rho_i h}{2}[P_{l+1}(1)-P_{l+1}(\cos\alpha)-P_{l-1}(1)+P_{l-1}(\cos\alpha)] \\
&= \text{(1.41)} = \frac{\rho_i h}{2}[1-P_{l+1}(\cos\alpha)-1+P_{l-1}(\cos\alpha)] \\
&= -\frac{\rho_i h}{2}[P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)] \quad \bullet \qquad (3.68)
\end{aligned}$$

3.3.3 Balanced disc load

The balanced disc load can be expressed in the LRF as:

$$L(t,\Theta) = f(t)\Big[\sigma^d(\Theta)+\sigma^c(\Theta)\Big], \qquad (3.69)$$

where $\sigma^d(\Theta)$ is the disc load function (3.60), and $\sigma^c(\Theta)$ is the complementary disc load function:

$$\sigma^c(\Theta) = \rho_i\begin{cases} 0 & \text{if} \quad 0 \leq \Theta \leq \alpha \\ h' & \text{if} \quad \alpha < \Theta \leq \pi, \end{cases} \qquad (3.70)$$

where the thickness h' is determined below, and ρ_i the ice density. From (3.60) and (3.70), the load function of the balanced disc load is:

$$\sigma^{cd}(\Theta) = \sigma^d(\Theta) + \sigma^c(\Theta) = \rho_i \begin{cases} h & \text{if} \quad 0 \leq \Theta \leq \alpha \\ h' & \text{if} \quad \alpha < \Theta \leq \pi, \end{cases} \tag{3.71}$$

where h is the primary load thickness. To make the constant h' in (3.71) explicit, we impose that the *total* static mass of the balanced load vanishes (see 3.10):

$$\begin{aligned} 0 = m_s^{cd} &= {\scriptstyle (3.37)} = \quad 2\pi a^2 \int_0^\pi \sigma^{cd}(\Theta) \sin\Theta d\Theta \\ &= {\scriptstyle (3.71)} = \quad 2\pi a^2 \rho_i \left[h \int_0^\alpha \sin\Theta d\Theta + h' \int_\alpha^\pi \sin\Theta d\Theta \right] \\ &= 2\pi a^2 \rho_i [h(-\cos\Theta)_0^\alpha + h'(-\cos\Theta)_\alpha^\pi] \\ &= 2\pi a^2 \rho_i [h(1-\cos\alpha) + h'(1+\cos\alpha)], \end{aligned}$$

hence

$$h' = h \left(\frac{\cos\alpha - 1}{\cos\alpha + 1} \right). \tag{3.72}$$

Proposition 37 *If m_s^d denotes the static mass the primary disc load and α its half–amplitude, the balanced disc load can also be described by the following load function:*

$$\sigma^{cd}(\Theta) = \frac{m_s^d}{2\pi a^2} \begin{cases} +\dfrac{1}{1-\cos\alpha} & \text{if} \quad 0 \leq \Theta \leq \alpha \\ \\ -\dfrac{1}{1+\cos\alpha} & \text{if} \quad \alpha < \Theta \leq \pi. \end{cases} \tag{3.73}$$

Proof. It suffices to use (3.61) in the first of (3.71) and (3.72) with (3.61) in the second •

Proposition 38 *The LEG coefficients of the expansion of the balanced disk load*

$$\sigma^{cd}(\Theta) = \sum_{l=0}^{\infty} \sigma_l^{cd} P_l(\cos\Theta) \tag{3.74}$$

are

$$\sigma_l^{cd} = \rho_i h \begin{cases} 0 & \text{if} \quad l = 0 \\ -\dfrac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{1+\cos\alpha} & \text{if} \quad l \geq 1. \end{cases} \tag{3.75}$$

Proof. The cases $l = 0$ and $l \geq 1$ are discussed separately.

1. The result $\sigma_0^{cd} = 0$ follows directly from the relationship between the degree 0 LEG coefficient of a generic AX load and its static mass (3.39) and from (3.10).

2. For $l \geq 1$ we start from the general expression valid for any AX surface load:

$$\begin{aligned}
\sigma_l^{cd} &= \text{(3.35)} = \frac{2l+1}{2}\int_0^\pi \sigma^{cd}(\Theta) P_l(\cos\Theta)\sin\Theta d\Theta = \\
&= \text{(3.71)} = \frac{2l+1}{2}\Bigg[\int_0^\alpha \rho_i h P_l(\cos\Theta)\sin\Theta d\Theta + \\
&\quad + \int_\alpha^\pi \rho_i h' P_l(\cos\Theta)\sin\Theta d\Theta\Bigg] \\
&= \text{(3.67)} = \sigma_l^d + \frac{2l+1}{2}\rho_i h' \int_{-1}^{\cos\alpha} P_l(x)dx \\
&= \text{(3.68, 1.46)} = -\frac{\rho_i h}{2}[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)] + \\
&\quad \frac{2l+1}{2}\rho_i h' \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1} \\
&= -\frac{\rho_i}{2}\Big[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\Big](h - h') \\
&= \text{(3.72)} = -\frac{\rho_i}{2}\Big[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\Big]\Big(h - h\frac{\cos\alpha - 1}{\cos\alpha + 1}\Big) \\
&= -\rho_i h \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{1+\cos\alpha} \quad (l \geq 1) \quad \bullet
\end{aligned} \tag{3.76}$$

3.3.4 Parabolic load

The surface load with parabolic cross–section constitutes an improvement with respect to the disc load, since it does not include unrealistic steep edges.

According with the general definition of surface load, we define the *parabolic surface load*[2] in the LRF as

$$L(t,\Theta) = f(t)\sigma^p(\Theta), \tag{3.77}$$

where $f(t)$ is the load time–history, and the load function appropriate for the parabolic load is

$$\sigma^p(\Theta) = \rho_i \begin{cases} h_o\sqrt{\dfrac{\cos\Theta - \cos\alpha}{1-\cos\alpha}} & \text{if} \quad 0 \le \Theta \le \alpha \\ \\ 0 & \text{if} \quad \alpha < \Theta \le \pi, \end{cases} \tag{3.78}$$

where ρ_i is the mass density of the load, α is the half–amplitude, and h_o is the load thickness for $\Theta = 0$, related to the load static and to α by (3.79). As it will be clear in the following, the particular form of $\sigma^p(\Theta)$ allows for a closed–form computation of the LEG expansion coefficients in the LRF and constitutes a quite realistic equilibrium profile for an ice cap.

Proposition 39 *The static mass of the parabolic surface load is*

$$m_s^p = \frac{4}{3}\pi a^2 \rho_i h_o (1-\cos\alpha). \tag{3.79}$$

Proof.

$$\begin{aligned} m_s^p &\equiv \text{(3.37)} = 2\pi a^2 \int_0^\pi \sigma^p(\Theta)\sin\Theta d\Theta \\ &= \text{(3.78)} = 2\pi a^2 \rho_i h_o \int_0^\alpha \sqrt{\frac{\cos\Theta-\cos\alpha}{1-\cos\alpha}}\sin\Theta d\Theta \\ &= \frac{2\pi a^2\rho_i h_o}{\sqrt{1-\cos\alpha}} \int_0^\alpha \sqrt{\cos\Theta - \cos\alpha}\sin\Theta \; d\Theta \\ &= \frac{2\pi a^2\rho_i h_o}{\sqrt{1-\cos\alpha}} \int_1^{\cos\alpha} \sqrt{x-\cos\alpha}\;(-dx) \\ &= \frac{2\pi a^2\rho_i h_o}{\sqrt{1-\cos\alpha}} \int_{\cos\alpha}^1 \sqrt{x-\cos\alpha}\; dx \end{aligned}$$

[2]As it is clear from (3.78) the profile of the surface load is parabolic in the variable $\cos\Theta$, where Θ is colatitude.

$$
\begin{aligned}
&= \frac{2\pi a^2 \rho_i h_o}{\sqrt{1-\cos\alpha}} \frac{2}{3}\Big(x - \cos\alpha\Big)^{3/2}\Big|_{\cos\alpha}^{1} \\
&= \frac{\frac{4}{3}\pi a^2 \rho_i h_o}{\sqrt{1-\cos\alpha}}(1-\cos\alpha)^{3/2} \\
&= \frac{4}{3}\pi a^2 \rho_i h_o (1-\cos\alpha) \quad \bullet
\end{aligned} \tag{3.80}
$$

Proposition 40 *The coefficients of the LEG expansion of the parabolic load function in the LRF*

$$
\sigma^p(\Theta) = \sum_{l=0}^{\infty} \sigma_l^p P_l(\cos\Theta) \tag{3.81}
$$

are

$$
\sigma_l^p = \frac{\rho_i h_o}{3}(1-\cos\alpha) \begin{cases} 1 & \text{if} \quad l=0 \\ \xi_l(\alpha) & \text{if} \quad l \geq 1, \end{cases} \tag{3.82}
$$

where

$$
\xi_l(\alpha) \equiv -\frac{\frac{3}{4}}{(1-\cos\alpha)^2}\left[\frac{T_{l+1}(\alpha) - T_{l+2}(\alpha)}{l+3/2} - \frac{T_{l-1}(\alpha) - T_l(\alpha)}{l-1/2}\right], \tag{3.83}
$$

and $T_l(\alpha)$ denotes the Chebichev polynomials of 2^{nd} kind (1.39).

Proof. The cases $l=0$ and $l \geq 1$ are considered separately.

1. In (3.82), the expression for $l=0$ follows from (3.39) and (3.79).

2. For $l \geq 1$ we start from the general expression (3.35), valid for any AX load:

$$
\begin{aligned}
\sigma_l^p &= \frac{2l+1}{2}\int_0^{\pi} \sigma^p(\Theta) P_l(\cos\Theta)\sin\Theta d\Theta \qquad (3.84) \\
&= {\scriptstyle (3.78)} = \quad \frac{2l+1}{2}\rho_i h_o \int_0^{\alpha} \sqrt{\frac{\cos\Theta - \cos\alpha}{1-\cos\alpha}} P_l(\cos\Theta)\sin\Theta d\Theta \\
&= \frac{2l+1}{2}\frac{\rho_i h_o}{\sqrt{1-\cos\alpha}}\int_0^{\alpha} \sqrt{\cos\Theta - \cos\alpha}\; P_l(\cos\Theta)\sin\Theta d\Theta \\
&= \frac{2l+1}{2}\frac{\rho_i h_o}{\sqrt{1-\cos\alpha}}\int_1^{\cos\alpha} \sqrt{x - \cos\alpha}\; P_l(x)(-dx)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2l+1}{2}\frac{\rho_i h_o}{\sqrt{1-\cos\alpha}}\int_{\cos\alpha}^{1}\sqrt{x-\cos\alpha}P_l(x)dx = \text{(1.40)} = \\
&= \frac{2l+1}{2}\frac{\rho_i h_o}{\sqrt{1-\cos\alpha}}\int_{\cos\alpha}^{1}\sqrt{x-\cos\alpha}\Big[\frac{P'_{l+1}(x)-P'_{l-1}(x)}{2l+1}\Big]dx \\
&= \frac{\rho_i h_o}{2\sqrt{1-\cos\alpha}}\Big[\int_{\cos\alpha}^{1}\sqrt{x-\cos\alpha}\frac{dP_{l+1}}{dx}dx - \\
&\quad \int_{\cos\alpha}^{1}\sqrt{x-\cos\alpha}\frac{dP_{l+1}}{dx}dx\Big] \\
&= \frac{\rho_i h_o}{2\sqrt{1-\cos\alpha}}\Big[\sqrt{x-\cos\alpha}P_{l+1}(x)\Big|_{\cos\alpha}^{1} - \frac{1}{2}\int_{\cos\alpha}^{1}\frac{P_{l+1}(x)dx}{\sqrt{x-\cos\alpha}} - \\
&\qquad \sqrt{x-\cos\alpha}P_{l-1}(x)\Big|_{\cos\alpha}^{1} + \frac{1}{2}\int_{\cos\alpha}^{1}\frac{P_{l-1}(x)dx}{\sqrt{x-\cos\alpha}}\Big] \\
&= \text{(1.41)} = -\frac{\rho_i h_o}{4\sqrt{1-\cos\alpha}}\Big[\int_{\cos\alpha}^{1}\frac{P_{l+1}(x)dx}{\sqrt{x-\cos\alpha}} - \int_{\cos\alpha}^{1}\frac{P_{l-1}(x)dx}{\sqrt{x-\cos\alpha}}\Big] \\
&= \text{(1.38)} = -\frac{\rho_i h_o}{4(1-\cos\alpha)}\Big[\frac{T_{l+1}(\alpha)-T_{l+2}(\alpha)}{l+3/2} - \frac{T_{l-1}(\alpha)-T_l(\alpha)}{l-1/2}\Big] \\
&= \text{(3.83)} = \frac{\rho_i h_o}{3}(1-\cos\alpha)\xi_l(\alpha) \quad \bullet
\end{aligned} \tag{3.85}
$$

