Solving the "Sea Level Equation"
Part I – *Theory*

Giorgio Spada and Paolo Stocchi

Istituto di Fisica – Università di Urbino
Via Santa Chiara 27
61029 Urbino (PU)
Italy

spada@fis.uniurb.it

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Preface

This booklet is devoted to the study of some theoretical and practical aspects of the so–called "Sea Level Equation" (SLE), an integral equation that predicts the time–dependent shape of the equipotential surface of a deformable body subject to surface forces. In the field of global geodynamics, the SLE serves as a tool for computing the postglacial sealevel variations and other observable quantities, taking as an input the shapes and chronology of the Pleistocene ice–sheets.

Our first purpose was simply to collect various sparse notes and to translate in simple words the theory of the SLE for the PhD students attending my lessons of "Global Geodynamics" at the University of Bologna. However, in Part II we also provide details on the numerical discretization of the SLE and a freely available Fortran 90 code (SELEN) that anyone can use to solve the SLE on his own computer. We hope that the material presented will facilitate the work of colleagues at their first approach to Glacial Isostatic Adjustment (GIA), and perhaps also more experienced geophysicists willing to benchmark their own codes. As far as we know, this is the first time that a sealevel equation solver is made freely publically available.

The development of the theory of the SLE is based on a number of approximations. First, the Earth is assumed to be radially stratified and incompressible, and the various layers are characterized by a linear viscoelastic rheology. This is a widely diffused approximation, but recent work has been done to include non–Newtonian rheologies and lateral viscosity variations in spherical Earth models (see e. g. [4, 26]). Second, it is assumed that the ocean function is constant, that implies fixed shorelines. Third, we totally neglect the effects of rotation on the GIA–induced sealevel variations. The reader is referred to [7] for the theoretical details concerning the rotational feedback and for the numerical evaluation of its consequences. In view of the approximations listed above, this booklet provides a zeroth–order model for the postglacial sealevel changes, that can be considerably refined in the future, hopefully with the aid and the contribution of other investigators.

The future releases of this document will benefit from the feedback of the readers of this first edition. Please feel free to write to `spada@fis.uniurb.it` for questions, comments, and suggestions.

GS, February 8, 2005.

Acknowledgments

This book mainly consists in the reorganization of previously published material. We have taken advantage from a number of recent and less recent works published in the field of the GIA by a number of investigators, which are acknowledged for their contribution. The bibliography, that is admittedly incomplete, only contains the basic reference material. We thank Max Tegmark for providing his pixelization code, and to Spina Cianetti and Carlo Giunchi for their very helpful suggestions.

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Chapter 1

Green's functions, surface loads, and the GIA problem

The first part of this chapter is devoted to the description of the Green's function (GF) approach in the context of the GIA theory. The GFs provide a way to quantify the displacements and the variations of the gravitational potential when a point–like, impulsive load is applied to the Earth's surface. Their usefulness is limited to the case of radially stratified Earth models based on elastic or linear viscoelastic rheologies. The GFs cannot be established by means of purely analytical methods, except for models with a very simple internal structure, for which the "load–deformation coefficients" (LDCs) are known explicitly [25]. For multi–layered models, cumbersome algebra prohibits any direct approach, and the solution can only be computed numerically [21]. For this reason, the program that we present in Part II can be used in conjunction with the companion software TABOO [15, 16], that provides the LDCs for a suite of radially layered models. Once the GFs relative to the relevant geophysical observables are constructed, the response of the Earth to surface loads characterized by a complex structure and time–evolution can be obtained by spatio–temporal convolutions, as it is explained in the second part of this chapter. The third and last section is devoted to the solution of what we call "simplified GIA problem", in which the ocean load is computed assuming eustasy.

1.1 Green's functions

Here we build the GFs for the displacement vector and the incremental gravitational potential. The analysis of the GFs for viscoelastic Earth models - which constitute the most general and useful tools for GIA studies - is preceded by separate subsections devoted to the GFs for a rigid and an elastic Earth, respectively.

1.1.1 Rigid Earth

We consider a localized point mass placed at the surface of a rigid, spherically symmetric Earth. The dynamic mass of the surface load is defined as

$$\mu(t) = f(t)\Delta m, \tag{1.1}$$

where $f(t)$ describes its time–evolution, and Δm represents its intrinsic (or static) mass. For an impulsive load, $f(t) = \delta(t)$ where $\delta(t)$ is Dirac's delta, so that in this specific case the dynamic mass

$$\mu(t) = \delta(t)\Delta m \tag{1.2}$$

has dimensions of a mass per unit time.

The gravitational potential per unit time exerted by the localized mass at a point P on the Earth surface is

$$\phi^r(d, t) = \frac{G\mu(t)}{d}, \tag{1.3}$$

where G is Newton's constant, d is the distance between the mass and P, and the superscript r recalls that we are dealing with a rigid Earth. Since ϕ^r adds to the background potential of the Earth, we will refer to it as to the *incremental* gravitational potential.

By simple trigonometry,

$$d(\alpha) = 2a\sin(\alpha/2), \tag{1.4}$$

where α is the colatitude of P with respect to the point load and a is the radius of the Earth. This allows to write

$$\phi^r(\alpha, t) = \frac{G\mu(t)}{2a\sin(\alpha/2)}, \tag{1.5}$$

which can be further transformed recalling that the surface gravity acceleration is

$$\gamma_o = \frac{Gm_e}{a^2}, \tag{1.6}$$

where m_e is the mass of the Earth. Hence

$$\phi^r(\alpha, t) = \frac{a\gamma_o \Delta m \delta(t)}{2m_e \sin(\alpha/2)}. \tag{1.7}$$

Since ϕ^r only results from the gravitational attraction of the imposed point mass, it is sometimes referred as to *direct* potential. From (1.7), the *Green function* (hereafter GF) for the incremental gravitational potential in the case of a rigid Earth is defined as

$$G^r_\phi(\alpha, t) \equiv \frac{\phi^r(\alpha, t)}{\Delta m} = \frac{a\gamma_o \delta(t)}{2m_e \sin(\alpha/2)}. \tag{1.8}$$

Notice that, by its own definition, $G^r_\phi(\alpha, t)$ has the dimensions of a gravitational potential per unit time and unit mass.

An equivalent expression for the GF can be obtained recalling the *Legendre sum*:

$$\sum_{l=0}^{\infty} P_l(\cos\alpha) = \frac{1}{2\sin(\alpha/2)}, \tag{1.9}$$

where $P_l(\cos\alpha)$ is the Legendre polynomial of harmonic degree l, defined in Appendix 4.1. This allows to write (1.8) in spectral form as

$$G^r_\phi(\alpha, t) = \delta(t) \sum_{l=0}^{\infty} \phi^r_l P_l(\cos\alpha), \tag{1.10}$$

where the coefficients of the expansion are

$$\phi^r_l \equiv \frac{a\gamma_o}{m_e}, \tag{1.11}$$

thus showing that the spectrum of G^r_ϕ is flat.

1.1.2 Elastic Earth

For an elastic Earth, the action of the impulsive mass (1.2) produces two related effects. First, the planet yields under the pressure exerted by the load. Second, there is a further variation in the gravitational potential following the change of the shape of the Earth. In analogy with (1.10), the corresponding GF may be written as

$$G_\phi^e(\alpha, t) = \delta(t) \sum_{l=0}^{\infty} \phi_l^e P_l(\cos\alpha), \tag{1.12}$$

which is in phase with G_ϕ^r as a consequence of elasticity. Furthermore, since the Earth responds linearly to the imposed forces, the spectral coefficients ϕ_l^e are proportional degree-by-degree to ϕ_l^r, that is:

$$\phi_l^e = k_l^e \phi_l^r, \tag{1.13}$$

where the non–dimensional number k_l^e is the *load–deformation coefficient* (hereafter LDC) for the incremental gravitational potential.

The total GF for incremental gravitational potential stems from a rigid and an elastic component

$$G_\phi(\alpha, t) = G_\phi^r(\alpha, t) + G_\phi^e(\alpha, t), \tag{1.14}$$

that, according to Equations (1.10-1.13) above, can be cast in the following spectral form

$$G_\phi(\alpha, t) = \delta(t) \frac{a\gamma_o}{m_e} \sum_{l=0}^{\infty} (1 + k_l^e) P_l(\cos\alpha). \tag{1.15}$$

At the surface of the Earth, the elastic displacement induced by the applied load can be expressed as

$$\vec{u}(\alpha, t) = G_u(\alpha, t)\hat{r} + G_v(\alpha, t)\hat{\alpha}, \tag{1.16}$$

where $\hat{r}$ and $\hat{\alpha}$ are unit vectors in the directions of increasing radius and colatitude, and G_u and G_v are the GFs relative to the vertical and horizontal components of displacement, respectively. By virtue of the spherical symmetry of the Earth, the transversal horizontal component of displacement - perpendicular to $\hat{\alpha}$ - vanishes identically.

In analogy with (1.14), we write the displacement GFs as

$$G_u(\alpha, t) = G_u^r(\alpha, t) + G_u^e(\alpha, t), \tag{1.17}$$

and

$$G_v(\alpha,t) = G_v^r(\alpha,t) + G_v^e(\alpha,t), \qquad (1.18)$$

where

$$G_u^r(\alpha,t) = G_v^r(\alpha,t) = 0 \qquad (1.19)$$

by virtue of the Earth rigidity.

The vertical and horizontal components of displacement can be expressed in spectral form by means of appropriate LDCs. Following (1.12) we write

$$G_u(\alpha,t) = \delta(t)\sum_{l=0}^{\infty} u_l P_l(\cos\alpha) \qquad (1.20)$$

$$G_v(\alpha,t) = \delta(t)\sum_{l=0}^{\infty} v_l \frac{\partial P_l(\cos\alpha)}{\partial\alpha}, \qquad (1.21)$$

where the derivative of $P_l(\cos\alpha)$ is used in the expression for G_v for the sake of convenience, and the coefficients u_l and v_l are proportional to the spectral amplitude of the direct potential:

$$u_l = h_l^e \frac{\phi_l^r}{\gamma_o} \qquad (1.22)$$

$$v_l = \ell_l^e \frac{\phi_l^r}{\gamma_o}, \qquad (1.23)$$

where the reference gravity at the Earth's surface γ_o (see 1.6) is introduced in order to keep the LDC h_l^e and ℓ_l^e dimensionless.