3.3.5 Balanced parabolic load

The balanced parabolic load can be expressed in the LRF as:

$$
L(t,\Theta) = f(t)\Big[\sigma^p(\Theta) + \sigma^c(\Theta)\Big], \tag{3.86}
$$

where $\sigma^p(\Theta)$ is the parabolic load function (3.78), and $\sigma^c(\Theta)$ is the load function of the complementary *disc* load (3.70). The load function of the balanced parabolic load is thus:

$$
\sigma^{cp}(\Theta) = \sigma^p(\Theta) + \sigma^c(\Theta) = \rho_i \begin{cases} h_o\sqrt{\frac{\cos\Theta-\cos\alpha}{1-\cos\alpha}} & \text{if } \ 0 \le \Theta \le \alpha \\ h' & \text{if } \ \alpha < \Theta \le \pi, \end{cases} \tag{3.87}
$$

where the parameter h_o is related to the static mass of the primary load by (3.79), and to make the constant h' explicit we impose that the static mass of the balanced parabolic load vanishes (see 3.10):

$$
0 = m_s^{cp} \ = \ \text{(3.37)} = \ 2\pi a^2\int_0^{\pi}\sigma^{cp}(\Theta)\sin\Theta d\Theta \tag{3.88}
$$

$$
\begin{aligned}
&= \ (3.87) = 2\pi a^2 \Big[\int_0^\alpha \sigma^p(\Theta) \sin\Theta d\Theta + \int_\alpha^\pi \rho_i h' \sin\Theta d\Theta \Big] \\
&= \ (3.80) = \quad \frac{4}{3}\pi a^2 \rho_i h_o (1-\cos\alpha) + 2\pi a^2 \rho_i h' [-\cos\Theta]_\alpha^\pi \\
&= \ \frac{4}{3}\pi a^2 \rho_i h_o (1-\cos\alpha) + 2\pi a^2 \rho_i h' (1+\cos\alpha) \\
&= \ 2\pi a^2 \rho_i \Big[\frac{2}{3} h_o (1-\cos\alpha) + h'(1+\cos\alpha) \Big], \qquad (3.89)
\end{aligned}
$$

hence

$$
h' = -\frac{2}{3} \left(\frac{1-\cos\alpha}{1+\cos\alpha} \right) h_o. \qquad (3.90)
$$

Proposition 41 *The LEG coefficients of the expansion of the balanced parabolic surface load function:*

$$
\sigma^{cp}(\Theta) = \sum_{l=0}^{\infty} \sigma_l^{cp} P_l(\cos\Theta) \qquad (3.91)
$$

are

$$
\sigma_l^{cp} = \rho_i h \begin{cases} 0 & \text{if} \quad l = 0 \\ \sigma_l^p + \dfrac{\rho_i h'}{2} [P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)] & \text{if} \quad l \geq 1. \end{cases} \qquad (3.92)
$$

where σ_l^p are the LEG coefficients of the expansion of the unbalanced parabolic load function (3.82) and h' is given by (3.90).

Proof. The proof is into two parts.

1. The case $l = 0$ in (3.92) follows from the condition of vanishing static load mass (see 3.88) and from (3.39).

2. For $l \geq 1$ we recall the general expression (3.35), valid for any AX surface load:

$$
\begin{aligned}
\sigma_l^{cp} &= \frac{2l+1}{2} \int_0^\pi \sigma^{cp}(\Theta) P_l(\cos\Theta) \sin\Theta d\Theta = (3.87) = \\
&= \frac{2l+1}{2} \Big[\int_0^\alpha \sigma^p(\Theta) P_l(\cos\Theta) \sin\Theta d\Theta + \\
&\quad + \int_\alpha^\pi \rho_i h' P_l(\cos\Theta) \sin\Theta d\Theta \Big]
\end{aligned}
$$

$$= \text{(3.84)} = \quad \sigma_l^p + \frac{2l+1}{2}\rho_i h' \int_{-1}^{\cos\alpha} P_l(x)dx$$

$$= \text{(1.46)} = \quad \sigma_l^p + \frac{2l+1}{2}\rho_i h' \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1}$$

$$= \sigma_l^p + \frac{\rho_i h'}{2}[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)] \quad \bullet \tag{3.93}$$

3.3.6 Harmonic load

The *harmonic surface load* is defined as

$$L(t, \Theta) = f(t)\sigma^h(\Theta), \tag{3.94}$$

where $f(t)$ is the load time–history, and the load function is

$$\sigma^h(\Theta) = K P_\ell(\cos\Theta), \tag{3.95}$$

where K is a constant, and ℓ is the harmonic degree of the load. Since $P_\ell(1) = 1$ (1.41), K represents the value of $\sigma^h(\Theta)$ at the pole of the load ($\Theta = 0$).

Proposition 42 *The static mass of the harmonic load is*

$$m_s^h = 4\pi a^2 K \delta_{\ell 0}. \tag{3.96}$$

As a consequence, for $\ell \neq 0$ the harmonic load is self–balanced (see §3.1.3).

Proof. From (3.39), $m_s^h = 4\pi a^2 \sigma_0^h$, and from (3.95): $\sigma_0^h = K\delta_{\ell 0}$. Hence, $m_s^h = 4\pi a^2 K \delta_{\ell 0}$. Since $m_s^h = 0$ for $\ell \neq 0$, the load is self–balanced for $\ell \neq 0$ $\bullet$

Chapter 4

Response to surface loads: displacement and geoid height

In this Chapter, the response of the Earth to surface loads is expressed in terms of surface displacement and geoid height at a given point of the Earth surface. In §4.1 the spectral coefficients of these physical quantities are axiomatically provided and employed to build the response of an elastic Earth to a generic NAX load in the GRF. As a particular case, the elastic response to an AX load in the LRF is also obtained.

In §4.1.1 we generalize the elastic solutions to the case of a viscoelastic Earth, and we subsequently introduce the viscoelastic load–deformation coefficients (LDC) by means of the response to an impulsive, unit load (§4.2.1). This allows for an explicit representation of the response to AX loads in the LRF (§4.3.1), which is subsequently employed in §4.3.2 to construct the GRF response to the AX load by means of simple geometrical arguments.

The viscoelastic response formulas valid for NAX loads are provided in §4.4. They are given both in complex and real forms. In §4.4.3 we view the response to an AX load as a particular case of the response to a NAX load. The resulting formulas are not computationally convenient, but are useful if spectral formulas are sought for an AX load in the GRF. In §4.5 we deal with the ocean corrections to the responses previously introduced.

4.1 Equilibrium of an elastic Earth

The link between the surface loads and the Earth response to the loads can be evidenced solving the equilibrium equations of a Spherically symmetric, Elastic, Incompressible, and Self–Gravitating Earth (in what follows, we will use the acronym SEISG to indicate such an Earth model, and SVISG to denote the its viscoelastic variant). The solution of the equilibrium equations requires a considerable amount of algebra, but the details can be omitted for an understanding of the basic functioning of `TABOO`. The main findings can be summarized as follows.

Proposition 43 *Consider a SEISG Earth model perturbed by a surface load* $L(t, \theta, \lambda) = f(t)\sigma(\theta, \lambda)$, *where* $f(t)$ *is the load time history and* $\sigma(\theta, \lambda)$ *the load function (§3.1.1). "It can be shown that:"*[1]

1. *The equilibrium equations can be split into two decoupled sets of linear, first order ordinary differential equations. The first involves the poloidal fields and the potential perturbation, and the second concerns the toroidal fields. The toroidal part of the displacement field vanishes identically due to the assumed spherical symmetry of the model and to the absence of toroidal terms in the load function expansion (3.13).*

2. *Due to the elastic behavior of the Earth, the response is proportional to the perturbing forces:*

$$\begin{bmatrix} u_{lm} \\ v_{lm} \\ \Phi_{lm} \end{bmatrix} (t, a) = \begin{bmatrix} c_l \\ d_l \\ e_l \end{bmatrix} \sigma_{lm} f(t), \quad (l \geq 2), \tag{4.1}$$

   where

   - u_{lm} *and* v_{lm} *are the poloidal CSH coefficients for the radial and horizontal components*[2] *of the displacement vector (2.6),*
   - Φ_{lm} *is the CSH coefficient of the potential perturbation (2.81),*
   - σ_{lm} *is the CSH coefficient of the load function expansion (3.13),*
   - $c_l, d_l,$ *and* e_l *are model-dependent constants.*

The special cases $l = 0$ *and* $l = 1$ *must be discussed separately, as indicated by Farrell [3].*

[1] The details will be reported in the next editions of this booklet.

[2] Notice that in (2.6) we have used the symbols $u_{lm}^{(1)}$ and $u_{lm}^{(2)}$ instead of u_{lm} and v_{lm}.

As already observed in §2.1.1, the degree 0 coefficients u_{00} *and* v_{00} *vanish identically due to incompressibility. From (2.97) and (2.99) we also have* $\Phi_{00} = 0$*, since we have assumed that the mass of the Earth is not altered by the perturbing process that has caused its deformation. Since the degree 0 coefficients vanish independently from the value of* σ_{00}*, the relationship (4.1) is valid also for* $l = 0$*, provided that* $c_l = d_l = e_l = 0$*.*

The CSH expansion of a given surface load contains, in general, a degree 1 term. At the time of this writing, the formulas required in order to describe the harmonic degree 1 responses have not yet been implemented in ***TABOO****. Thus, in what follows the responses are computed as if* $\sigma_{1m} = 0$ $(m = 0, 1)$*. Since we acknowledge that the degree 1 term may produce non–negligible effects, it will be implemented in the future releases of the software.*

Proposition 44 *The components of surface displacement and the potential perturbation in response to a generic NAX surface load acting on a SEISG Earth model are:*

$$\left\{ \begin{array}{c} u_r \\ u_\theta \\ u_\lambda \\ \Phi \end{array} \right\}^{nax} (t, a, \theta, \lambda) =$$

$$= \sum_{lm} \left\{ \begin{array}{c} u_{lm} \\ v_{lm} \\ v_{lm} \\ \Phi_{lm} \end{array} \right\} (t, a) \cdot \left\{ \begin{array}{c} 1 \\ \partial_\theta \\ \frac{\partial_\lambda}{\sin\theta} \\ 1 \end{array} \right\} Y_{lm}(\theta, \lambda), \tag{4.2}$$

where u_{lm}*,* v_{lm} *and* Φ_{lm} *are given by (4.1).*

Proof. The first three lines of (4.2) are a direct consequence of the general toroidal–poloidal decomposition (2.6), in which the toroidal terms are absent due to proposition 43 above. The fourth derives from (2.81). The expansion (4.2) formally contains all of the harmonic degrees with $l \geq 0$. Actually, since the degree 0 responses vanish for an incompressible Earth, and since the degree 1 components of the load functions are simply ignored (see proposition 43), the expansion begins with the term $l = 2$. This applies to all of the results presented in this Chapter •

Proposition 45 *Here we consider the particular case of an AX load with its axis of symmetry coincident with the z–axis of the GRF. In this configuration, the LRF and the GRF are superimposed, so that $R = r$, $\Theta = \theta$ and $\Lambda = \lambda$, where (R, Θ, Λ) and (r, θ, λ) are the spherical coordinates of a given point in the LRF and GRF, respectively. We show that in this particular geometrical configuration the formulas (4.2), which describe the response of a SEISG Earth, degenerate into*

$$\left\{\begin{array}{c} u_R \\ u_\Theta \\ u_\Lambda \\ \Phi \end{array}\right\}^{AX}(t,a,\Theta) = \sum_{l=0}^{\infty}\left\{\begin{array}{c} u_l \\ v_l \\ 0 \\ \Phi_l \end{array}\right\}^{AX}(t,a)\cdot\left\{\begin{array}{c} 1 \\ \partial_\Theta \\ 0 \\ 1 \end{array}\right\}P_l(\cos\Theta), \tag{4.3}$$

with

$$\left[\begin{array}{c} u_l \\ v_l \\ \Phi_l \end{array}\right]^{AX}(t,a) = \left[\begin{array}{c} c_l \\ d_l \\ e_l \end{array}\right]\sigma_l^{AX} f(t), \tag{4.4}$$

where σ_l^{AX} is the LEG coefficient of the AX load (3.34), and the constants c_l, d_l, and e_l are the same as in (4.1).