From above, the GFs pertaining to vertical and horizontal displacement are

$$G_u(\alpha,t) = \delta(t)\frac{a}{m_e}\sum_{l=0}^{\infty} h_l^e P_l(\cos\alpha) \qquad (1.24)$$

and

$$G_v(\alpha,t) = \delta(t)\frac{a}{m_e}\sum_{l=0}^{\infty} \ell_l^e \frac{\partial P_l(\cos\alpha)}{\partial\alpha}, \qquad (1.25)$$

that along with (1.15) constitute the basic set of GFs in the case of an elastic Earth. We observe that the GFs G_u and G_v have dimensions of displacements per unit time and per unit mass.

1.1.3 Viscoelastic Earth

Viscoelasticity introduces a delayed response of the Earth to the surface load. As first shown by [9] and later discussed in a number of papers, for a spherically symmetric layered and linear viscoelastic Earth, the GF relative to the total incremental gravitational potential reads

$$G_\phi(\alpha, t) = G^r_\phi(\alpha, t) + G^e_\phi(\alpha, t) + G^v_\phi(\alpha, t), \tag{1.26}$$

where the rigid and elastic components are given by (1.8) and (1.12), respectively, while the viscous part G^v_ϕ is

$$G^v_\phi(\alpha, t) = H(t)\frac{a\gamma_o}{m_e}\sum_{l=0}^{\infty}\Big(\sum_{i=1}^{M} k_{li}e^{s_{li}t}\Big)P_l(\cos\alpha), \tag{1.27}$$

where

$$H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{1.28}$$

is the Heaviside step function, k_{li} are the *viscoelastic LDCs* for the incremental potential and $s_{li} = -1/\tau_{li}$ where τ_{li} are the *relaxation times* of the Earth model that is being adopted. The couple $\{k_{li}, s_{li}\}$ is referred as to the $i-th$ *viscoelastic mode* of harmonic degree l. The number of modes M increases with increasing number of elastic or viscoelastic layers and also depends on the nature of the interfaces between the layers [10, 12, 14]. The reader is referred to [16] for more details on how the modes can be numerically determined. Using (1.15) and (1.27) in (1.26), the explicit form of the viscoelastic GF for the incremental potential turns out to be

$$G_\phi(\alpha, t) = \frac{a\gamma_o}{m_e}\sum_{l=0}^{\infty}\Big(\delta(t)(1 + k^e_l) + H(t)\sum_{i=1}^{M} k_{li}e^{s_{li}t}\Big)P_l(\cos\alpha). \tag{1.29}$$

The GFs pertaining to the vertical and horizontal components of displacement can be similarly written as the sum of elastic and viscoelastic parts, with

$$G_u(\alpha, t) = G^e_u(\alpha, t) + G^v_u(\alpha, t), \tag{1.30}$$

and

$$G_v(\alpha, t) = G^e_v(\alpha, t) + G^v_v(\alpha, t), \tag{1.31}$$

where we have used (1.19), the elastic components G_u^e and G_v^e are given by (1.24) and (1.25), respectively, and in analogy with (1.27) the viscous components are

$$G_u(\alpha,t) = H(t)\frac{a}{m_e}\sum_{l=0}^{\infty}\Big(\sum_{i=1}^{M} h_{li}e^{s_{li}t}\Big)P_l(\cos\alpha) \tag{1.32}$$

and

$$G_v(\alpha,t) = H(t)\frac{a}{m_e}\sum_{l=0}^{\infty}\Big(\sum_{i=1}^{M} \ell_{li}e^{s_{li}t}\Big)\frac{P_l(\cos\alpha)}{\partial\alpha}, \tag{1.33}$$

where h_{li} and ℓ_{li} are the viscoelastic LDCs relative to the radial and horizontal components of displacement, respectively (notice that the relaxation times are common to all of the three viscous GFs so far introduced). The total GFs for the components of displacement are thus

$$G_u(\alpha,t) = \frac{a}{m_e}\sum_{l=0}^{\infty}\Big(\delta(t)h_l^e + H(t)\sum_{i=1}^{M} h_{li}e^{s_{li}t}\Big)P_l(\cos\alpha) \tag{1.34}$$

and

$$G_v(\alpha,t) = \frac{a}{m_e}\sum_{l=0}^{\infty}\Big(\delta(t)\sum_{l=0}^{\infty}\ell_l^e + H(t)\sum_{i=1}^{M} \ell_{li}e^{s_{li}t}\Big)\frac{P_l(\cos\alpha)}{\partial\alpha}. \tag{1.35}$$

A more compact form for the GFs can be established introducing the time–dependent LDCs

$$\left\{\begin{array}{c} k_l \\ h_l \\ \ell_l \end{array}\right\}(t) = \left\{\begin{array}{c} 1+k_l^e \\ h_l^e \\ \ell_l^e \end{array}\right\}\delta(t) + \sum_{i=1}^{M} H(t)\left\{\begin{array}{c} k_{li} \\ h_{li} \\ \ell_{li} \end{array}\right\}\mathrm{e}^{s_{li}t}, \tag{1.36}$$

that allows to write

$$\left\{\begin{array}{c} \frac{1}{\gamma_o}G_\phi \\ G_u \\ G_v \end{array}\right\}(\alpha,t) = \frac{a}{m_e}\sum_{l=0}^{\infty}\left\{\begin{array}{c} k_l \\ h_l \\ \ell_l \end{array}\right\}(t) \times \left\{\begin{array}{c} 1 \\ 1 \\ \partial_\alpha \end{array}\right\}P_l(\cos\alpha), \tag{1.37}$$

with

$$\partial_\alpha \equiv \frac{\partial}{\partial\alpha}. \tag{1.38}$$

1.2 Surface loads

The response of the Earth to a localized impulsive load (i. e. the GFs), can be used to build the response to finite–size, time–evolving surface loads. In general, we can write the total load as

$$\mathcal{L}(\omega, t) = L_i(\omega, t) + L_o(\omega, t), \tag{1.39}$$

where $\omega = (\theta, \lambda)$, θ and λ are colatitude and longitude, t is time, and L_i and L_o are the surface loads associated with changes of the weight of the ice sheets and of the ocean mass, respectively. These two terms are separately studied in the following sections.

1.2.1 Ice load

Given the ice thickness $T(\omega, t)$ at a point P of coordinates ω and at a given time t, the *ice thickness variation* is defined as

$$I(\omega, t) = T(\omega, t) - T_0, \tag{1.40}$$

where T_0 is the ice thickness at P at the remote reference time t_0. The *ice load* is

$$L_i(\omega, t) = \rho_i I(\omega, t), \tag{1.41}$$

where ρ_i is the ice density. By its own definition, L_i has units of mass per unit surface. The change in the mass of the whole ice sheet can be obtained by

$$m_i(t) = a^2 \int_i L_i(\omega, t) d\omega, \tag{1.42}$$

where the integral is over the region i where $I(\omega, t) \neq 0$, and

$$d\omega = \sin\theta d\theta d\lambda \tag{1.43}$$

is the element of area on the unit sphere.

In the case of an ice load with fixed margins, the following factorization is possible

$$L_i(\omega, t) = \sigma(\omega) f(t), \tag{1.44}$$

where $\sigma(\omega)$ is the *load function*, which describes the spatial distribution of the load, and $f(t)$ is the *load time–history*. The load function is defined for $\omega \in i$, and vanishes outside the ice margins. The more general case of a complex ice–load with time–evolving margins can be dealt with by a combination of loads of the form (1.44).

The load function can be decomposed in spectral form as

$$\sigma(\omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sigma_{lm} Y_{lm}(\omega), \tag{1.45}$$

where $Y_{lm}(\omega)$ are the complex spherical harmonics with the orthonormality relationship given by (4.5) and the harmonic coefficients of the load function are

$$\sigma_{lm} = \int_{\Omega} \sigma(\omega) Y_{lm}^{*}(\omega) d\omega. \tag{1.46}$$

The change of the ice mass can be expressed by means of the load function $\sigma(\omega)$. Using (1.44) and (1.42), we in fact obtain

$$m_i(t) = m_s f(t) \tag{1.47}$$

where the *static mass* of the load is

$$m_s = a^2 \int_i \sigma(\omega) d\omega. \tag{1.48}$$

1.2.2 Ocean load

The growth and the melting of the ice sheets is accompanied by a variation of the mass of the oceans. At a given point ω and time t, the *sealevel* is defined as

$$SL(\omega, t) = r'_g(\omega, t) - r'_t(\omega, t), \tag{1.49}$$

where r'_g and r'_t represent the distances between the Earth's center of mass and the geoid and the solid surface of the Earth, respectively. The *sealevel change* $S(\omega, t)$ is defined as

$$S(\omega, t) = SL(\omega, t) - SL_0, \tag{1.50}$$

where SL_0 is the sealevel measured at the same point at the reference time t_0.

The *ocean load* is

$$L_o(\omega, t) = \rho_o S(\omega, t) \mathcal{O}(\omega), \tag{1.51}$$

where ρ_o is the density of water, and $\mathcal{O}(\omega)$ is the ocean function

$$\mathcal{O}(\omega) = \begin{cases} 1, & \text{if} \quad \omega \in \text{oceans} \\ 0, & \text{if} \quad \omega \in \text{continents}, \end{cases} \tag{1.52}$$

that can be expanded in series of spherical harmonics as

$$\mathcal{O}(\omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} o_{lm} Y_{lm}(\omega), \tag{1.53}$$

with coefficients

$$o_{lm} = \int_{\Omega} o(\omega) Y_{lm}^*(\omega) d\omega = \int_{o} Y_{lm}^*(\omega) d\omega, \tag{1.54}$$

where o is the ocean domain.