Proof. From proposition 9, the CSH coefficients of an AX load function are

$$\sigma_{lm}^{ax} = \frac{1}{\mu_{lm}}\delta_{m0}\sigma_l^{AX}, \tag{4.5}$$

where σ_l^{AX} are the LEG coefficients of the load function (3.34). Thus, from (4.1), the CSH coefficients associated with the radial displacement are:

$$\begin{aligned} u_{lm}(t,a) &= c_l\sigma_{lm}^{ax}f(t) \\ &= {\scriptstyle (4.5)} = \quad c_l\frac{1}{\mu_{lm}}\delta_{m0}\sigma_l^{AX}f(t), \end{aligned} \tag{4.6}$$

so that from the first of (4.2) (recall that $R = r$, $\Theta = \theta$, and $\Lambda = \lambda$), we obtain:

$$\begin{aligned} u_R^{AX}(t,a,\Theta,\Lambda) &= \Sigma_{lm}u_{lm}(t,a)Y_{lm}(\Theta,\Lambda) \\ &= {\scriptstyle (4.6,\,1.21)} = \Sigma_{lm}c_l\frac{1}{\mu_{lm}}\delta_{m0}\sigma_l^{AX}f(t)\mu_{lm}P_{lm}(\cos\Theta)e^{im\Lambda} \\ &= \sum_{l=0}^{\infty}u_l^{AX}(t,a)P_l(\cos\Theta) = \\ &= u_R^{AX}(t,a,\Theta), \end{aligned} \tag{4.7}$$

with

$$u_l^{AX}(t,a) = c_l \sigma_l^{AX} f(t), \tag{4.8}$$

where c_l is the same as in (4.1). The demonstration is similar for the remaining equations in (4.3) •

4.1.1 Extension to viscoelasticity

Till now, we have limited our attention to the elastic response to an applied surface load. The extension to the *linear viscoelastic response* of (4.2) and (4.3) is simple by virtue of the *correspondence principle* of linear viscoelasticity. It states that the equilibrium equations for a viscoelastic body with linear rheology can be obtained from the elastic ones substituting the elastic moduli with appropriate complex moduli, and the field variables with their Laplace–transformed [6].

A general demonstration of the correspondence principle is beyond our purposes. However, it may be useful to illustrate it in the simple case of a Maxwell body, with the aid of one–dimensional mechanical analogies. The elastic and viscous components of the Maxwell rheology are described by

$$\epsilon_e = \frac{\sigma}{2G}, \tag{4.9}$$

and

$$\dot{\epsilon}_v = \frac{\sigma}{2V}, \tag{4.10}$$

where ϵ_e is the elastic strain, $\dot{\epsilon}_v$ is the viscous strain rate, σ is the applied stress (not to be confused with the load function), G is the shear modulus, V is the Maxwell viscosity, and the dot indicates the time derivative. When the elastic and the viscous elements are arranged in series, the total strain rate

$$\dot{\epsilon} = \dot{\epsilon}_e + \dot{\epsilon}_v \tag{4.11}$$

is

$$\dot{\epsilon} = \frac{\dot{\sigma}}{2G} + \frac{\sigma}{2V}, \tag{4.12}$$

which constitutes the rheological law for a Maxwell viscoelastic body in one dimension. Taking the Laplace transform of both sides of (4.12), using (1.97), and assuming vanishing strain and stress at time $t = 0$, we obtain:

$$\epsilon(s) = \frac{\sigma(s)}{2G(s)} \tag{4.13}$$

where $\sigma(s)$ and $\epsilon(s)$ are the Laplace transforms of $\sigma(t)$ and $\epsilon(t)$, respectively, and

$$G(s) = \frac{Gs}{s + G/V} \tag{4.14}$$

is the *complex shear modulus* appropriate for the Maxwell rheology.

A comparison between (4.13) and (4.9) reveals that in the Laplace transformed space the Maxwell constitutive equation is formally identical to the elastic equation in the time domain, provided that the shear modulus is replaced by the s–dependent modulus given by (4.14), and the strain and stress are replaced by their LTs. This statement is not restricted to one–dimensional problems, and can be extended to other linear viscoelastic rheologies. The s–dependence of $G(s)$ determines the form of the s–dependent constants c_l, d_l, and e_l in (4.1).

On the basis of the correspondence principle outlined above, the results (4.2) and (4.3) can be generalized to the linear viscoelastic case as follows.

Proposition 46 *The Laplace–transformed components of surface displacement and potential perturbation induced by a generic NAX surface load acting on a SVISG Earth model are:*

$$\left\{ \begin{array}{c} u_r \\ u_\theta \\ u_\lambda \\ \Phi \end{array} \right\}^{nax} (s, a, \theta, \lambda) = \tag{4.15}$$

$$= \sum_{lm} \left\{ \begin{array}{c} u_{lm} \\ v_{lm} \\ v_{lm} \\ \Phi_{lm} \end{array} \right\} (s, a) \cdot \left\{ \begin{array}{c} 1 \\ \partial_\theta \\ \frac{\partial_\lambda}{\sin\theta} \\ 1 \end{array} \right\} Y_{lm}(\theta, \lambda),$$

with (see 4.1):

$$\left[\begin{array}{c} u_{lm} \\ v_{lm} \\ \Phi_{lm} \end{array} \right] (s, a) = \left[\begin{array}{c} c_l \\ d_l \\ e_l \end{array} \right] (s) \;\; \sigma_{lm} f(s), \tag{4.16}$$

where $f(s)$ is the Laplace–transformed time–history of the surface load, and σ_{lm} are the CSH coefficients of the load function (3.13).

Proposition 47 *The Laplace–transformed components of surface displacement and potential perturbation induced by an AX surface load in the LRF acting on a SVISG Earth model are:*

$$\left\{\begin{array}{c} u_R \\ u_\Theta \\ u_\Lambda \\ \Phi \end{array}\right\}^{AX}(s,a,\Theta) = \sum_{l=0}^{\infty}\left\{\begin{array}{c} u_l \\ v_l \\ 0 \\ \Phi_l \end{array}\right\}^{AX}(s,a)\cdot\left\{\begin{array}{c} 1 \\ \partial_\Theta \\ 0 \\ 1 \end{array}\right\}P_l(\cos\Theta), \tag{4.17}$$

with (see 4.4):

$$\left[\begin{array}{c} u_l \\ v_l \\ \Phi_l \end{array}\right]^{AX}(s,a) = \left[\begin{array}{c} c_l \\ d_l \\ e_l \end{array}\right](s)\ \ \sigma_l^{AX} f(s). \tag{4.18}$$

4.2 Response to an impulsive unit load

The impulsive unit load is a particular AX unit load defined as

$$L(t,\Theta) = \delta(t)\sigma^\delta(\Theta), \tag{4.19}$$

where $\delta(t)$ is the Dirac delta (hence the attribute *impulsive*), and $\sigma^\delta(\Theta)$ is the unit load function (3.48) with LEG expansion coefficients σ_l^δ given by (3.56). From (4.18), valid for any AX load, the LEG coefficients of the response are:

$$\left[\begin{array}{c} u_l \\ v_l \\ \Phi_l \end{array}\right]^{\delta}(s,a) = \left[\begin{array}{c} c_l \\ d_l \\ e_l \end{array}\right](s)\ \ \sigma_l^{\delta}, \tag{4.20}$$

since $\mathrm{LT}[\delta(t)] = 1$ (see Table 1.9).

4.2.1 Load–deformation coefficients

The s–dependent *load–deformation coefficients* (LDC) $h_l(s)$, $l_l(s)$, and $k_l(s)$ are defined with reference to the response to the impulsive unit surface load (4.20):

$$\frac{1}{a}\left[\begin{array}{c} u_l \\ v_l \\ \frac{\Phi_l}{\gamma_o} \end{array}\right]^{\delta}(s,a) \equiv \frac{m_s^\delta}{m_e}\left[\begin{array}{c} h_l \\ l_l \\ 1+k_l \end{array}\right](s), \quad (l \geq 2), \tag{4.21}$$

where a is the reference Earth radius, $\gamma_o = Gm_e/a^2$ is the gravity acceleration at $r = a$ in the unperturbed state (see 2.2.6), m_s^δ is the static mass of the unit load, and m_e is the mass of the Earth. From (4.21) we see that the LDC $h_l(s)$ and $l_l(s)$ are those non–dimensional quantities by which the ratio m^δ/m_e must be multiplied to give the ratios $u_l^\delta(s)/a$ and $l_l^\delta(s)/a$.

The definition of $k_l(s)$ deviates from that of the other two LDC, but it can be reconciled with intuition observing that:

$$\begin{aligned}\Phi_l^\delta(s,a) &= \text{(4.21)} = & a\gamma_o \frac{m_s^\delta}{m_e}[1+k_l(s)] \\ &= \text{(2.72)} = & \frac{Gm_s^\delta}{a}[1+k_l(s)] \\ &\equiv \text{(3.54)} = & \phi_l^p(a)[1+k_l(s)],\end{aligned} \quad (4.22)$$

where $\phi_l^p(a)$ is the degree l LEG coefficient of the potential perturbation due to the presence of a point source on the unperturbed Earth surface. Hence, from above:

$$\begin{aligned}\Phi_l^\delta(s,a) &= \phi_l^p(a) + k_l(s)\phi_l^p(a) \\ &= \phi_l^p(a) + \phi_l^{def}(s,a),\end{aligned} \quad (4.23)$$

where the first term represents the perturbation which would be produced if the Earth were rigid, and the second represents the perturbation which arises from the deformation of the Earth under the load. The latter term is proportional to the former, as it is expected from a linear response to the applied load. The decomposition (4.23) is the counterpart, in the Legendre and Laplace–transformed space, of our previous decomposition of the potential perturbation (2.78). From (4.23), a definition of $k_l(s)$ which better clarifies its meaning is thus

$$k_l(s) = \frac{\phi_l^{def}(s,a)}{\phi_l^p(a)}. \quad (4.24)$$

4.2.2 Form of the LDC

Explicit solution of the equilibrium equations for a linear viscoelastic body subject to an impulsive unit load show that the s–dependent LDC have the form:

$$\left\{\begin{array}{c} h_l \\ l_l \\ k_l \end{array}\right\}(s) = \left\{\begin{array}{c} h_l \\ l_l \\ k_l \end{array}\right\}^E + \sum_{i=1}^{M} \frac{1}{s - s_{li}} \left\{\begin{array}{c} h_{li} \\ l_{li} \\ k_{li} \end{array}\right\}^V, \quad (4.25)$$

where

1. The dimensionless terms h_l^E, l_l^E, and k_l^E are called *elastic LDC*, since they describe the response to the impulsive unit load in the limit of infinite frequency ($s \mapsto -\infty$). Their amplitude does not depend on the viscosity profile of the mantle, but only on the density and shear modulus profile.

2. The terms h_{li}^V, l_{li}^V, and k_{li}^V (i=1,..., M) are the *viscous amplitudes* (or *viscous residues*) of the LDC. They have the physical dimensions of a frequency, and their value depends on the viscosity, density, and rigidity profile.

3. The terms s_{li} (i=1,..., M) describe the relaxation of the Earth to the imposed impulsive unit load. The numerical solution of the equilibrium equation indicates that the quantities s_{li} are real and negative, even if a rigorous proof of this statement valid for any Earth model is still to come. In the case of an incompressible viscoelastic body, the terms s_{li} are the roots of an algebraic equation of degree M, with M depending on the number of layers of the Earth model employed and on the nature of the interfaces between the layers. The reader is referred to [9] and [13] for more insight on this point. The parameters

$$\tau_{li} = -\frac{1}{s_{li}}, \quad (i = 1, \ldots, M) \tag{4.26}$$

are the *relaxation times* of the Earth model.

According to (4.25) and to the points above, the LDC have the following *multi–exponential form* (1.93) in the time domain:

$$\left\{ \begin{array}{c} h_l \\ l_l \\ k_l \end{array} \right\}(t) = \left\{ \begin{array}{c} h_l \\ l_l \\ k_l \end{array} \right\}^E \delta(t) + \sum_{i=1}^{M} e^{s_{li}t} \left\{ \begin{array}{c} h_{li} \\ l_{li} \\ k_{li} \end{array} \right\}^V . \tag{4.27}$$

The readers are referred to [12] and references therein for more detailed discussion about the expansion (4.27).

4.3 Viscoelastic response formulas for AX loads

Here we provide the explicit expression for the Earth response to AX loads. Three forms of the response are given. The first is Laplace-transformed and written in the LRF, while the second is the time domain version of the first. The third form, written in the GRF, includes the second as a particular case.

4.3.1 Response to AX loads in the LRF

The introduction of the LDC by (4.21) allows to rephrase the response of the Earth to an AX load when the LRF coincides with the GRF. We can summarize the main results as follows.

Proposition 48 *The Laplace–transformed response of SVISG Earth model to an AX load in the LRF is:*

$$\left\{ \begin{array}{c} u_R \\ u_\Theta \\ \frac{\Phi}{\gamma_o} \end{array} \right\}^{AX} (s, a, \Theta) = \frac{3}{\bar{\rho}_e} \sum_{l=0}^{\infty} \left\{ \begin{array}{c} h_l \\ l_l \\ 1 + k_l \end{array} \right\} (s) f(s) \frac{\sigma_l^{AX}}{2l+1} \left\{ \begin{array}{c} 1 \\ \partial_\Theta \\ 1 \end{array} \right\} P_l(\cos\Theta), \tag{4.28}$$

where

$$\bar{\rho}_e = \frac{3m_e}{4\pi a^3} \tag{4.29}$$

is the average density of the Earth.

Proof. From the first of (4.17):

$$\begin{aligned} u_R^{AX}(s, a, \Theta) &\equiv \sum_{l=0}^{\infty} u_l^{AX}(s, a) P_l(\cos\Theta) \\ &= \text{(4.18)} = \sum_{l=0}^{\infty} c_l(s) \sigma_l^{AX} f(s) P_l(\cos\Theta) \\ &= \text{(4.20)} = \sum_{l=0}^{\infty} \frac{u_l^\delta(s, a)}{\sigma_l^\delta} \sigma_l^{AX} f(s) P_l(\cos\Theta) \end{aligned}$$

$$= \;{\scriptstyle (4.21,\,3.56)} = \sum_{l=0}^{\infty} a \frac{m_s^\delta}{m_e} \frac{\sigma_l^{AX}}{m_s^\delta \left(\frac{2l+1}{4\pi a^2} \right)} h_l(s) f(s) P_l(\cos\Theta)$$

$$= \frac{4\pi a^3}{m_e} \sum_{l=0}^{\infty} h_l(s) f(s) \frac{\sigma_l^{AX}}{2l+1} P_l(\cos\Theta)$$

$$= \;{\scriptstyle (4.29)} = \frac{3}{\bar{\rho}_e} \sum_{l=0}^{\infty} h_l(s) f(s) \frac{\sigma_l^{AX}}{2l+1} P_l(\cos\Theta), \qquad (4.30)$$

The second and the third of (4.28) can be obtained from the second and the fourth of (4.17) in a similar way •

Proposition 49 *The time–domain response of a SVISG Earth model to an AX load in the LRF is:*

$$\left\{ \begin{array}{c} u_R \\ u_\Theta \\ \frac{\Phi}{\gamma_o} \end{array} \right\}^{AX} (t, a, \Theta) = \frac{3}{\bar{\rho}_e} \sum_{l=0}^{\infty} \left\{ \begin{array}{c} \bar{h}_l \\ \bar{l}_l \\ \bar{k}_l \end{array} \right\} (t) \frac{\sigma_l^{AX}}{2l+1} \left\{ \begin{array}{c} 1 \\ \partial_\Theta \\ 1 \end{array} \right\} P_l(\cos\Theta),$$

where $\bar{\rho}_e = \frac{3m_e}{4\pi a^3}$ *is the average density of the Earth, and*

$$\left[\begin{array}{c} \bar{h}_l \\ \bar{l}_l \\ \bar{k}_l \end{array} \right] (t) \equiv \left[\begin{array}{c} h_l(t) \\ l_l(t) \\ \delta(t) + k_l(t) \end{array} \right] \otimes f(t). \qquad (4.31)$$

Proof. It is sufficient to take the inverse Laplace transform of (4.28) and to recall (1.100). The time convolutions (4.31) will be made explicit in §7.2 for the time–histories of §7.1 •

4.3.2 Response to AX loads in the GRF

When the axis of symmetry of the AX load does not coincide with the z–axis of the GRF, the response can be computed taking advantage of the load symmetry. The time–domain response is first computed in the LRF using (4.31), then it is projected along the unit vectors of the GRF. We use the following notation:

1. (θ, λ) = colatitude and longitude of a point P on the Earth surface in the GRF.

2. $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\lambda)$ = unit vectors at P along the directions of increasing radius, colatitude and longitude in the GRF.

3. (θ_c, λ_c) = colatitude and the longitude of the pole of the load in the GRF (we recall that the pole of the load is the point in which the axis of symmetry of the AX load pierces the Earth surface).

4. Θ = colatitude of P with respect to the pole of the load (i.e., colatitude of P in the LRF).

5. $(\hat{e}_R, \hat{e}_\Theta)$ = unit vectors at P along the directions of increasing radius and colatitude. Notice that $\hat{e}_r = \hat{e}_R$, but $\hat{e}_\theta \neq \hat{e}_\Theta$.

6. X = angle between $\hat{\Theta}$ and $\hat{\theta}$.

Proposition 50 *The time–domain response of SVISG Earth to an AX load in the GRF is:*

$$\left\{ \begin{array}{c} u_r \\ u_\theta \\ u_\lambda \\ \frac{\Phi}{\gamma_o} \end{array} \right\}^{ax} (t, a, \theta, \lambda) = \left\{ \begin{array}{c} u_R \\ \cos X u_\Theta \\ \sin X u_\Theta \\ \frac{\Phi}{\gamma_o} \end{array} \right\}^{AX} (t, a, \Theta), \tag{4.32}$$

where u_R^{AX}, u_Θ^{AX}, and Φ^{AX} are the components of displacements vector and the potential perturbation computed in the LRF by means of (4.31). To evaluate explicitely (4.32) we need to write $\cos X$, $\sin X$ *and* $\cos\Theta$ *in terms of the known quantities $(\theta, \lambda, \theta_c, \lambda_c)$. This can be done with the aid of:*

$$\left\{ \begin{array}{rcl} \cos\Theta & = & \cos\theta\cos\theta_c + \sin\theta\sin\theta_c\cos(\lambda - \lambda_c) \\ \cos X & = & \dfrac{\cos\theta_c - \cos\theta\cos\Theta}{\sin\theta\sqrt{1-\cos^2\Theta}} \\ \sin X & = & \dfrac{\sin(\lambda-\lambda_c)\sin\theta_c}{\sqrt{1-\cos^2\Theta}}, \end{array} \right. \tag{4.33}$$

where we notice that for $\theta_c = 0$ (i.e., when the LRF coincides with the GRF), from above we obtain $\cos X = 1$, $\sin X = 0$, *and* $\cos\Theta = \cos\theta$, *respectively, so that (4.32) reduce to (4.31).*

Proof. The proof is in three steps.

1. We first observe that since the potential perturbation is a scalar quantity, its value at a given point P on the Earth surface is the same in the LFR and in the GRF. This proves the fourth of (4.32).

2. The displacement vector at P can be equivalently expressed into two different ways:

$$\hat{e}_r u_r + \hat{e}_\theta u_\theta + \hat{e}_\lambda u_\lambda \equiv \hat{e}_R u_R + \hat{e}_\Theta u_\Theta, \tag{4.34}$$

where we have used an abbreviated notation for the sake of simplicity. Dotting both sides of (4.34) by $\hat{e}_\theta$, $\hat{e}_\lambda$, and $\hat{e}_r$ we obtain:

$$\begin{aligned} u_\theta &= u_\Theta \ \hat{e}_\Theta \cdot \hat{e}_\theta = u_\Theta \ \cos(\hat{e}_\Theta, \hat{e}_\theta) = u_\Theta \ \cos X \\ u_\lambda &= u_\Theta \ \hat{e}_\Theta \cdot \hat{e}_\lambda = u_\Theta \ \cos(\hat{e}_\Theta, \hat{e}_\lambda) = u_\Theta \ \sin X \\ u_r &= u_R \ \hat{e}_R \cdot \hat{e}_r = u_R, \end{aligned} \tag{4.35}$$

which demonstrate the first three lines of (4.32).

3. The relationships (4.33) can be obtained recalling the basic formulas of the spherical trigonometry. In particular, the first and the second follow from the cosines theorem applied to the spherical triangle of sides Θ, θ, and θ_c, observing that the angle opposite to Θ is $\lambda - \lambda_c$ and that $\sin\Theta = +\sqrt{1-\cos^2\Theta}$. The third is a consequence of the sines theorem applied to the same triangle as above •

4.4 Viscoelastic response formulas for NAX loads

The objective of this section is to the provide the time–domain response formulas for a generic NAX load, which were introduced by (4.15) in the Laplace domain without the aid of the LDC. This will be done in two steps. First, we will derive the expansions in CSH form, then this will be converted into a RSH form which is more convenient for computational purposes.

4.4.1 Response to NAX loads in complex form

Proposition 51 *The time–domain response of a SVISG Earth to a generic NAX load can be expressed as follows in the CSH basis:*

$$\begin{Bmatrix} u_r \\ u_\theta \\ u_\lambda \\ \frac{\Phi}{\gamma_o} \end{Bmatrix}^{nax} (t, a, \theta, \lambda) = \frac{3}{\bar{\rho}_e} \sum_{lm} \begin{Bmatrix} \bar{h}_l \\ \bar{l}_l \\ \bar{l}_l \\ \bar{k}_l \end{Bmatrix} (t) \frac{\sigma_{lm}}{2l+1} \begin{Bmatrix} 1 \\ \partial_\theta \\ \frac{\partial_\lambda}{\sin\theta} \\ 1 \end{Bmatrix} Y_{lm}(\theta, \lambda) \quad (4.36)$$

where $\bar{\rho}_e = \frac{3m_e}{4\pi a^3}$ *is the average density of the Earth (4.29),* σ_{lm} *is the CSH coefficient of the load function (3.14), and the convolutions* $\bar{h}_l(t)$*,* $\bar{l}_l(t)$*, and* $\bar{k}_l(t)$ *are given by (4.31).*

Proof.

$$\begin{aligned} u_r^{nax}(s, a, \theta, \lambda) &= \text{(4.15)} = \sum_{lm} u_{lm}(s, a) Y_{lm} \\ &= \text{(4.16)} = \sum_{lm} c_l(s) f(s) \sigma_{lm} Y_{lm} \\ &= \text{(4.20)} = \sum_{lm} \frac{u_l^\delta(s, a)}{\sigma_l^\delta} f(s) \sigma_{lm} Y_{lm} \\ &= \text{(4.21, 3.56)} = \sum_{lm} \sigma_{lm} a \frac{m_s^\delta}{m_e} \frac{\sigma_{lm}}{m_s^\delta \left(\frac{2l+1}{4\pi a^2} \right)} h_l(s) f(s) Y_{lm} \\ &= \text{(4.29)} = \frac{3}{\bar{\rho}_e} \sum_{lm} \frac{\sigma_{lm}}{2l+1} h_l(s) f(s) Y_{lm}, \end{aligned} \quad (4.37)$$

which can be easily converted to the time domain:

$$\begin{aligned} u_r^{nax}(t, a, \theta, \lambda) &= \text{(1.100)} = \frac{3}{\bar{\rho}_e} \sum_{lm} \frac{\sigma_{lm}}{2l+1} [h_l(t) \otimes f(t)] Y_{lm} \\ &= \text{(4.31)} = \frac{3}{\bar{\rho}_e} \sum_{lm} \frac{\sigma_{lm}}{2l+1} \bar{h}_l(t) Y_{lm}. \end{aligned} \quad (4.38)$$

The remaining three rows of (4.36) can be demonstrated by the same reasoning as above •

4.4.2 Response to NAX loads in real form

Proposition 52 *The time–domain response of a SVISG Earth to a generic NAX load can be expressed in the following RSH forms:*

$$\left\{\begin{array}{c} u_r \\ u_\theta \\ u_\lambda \\ \frac{\Phi}{\gamma_o} \end{array}\right\}^{nax}(t,a,\theta,\lambda) = \frac{3}{\bar\rho_e}\Sigma'_{lm}\left\{\begin{array}{c} \bar h_l \\ \bar l_l \\ \bar l_l \\ \bar k_l \end{array}\right\}(t)\frac{1}{2l+1}\cdot \tag{4.39}$$

$$\cdot\left\{\begin{array}{c} c^\sigma_{lm}\cos m\lambda + s^\sigma_{lm}\sin m\lambda \\ c^\sigma_{lm}\cos m\lambda + s^\sigma_{lm}\sin m\lambda \\ s^\sigma_{lm}\cos m\lambda - c^\sigma_{lm}\sin m\lambda \\ c^\sigma_{lm}\cos m\lambda + s^\sigma_{lm}\sin m\lambda \end{array}\right\}\cdot$$

$$\cdot\left\{\begin{array}{c} 1 \\ \partial_\theta \\ \frac{m}{\sin\theta} \\ 1 \end{array}\right\}P_{lm}(\cos\theta),$$

where c^σ_{lm} and s^σ_{lm} are the cosine and sine coefficients of the RSH expansion of the load function (3.15), and $\bar\rho_e = \frac{3m_e}{4\pi a^3}$ is the average density of the Earth.