The sealevel changes defined by (1.50) result from (i) variations of the surface topography caused by vertical displacements, (ii) alterations of the shape of the geoid and (iii) modifications of the amount of water contained within the ocean basins. The quantitative law that describes these dependencies is the *Sea Level Equation*, that will be introduced in the next chapter. A rough estimate of the amplitude of the ocean load can be obtained neglecting the effects (i) and (ii) above, i.e. assuming that the Earth does not deform under the action of the loads (rigid Earth hypothesis), and that no geoid changes occur (Newton's gravitational constant is $G = 0$). In this simplified ideal case, the sealevel changes are called *eustatic*, a word coined by Suess [17]. Given a change $m_i(t)$ in the ocean mass, eustasy imposes spatially uniform sealevel variations

$$S(\omega, t) = -\frac{m_i(t)}{\rho_o A_o} \mathcal{O}(\omega), \tag{1.55}$$

and, according to (1.51) a spatially uniform load as well, with

$$L_o(\omega, t) = -\frac{m_i(t)}{A_o} \mathcal{O}(\omega). \tag{1.56}$$

1.3 The simplified GIA problem

Once the ice and the ocean loads are determined from (1.44) and (1.51), the total load can be retrieved from (1.39), that provides:

$$\mathcal{L}(\omega, t) = \rho_i I(\omega, t) + \rho_o S(\omega, t) \mathcal{O}(\omega). \tag{1.57}$$

The vertical displacement U and the change in the gravitational potential Φ at a given point ω and time t result from the displacements and potentials due to changes of the ice and of the ocean mass distributions at any point ω' and times $t' \leq t$. This involves a spatial integration over the whole surface of the Earth and a time convolution which accounts for the time–dependent response of the viscoelastic mantle. Since the GFs for vertical displacement G_u and gravitational potential G_ϕ have similar forms[1] the spatio–temporal convolutions can be compacted as

$$\left\{ \begin{array}{c} U \\ \Phi \end{array} \right\}(\omega, t) = \int_{-\infty}^{t} dt' \int_{\Omega} \left\{ \begin{array}{c} G_u \\ G_\phi \end{array} \right\}(\alpha, t - t')\mathcal{L}(\omega', t')dA', \tag{1.58}$$

where Ω denotes the whole unit sphere,

$$dA = a^2 \sin\theta d\theta d\lambda, \tag{1.59}$$

and α implicitly depends from $\omega = (\theta, \lambda)$ and $\omega' = (\theta', \lambda')$ via the cosines formula of spherical geometry

$$\cos\alpha = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\lambda' - \lambda). \tag{1.60}$$

To simplify the formulas that follow, it is useful to introduce the notation

$$(G \otimes_Q L)(\omega, t) \equiv \int_{-\infty}^{t} dt' \int_{Q} G(\alpha, t - t')L(\omega', t')dA', \tag{1.61}$$

where G and L represent a generic GF and a generic load, respectively, and $Q \subseteq \Omega$. Hence

$$\left\{ \begin{array}{c} U \\ \Phi \end{array} \right\}(\omega, t) = \left\{ \begin{array}{c} G_u \\ G_\phi \end{array} \right\} \otimes_E \mathcal{L} = \left\{ \begin{array}{c} U_i + U_o \\ \Phi_i + \Phi_o \end{array} \right\}, \tag{1.62}$$

with

$$\left\{ \begin{array}{c} U_i \\ \Phi_i \end{array} \right\}(\omega, t) = \left\{ \begin{array}{c} G_u \\ G_\phi \end{array} \right\} \otimes_i \rho_i I, \tag{1.63}$$

and

$$\left\{ \begin{array}{c} U_o \\ \Phi_o \end{array} \right\}(\omega, t) = \left\{ \begin{array}{c} G_u \\ G_\phi \end{array} \right\} \otimes_o \rho_o S, \tag{1.64}$$

[1]The computation of the horizontal component of displacement requires further considerations, that will be illustrated in §2.5.2.

where $\otimes_i$ and $\otimes_o$ indicate spatial integrations over the ice sheets and the oceans, respectively, and the dependence from (ω, t) is implicit in the right hand side of the last two equations.

The method outlined above poses a conceptual difficulty. This can be appreciated observing that the knowledge of the total load $\mathcal{L}$ depends from both the ice thickness and sealevel variations (see 1.57). While the former can be reasonably *assumed* to be known from geological or geophysical evidence, the latter will depend, beside on the amount of water exchanged between the ice sheets an the oceans, also on vertical displacement of the solid Earth and geoid height changes, which can only be determined once the sealevel changes themselves are known! Here we will escape to this circularity assuming an eustatic ocean load, which leads to an attractive - but possibly inaccurate - analytical solution for the responses. Once the sealevel equation will be derived in chapter 2, the eustatic approximation will be abandoned.

1.3.1 Response to the ice load

Here we provide the details of the computations involved in the "ice component" of the response of the Earth to the surface loads. Recalling (1.63) and using the factorized form of the load $\sigma(\omega)$ (see 1.44), we obtain

$$\left\{ \begin{array}{c} U_i \\ \Phi_i \end{array} \right\} (\omega, t) = \int_{-\infty}^{t} dt' \int_i \left\{ \begin{array}{c} G_u \\ G_\phi \end{array} \right\} (\alpha, t - t') \sigma(\omega') f(t') dA', \tag{1.65}$$

where U_i and Φ_i represent vertical displacement and incremental gravitational potential, respectively. We now make use of the explicit expressions for the GFs G_u and G_ϕ given by (1.37) and of the spectral decomposition of the load function (1.45) to write

$$\begin{aligned} \left\{ \begin{array}{c} U_i \\ \frac{1}{\gamma_o} \Phi_i \end{array} \right\} (\omega, t) &= \frac{a^3}{m_e} \int_{-\infty}^{t} dt' f(t') \int_i \sum_{l=0}^{\infty} \left\{ \begin{array}{c} h_l \\ k_l \end{array} \right\} (t - t') P_l(\cos\alpha) \times \\ &\times \sum_{l'm'} \sigma_{l'm'} Y_{l'm'}(\omega') d\omega', \end{aligned} \tag{1.66}$$

where we have used the notation

$$\sum_{lm} \equiv \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} . \tag{1.67}$$

We now substitute to $P_l(\cos\alpha)$ the equivalent form provided by the addition theorem for spherical harmonics (4.9) and integrate over the whole sphere -

this is possible since $\sigma(\omega) = 0$ outside the ice margins - to obtain:

$$\left\{ \begin{array}{c} U_i \\ \frac{1}{\gamma_o}\Phi_i \end{array} \right\}(\omega, t) = \frac{a^3}{m_e}\int_{-\infty}^{t} dt' f(t') \int_{\Omega} \sum_{l=0}^{\infty} \left\{ \begin{array}{c} h_l \\ k_l \end{array} \right\}(t - t') \times \times \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} Y^*_{lm}(\omega') Y_{lm}(\omega) \sum_{l'm'} \sigma_{l'm'} Y_{l'm'}(\omega') d\omega', \quad (1.68)$$

which can be further manipulated to give

$$\left\{ \begin{array}{c} U_i \\ \frac{1}{\gamma_o}\Phi_i \end{array} \right\}(\omega, t) = \frac{3}{\bar{\rho}_e} \sum_{lm} \sum_{l'm'} \frac{\sigma_{l'm'}}{2l+1} \left\{ \begin{array}{c} \bar{h}_l \\ \bar{k}_l \end{array} \right\}(t - t') Y_{lm}(\omega) \times \times \int_{\Omega} Y^*_{lm}(\omega') Y_{l'm'}(\omega') d\omega', \quad (1.69)$$

where we have defined

$$\left\{ \begin{array}{c} \bar{h}_l \\ \bar{k}_l \end{array} \right\}(t) \equiv \int_{-\infty}^{t} \left\{ \begin{array}{c} h_l \\ k_l \end{array} \right\}(t - t') f(t') dt', \quad (1.70)$$

and

$$\bar{\rho}_e = \frac{3 m_e}{4\pi a^3} \quad (1.71)$$

is the average density of the Earth. Also taking into account the orthonormality relationship for the spherical harmonics (4.5), from (1.69) we obtain

$$\left\{ \begin{array}{c} U_i \\ \frac{1}{\gamma_o}\Phi_i \end{array} \right\}(\omega, t) = \frac{3}{\bar{\rho}_e} \sum_{lm} \frac{\sigma_{lm}}{2l+1} \left\{ \begin{array}{c} \bar{h}_l \\ \bar{k}_l \end{array} \right\}(t) Y_{lm}(\omega), \quad (1.72)$$

that represents the response of the Earth to the ice load in terms of vertical displacement and incremental gravitational potential.

1.3.2 Response to the ocean load

Using the eustatic approximation and (1.56), the "ocean component" of the responses (1.64) reads

$$\left\{ \begin{array}{c} U_o \\ \Phi_o \end{array} \right\}(\omega, t) = -\int_{-\infty}^{t} dt' \int_{o} \left\{ \begin{array}{c} G_u \\ G_\phi \end{array} \right\}(\alpha, t - t') \frac{m_i(t')}{A_o} f(t') dA', \quad (1.73)$$

that can be rephrased recalling the definition of static mass (see 1.48) and the explicit expression of the GFs given by (1.37):

$$\begin{aligned}\left\{\begin{array}{c} U_o \\ \frac{1}{\gamma_o}\Phi_o \end{array}\right\}(\omega,t) &= -\frac{m_s}{m_e}\frac{a^3}{A_o}\int_{-\infty}^{t} dt' f(t') \times \\ &\times \int_o \sum_{l=0}^{\infty}\left\{\begin{array}{c} h_l \\ k_l \end{array}\right\}(t-t')P_l(\cos\alpha)d\omega'. \end{aligned} \quad (1.74)$$

Using (1.70) and the addition theorem for spherical harmonics (4.9), we obtain

$$\begin{aligned}\left\{\begin{array}{c} U_o \\ \frac{1}{\gamma_o}\Phi_o \end{array}\right\}(\omega,t) &= -\frac{m_s}{m_e}\frac{4\pi a^3}{A_o}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left\{\begin{array}{c} \bar{h}_l \\ \bar{k}_l \end{array}\right\}(t) \times \\ &\times \sum_{m=-l}^{+l} Y_{lm}(\omega)\int_o Y^*_{lm}(\omega')d\omega', \end{aligned} \quad (1.75)$$

that reduces to

$$\left\{\begin{array}{c} U_o \\ \frac{1}{\gamma_o}\Phi_o \end{array}\right\}(\omega,t) = -\frac{3}{\bar{\rho}_e}\frac{m_s}{A_o}\sum_{lm}\left\{\begin{array}{c} \bar{h}_l \\ \bar{k}_l \end{array}\right\}(t)\frac{o_{lm}}{2l+1}Y_{lm}(\omega), \quad (1.76)$$

where we have also used (1.54), (1.67) and (1.71).

1.3.3 Solution of the simplified GIA problem

According to (1.72) and (1.76), the vertical displacement and the change in the gravitational potential due to the combined action of ice and ocean loads is

$$\left\{\begin{array}{c} U \\ \frac{1}{\gamma_o}\Phi \end{array}\right\}(\omega,t) = \frac{3}{\bar{\rho}_e}\sum_{lm}\frac{\sigma'_{lm}}{2l+1}\left\{\begin{array}{c} \bar{h}_l \\ \bar{k}_l \end{array}\right\}(t)Y_{lm}(\omega), \quad (1.77)$$

where

$$\sigma'_{lm} = \sigma_{lm} - \frac{m_s}{A_o}o_{lm} \quad (1.78)$$

are the harmonic coefficients of the expansion

$$\sigma'(\omega) = \sum_{lm}\sigma'_{lm}Y_{lm}(\omega), \quad (1.79)$$

where

$$\sigma'(\omega) = \sigma(\omega) - \frac{m_s}{A_o}o(\omega) \quad (1.80)$$

is the total load function.

The expansion (1.77), based on the complex spherical harmonics, are not suitable for numerical implementations. It is by far more economical to employ a real expansion such as

$$\left\{ \begin{array}{c} U \\ \frac{1}{\gamma_o}\Phi \end{array} \right\}(\omega, t) = a \sum_{lm}{}' \left(\left\{ \begin{array}{c} c_u \\ c_\phi \end{array} \right\}_{lm} \cos m\lambda + \left\{ \begin{array}{c} s_u \\ s_\phi \end{array} \right\}_{lm} \sin m\lambda \right) P_{lm}(\cos\theta), \quad (1.81)$$

where a is the reference Earth radius, and

$$\sum_{lm}{}' \equiv \sum_{l=0}^{\infty} \sum_{m=0}^{l} . \quad (1.82)$$

The real, non–dimensional coefficients of the expansion (1.81) are given by

$$\left\{ \begin{array}{c} c_u \\ s_u \end{array} \right\}_{lm}(t) = \frac{3}{a\bar{\rho}_e} \frac{\bar{h}_l(t)}{(2l+1)} \left\{ \begin{array}{c} c_{\sigma'} \\ s_{\sigma'} \end{array} \right\}_{lm} \quad (1.83)$$

and

$$\left\{ \begin{array}{c} c_\phi \\ s_\phi \end{array} \right\}_{lm}(t) = \frac{3}{a\bar{\rho}_e} \frac{\bar{k}_l(t)}{(2l+1)} \left\{ \begin{array}{c} c_{\sigma'} \\ s_{\sigma'} \end{array} \right\}_{lm}, \quad (1.84)$$

respectively, where $c_{\sigma'}$ and $s_{\sigma'}$ represent the real coefficients of the expansion of the total load function

$$\sigma'(\omega) = \sum_{lm}{}' (c_{\sigma'} \cos m\lambda + s_{\sigma'} \sin m\lambda) P_{lm}(\cos\theta), \quad (1.85)$$

with

$$\left\{ \begin{array}{c} c_{\sigma'} \\ s_{\sigma'} \end{array} \right\}_{lm} = (2 - \delta_{0m})\mu_{lm} \left\{ \begin{array}{c} +\mathrm{Re}(\sigma') \\ -\mathrm{Im}(\sigma') \end{array} \right\}_{lm}. \quad (1.86)$$

where we have used proposition (5) of [15], with μ_{lm} given by (4.4).

The expansions above, that solve what we have called simplified GIA problem, are implemented in the software TABOO to account for various ice load time–histories and shapes [15, 16]. Since they assume an extremely simplified ocean load, we expect that these solutions may be somewhat inaccurate especially in the far field with respect to the former ice sheets, where the effects of the water load may be comparable to that of the ice load. However, the most serious weakness of the solution that we have found is that it does not directly provide the most basic geophysical observation in this context, i. e., the postglacial sealevel variations. This difficulty may be circumvented by introducing the sealevel equation (SLE), that is done in the next chapter.

Chapter 2

The Sea Level Equation

This chapter is devoted to the study of the sealevel equation (SLE). The theoretical aspects of our analysis are based on the manuscript by Farrell and Clark [3], and Wu and Peltier [24], while for the numerical aspects that will be reported in next chapter we have largely followed the study of Peltier et al. [11], who first introduced the "finite–elements" approach in this context. However, to help the reader – and particularly the students and the non–specialist geophysicists – our discussion is much more detailed (and perhaps boring) in some parts.

The basic idea of the SLE dates back to 1888, when Woodward published his pioneering work on the *form and position of mean sea level* [23], and only later has been refined by Platzman [13] and Farrell [2] in the context of the study of the ocean tides. In the words of Wu and Peltier [24], the solution of the SLE *yields the space– and time–dependent change of ocean bathymetry which is required to keep the gravitational potential of the sea surface constant for a specific deglaciation chronology and viscoelastic earth model.* This constitutes a significant improvement with respect to the approach of the last chapter, where the ocean load has been assumed to be uniform – or eustatic – in order to simplify the discussion.

2.1 Background

Before showing how the SLE can be obtained, here we provide a few definitions and statements that will be useful in the following.

Let us define by $r = r'_t$ the radius of a point P of the Earth's solid surface with spherical coordinates ω at present time, and with $r = r'_g$ the radius of the projection of P drawn to the geoid, at the same time. These radii are measured with respect to the Earth's center of mass. In ocean areas, the *sealevel* is defined as the offset between the surface of the geoid and that of the solid Earth

$$SL = SL(\omega, t) = r'_g - r'_s, \tag{2.1}$$

while the *sealevel change* is

$$S(\omega, t) = SL - SL_0, \tag{2.2}$$

where SL_0 is a reference sealevel, measured at the same point P but at a remote time $t = t_0$:

$$SL_0 = r_g - r_s. \tag{2.3}$$

The quantity provided by (2.2), that is consistent with the definition already given in §1.2.2, represents exactly the sealevel variation that would be measured by a stick–meter at P, as described by Farrell and Clark [3]. A particularly illuminating expression for $S(\omega, t)$ can be obtained combining (2.1) with (2.3):

$$S(\omega, t) = N - U, \tag{2.4}$$

where we have introduced the *geoid height change*

$$N = r'_g - r_g \tag{2.5}$$

and the *vertical displacement* of the solid surface of the Earth

$$U = r'_s - r_s. \tag{2.6}$$

Equation (2.4) shows explicitly that the sealevel variations at a given place are determined by changes in the shape of the two surfaces that define the geoid and the solid Earth.

In our ensuing discussion, it is assumed that the surface of the solid Earth and that of the geoid always slightly depart from the perfectly spherical surface of an ideal Earth having radius $r = r_0$ and the same mass m_e of the real planet, but characterized by a radially varying density. In this quasi–spherical approximation, the gravity acceleration is

$$\vec{g} = -g_r \hat{r} + g_t \hat{t} \tag{2.7}$$

with vertical and horizontal components given by

$$g_r = \gamma(r) + \delta g_r \tag{2.8}$$

and

$$g_t = \delta g_t, \tag{2.9}$$

where δg_r and δg_t are small quantities, and

$$\gamma(r) = \frac{Gm_e}{r^2} \tag{2.10}$$

is the modulus of the gravity field of the spherically symmetric ideal Earth.

The total gravitational potential W at given time includes contributions from the solid Earth, the oceans, and the ice sheets. It is related to the gravity field by $\vec{g} = \nabla W$, that implies

$$g_r = -\frac{\partial W}{\partial r}. \tag{2.11}$$

Since the free surface of the oceans is an equipotential surface for the gravitational potential, we write

$$W'(r'_g) = c' \tag{2.12}$$

and

$$W_0(r_g) = c_0, \tag{2.13}$$

where W' and W_0 are the total potentials at present time and in the remote reference state, respectively, and c' and c_0 are two constants.

If $r = r_{ep}$ is an equipotential surface close to the sphere $r = r_0$ and $\epsilon = \epsilon(\omega)$ is a small height change ($|\epsilon| \ll r_0$), the gravitational potential on the perturbed surface is, to first order in ϵ:

$$W(r_{ep} + \epsilon) = W(r_{ep}) - \epsilon\gamma_o, \tag{2.14}$$

where

$$\gamma_o = \gamma(r_0). \tag{2.15}$$

The statement (2.14) follows from

$$\begin{aligned} W(r_{ep} + \epsilon) &\simeq W(r_{ep}) + \epsilon \frac{\partial W}{\partial r}\Big|_{r_{ep}} \\ &= W(r_{ep}) - \epsilon g_r(r_{ep}) \\ &= W(r_{ep}) - \epsilon(\gamma(r_{ep}) + \delta g_r(r_{ep})) \\ &\simeq W(r_{ep}) - \epsilon\gamma(r_0), \end{aligned} \tag{2.16}$$

where we have used (2.8) and (2.11), we have neglected terms of second order in the small quantities ϵ and δg_r, and we have approximated $\epsilon\gamma(r_{ep})$ with $\epsilon\gamma(r_0)$. An interesting consequence of (2.14) is that in the particular case $\epsilon = c =$ constant we obtain

$$W(r_{ep} + c) = W(r_{ep}) - c\gamma_o = \text{constant}, \tag{2.17}$$

showing that if $r = r_{ep}$ is an equipotential surface, a further equipotential surface is at radius $r = r_{ep} + c$, provided that $c \ll r_0$ is a small constant.

2.2 Obtaining the SLE

To obtain the SLE, we need first to evaluate how the surface of the geoid has changed from the reference state to the current state. Both states are characterized by an arbitrary distribution of ice and ocean masses, and they differ by the amounts of mass contained in these reservoirs – the total mass being unaltered. Here we denote with $m_i(t)$ the change in the ice mass, where conventionally a positive change denotes accretion. We recall that we are assuming fixed shorelines, so that the shape of the continents is unaltered by the exchange of mass between the ice sheets and the oceans. Of course, the shapes of the solid Earth and of the geoid will differ in the two states, due to the global deformations induced by the mass redistribution at the surface.