Proof.

$$\begin{aligned} u_r^{nax}(t,a,\theta,\lambda) &= (4.38) = \frac{3}{\bar\rho_e}\sum_{lm}\frac{\sigma_{lm}}{2l+1}\bar h_l(t)Y_{lm} \\ &= \Sigma'_{lm}(\bar c_{lm}\cos m\lambda + \bar s_{lm}\sin m\lambda)P_{lm}(\cos\theta), \end{aligned} \tag{4.40}$$

where:

$$\begin{aligned} \left\{\begin{array}{c} \bar c_{lm} \\ \bar s_{lm} \end{array}\right\} &= (1.63) = (2-\delta_{0m})\mu_{lm}\frac{3\bar h_l(t)}{\bar\rho_e(2l+1)}\left\{\begin{array}{c} \mathrm{Re}(\sigma_{lm}) \\ -\ \mathrm{Im}(\sigma_{lm}) \end{array}\right\} \\ &= (3.16) = \frac{3\bar h_l(t)}{\bar\rho_e(2l+1)}\left\{\begin{array}{c} c^\sigma_{lm} \\ s^\sigma_{lm} \end{array}\right\}. \end{aligned} \tag{4.41}$$

Hence we obtain:

$$u_r^{nax}(t,a,\theta,\lambda) = \frac{3}{\bar\rho_e}\Sigma'_{lm}\frac{\bar h(t)}{2l+1}(c^\sigma_{lm}\cos m\lambda + s^\sigma_{lm}\sin m\lambda)P_{lm}(\cos\theta), \tag{4.42}$$

which coincides with the first of (4.39). The demonstration is similar for the other components of the displacement and for the potential perturbation •

4.4.3 AX response as a particular NAX response

Here the general NAX formulas (4.39) are used to compute the response to an AX load in the GRF. This problem has been already solved before by means of a direct approach (see 4.32), but we will see here that when the AX load is viewed as a particular NAX load, it is possible to access directly to the the RSH coefficients of the response, which otherwise are not available.

Proposition 53 *The time–domain RSH response of a SVISG Earth model to an AX load in the GRF is:*

$$\left\{\begin{array}{c} u_r \\ u_\theta \\ u_\lambda \\ \frac{\Phi}{\gamma_o} \end{array}\right\}^{ax}(t,a,\theta,\lambda) = \frac{3}{\bar\rho_e}\Sigma'_{lm}\left\{\begin{array}{c} \bar h_l(t) \\ \bar l_l(t) \\ \bar l_l(t) \\ \bar k_l(t) \end{array}\right\}\frac{1}{2l+1}\cdot \tag{4.43}$$

$$\cdot(2-\delta_{0m})\frac{(l-m)!}{(l+m)!}\sigma_l^{AX}P_{lm}(\cos\theta_c)\cdot$$

$$\cdot\left\{\begin{array}{c} \cos m(\lambda-\lambda_c) \\ \cos m(\lambda-\lambda_c) \\ -\ \sin m(\lambda-\lambda_c) \\ \cos m(\lambda-\lambda_c) \end{array}\right\}\cdot$$

$$\cdot\left\{\begin{array}{c} 1 \\ \partial_\theta \\ \frac{m}{\sin\theta} \\ 1 \end{array}\right\}P_{lm}(\cos\theta),$$

where θ_c and λ_c are the coordinates of the pole of the AX load in the GRF, $\bar\rho_e = \frac{3m_e}{4\pi a^3}$ is the average Earth density, and σ_l^{AX} are the LEG coefficients of the AX load function (3.35).

Proof. We recall from (3.44) that the CSH coefficients of an AX load in the GRF are

$$\sigma_{lm}^{ax} = \frac{4\pi Y_{lm}^*(\theta_c,\lambda_c)}{2l+1}\sigma_l^{AX}, \tag{4.44}$$

where (θ_c,λ_c) are the spherical coordinates of the pole of the load in the GRF, and σ_l^{AX} is the LEG coefficient of the load in the LRF (3.35). Thus, in the particular case of an AX load, the RSH coefficients to be used in (4.39) are:

$$\left\{\begin{array}{c} c^{\sigma}_{lm} \\ s^{\sigma}_{lm} \end{array}\right\} = \text{(3.16)} = (2-\delta_{0m})\mu_{lm}\left\{\begin{array}{c} \text{Re}(\sigma^{ax}_{lm}) \\ -\ \text{Im}(\sigma^{ax}_{lm}) \end{array}\right\}$$
$$= \text{(4.44)} = (2-\delta_{0m})\mu_{lm}\frac{4\pi\sigma^{AX}_{l}}{2l+1}\mu_{lm}P_{lm}(\cos\theta_c)\left\{\begin{array}{c} \cos m\lambda_c \\ \sin m\lambda_c \end{array}\right\}$$
$$= \text{(1.23)} = (2-\delta_{0m})\frac{(l-m)!}{(l+m)!}\sigma^{AX}_{l}P_{lm}(\cos\theta_c)\left\{\begin{array}{c} \cos m\lambda_c \\ \sin m\lambda_c \end{array}\right\}, \tag{4.45}$$

which substituted into (4.39) provides the result (4.43) •

4.5 Ocean corrections

In this section we discuss the effect of a uniform ocean load on the components of the displacement vector and on the potential perturbation. The ocean correction is done introducing an ad–hoc NAX secondary load such that at any time t the total static mass of the system (primary + secondary load) is zero (see §3.1.3). The secondary load has the form:

$$L(t,\theta,\lambda) = f(t)\sigma^{\mathcal{O}}(\theta,\lambda), \tag{4.46}$$

with load function

$$\sigma^{\mathcal{O}}(\theta,\lambda) = -\frac{m_s}{A_{oc}}\mathcal{O}(\theta,\lambda), \tag{4.47}$$

where m_s and $f(t)$ are the static mass of the *primary load* and its time–history, respectively, A_{oc} is the area of the oceans, and $\mathcal{O}(\theta,\lambda)$ is the ocean function (1.80).

According to (4.39), the secondary load so introduced produces the response:

$$\left\{\begin{array}{c} u_r \\ u_\theta \\ u_\lambda \\ \frac{\Phi}{\gamma_o} \end{array}\right\}^{oc}(t,a,\theta,\lambda) = -\frac{3}{\bar{\rho}_e}\left(\frac{m_s}{A_{oc}}\right)\Sigma'_{lm}\left\{\begin{array}{c} \bar{h}_l \\ \bar{l}_l \\ \bar{l}_l \\ \bar{k}_l \end{array}\right\}(t)\frac{1}{2l+1}\,\cdot$$

$$\cdot \left\{ \begin{array}{c} c^{\mathcal{O}}_{lm} \cos m\lambda + s^{\mathcal{O}}_{lm} \sin m\lambda \\ c^{\mathcal{O}}_{lm} \cos m\lambda + s^{\mathcal{O}}_{lm} \sin m\lambda \\ s^{\mathcal{O}}_{lm} \cos m\lambda - c^{\mathcal{O}}_{lm} \sin m\lambda \\ c^{\mathcal{O}}_{lm} \cos m\lambda + s^{\mathcal{O}}_{lm} \sin m\lambda \end{array} \right\} \cdot$$

$$\cdot \left\{ \begin{array}{c} 1 \\ \partial_\theta \\ \frac{m}{\sin\theta} \\ 1 \end{array} \right\} P_{lm}(\cos\theta), \tag{4.48}$$

where $c^{\mathcal{O}}_{lm}$ and $s^{\mathcal{O}}_{lm}$ are the cosine and sine coefficients of the RSH expansion of the ocean function (1.84), respectively.

The ratio (m_s/A_{oc}) in (4.48) can be transformed in a more meaningful form. In fact, the area of the surface of the oceans can be written as $A_{oc} = 4\pi a^2 c^{\mathcal{O}}_{00}$ (see 1.85), and the static mass of NAX loads is $m_s = 4\pi a^2 c^{\sigma}_{00}$ (3.18). Hence:

$$\frac{m_s}{A_{oc}} = \frac{c^{\sigma}_{00}}{c^{\mathcal{O}}_{00}} \quad \text{(NAX loads)}. \tag{4.49}$$

In the case of AX loads, from Table (1.7) we have $c^{\sigma}_{00} = \sigma^{AX}_0$, where σ^{AX}_0 is the degree 0 LEG coefficient of the primary load function expansion, so that:

$$\frac{m_s}{A_{oc}} = \frac{\sigma^{AX}_0}{c^{\mathcal{O}}_{00}} \quad \text{(AX loads)}. \tag{4.50}$$

The above results are summarized in the two following propositions, for NAX and AX loads, respectively.

Proposition 54 *The time–domain response of a SVISG Earth model to a NAX load balanced by means of a secondary ocean load is:*

$$\left\{ \begin{array}{c} u_r \\ u_\theta \\ u_\lambda \\ \frac{\Phi}{\gamma_o} \end{array} \right\}^{nax+oc} (t,a,\theta,\lambda) = \left\{ \begin{array}{c} u_r \\ u_\theta \\ u_\lambda \\ \frac{\Phi}{\gamma_o} \end{array} \right\}^{nax} (t,a,\theta,\lambda) + \left\{ \begin{array}{c} u_r \\ u_\theta \\ u_\lambda \\ \frac{\Phi}{\gamma_o} \end{array} \right\}^{oc} (t,a,\theta,\lambda), \tag{4.51}$$

where the first term on the righthand side is given by (4.39), while the second is given by (4.48) with (4.49).

Proposition 55 *The time–domain response of a SVISG Earth model to an AX load balanced by means of a secondary ocean load is:*

$$\left\{\begin{array}{c} u_r \\ u_\theta \\ u_\lambda \\ \frac{\Phi}{\gamma_o} \end{array}\right\}^{ax+oc}(t,a,\theta,\lambda) = \left\{\begin{array}{c} u_r \\ u_\theta \\ u_\lambda \\ \frac{\Phi}{\gamma_o} \end{array}\right\}^{ax}(t,a,\theta,\lambda) + \left\{\begin{array}{c} u_r \\ u_\theta \\ u_\lambda \\ \frac{\Phi}{\gamma_o} \end{array}\right\}^{oc}(t,a,\theta,\lambda), \qquad (4.52)$$

where the first term on the righthand side is given by (4.32) or by (4.43), and the second is given (4.48) with (4.50).

Chapter 5

Response to surface loads: Stokes coefficients and inertia variations

In this Chapter we provide the expressions for the variations of the Stokes coefficients and of the inertia tensor in response to surface loads.

5.1 Stokes coefficients variations

Proposition 56 *The variations of the Stokes coefficients due to the action of a NAX load on a SVISG Earth model are:*

$$\left\{ \begin{array}{c} \delta c_{lm} \\ \delta s_{lm} \end{array} \right\}^{nax}(t) = \frac{4\pi a^2}{m_e} \frac{\bar{k}_l(t)}{2l+1} \left\{ \begin{array}{c} c^{\sigma}_{lm} \\ s^{\sigma}_{lm} \end{array} \right\}, \tag{5.1}$$

where m_e is the mass of the Earth, a is the reference Earth radius, and $\bar{k}_l(t)$ is given by the third of (4.31). Based on the arguments presented in §2.2.7, the result above is valid for $l \geq 2$. Its fully normalized form can be obtained from (1.71).

Proof. From (2.89), the RSH expansion of the geoid height is:

$$N(t, \theta, \lambda) = a\ \Sigma'_{lm}(\delta c_{lm}(t) \cos m\lambda + \delta s_{lm}(t) \sin m\lambda) P_{lm}(\cos\theta), \tag{5.2}$$

where a is the radius of the Earth in the reference state (see §2.2.6), $\delta c_{lm}(t)$ and $\delta s_{lm}(t)$ ($l \geq 2$) are the variations of the Stokes coefficients in response to generic perturbing forces (see 2.90). If these forces are associated with a general NAX load acting at the Earth surface, using the fourth of (4.39) and recalling (2.83) we can also write:

$$N(t,\theta,\lambda) = \frac{3}{\bar{\rho}_e}\Sigma'_{lm}\frac{\bar{k}_l(t)}{2l+1}\cdot(c^{\sigma}_{lm}\cos m\lambda + s^{\sigma}_{lm}\sin m\lambda)P_{lm}(\cos\theta), \quad (5.3)$$

which can be compared with (5.2) term by term to provide the result (5.1) •

Proposition 57 *The variations of the Stokes coefficients due to the action of a NAX load balanced on a secondary ocean load with realistic shape on a SVISG Earth model are:*

$$\left\{\begin{array}{c}\delta c_{lm}\\ \delta s_{lm}\end{array}\right\}^{nax+oc}(t) = \left\{\begin{array}{c}\delta c_{lm}\\ \delta s_{lm}\end{array}\right\}^{nax}(t) + \left\{\begin{array}{c}\delta c_{lm}\\ \delta s_{lm}\end{array}\right\}^{oc}(t), \quad (5.4)$$

where the first term on the righthand side is given by (5.1), and:

$$\left\{\begin{array}{c}\delta c_{lm}\\ \delta s_{lm}\end{array}\right\}^{oc}(t) = -\frac{4\pi a^2}{m_e}\frac{c^{\sigma}_{00}}{c^{\mathcal{O}}_{00}}\frac{\bar{k}_l(t)}{2l+1}\left\{\begin{array}{c}c^{\mathcal{O}}_{lm}\\ s^{\mathcal{O}}_{lm}\end{array}\right\}, \quad (5.5)$$

where $(c^{\mathcal{O}}_{lm}, s^{\mathcal{O}}_{lm})$ are the RSH coefficients of the ocean function (1.84).