We first observe that it is possible to determine a (non constant) small height $h = h(\omega)$ with $|h| \ll r_0$ such that the current gravitational potential at radius $r_g + h$ equals the potential in the reference state at $r = r_g$, i. e.:

$$W'(r_g + h) = W_0(r_g) = \text{constant}, \tag{2.18}$$

where the second equality holds because the surface $r = r_g$ is the geoid in the reference state. The expression for $h(\omega)$ can be obtained observing that, from (2.14) we have

$$W'(r_g + h) \simeq W'(r_g) - h\gamma_o, \tag{2.19}$$

so that

$$h = \frac{W'(r_g) - W_0(r_g)}{\gamma_o} = \frac{\Phi(r_g)}{\gamma_o}, \tag{2.20}$$

where Φ indicates the variation of the total gravitational potential with respect to the reference state, computed on the former geoid $r = r_g$. The potential Φ can be considered as a small quantity since the masses involved in the process of glacio–isostacy are small if compared to the Earth mass. Therefore, to a high degree of approximation we can write $\Phi(r_g) \simeq \Phi(r_0)$, that gives

$$h = \frac{\Phi(r_0)}{\gamma_o}, \tag{2.21}$$

where the general expression for $\Phi(r_0)$ is given by the first of (1.62) as a spatio–temporal convolution between the GF and the surface load. It results from variations of the ice load, of the ocean load, and from the deformations of the solid Earth in response to the varying surface loads.

The surface $r = r_g + h$ determined above is certainly an equipotential surface, but it is not necessarily coincident with the geoid. In fact, beside being an equipotential surface, the geoid must be constrained by the requirement of mass conservation, that has not been imposed so far. Since from (2.17) we know that a new equipotential surface can be obtained by a uniform shift of an existing equipotential surface, we seek the current geoid in the form

$$r'_g = r_g + h + c, \tag{2.22}$$

where the constant c will be determined below imposing mass conservation. As a consequence of (2.22) and (2.21), the geoid height variation given by (2.5) can also be expressed as

$$N = \frac{\Phi}{\gamma_o} + c. \tag{2.23}$$

According to the definition (2.2), the sealevel change is

$$\begin{aligned} S &= SL - SL_0 \\ &= (r'_g - r'_s) - (r_g - r_s) \\ &= r_g + h + c - r'_s - r_g + r_s \\ &= h + c - (r'_s - r_s) \\ &= \frac{\Phi}{\gamma_o} - U + c, \end{aligned} \tag{2.24}$$

where we have used (2.22), and we have introduced the vertical displacement of the solid surface of the Earth using the definition (2.6). Its general form is expressed by the second of (1.62) as a convolution between the appropriate GF and the surface load.

The requirement of mass conservation can be stated as

$$\Delta M = 0, \tag{2.25}$$

where M is the mass of the system ice+oceans. Since

$$\Delta M = m_i(t) + \int_o S\rho_o dA, \tag{2.26}$$

using (2.25) with (2.26) and (2.24) we obtain

$$m_i(t) + \int_o \rho_o \Big(\frac{\Phi}{\gamma_o} - U + c\Big) dA = 0, \tag{2.27}$$

that gives

$$m_i(t) + c\rho_o \int_o dA + \rho_o \int_o \Big(\frac{\Phi}{\gamma_o} - U\Big) dA = 0, \tag{2.28}$$

where we have assumed a constant water density. Hence

$$m_i(t) + c\rho_o A_o + \rho_o A_o \overline{\Big(\frac{\Phi}{\gamma_o} - U\Big)} = 0, \tag{2.29}$$

where the bar denotes the spatial average over the oceans surface:

$$\overline{(\cdot)} \equiv \frac{1}{A_o} \int_o (\cdot) dA. \tag{2.30}$$

Solving for the yet unknown constant c we obtain

$$c = -\frac{m_i(t)}{\rho_o A_o} - \overline{\Big(\frac{\Phi}{\gamma_o} - U\Big)}, \tag{2.31}$$

that substituted into (2.24) finally provides

$$S = \Big(\frac{\Phi}{\gamma_o} - U\Big) + S_E - \overline{\Big(\frac{\Phi}{\gamma_o} - U\Big)}, \tag{2.32}$$

which represents the "sea level equation" (SLE), where

$$S_E = -\frac{m_i(t)}{\rho_o A_o} \tag{2.33}$$

is the *eustatic term*.

2.3 "Gravitationally self–consistent" SLE

The SLE is an implicit equation. This can be seen recalling from §1.3 the general expressions for the vertical displacement and the incremental gravitational potential

$$\left\{ \begin{array}{c} U \\ \Phi \end{array} \right\} (\omega, t) = \rho_o \left\{ \begin{array}{c} G_u \\ G_\phi \end{array} \right\} \otimes_o S + \rho_i \left\{ \begin{array}{c} G_u \\ G_\phi \end{array} \right\} \otimes_i I, \tag{2.34}$$

that inserted into (2.32) provides

$$\frac{\Phi}{\gamma_o} - U = \frac{\rho_i}{\gamma_o} G_s \otimes_i I + \frac{\rho_o}{\gamma_o} G_s \otimes_o S, \tag{2.35}$$

where G_s is the *sealevel GF*, with

$$\frac{G_s}{\gamma_o} = \frac{G_\phi}{\gamma_o} - G_u, \tag{2.36}$$

which according to (1.26) and (1.30) can also be expressed as

$$\frac{G_s}{\gamma_o} = \frac{G^r_\phi + G^e_\phi + G^v_\phi}{\gamma_o} - (G^e_u + G^v_u). \tag{2.37}$$

From (2.32) and (2.35), the SLE can be finally arranged as

$$S \;=\; \frac{\rho_i}{\gamma_o} G_s \otimes_i I + \frac{\rho_o}{\gamma_o} G_s \otimes_o S + S_E - \frac{\rho_i}{\gamma_o} \overline{G_s \otimes_i I} - \frac{\rho_o}{\gamma_o} \overline{G_s \otimes_o S}, \tag{2.38}$$

with the eustatic term given by (2.33). In the literature, the solution of the SLE in the form (2.38) is often referred as to "gravitationally self–consistent" solution, since it has been obtained imposing that the geoid is that equipotential surface of the gravity field determined by the requirement of mass conservation; the ocean load is thus "consistent" with the gravity field.

The unknown sealevel variation S appears explicitly to left–hand side of (2.38), but it is embedded in convolution products on the right–hand side. This makes the SLE an integral equation. We observe that, differently from the simplified rebound problem described in §1.3, the solution of (2.38) directly provides the sealevel variations, that allows to make direct comparisons with geophysical observations, as discussed below in §2.5. Once S has been determined, the vertical displacement of the solid surface of the Earth and the incremental gravitational potential can be obtained from (2.34), that do not rely on the assumption of an eustatic ocean load, as the approximate solutions (1.77) do.

The solution of the SLE is a particularly challenging task and cannot be undertaken by means of analytical methods, except in an extremely particular case that will be described in §2.4.2. The finite–elements method that we will discuss in Chapter 3 takes advantage from an iterative approach, similar to the Neumann's method that is used for the Fredholm integral equations (see e.g. [5]). As a first guess, it is assumed that the sealevel variations are eustatic and the approximate solution $S^{(0)} = S_E$ is substituted to S on the right–hand side of (2.38). In this way, a new estimate $S^{(1)}$ is obtained at the left–hand side, and the process is iterated. The practice has shown that this procedure converges after a few steps to a stable solution [11]. However, as far as we know, no investigation has been performed to date in order to ascertain the convergence conditions nor to study other theoretical aspects of the SLE.

2.4 Approximate solutions of the SLE

The feature that makes the solution of the SLE a particularly difficult task is its integral character. Here we show that under some *special* conditions the SLE can be reduced to a non–integral equation, that can be solved without invoking the Neumann iterative procedure. In general, the assumptions that we will introduce are quite restrictive, so that these non–integral versions of the SLE should only be used in special circumstances, e. g., when the traditional Green functions approach outlined in the first chapter is not viable due to the presence of lateral viscosity variations or non–Newtonian rheologies. The approximate solutions presented here may be also useful for didactic purposes, since they allow to better appreciate the physical meaning of the various terms that appear in the SLE.

2.4.1 Eustatic solution

The term of the SLE

$$S_E = -\frac{m_i}{\rho_o A_o} \tag{2.39}$$

is called *eustatic term*, and plays a relevant role in our discussion. From (2.32), we observe that the SLE has solution

$$S = S_E \tag{2.40}$$

provided that

$$U = 0, \tag{2.41}$$

and

$$\Phi = 0. \tag{2.42}$$

According to (2.34), (1.30), and (1.26), the first implies

$$G_u^e = G_u^v = 0, \tag{2.43}$$

while the second yields

$$G_\phi^e = G_\phi^v = 0 \tag{2.44}$$

and

$$G_\phi^r = 0. \tag{2.45}$$

Equations (2.43) and (2.44) state that the Earth is rigid, while (2.45) imply that the direct changes of gravitational potential due to the rearrangement of the ice and ocean masses is neglected. From (2.36) we observe that the sealevel GF is $G_s = 0$ in the eustatic approximation. Since the real Earth is deformable and the Newton's constant is $\neq 0$, we must expect that the melting of the ice sheets produces spatially non–uniform sealevel variations, which may significantly depart from eustasy. This is widely confirmed by the results published in the literature and will be fully appreciated in Part II of this book, where the SLE is solved numerically.

More insight in the meaning of the eustatic approximation can be gained taking the ocean–average of both sides of the SLE in the form (2.32). Since $\overline{c} = c$ for any constant c, we simply have

$$\overline{S} = \overline{\left(\frac{\Phi}{\gamma_o} - U\right)} + \overline{S_E} - \overline{\overline{\left(\frac{\Phi}{\gamma_o} - U\right)}} = S_E, \tag{2.46}$$

showing that the spatially–averaged sealevel variations coincide with the eustatic term. This property holds independently from the assumed variations of the ice thickness and the shape of the ocean–continents distribution.

2.4.2 Woodward's solution

Woodward (1888) has found an analytical solution of the SLE assuming that (i) the Earth is rigid, (ii) the mass load is localized and impulsive - so that in this problem we are faced with is the instantaneous freezing of a point load -, (iii) the oceans cover uniformly the Earth, and (iv) changes in the oceans distribution do not affect the gravity field - this last condition is often referred as to "non–self–gravitating oceans" condition. Although the assumptions of Woodward are very severe, the solution of the SLE that can be obtained in this case is instructive, since it shows that the sealevel changes can significantly depart from eustasy even when essential physical ingredients - such as the deformations of the solid Earth and the shape and self-gravitation of the oceans - are totally disregarded. The Woodward's approach can be extended to the case of realistic oceans, for which an analytical solution cannot be determined. The numerical code described in Part II of this book allows to deal with this case.