Proof. From (4.48) and (2.83), the ocean correction to the geoid height is:

$$N^{oc}(t,\theta,\lambda) = -\frac{3}{\bar{\rho}_e}\frac{c^{\sigma}_{00}}{c^{\mathcal{O}}_{00}}\Sigma'_{lm}\frac{\bar{k}_l(t)}{2l+1}(c^{\mathcal{O}}_{lm}\cos m\lambda + s^{\mathcal{O}}_{lm}\sin m\lambda)P_{lm}(\cos\theta), \quad (5.6)$$

which can be rewritten as

$$N^{oc}(t,\theta,\lambda) = \Sigma'_{lm}(\delta c^{oc}_{lm}(t)\cos m\lambda + \delta s^{oc}_{lm}(t)\sin m\lambda)P_{lm}(\cos\theta) \quad (5.7)$$

with coefficients given by (5.5), where we have recalled that the average Earth density is $\bar{\rho}_e = \frac{3m_e}{4\pi a^3}$ •

Proposition 58 *The variations of the Stokes coefficients due to the action of an AX load on a SVISG Earth model are:*

$$\left\{\begin{array}{c}\delta c_{lm}\\ \delta s_{lm}\end{array}\right\}^{ax}(t) = \frac{4\pi a^2}{m_e}\frac{\sigma^{AX}_l\bar{k}_l(t)}{2l+1}(2-\delta_{0m})\frac{(l-m)!}{(l+m)!}P_{lm}(\cos\theta_c)\left\{\begin{array}{c}\cos m\lambda_c\\ \sin m\lambda_c\end{array}\right. \quad (5.8)$$

where m_e *is the mass of the Earth,* a *is the reference Earth radius,* (θ_c, λ_c) *are the coordinates of the pole of the load in the GRF,* $\bar{k}_l(t)$ *is given by the third of (4.31), and* σ_l^{AX} *is the degree* l *coefficient of the LEG expansion of the load function in the LRF (3.35). Based on the arguments presented in §2.2.7, the result above is valid for* $l \geq 2$*. Its fully normalized form can be obtained from (1.71).*

Proof. According to the arguments of §4.4.3, the AX load can be viewed as a particular NAX load. By substitution of (4.45) into (5.1) we directly obtain (5.8) •

Proposition 59 *The variations of the Stokes coefficients due to the action of a AX load balanced on a secondary ocean load with realistic shape on a SVISG Earth model are:*

$$\left\{ \begin{array}{c} \delta c_{lm} \\ \delta s_{lm} \end{array} \right\}^{ax+oc}(t) = \left\{ \begin{array}{c} \delta c_{lm} \\ \delta s_{lm} \end{array} \right\}^{ax}(t) + \left\{ \begin{array}{c} \delta c_{lm} \\ \delta s_{lm} \end{array} \right\}^{oc}(t), \tag{5.9}$$

where the first term on the righthand side is given by (5.8), and:

$$\left\{ \begin{array}{c} \delta c_{lm} \\ \delta s_{lm} \end{array} \right\}^{oc}(t) = -\frac{4\pi a^2}{m_e} \frac{\sigma_0^{AX}}{c_{00}^{\mathcal{O}}} \frac{\bar{k}_l(t)}{2l+1} \left\{ \begin{array}{c} c_{lm}^{\mathcal{O}} \\ s_{lm}^{\mathcal{O}} \end{array} \right\}, \tag{5.10}$$

where $(c_{lm}^{\mathcal{O}}, s_{lm}^{\mathcal{O}})$ *are the RSH coefficients of the ocean function (1.84).*

Proof. The ocean correction for an AX load has exactly the same form of that valid for a NAX load, given by (5.5). We only notice that for an AX load, $c_{00}^{\sigma}=\sigma_0^{AX}$ (see also §4.5), which proves (5.10) •

5.2 Inertia variations

Proposition 60 *The change of the (normalized) inertia tensor due to the action of a NAX load on a SVISG Earth model is:*

$$\left\{ \begin{array}{c} \delta\bar{i}_{xx} \\ \delta\bar{i}_{yy} \\ \delta\bar{i}_{zz} \\ \delta\bar{i}_{xz} \\ \delta\bar{i}_{yz} \\ \delta\bar{i}_{xy} \end{array} \right\}^{nax}(t) \;=\; \frac{4\pi a^2}{5 m_e} \bar{k}_2(t) \left\{ \begin{array}{c} c_{20}^{\sigma}/3 - 2c_{22}^{\sigma} \\ c_{20}^{\sigma}/3 + 2c_{22}^{\sigma} \\ -2c_{20}^{\sigma}/3 \\ c_{21}^{\sigma} \\ s_{21}^{\sigma} \\ -2s_{22}^{\sigma} \end{array} \right\}, \tag{5.11}$$

where a is the reference radius of the Earth (§2.2.6), m_e is its mass, and $\bar{k}_2(t)$ is given by the third of (4.31) computed for degree $l = 2$. Due to incompressibility, the trace of the inertia tensor is unchanged: $\delta\bar{i}_{xx}+\delta\bar{i}_{yy}+\delta\bar{i}_{zz} = 0$ (see 2.104 and [10]).

Proof. The variation of the (normalized) inertia tensor in response to a generic perturbing force is expressed by (2.105) in terms of the variations of the degree 2 Stokes coefficients. In the particular case of a NAX surface load, the latter can be obtained by (5.1) and substituted into (2.105) to demonstrate (5.11) •

Proposition 61 *The change of the (normalized) inertia tensor due to the action of a NAX load balanced on a secondary ocean load with realistic shape on a SVISG Earth model is:*

$$\left\{\begin{array}{c}\delta\bar{i}_{xx}\\ \delta\bar{i}_{yy}\\ \delta\bar{i}_{zz}\\ \delta\bar{i}_{xz}\\ \delta\bar{i}_{yz}\\ \delta\bar{i}_{xy}\end{array}\right\}^{nax+oc}(t) \quad = \left\{\begin{array}{c}\delta\bar{i}_{xx}\\ \delta\bar{i}_{yy}\\ \delta\bar{i}_{zz}\\ \delta\bar{i}_{xz}\\ \delta\bar{i}_{yz}\\ \delta\bar{i}_{xy}\end{array}\right\}^{nax}(t) + \left\{\begin{array}{c}\delta\bar{i}_{xx}\\ \delta\bar{i}_{yy}\\ \delta\bar{i}_{zz}\\ \delta\bar{i}_{xz}\\ \delta\bar{i}_{yz}\\ \delta\bar{i}_{xy}\end{array}\right\}^{oc}(t), \tag{5.12}$$

where the first term on the righthand side is given by (5.11), and

$$\left\{\begin{array}{c}\delta\bar{i}_{xx}\\ \delta\bar{i}_{yy}\\ \delta\bar{i}_{zz}\\ \delta\bar{i}_{xz}\\ \delta\bar{i}_{yz}\\ \delta\bar{i}_{xy}\end{array}\right\}^{oc}(t) = -\frac{4\pi a^2}{5m_e}\frac{c_{00}^{\sigma}}{c_{00}^{\mathcal{O}}}\bar{k}_2(t)\left\{\begin{array}{c}c_{20}^{\mathcal{O}}/3 - 2c_{22}^{\mathcal{O}}\\ c_{20}^{\mathcal{O}}/3 + 2c_{22}^{\mathcal{O}}\\ -2c_{20}^{\mathcal{O}}/3\\ c_{21}^{\mathcal{O}}\\ s_{21}^{\mathcal{O}}\\ -2s_{22}^{\mathcal{O}}\end{array}\right\}. \tag{5.13}$$

Proof. The ocean correction to the variations of the Stokes coefficients is given by (5.5). To demonstrate (5.13), it suffices to compute the degree 2 variations and to recall (2.105) •

Proposition 62 *The change of the (normalized) inertia tensor due to the* *action of an AX load on a SVISG Earth model is:*

$$\begin{Bmatrix} \delta\bar{i}_{xx} \\ \delta\bar{i}_{yy} \\ \delta\bar{i}_{zz} \\ \delta\bar{i}_{xz} \\ \delta\bar{i}_{yz} \\ \delta\bar{i}_{xy} \end{Bmatrix}^{ax}(t) = \frac{2\pi a^2}{15m_e}\sigma_2^{AX}\bar{k}_2(t)\begin{Bmatrix} 2P_{20}(\cos\theta_c) - P_{22}(\cos\theta_c)\cos 2\lambda_c \\ 2P_{20}(\cos\theta_c) + P_{22}(\cos\theta_c)\cos 2\lambda_c \\ -4P_{20}(\cos\theta_c) \\ 2P_{21}(\cos\theta_c)\cos\lambda_c \\ 2P_{21}(\cos\theta_c)\sin\lambda_c \\ -P_{22}(\cos\theta_c)\sin 2\lambda_c \end{Bmatrix}, \tag{5.14}$$

where a is the reference radius of the Earth, m_e is its mass, σ_2^{AX} is the degree 2 LEG coefficient of the load function, $\bar{k}_2(t)$ is given by the third of (4.31) computed for harmonic degree $l = 2$, and (θ_c, λ_c) are the coordinates of the pole of the load in the GRF. Due to incompressibility, the trace of the inertia tensor is unchanged: $\delta\bar{i}_{xx}+\delta\bar{i}_{yy}+\delta\bar{i}_{zz} = 0$ (see 2.104 and [10]).

Proof. We recall that an AX load can always be viewed as a special NAX load. In particular, the c_{2m}^{σ} and s_{2m}^{σ} coefficients in (5.11) can be replaced by their equivalent AX expressions given by (4.45) to obtain the result (5.14) in a straightforward manner •

Proposition 63 *The change of the (normalized) inertia tensor due to the action of an AX load balanced on a secondary ocean load with realistic shape on a SVISG Earth model is:*

$$\begin{Bmatrix} \delta\bar{i}_{xx} \\ \delta\bar{i}_{yy} \\ \delta\bar{i}_{zz} \\ \delta\bar{i}_{xz} \\ \delta\bar{i}_{yz} \\ \delta\bar{i}_{xy} \end{Bmatrix}^{ax+oc}(t) = \begin{Bmatrix} \delta\bar{i}_{xx} \\ \delta\bar{i}_{yy} \\ \delta\bar{i}_{zz} \\ \delta\bar{i}_{xz} \\ \delta\bar{i}_{yz} \\ \delta\bar{i}_{xy} \end{Bmatrix}^{ax}(t) + \begin{Bmatrix} \delta\bar{i}_{xx} \\ \delta\bar{i}_{yy} \\ \delta\bar{i}_{zz} \\ \delta\bar{i}_{xz} \\ \delta\bar{i}_{yz} \\ \delta\bar{i}_{xy} \end{Bmatrix}^{oc}(t), \tag{5.15}$$

where the first term on the righthand side is given by (5.14), and the ocean correction is:

$$\begin{Bmatrix} \delta\bar{i}_{xx} \\ \delta\bar{i}_{yy} \\ \delta\bar{i}_{zz} \\ \delta\bar{i}_{xz} \\ \delta\bar{i}_{yz} \\ \delta\bar{i}_{xy} \end{Bmatrix}^{oc}(t) = -\frac{4\pi a^2}{5m_e}\bar{k}_2(t)\frac{\sigma_0^{AX}}{c_{00}^{\mathcal{O}}}\begin{Bmatrix} c_{20}^{\mathcal{O}}/3 - 2c_{22}^{\mathcal{O}} \\ c_{20}^{\mathcal{O}}/3 + 2c_{22}^{\mathcal{O}} \\ -2c_{20}^{\mathcal{O}}/3 \\ c_{21}^{\mathcal{O}} \\ s_{21}^{\mathcal{O}} \\ 2s_{22}^{\mathcal{O}} \end{Bmatrix}. \tag{5.16}$$

Proof. The ocean correction for an AX load has the same form of that valid for a NAX load, given by (5.13). We only notice that for an AX load, $c_{00}^{\sigma}=\sigma_0^{AX}$ (see also §4.5), which proves (5.16) •

Chapter 6

Response to surface loads: baselines variations

This short Chapter is devoted to the study of the response of the Earth to surface loads in terms of baselines evolutions. Our purpose is to provide a tool for comparing model predictions with actual GPS or VLBI observations. As explained in the `TABOO` user guide, the software is particularly designed to deal with the NASA GSFC VLBI baselines network[1], but it can be also adapted to study the time evolution of baselines connecting sites belonging to other geodetic networks, being them real or built *ad hoc* by the user. The mathematics employed here is quite straightforward, but as far as I know it is not reported elsewhere. Any comment is appreciated.

6.1 Baseline unit vectors

We consider two points P_1 and P_2 on the Earth surface, with position vectors $\vec{r}_i = (x_i, y_i, z_i)$ $(i = 1, 2)$ in a Cartesian orthogonal reference frame with origin in the CM of the Earth, and unit vectors $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_z$. The points P_1 and P_2 correspond to two specific *sites* (e. g. VLBI stations), connected by a *rectilinear segment* called *baseline*. In the following, we will denote with (λ_i, θ_i) $(i = 1, 2)$ the longitude and colatitude of the two sites, respectively. Using (1.1), the Cartesian coordinates (x_i, y_i, z_i) $(i = 1, 2)$ can be re-written

[1]NASA Goddard Space Flight Center VLBI Group, 1999. Data products available electronically at `http://lupus.gsfc.nasa.gov/vlbi.html`.

in terms of the spherical coordinates:

$$\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = a \begin{bmatrix} \sin\theta_i \cos\lambda_i \\ \sin\theta_i \sin\lambda_i \\ \cos\theta_i \end{bmatrix} \quad (i = 1, 2), \tag{6.1}$$

where a is the reference radius of the Earth (see §2.2.6).

In order to describe the motion of the two sites it is conventional to introduce a new Cartesian orthogonal reference frame with origin in P_2 and unit vectors defined as:

$$\hat{l} = \frac{\vec{r}_2 - \vec{r}_1}{\|\vec{r}_2 - \vec{r}_1\|} \tag{6.2}$$

$$\hat{t} = \frac{\vec{r}_2 \times \vec{r}_1}{\|\vec{r}_2 \times \vec{r}_1\|} \tag{6.3}$$

$$\hat{\nu} = \hat{l} \times \hat{t}, \tag{6.4}$$

which are called *length*, *transverse*, and *vertical* baseline (unit) vectors. They can be decomposed as follows along the axes of the Oxyz frame:

$$\begin{aligned} \hat{l} &= l_x\hat{e}_x + l_y\hat{e}_y + l_z\hat{e}_z \\ \hat{t} &= t_x\hat{e}_x + t_y\hat{e}_y + t_z\hat{e}_z \\ \hat{\nu} &= \nu_x\hat{e}_x + \nu_y\hat{e}_y + \nu_z\hat{e}_z \end{aligned} \tag{6.5}$$

where:

$$\begin{bmatrix} l_x \\ l_y \\ l_z \end{bmatrix} = \frac{1}{C} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} \tag{6.6}$$

$$\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = \frac{1}{D} \begin{bmatrix} y_2 z_1 - y_1 z_2 \\ z_2 x_1 - z_1 x_2 \\ x_2 y_1 - x_1 y_2 \end{bmatrix} \tag{6.7}$$

$$\begin{bmatrix} \nu_x \\ \nu_y \\ \nu_z \end{bmatrix} = \begin{bmatrix} l_y t_z - l_z t_y \\ l_z t_x - l_x t_z \\ l_x t_y - l_y t_x \end{bmatrix} \tag{6.8}$$

with

$$C = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}, \tag{6.9}$$

and

$$D = \sqrt{(z_1 y_2 - y_1 z_2)^2 + (x_1 z_2 - z_1 x_2)^2 + (x_2 y_1 - x_1 y_2)^2}. \tag{6.10}$$

6.2 Baseline rates

We denote by $\vec{v}(i;t)$ $(i = 1, 2)$ the velocity of the two sites connected by the baseline in the Oxyz reference frame at time t. The vector $\vec{v}(i;t)$ $(i = 1, 2)$ can be equivalently decomposed along the axes of the Oxyz frame and the axes of the baseline reference frame:

$$\vec{v}(i;t) = v_x\hat{e}_x + v_y\hat{e}_y + v_z\hat{e}_z \tag{6.11}$$

$$= v_l\hat{l} + v_t\hat{t} + v_\nu\hat{\nu} = {\scriptstyle (6.5)} = \tag{6.12}$$

$$\begin{array}{ccccccccc} = & v_l\,(l_x\hat{e}_x & + & l_y\hat{e}_y & + & l_z\hat{e}_z) & + \\ + & v_t\,(t_x\hat{e}_x & + & t_y\hat{e}_y & + & t_z\hat{e}_z) & + \\ + & v_\nu\,(\nu_x\hat{e}_x & + & \nu_y\hat{e}_y & + & \nu_z\hat{e}_z) & = \end{array} \tag{6.13}$$

$$\begin{array}{ccccccccc} = & \hat{e}_x\,(l_x v_l & + & t_x v_t & + & \nu_x v_\nu) & + \\ + & \hat{e}_y\,(l_y v_l & + & t_y v_t & + & \nu_y v_\nu) & + \\ + & \hat{e}_z\,(l_z v_l & + & t_z v_t & + & \nu_z v_\nu), & \end{array} \tag{6.14}$$

hence, comparing (6.11) with (6.14), we obtain:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}(i;t) = B(1,2;t)\begin{bmatrix} v_l \\ v_t \\ v_\nu \end{bmatrix}(i;t), \quad (i = 1, 2), \tag{6.15}$$

where the elements of the array

$$B(1,2) = \begin{bmatrix} l_x & t_x & \nu_x \\ l_y & t_y & \nu_y \\ l_z & t_z & \nu_z \end{bmatrix} \tag{6.16}$$

only depend on the longitude and colatitude of points P_1 and P_2. We can easily invert (6.15) observing that $B(1,2)$ is orthogonal, since it describes a rotation from the Oxyz reference frame to the baseline reference frame:

$$\begin{bmatrix} v_l \\ v_t \\ v_\nu \end{bmatrix}(i;t) = B^t(1,2)\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}(i;t), \quad (i = 1, 2) \tag{6.17}$$

where $B^t(1,2)$ is the transpose of $B(1,2)$.

It is now convenient to introduce the spherical components of $\vec{v}(i;t)$, since the response formulas of Chapter 4 are all given in spherical form. This can be done recalling (1.7):

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}(i;t) = G(i) \begin{bmatrix} v_r \\ v_\theta \\ v_\lambda \end{bmatrix}(i;t), \quad (i = 1, 2) \tag{6.18}$$

with

$$G(i) = \begin{bmatrix} \sin\theta_i \cos\lambda_i & \cos\theta_i \cos\lambda_i & -\sin\lambda_i \\ \sin\theta_i \sin\lambda_i & \cos\theta_i \sin\lambda_i & \cos\lambda_i \\ \cos\theta_i & -\sin\lambda_i & 0 \end{bmatrix}, \quad (i = 1, 2) \tag{6.19}$$

where θ_i and λ_i denote the colatitude and the longitude of the site i in the GRF. We therefore obtain the final result:

$$\begin{bmatrix} v_l \\ v_t \\ v_\nu \end{bmatrix}(i;t) = B^t(1,2)G(i) \begin{bmatrix} v_r \\ v_\theta \\ v_\lambda \end{bmatrix}(i;t), \quad (i = 1, 2) \tag{6.20}$$

which allows to convert the spherical components of velocity into the baseline components.

Proposition 64 *We consider a baseline connecting two sites placed at points P_1 and P_2 on the Earth surface. The evolution of the baseline P_1-P_2 at a given time t is determined specifying the velocity of the site (2) relative to site (1). This can be done introducing the three baselines components rates defined as*

$$\begin{Bmatrix} \dot{L} \\ \dot{T} \\ \dot{V} \end{Bmatrix} = \begin{Bmatrix} v_l(2;t) & - & v_l(1;t) \\ v_t(2;t) & - & v_t(1;t) \\ v_\nu(2;t) & - & v_\nu(1;t) \end{Bmatrix}, \tag{6.21}$$

where $v_l(i;t)$, $v_t(i;t)$, and $v_\nu(i;t)$ are computed using (6.20).

Chapter 7

Appendices

7.1 Time–histories and their derivatives

`TABOO` can deal with AX and NAX loads characterized by various kinds of time–histories. For *load time–history* we indicate the function $f(t)$ which allows to write

$$L(\theta, \lambda, t) = f(t)\sigma(\theta, \lambda), \tag{7.1}$$

where t is time, and the surface load function $\sigma(\theta, \lambda)$ has been introduced in §3.1.1 and made explicit in various forms in the ensuing sections. In the following we define the set of time–histories available in `TABOO`, together with ptheir time–derivatives.

Since the definition of the time–histories is often made easy by the use of the step function $H(t)$ (1.91), their time–derivatives will contain delta–like terms (see 1.92). In the formulas that follow and (obviously) in their implementation in `TABOO`, these terms *are not included*. So the reader is warned that the derivatives given here differ from the 'true' ones from functions equal to zero almost everywhere.

7.1.1 $f_0(t)$: Instantaneous loading

The load is absent for times $-\infty \leq t < 0$, and constant for time $t \geq 0$:

$$\begin{cases} f_0(t) &= H(t) \\ f_0'(t) &= 0. \end{cases} \tag{7.2}$$

7.1.2 $f_1(t)$: Instantaneous un–loading

The load is constant for $-\infty \leq t < 0$, and absent for $t \geq 0$:

$$\left\{ \begin{array}{lcl} f_1(t) & = & 1 - H(t) \\ f_1'(t) & = & 0. \end{array} \right. \tag{7.3}$$

7.1.3 $f_2(t)$: Instantaneous loading and un–loading

The load is absent for $-\infty \leq t < -\tau$, constant for $-\tau \leq t < 0$, and again absent for $t \geq 0$, where $\tau > 0$:

$$\left\{ \begin{array}{lcl} f_2(t) & = & H(t+\tau) - H(t) \\ f_2'(t) & = & 0. \end{array} \right. \tag{7.4}$$

7.1.4 $f_3(t)$: Simple deglaciation

The load is constant for $-\infty \leq t < 0$, it is turned off at a constant rate for $0 \leq t < \tau$, and it is absent for $t \geq \tau$, where $\tau > 0$:

$$\left\{ \begin{array}{lcl} f_3(t) & = & 1 - H(t) + (1 - t/\tau)[H(t) - H(t-\tau)] \\ f_3'(t) & = & -(1/\tau)[H(t) - H(t-\tau)]. \end{array} \right. \tag{7.5}$$

7.1.5 $f_4(t)$: Saw-tooth

The load is characterized by a periodic saw–tooth time–history in which loading and unloading phases occur at constant rates. The length of each loading phase is τ, and the length of the unloading phase is δ. The time-history includes N_r phases of loading and unloading in addition to the more recent, so that their total number $N_r + 1$. The end of the last loading phase (i.e. the beginning of the last unloading phase) occurs at time t=0.

It can be easily realized that the time–history restricted to the n^{th} phase is

$$\begin{aligned} \varphi_n(t) \;=\; & [H(t+n\theta+\tau) - H(t+n\theta)]\varphi_n^{\uparrow}(t) + \\ & [H(t+n\theta) - H(t+n\theta-\delta)]\varphi_n^{\downarrow}(t) \\ & (n = 0, 1, \ldots, N_r), \quad -n\theta - \tau \leq t \leq -n\theta + \delta, \end{aligned} \tag{7.6}$$

where

$$\theta \equiv \tau + \delta, \tag{7.7}$$

and the functions

$$\varphi_n^{\uparrow}(t) = +\frac{t}{\tau} + \frac{n\theta + \tau}{\tau} \tag{7.8}$$

$$\varphi_n^{\downarrow}(t) = -\frac{t}{\delta} - \frac{n\theta - \delta}{\delta} \tag{7.9}$$

describe the phases of loading and of unloading, respectively. The time–history and its time derivative are

$$\begin{cases} f_4(t) = \sum_{n=0}^{N_r} \varphi_n(t) \\ f_4'(t) = \sum_{n=0}^{N_r} \varphi_n'(t), \end{cases} \tag{7.10}$$

where

$$\begin{aligned} \varphi_n'(t) &= [H(t + n\theta + \tau) - H(t + n\theta)]\varphi_n'^{\uparrow}(t) + \\ &\quad [H(t + n\theta) - H(t + n\theta - \delta)]\varphi_n'^{\downarrow}(t) \\ &\quad (n = 0, 1, \dots, N_r), \quad -n\theta - \tau \leq t \leq -n\theta + \delta. \end{aligned} \tag{7.11}$$

and

$$\varphi_n'^{\uparrow}(t) = +\frac{1}{\tau} \tag{7.12}$$

$$\varphi_n'^{\downarrow}(t) = -\frac{1}{\delta}. \tag{7.13}$$

7.1.6 $f_5(t)$: Sinusoidal loading

For $-\infty \leq t \leq +\infty$ the load evolves according to

$$\begin{cases} f_5(t) = \frac{1}{2}(1 + \sin \omega t) \\ f_5'(t) = \frac{\omega}{2} \cos \omega t, \end{cases} \tag{7.14}$$

where $\omega \equiv \frac{2\pi}{T}$ and T is the period of the sinusoid ($T > 0$).