Since the Earth is assumed to be rigid, the vertical displacement vanishes ($U = 0$). Furthermore, since we are dealing with an impulsive point mass, we can identify the incremental potential Φ with the GF G^r_ϕ given by (1.8). Accordingly, from (2.32) we write the SLE as

$$S^* = \frac{G^r_\phi}{\gamma_o} + S^*_E - \frac{\overline{G^r_\phi}}{\gamma_o}, \tag{2.47}$$

where the asterisk indicates that the sealevel changes are referred to a unit mass and unit time.

The three terms in (2.47) can be worked out analytically. Although the mathematics is straightforward, we report here the details. From (1.8) the first term of the SLE is simply

$$\frac{G^r_\phi}{\gamma_o} = \delta(t)\frac{a}{m_e}\frac{1}{2\sin(\alpha/2)}. \tag{2.48}$$

The second term - that represents the eustatic component of the SLE - can be transformed as follows

$$S^*_E = -\frac{\delta(t)}{\rho_o A_o} = -\frac{\delta(t)}{4\pi a^2 \rho_o} = -\frac{\delta(t)}{\dfrac{3m_e}{a\bar{\rho}_e}\rho_o} - \delta(t)\frac{a}{m_e}\frac{\bar{\rho}_e}{3\rho_o}, \tag{2.49}$$

where we have taken into account that the oceans cover uniformly the Earth ($A_o = 4\pi a^2$), and we have used the expression for the Earth's average density given by (1.71). The third and last term of the SLE can be expressed as

$$
\begin{aligned}
-\frac{\overline{G^r_\phi}}{\gamma_o} &\equiv -\frac{1}{A_o}\int_o \frac{G^r_\phi}{\gamma_o} dA \\
&= -\delta(t)\frac{1}{4\pi a^2 m_e} a^3 \int_0^{2\pi} d\lambda \int_0^{\pi} \frac{\sin\alpha d\alpha}{2\sin(\alpha/2)} \\
&= -\delta(t)\frac{a}{2m_e}\int_0^{\pi}\cos(\alpha/2)d\alpha \\
&= -\delta(t)\frac{a}{m_e},
\end{aligned}
\tag{2.50}
$$

where we have written the element of area as $dA = a^2 \sin\alpha d\alpha d\lambda$ - where λ is longitude -, and we have again integrated over the uniform oceans.

Thus, from (2.47), (2.48), (2.49), and (2.50), the solution of the Woodward's problem for a uniform ocean is

$$
S^*(\alpha, t) = \delta(t)\frac{a}{m_e}\Big(\frac{\Delta m}{m_e}\Big)\Big(\frac{1}{2\sin(\alpha/2)} - \frac{\bar{\rho}_e}{3\rho_o} - 1\Big). \tag{2.51}
$$

Since we are using a point source and Earth is spherically symmetric, the sealevel variations only depend on the angular distance α from the load.

It is clear from (2.51) that Woodward's solution departs from eustasy. This is merely a consequence of the gravitational attraction exerted by the point mass on the uniform oceans. We observe that the sealevel changes exceed the eustatic sealevel drop for $(2\sin(\alpha/2) - 1)^{-1} \geq 0$, that is for colatitudes $\alpha \leq 60°$. It is also noticeable that the sealevel variations are positive for $\sin(\alpha/2) \leq (2(1 + \bar{\rho}_e/3\rho_o))^{-1}$, that implies $\alpha \leq 10.2°$ where we have used $\bar{\rho}_e = 5511.57$ kg m^{-3} and $\rho_o = 1000$ kg m^{-3}. The singularity shown by (2.51) for $\alpha \mapsto 0$ is due to the fact that we are dealing with a point mass; it disappears as soon as a finite–sized ice sheets are employed.

The point–source approximation can be easily relaxed in order to deal with finite–sized ice sheets. Of course, since the ice sheet structure is in general spatially complex, the sealevel variations will depend on both colatitude and longitude, and in general we cannot obtain a closed form–solution. However, given that the oceans are still assumed to be non–self–gravitating, the right–hand side of the SLE is explicit and there is no need to perform iterations in order to obtain the solution. In such approximation, the SLE has still the form (2.47), but according to (2.34) the incremental gravitational potential is $\Phi = \rho_i G^r_\phi \otimes_i I$. As a final remark, we observe that the self–gravitation of the oceans can be introduced within the Woodward approximation just writing the incremental gravitational potential as $\Phi = \rho_i G^r_\phi \otimes_i I + \rho_o G^r_\phi \otimes_o S$.

However, in such *"modified Woodward approximation"* the SLE clearly becomes an integral equation.

2.4.3 An ice–free approach

In their 1976 manuscript, Farrell and Clark [3] have considered the variations of sealevel that would be observed in the absence of ice loads, when the Earth is considered as an elastic body. In their own words, *... the mass is added* (to the oceans) *from outside the Earth, rather than coming from a melting ice sheet.* With $I = 0$, the SLE has the form

$$S = \frac{\rho_o}{\gamma_o} G_s \otimes_o S + S_E - \frac{\rho_o}{\gamma_o} \overline{G_s \otimes_i S}, \tag{2.52}$$

with the sealevel GF given by

$$\frac{G_s}{\gamma_o} = \frac{G_\phi^r + G_\phi^e}{\gamma_o} - G_u^e, \tag{2.53}$$

where the viscoelastic terms are neglected. This ice–free version of the SLE was developed in order to show how the changes of sealevel depart from eustasy, even when the deformations induced by the ice–sheets are neglected. The readers are referred to [3] for a discussion of the outcomes of this approximate approach to the SLE. The numerical code described in Part II does not include this particular solution of the SLE.

2.4.4 An explicit approach

Here we show that it is possible reduce the SLE to an explicit form also in the general case of a viscoelastic Earth and of spatially complex surface loads. This approach can only be employed to describe the sealevel variations after the end of the melting of the ice sheets, and is based on the assumption of negligible ocean loads. Due to the simple solution that we find in this case - that takes advantage from the approximate solution of the GIA problem that we have already described in §1.3 - this approach constitutes a tool useful to obtain first guesses of postglacial sealevel changes.

We recall from (2.26) that the net mass variation of the ice+ocean system can be expressed by

$$\Delta M = m_i(t) + \int_o S \rho_o dA, \tag{2.54}$$

where the terms on the right–hand side account for the mass variation of the ice sheets and of the oceans, respectively. In §2.2 we have determined that particular constant c that allows to write the SLE in the form

$$S = \frac{\Phi}{\gamma_o} - U + c, \tag{2.55}$$

so that mass conservation is ensured, with $\Delta M = 0$ in (2.54). The expression that we have found for c is given by (2.31). From (2.54), here we observe that the constraint $\Delta M = 0$ can be also satisfied – for any value of the constant c – provided that

$$m_i(t) = 0, \tag{2.56}$$

and

$$\rho_o = 0, \tag{2.57}$$

where (2.56) states that the mass of the ice load has remained unaltered since the remote reference time, while with (2.57) the weight of the extra water that has filled the ocean basins is neglected. The indeterminacy of the constant c indicates that the geoid cannot be unequivocally determined for a weightless ocean. For the sake of convenience, we choose $c = 0$, that according to (2.22) minimizes the offset between the new and the former geoid. From (2.23), this gives

$$N = \frac{\Phi}{\gamma_o}. \tag{2.58}$$

Furthermore, from (2.56) and (2.57), the SLE reduces to

$$S = \frac{\Phi}{\gamma_o} - U, \tag{2.59}$$

that is still an implicit equation, since vertical displacement and the incremental gravitational potential will depend, according to (2.34), on S itself. However, since here the water is assumed to be weightless, the ocean components of both U and Φ simply vanish, and we can write

$$S = \frac{\Phi_i}{\gamma_o} - U_i, \tag{2.60}$$

that reduces the SLE to a non–integral form that can be solved without invoking iterative techniques.

Due to its intrinsic limitations, the version of the SLE that we have determined should only be used to model the process of rebound that follows the melting of the ice sheets, when the Earth surface is subject to a free relaxation. Since the deformations and potential perturbations associated with the ocean load are likely that this approximation fails to predict correctly the sealevel variations far from the former ice sheets, where the signals due to ice melting and to the ocean bottom deformations may be of comparable magnitude.

An even more drastic approach to the SLE consists in neglecting the variation in the incremental gravitational potential in front of the vertical displacement, that from (2.60) leads to

$$S = -U_i, \tag{2.61}$$

and, from (2.58)

$$N = 0. \tag{2.62}$$

This further approximation may constitute a valid tool for modelers that employ finite–elements techniques, that often do not allow for a straightforward implementation of the gravitational interactions. The error that is implicit in (2.61) is generally not severe, but may vary from place to place. As a rule–of–thumb, there is a ratio of 1 to 10 between the 'geoid variation' Φ/γ_o and the vertical displacement U (this can be verified in practice a number of situations employing the software TABOO by Spada et al. [16]). In this respect, the degree of approximation implicit in (2.61) may be acceptable for testing, awaiting for the implementation of the gravitational forces within the FE approach.

2.5 By–products of the SLE

Once the SLE has been solved, it is possible to compute a number of relevant geophysical quantities in addition to the sealevel change $S(\omega, t)$. In the following we discuss how they can be derived. We first introduce the *relative sealevel* and the rate of sealevel change. Then, we discuss the velocity field and the time–variation of the Stokes coefficients of the gravity field.

2.5.1 Observed sealevel variations

The predictions made using the SLE can be directly compared with two important data sets, namely the Holocene relative sealevel variations evidenced by geological investigations, and the present–day rates of sealevel variations measured by tide–gauges.

As previously discussed, the SLE does not provide absolute sealevel variations, but rather it allows to compute sealevel changes referred to a remote time. More specifically, if t_{BP} denotes a given time before present, the SLE directly provides the difference

$$S(\omega, t_{BP}) = SL(\omega, t_{BP}) - SL(\omega, t_r), \tag{2.63}$$

where $\omega = (\theta, \lambda)$ are the spherical coordinates of a specific site, SL is the sealevel, and t_r is a remote (and arbitrary) time BP. At the present time $t = t_p$, we can similarly write

$$S(\omega, t_p) = SL(\omega, t_p) - SL(\omega, t_r), \tag{2.64}$$

that combined with (2.63) provides

$$RSL(\omega, t_{BP}) \equiv SL(\omega, t_{BP}) - SL(\omega, t_p) = S(\omega, t_{BP}) - S(\omega, t_p), \tag{2.65}$$

where the left equality defines the *relative sealevel variations* with respect to the present level and the right–hand term can be obtained solving the SLE. As discussed in e.g. [3], the RSL observations are typically obtained by the analysis of elevated beach terraces formed as a consequence of the vertical uplift in response to the melting of the late–Pleistocene ice sheets and from the submerged archaeological observations that are sometimes available in anciently populated areas. In Part II we will provide RSL predictions for the 392 sites contained in the widely employed global database of Tushingham and Peltier [19, 20] for which radiocarbon–controlled data are available.

The currently observed rate of sealevel change constitutes another important source of information in addition to the Holocene relative sealevel variations. While the latter data set spans several thousand years, the former only provides a snapshot of present–day changes. More precisely, the rate of sealevel change simply represents the steepness of the RSL curve at time t_p as

$$\dot{S}(\omega, t_p) = \frac{S(\omega, t_p - \delta t) - S(\omega, t_p)}{\delta t}. \tag{2.66}$$

In Part II we will present predictions of the rate $\dot{S}$ for all of the tide–gauges sites that are currently included in the database of the Permanent Service for the Mean Sea Level (PSMSL).

2.5.2 Velocity fields

From §1.3 we recall that the vertical displacement is given by

$$U(\omega, t) = \rho_i G_u \otimes_i I + \rho_o G_u \otimes_o S, \tag{2.67}$$

that can be explicitly evaluated once $S(\omega, t)$ has been retrieved from the SLE. When S and U have been determined, the change in the geoid height N - and its rate of variation - can be easily deduced by (2.4).

The horizontal components of displacements have not been discussed so far. They require a special treatment, since the GF G_v that we have introduced in §1.1.3 only allows for the computation of the component along the unit vector $\hat{\alpha}$, so that a specific projection is needed to represent the horizontal displacements along the usual unit vectors $\hat{\theta}$ and $\hat{\lambda}$ of the geographical reference system. According to [15], we write

$$\left\{ \begin{array}{c} U_\theta \\ U_\lambda \end{array} \right\} (\omega, t) = \rho_i G_v \left\{ \begin{array}{c} \cos X \\ \sin X \end{array} \right\} \otimes_i I + \rho_o G_v \left\{ \begin{array}{c} \cos X \\ \sin X \end{array} \right\} \otimes_o S, \tag{2.68}$$

with

$$\cos X = \frac{\cos\theta' - \cos\theta\cos\alpha}{\sin\theta\sqrt{(1-\cos^2\alpha)}} \tag{2.69}$$

and

$$\sin X = \frac{\sin(\lambda - \lambda')\sin\theta'}{\sqrt{(1-\cos^2\alpha)}}, \tag{2.70}$$

where (θ', λ') and (θ, λ) are the integration variables and the coordinates of the observer, respectively, while $\cos\alpha$ is given by (1.60).

It should be observed that the components of displacement by themselves do not constitute a set of geophysical observables. From (2.67) and (2.68) the three components of the *velocity field* ($\dot{U}$, $\dot{U}_\theta$, $\dot{U}_\lambda$) can be easily obtained by time differentiation. These variables are observable, since modern geodetic techniques - such as GPS and VLBI - allow for the determination of trends of displacements (i.e. velocities) based on observations over finite time periods. If needed, the velocity field can be easily projected along the conventional unit vectors $(\hat{l}, \hat{t}, \hat{v})$ of the geodetic baseline that connects two sites. The relevant formulas are given by [15].

2.5.3 Stokes coefficients

According to our discussion of §1.3, the incremental gravitational potential is determined by

$$\Phi(\omega,t) = \rho_i G_\phi \otimes_i I + \rho_o G_\phi \otimes_o S, \tag{2.71}$$

where the GF G_ϕ is given by (1.29). It is useful to expand Φ as

$$\frac{\Phi(\omega,t)}{\gamma_o} = \sum_{lm,l\geq 2} n_{lm} Y_{lm}(\omega), \tag{2.72}$$

where $Y_{lm}(\omega)$ are the complex spherical harmonics (see 4.3) and the coefficients n_{lm} have the dimensions of a length.

Since mass is conserved and we assume that the origin of the reference system always coincides with the center of mass of the Earth, in (2.72) the sum is restricted to degrees $l \geq 2$ (see [16] for further details). Because Φ/γ_o differs from the geoid height change N by an additive constant, that represents a term of harmonic degree zero (see 2.23), the coefficients n_{lm} coincide with the harmonic coefficients of N in this range of degrees. Using (2.72) and the orthogonality relationship for the spherical harmonics (see 4.5), they are given by

$$n_{lm} = \int_\Omega \frac{\Phi(\omega,t)}{\gamma_o} Y^*_{lm}(\omega) d\omega, \tag{2.73}$$

where the integration is over the whole unit sphere.

The incremental potential can be equivalently expanded in series of real spherical harmonics as

$$\frac{\Phi(\omega,t)}{\gamma_o} = a \sum_{lm,l\geq 2}{}' (\delta c_{lm}(t)\cos m\lambda + \delta s_{lm}(t)\sin m\lambda) P_{lm}(\cos\theta), \tag{2.74}$$

where $P_{lm}(\cos\theta)$ are the associated Legendre functions, the non dimensional quantities $(\delta c_{lm}, \delta s_{lm})$ represent the variations of the Stokes coefficients, and the sum is for orders $m \geq 0$. By proposition (5) of [15], the following relationship holds between complex and real coefficients

$$\begin{Bmatrix} \delta c \\ \delta s \end{Bmatrix}_{lm} (t) = \frac{(2-\delta_{0m})\mu_{lm}}{a} \begin{Bmatrix} +\mathrm{Re}(n) \\ -\mathrm{Im}(n) \end{Bmatrix}_{lm}, \tag{2.75}$$

where the normalization constant μ_{lm} is given by (4.4). The trend of variation of the changes of the Stokes coefficients - that can be observed by means of satellite geodetic techniques for low harmonic degrees - can be obtained taking the time–derivative of (2.75). Using (4.3), (4.4), (2.73) and (2.74) we obtain

$$\left\{ \begin{matrix} \dot{\delta c} \\ \dot{\delta s} \end{matrix} \right\}_{lm} (t) = \frac{(2-\delta_{0m})}{a^3} \mu_{lm}^2 \int_\Omega \dot{N} P_{lm}(\cos\theta) \left\{ \begin{matrix} \cos m\lambda \\ \sin m\lambda \end{matrix} \right\} dA, \qquad (2.76)$$

with $\dot{N} = \dot{S} + \dot{U}$ and $(l \geq 2, m = 0, 1, ..., l)$

According to e. g. [15], the "fully–normalized form" of the rates of change of the Stokes coefficients given by (2.76) is

$$\left\{ \begin{matrix} \dot{\overline{\delta c}} \\ \dot{\overline{\delta s}} \end{matrix} \right\}_{lm} (t) = h_{lm} \left\{ \begin{matrix} \dot{\delta c} \\ \dot{\delta s} \end{matrix} \right\}_{lm} (t), \qquad (2.77)$$

where the conversion factor h_{lm} is

$$h_{lm} = \sqrt{\frac{1}{(2-\delta_{0m})(2l+1)} \frac{(l+m)!}{(l-m)!}}. \qquad (2.78)$$

Chapter 3

Numerical approach to the SLE

This chapter is devoted to the temporal and spatial discretization of the SLE. While for the temporal aspect we strictly follow [11], the spatial part of the problem is approached by means of a technique which consists in a pixelization of the sphere in a set of approximately equal–area hexagons. This method is yet unused in the field of global geodynamics, but it has been successfully employed in astrophysics [18]. With respect to the spatial discretization of [11], based on rectangular "finite elements" of various sizes, the advantage of using the hexagonal pixelization consists in the easy quadrature formulas available, and in a better resolution of the oceans–continents margins. In addition, since an automatic pixelization algorithm is freely available and the resolution can be easily modified, the subdivision of the surface of the sphere is less prone to subjective choices. Other possible choices, such as the discs–based discretization [19], offer the further disadvantage of producing a non–uniform coverage, with overlaps and interstices that can cause significant alterations of the ice distribution.

The 'finite elements' method adopted here is by no means the only technique available for solving the SLE. Another widely employed – and possibly superior – technique is the so called 'pseudo–spectral method', that is exhaustively described in [8] and [7]. It is not our intention to produce a comparison between these two solution methods. This is left to the readers, who can actively contribute to this discussion proposing their own numerical solutions (and perhaps writing a new chapter of this booklet).

In the first section we describe the discretization scheme. In the two ensuing sections, the scheme will be applied to the SLE, while in the last one we will describe the iterative procedure that is used to retrieve the solution.

3.1 Discretization scheme

In this context, the time–discretization of any scalar quantity $F(\omega, t)$ is accomplished according to this scheme

$$F(\omega, t) = \begin{cases} F_0(\omega), & t < t_0 \\ F_l(\omega), & t_{l-1} \leq t < t_l, \quad l = 1, 2, ..., N \\ F_{N+1}(\omega), & t \geq t_N, \end{cases} \tag{3.1}$$

where $N \geq 1$ is an integer, and the times t_l are equally spaced, with

$$t_l - t_{l-1} = \Delta = \frac{t_N}{N}. \tag{3.2}$$

It is straightforward to verify that we can equivalently write

$$F(\omega, t) = F_0(\omega) + \sum_{l=0}^{N} \delta F_l(\omega) H(t - t_l), \tag{3.3}$$

where $H(t)$ is the Heaviside step function:

$$H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0, \end{cases} \tag{3.4}$$

and

$$\delta F_l(\omega) = F_{l+1}(\omega) - F_l(\omega), \qquad l = 0, 1, ..., N. \tag{3.5}$$

Since in our discussion we are mainly concerned with the *time variations* of geophysical quantities - such as the ice thickness and the sealevel - rather than with their absolute values, we introduce the variation of F with respect to the remote time t_0 as

$$DF(\omega, t) = F(\omega, t) - F_0(\omega), \tag{3.6}$$

that from (3.3) gives

$$DF(\omega, t) = \sum_{l=0}^{N} \delta F_l(\omega) H(t - t_l). \tag{3.7}$$

The spatial discretization is built according to the icosahedron–based method proposed by Max Tegmark for astrophysical applications [18][1], that

[1]A FORTRAN code is available at http://www.sns.ias.edu/∼max/icosahedron.html.

offers the advantage of providing a quasi–regular spherical grid. It is assumed that the relevant quantities – such as the ice thickness and the sealevel – are constant within each of the grid pixels. If $\omega_j = (\theta_j, \lambda_j)$ denotes the coordinates of the centroid of a given pixel, and N_p is the number of pixels, the spatial discretization is accomplished introducing the two–dimensional arrays

$$DF_{jl} = DF(\omega_j, t_l), \tag{3.8}$$

and

$$\delta F_{jl} \equiv \delta F_l(\omega_j) = F_{l+1}(\omega_j) - F_l(\omega_j), \tag{3.9}$$

with $(j = 1, .., N_p; l = 0, ..., N)$.