7.1.7 $f_6(t)$: Piecewise linear

The load is characterized by a piecewise continuous, linear time-history. For $0 \equiv t_0 \leq t < t_N$ (piecewise linear phase), the time–history is linear over the (non necessarily identical) intervals $t_{k-1} \leq t < t_k$, with $k = 1, 2, ..., N$, and a_k is the value taken at time t_k. For $-\infty \leq t < 0$ the time–history has the constant value a_0, whereas for $t > 0$ it takes the constant value a_N.

The time–history and its time–derivative are:

$$\begin{cases} f_6(t) &= a_0 + \sum_{j=0}^{N} (\alpha_j + \beta_j t) H(t - t_j) \\ f_6'(t) &= \sum_{j=0}^{N} \beta_j H(t - t_j), \end{cases} \tag{7.15}$$

where

$$\begin{cases} \alpha_0 &= (a_1 - a_0) - r_1 t_1 \\ \alpha_j &= (a_{j+1} - a_j) - r_{j+1} t_{j+1} + r_j t_j \quad (1 \leq j \leq N-1) \\ \alpha_N &= r_N t_N, \end{cases} \tag{7.16}$$

and

$$\begin{cases} \beta_0 &= r_1 \\ \beta_j &= r_{j+1} - r_j \quad (1 \leq j \leq N-1) \\ \beta_N &= -r_N, \end{cases} \tag{7.17}$$

with

$$r_j = \frac{a_j - a_{j-1}}{t_j - t_{j-1}} \quad (1 \leq j \leq N). \tag{7.18}$$

7.1.8 $f_7(t)$: Piecewise constant

For $0 \equiv t_0 \leq t < t_N$ (piecewise constant phase), the time–history has the constant value a_k over the identical time intervals $t_{k-1} \leq t < t_k$, with $k = 1, 2, ..., N$. For $-\infty \leq t < 0$ the load has the constant value a_0. Finally, for $t \geq t_N$ the load has the constant value $a_{N+1} \equiv a_N$. The time–history and its time–derivative are:

$$\begin{cases} f_7(t) &= a_0 + \sum_{k=0}^{N} (a_{k+1} - a_k) H(t - t_k) \\ f_7'(t) &= 0. \end{cases} \tag{7.19}$$

7.1.9 $f_8(t)$: Piecewise constant with loading phase

The time–history is identical to the previous for $0 \equiv t_0 \leq t < t_N$ and $t \geq t_N$. For $-\infty \leq t < 0$ the constant phase of time–history $f_7(t)$ is replaced by a linear loading phase of duration τ. At the end of this loading phase the time–history takes the value a_0. The time–history and its time–derivative are

$$\begin{cases} f_8(t) = f_7(t) + a_0 \left\{ \dfrac{t}{\tau}[H(t+\tau) - H(t)] + H(t+\tau) - 1 \right\} \\ \\ f_8'(t) = f_7'(t) + a_0 \dfrac{1}{\tau}[H(t+\tau) - H(t)]. \end{cases} \quad (7.20)$$

7.2 Time convolutions and their derivatives

Here we provide the expressions of the time convolutions between each of the time–histories listed in §7.1 and the LDCs (§4.2.2). No demonstration is given, since the results given here may be obtained by simple (but admittedly tedious) algebra. We use the following notation and conventions:

1. With $h(t)$ we indicate one of the LDCs $h_l(t)$, $l_l(t)$ or $\delta(t)+k_l(t)$ (§4.2.2), and we use the symbol h_i to denote the viscous amplitude of $h(t)$. The dependence on the harmonic degree is implicit to simplify the notation.

2. We indicate with s_i ($i = 1, \ldots, M$) the negative of the inverse of the relaxation times (4.26). As above, the l–dependence is implicit in s_i.

3. According to the conventions above, and to the statements of §4.2.2, here we assume a multi–exponential form for the LDC:

$$h(t) = h^E \delta(t) + \sum_i e^{s_i t} h_i, \quad (7.21)$$

where h^E is the elastic part of the LDC (implicitly dependent on the harmonic degree), and $\sum_i$ stands for $\sum_{i=1}^{M}$, where M is the total number of viscoelastic relaxation modes.

4. We define the *fluid LDC* as

$$h^F = h^E - \sum_i \frac{h_i}{s_i}. \quad (7.22)$$

Convolution $c_0(t)$ and its derivative $c_0'(t)$

$$c_0(t) = H(t)\Big(h^F + \sum_i \frac{h_i}{s_i} e^{s_i t}\Big) \tag{7.23}$$

$$c_0'(t) = H(t) \sum_i h_i e^{s_i t} \quad \bullet \tag{7.24}$$

Convolution $c_1(t)$ and its derivative $c_1'(t)$

$$c_1(t) = h^F - H(t)\Big(h^F + \sum_i \frac{h_i}{s_i} e^{s_i t}\Big) \tag{7.25}$$

$$c_1'(t) = -H(t) \sum_i h_i e^{s_i t} \quad \bullet \tag{7.26}$$

Convolution $c_2(t)$ and its derivative $c_2'(t)$

$$c_2(t) = H(t+\tau)\Big[h^F + \sum_i \frac{h_i}{s_i} e^{s_i(t+\tau)}\Big] - H(t)\Big[h^F + \sum_i \frac{h_i}{s_i} e^{s_i t}\Big] \tag{7.27}$$

$$c_2'(t) = H(t+\tau) \sum_i h_i e^{s_i(t+\tau)} - H(t) \sum_i h_i e^{s_i t} \quad \bullet \tag{7.28}$$

Convolution $c_3(t)$ and its derivative $c_3'(t)$

$$\begin{aligned} c_3(t) = {} & h^F - \\ & H(t)\Big[h^E \frac{t}{\tau} - \sum_i \frac{h_i}{s_i}\Big(\frac{t}{\tau} + \frac{1 - e^{s_i t}}{s_i \tau}\Big)\Big] \\ & + H(t-\tau)\Big[h^F\Big(\frac{t}{\tau} - 1\Big) - \sum_i \frac{h_i}{s_i} \frac{1 - e^{s_i(t-\tau)}}{s_i \tau}\Big] \end{aligned} \tag{7.29}$$

$$\begin{aligned} c_3'(t) = {} & -H(t)\Big[\frac{h^E}{\tau} - \sum_i \frac{h_i}{s_i}\Big(\frac{1}{\tau} - \frac{e^{s_i t}}{\tau}\Big)\Big] \\ & + H(t-\tau)\Big[\frac{h^F}{\tau} + \sum_i \frac{h_i}{s_i} \frac{e^{s_i(t-\tau)}}{\tau}\Big] \quad \bullet \end{aligned} \tag{7.30}$$

Convolution $c_4(t)$ and its derivative $c_4'(t)$

$$
\begin{aligned}
c_4(t) &= h^F f_4(t) + \sum_{n=0}^{N_r} \sum_i \frac{h_i}{s_i} \Bigg\{ \\
&\left[\left(\frac{1}{\tau} + \frac{1}{\delta} \right) \left(t + n\theta + \frac{1 - e^{s_i(t+n\theta)}}{s_i} \right) \right] H(t + n\theta) \\
&- \left[\frac{t + n\theta + \tau}{\tau} + \frac{1 - e^{s_i(t+n\theta+\tau)}}{s_i \tau} \right] H(t + n\theta + \tau) \\
&- \left[\frac{t + n\theta - \delta}{\delta} + \frac{1 - e^{s_i(t+n\theta-\delta)}}{s_i \delta} \right] H(t + n\theta - \delta) \Bigg\}
\end{aligned} \tag{7.31}
$$

$$
\begin{aligned}
c_4'(t) &= h^F f_4'(t) + \sum_{n=0}^{N_r} \sum_i \frac{h_i}{s_i} \Bigg\{ \\
&\left[\left(\frac{1}{\tau} + \frac{1}{\delta} \right) \left(1 - e^{s_i(t+n\theta)} \right) \right] H(t + n\theta) \\
&- \left[\frac{1}{\tau} - \frac{e^{s_i(t+n\theta+\tau)}}{\tau} \right] H(t + n\theta + \tau) \\
&- \left[\frac{1}{\delta} - \frac{e^{s_i(t+n\theta-\delta)}}{\delta} \right] H(t + n\theta - \delta) \Bigg\} \quad \bullet
\end{aligned} \tag{7.32}
$$

Convolution $c_5(t)$ and its derivative $c_5'(t)$

$$c_5(t) = h^F f_5(t) + A_\omega \cos \omega t + B_\omega \sin \omega t \tag{7.33}$$

$$c_5'(t) = h^F f_5'(t) - \omega A_\omega \sin \omega t + \omega B_\omega \cos \omega t \tag{7.34}$$

where

$$A_\omega = +\frac{1}{2} \sum_i \frac{h_i}{s_i} \frac{\omega^2}{s_i^2 + \omega^2} \tag{7.35}$$

$$B_\omega = -\frac{1}{2} \sum_i \frac{h_i}{s_i} \frac{\omega s_i}{s_i^2 + \omega^2} \quad \bullet \tag{7.36}$$

Convolution $c_6(t)$ and its derivative $c_6'(t)$

$$c_6(t) = a_0 h^F + \tag{7.37}$$

$$\sum_{j=0}^{N} \left[h^E(\alpha_j + \beta_j t) + \sum_i \frac{h_i}{s_i} Q_{ij}(t) \right] H(t - t_j) \tag{7.38}$$

$$c_6(t) \;=\; \sum_{j=0}^{N}\left[h^E\beta_j t + \sum_i \frac{h_i}{s_i}Q'_{ij}(t)\right]H(t-t_j), \tag{7.39}$$

where α_j and β_j are given by (7.16) and (7.17), and

$$Q_{ij}(t) \;=\; \alpha_j + \beta_j t_j + (\beta_j/s_i)[e^{s_i(t-t_j)} - 1] - \beta_j(t-t_j) \tag{7.40}$$

$$Q'_{ij}(t) \;=\; \beta_j[e^{s_i(t-t_j)} - 1] \quad \bullet \tag{7.41}$$

Convolution $c_7(t)$ and its derivative $c_7'(t)$

$$c_7(t) \;=\; a_0 h^F + \sum_{k=0}^{N}\delta a_k\left[h^F + \sum_i \frac{h_i}{s_i}e^{s_i(t-t_k)}\right]H(t-t_k) \tag{7.42}$$

$$c_7'(t) \;=\; \sum_{k=0}^{N}\delta a_k\left[\sum_i h_i e^{s_i(t-t_k)}\right]H(t-t_k) \tag{7.43}$$

where

$$\delta a_k \equiv (a_{k+1} - a_k) \quad \bullet \tag{7.44}$$

Convolution $c_8(t)$ and its derivative $c_8'(t)$

$$\begin{aligned} c_8(t) \;=\; & c_7(t) + a_0\cdot\Bigg\{ -h^F \\ & + \left[h^F\left(1+\frac{t}{\tau}\right) - \sum_i \frac{h_i}{s_i}\frac{1-e^{s_i(t+\tau)}}{s_i\tau}\right]H(t+\tau) \\ & - \left[h^F\frac{t}{\tau} - \sum_i \frac{h_i}{s_i}\frac{1-e^{s_i t}}{s_i\tau}\right]H(t)\Bigg\} \end{aligned} \tag{7.45}$$

$$\begin{aligned} c_8'(t) \;=\; & c_7'(t) + a_0\cdot\Bigg\{ \\ & + \left[h^F\frac{1}{\tau} + \sum_i \frac{h_i}{s_i}\frac{e^{s_i(t+\tau)}}{\tau}\right]H(t+\tau) \\ & - \left[h^F\frac{1}{\tau} + \sum_i \frac{h_i}{s_i}\frac{e^{s_i t}}{\tau}\right]H(t)\Bigg\} \end{aligned} \tag{7.46}$$

where $c_7(t)$ and $c_7'(t)$ are given by (7.42) and (7.43), respectively $\bullet$

7.3 Glossary

Here we list some keywords and the page where they are defined first.

- CSH = Complex Spherical Harmonics (page 4).
- RSH = Real Spherical Harmonics (10).
- FNSH = Fully Normalized Spherical Harmonics (10).
- LT = Laplace Transform (22).
- EP = EquiPotential surface (28).
- CM = Center of Mass (30).
- AX = AXis–symmetric load (48).
- NAX = Non AXis–symmetric load (48).
- GRF = Geographical Reference Frame (48).
- LRF = Load Reference Frame (48).
- LDC = Load–Deformation Coefficient (67).
- SEISG = Spherically symmetric, Elastic, Incompressible, Self-Gravitating (67).
- SVISG = Sphericall symmetric, Viscoelastic, Incompressible, Self-Gravitating (68).

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