From (3.7) and (3.9), the discretized form of $DF(\omega, t)$ is

$$\begin{aligned} DF_{jl} &= \sum_{k=0}^{N} \delta F_k(\omega_j) H(t_l - t_k) \\ &= \sum_{k=0}^{l} \delta F_k(\omega_j) \\ &= \sum_{k=0}^{l} \delta F_{jk} \\ &= \sum_{k=0}^{l} [F_{k+1}(\omega_j) - F_k(\omega_j)] \\ &= F_{l+1}(\omega_j) - F_0(\omega_j). \end{aligned} \tag{3.10}$$

Furthermore, the following relationship holds between the arrays introduced above:

$$\begin{aligned} \delta F_{jl} &= [F_{l+1}(\omega_j) - F_0(\omega_j)] - [F_l(\omega_j) - F_0(\omega_j)] \\ &= DF_{jl} - DF_{j,l-1}. \end{aligned} \tag{3.11}$$

The discretization scheme outlined above will be directly applied to the most relevant geophysical quantities in this context. Namely, it suffices to identify the generic function F with the ice thickness T, and DF with the change in the ice thickness I. In a similar way, F and DF can be identified with the sealevel SL and with the sealevel change S, respectively.

The pixelization scheme adopted here allows for a straightforward equal–weights spherical quadrature [18]. Thus, given any time–discretized function $G = G(\omega, t_l)$, we write

$$\int_D G(\omega', t_l) dA = \sum_{j:\omega_j \in D} a_j G_{jl}, \tag{3.12}$$

where $D \subseteq \Omega$, ω_j are the coordinates of the centroids of the pixels, $G_{jl} = G(\omega'_j, t_l)$, and the (equal) weights are

$$a_j = \xi \equiv \frac{4\pi a^2}{N_p}, \tag{3.13}$$

where N_p is the total number of pixels over the surface of the sphere.

3.2 Discretizing the SLE

In this section we apply to the SLE the discretization scheme that we have described in §3.1. Although we largely follow [11], here we employ a slightly different notation and we give more details with the aim to help the non–specialist readers and the comprehension of the Fortran 90 codes described in Part II.

To facilitate the analysis that follows, it convenient to write the SLE in the form

$$S = S_1 + S_2 + S_E - \overline{S_1} - \overline{S_2}, \tag{3.14}$$

where, according to (2.38) and (2.33), we have

$$S_1 = \rho_i \frac{G_s}{\gamma_o} \otimes_i I, \tag{3.15}$$

$$S_2 = \rho_o \frac{G_s}{\gamma_o} \otimes_o S, \tag{3.16}$$

$$S_E = -\frac{m_i}{\rho_o A_o}. \tag{3.17}$$

Furthermore, we write the sealevel GF as

$$\frac{G_s}{\gamma_o} = \beta[\psi^\delta(\alpha)\delta(t) + \psi^h(\alpha, t)H(t)], \tag{3.18}$$

with

$$\psi^{\delta}(\alpha) = \sum_{n=0}^{\infty}(1 + k_n^e - h_n^e)P_n(\cos\alpha), \tag{3.19}$$

$$\psi^{h}(\alpha, t) = \sum_{n=0}^{\infty}\left[\sum_{m=1}^{M}(k_{nm} - h_{nm})e^{s_{nm}t}\right]P_n(\cos\alpha), \tag{3.20}$$

and

$$\beta = \frac{a}{m_e}, \tag{3.21}$$

where we have used the definitions (2.36) and (2.37) and the expressions of the viscoelastic GFs (1.29) and (1.34).

The discretization of the SLE is accomplished into two steps, which concern the temporal and the spatial facets of the problem, respectively.

3.2.1 Time

Here we consider separately the three terms of the SLE given by (3.15), (3.16) and (3.17), showing in detail how they can be reduced to a time–discretized form.

The terms S_1 and S_2

From (3.15), the first term of the SLE is

$$S_1(\omega, t) = \frac{G_s}{\gamma_o} \otimes_i \rho_i I = S_{1A} + S_{1B}, \tag{3.22}$$

where, using (3.18):

$$S_{1A} = \beta\psi^{\delta}(\alpha)\delta(t) \otimes_i \rho_i I \tag{3.23}$$

and

$$S_{1B} = \beta\psi^{h}(\alpha, t)H(t) \otimes_i \rho_i I. \tag{3.24}$$

According to (1.61), S_{1A} can be written as

$$\begin{aligned} S_{1A} &= \beta\int_{-\infty}^{t}dt'\int_i dA'\psi^{\delta}(\alpha)\delta(t-t')\rho_i I(\omega', t') \\ &= \beta\rho_i\int_i dA'\psi^{\delta}(\alpha)I(\omega', t), \end{aligned} \tag{3.25}$$

where we have used the basic property of the Dirac delta.

The term S_{1B} requires a more elaborate treatment. From (3.24) and (1.61) we obtain

$$\begin{aligned} S_{1B} &= \beta \int_{-\infty}^{t} dt' \int_i \psi^h(\alpha, t-t') H(t-t') \rho_i I(\omega', t') dA' \\ &= \beta \rho_i \int_{-\infty}^{t} dt' \int_i \psi^h(\alpha, t-t') I(\omega', t') dA', \end{aligned} \tag{3.26}$$

where we have used the definition of the Heaviside step function (3.4). According to (3.7), we expand the ice thickness variation as follows

$$I(\omega, t) = \sum_{l=0}^{N} \delta T_l(\omega) H(t - t_l), \tag{3.27}$$

with

$$\delta T_l(\omega) = T_{l+1}(\omega) - T_l(\omega), \tag{3.28}$$

where $T_l(\omega)$ is the ice thickness at ω for time $t_{l-1} \leq t < t_l$. Hence

$$\begin{aligned} S_{1B} &= \beta \rho_i \int_{-\infty}^{t} dt' \int_i \psi^h(\alpha, t-t') \sum_{l=0}^{N} \delta T_l(\omega') H(t'-t_l) dA' \\ &= \beta \rho_i \sum_{l=0}^{N} \int_i dA' \delta T_l(\omega') \int_{-\infty}^{t} dt' \psi^h(\alpha, t-t') H(t'-t_l), \end{aligned} \tag{3.29}$$

where we have exchanged the order of integration. Introducing the new variable $\tau = t' - t_l$, we obtain

$$S_{1B} = \beta \rho_i \sum_{l=0}^{N} \int_i dA' \delta T_l(\omega') \int_{-\infty}^{t-t_l} d\tau \psi^h(\alpha, t - t_l - \tau) H(\tau), \tag{3.30}$$

where, because of the term $H(\tau)$, the integral over time vanishes identically for $t - t_l < 0$ and can be otherwise restricted to the interval $0 \leq \tau \leq t - t_l$. Hence

$$\begin{aligned} S_{1B} &= \beta \rho_i \sum_{l=0}^{N} \int_i dA' \delta T_l(\omega') H(t-t_l) \int_{0}^{t-t_l} d\tau \psi^h(\alpha, t - t_l - \tau) \\ &= \beta \rho_i \sum_{l=0}^{N} \int_i dA' \delta T_l(\omega') \tilde{\psi}^h(\alpha, t - t_l), \end{aligned} \tag{3.31}$$

where

$$\tilde{\psi}^h(\alpha,t) \equiv H(t)\int_0^t d\tau \psi^h(\alpha,t-\tau). \tag{3.32}$$

Thus, according to (3.25) and (3.31), the time–discretized version of the first term of the SLE is

$$\begin{aligned} S_1(\omega,t_p) &\equiv S_{1A}(\omega,t_p)+S_{1B}(\omega,t_p) = \\ &= \beta\rho_i \int_i dA' \psi^\delta(\alpha) I(\omega',t_p) + \\ &+ \beta\rho_i \sum_{l=0}^{N} \int_i dA' \delta T_l(\omega') \tilde{\psi}^h(\alpha,t_p-t_l), \end{aligned} \tag{3.33}$$

where $t = t_p$ indicates a specific time, and we observe that S_1 depends from ω via the relationship $\alpha = \alpha(\omega,\omega')$ (see 1.60).

As it can be observed comparing (3.15) with (3.16), the structure of S_2 is formally identical to that of S_1, but it involves "ocean variables" instead of "ice variables". Thus, substituting in (3.33) ρ_i with ρ_o, I with S, δT_l with δS_l, and integrating over the oceans, we obtain

$$\begin{aligned} S_2(\omega,t_p) &\equiv S_{2A}(\omega,t_p)+S_{2B}(\omega,t_p) = \\ &= \beta\rho_o \int_o dA' \psi^\delta(\alpha) S(\omega',t_p) + \\ &+ \beta\rho_o \sum_{l=0}^{N} \int_o dA' \delta S_l(\omega') \tilde{\psi}^h(\alpha,t_p-t_l), \end{aligned} \tag{3.34}$$

where according to (3.7) we have expanded the sealevel variation as

$$S(\omega,t) = \sum_{l=0}^{N} \delta S_l(\omega) H(t-t_l), \tag{3.35}$$

with

$$\delta S_l(\omega) = SL_{l+1}(\omega) - SL_l(\omega), \tag{3.36}$$

where $SL_l(\omega)$ is the sealevel at ω for time $t_{l-1} \leq t < t_l$.

The term S_E

This term represents the eustatic component of the SLE. Using (1.42) in (2.33) we have

$$S_E(t_p) = -\frac{\rho_i}{\rho_o A_o} \int_i dA I(\omega, t_p). \tag{3.37}$$

The terms $\overline{S_1}$ and $\overline{S_2}$

From (3.22):

$$\overline{S_1} = \frac{1}{A_o} \int_o dA S_{1A} + \frac{1}{A_o} \int_o dA S_{1B}. \tag{3.38}$$

Hence, using (3.25) and (3.31):

$$\begin{aligned} \overline{S_1}(t_p) &\equiv \overline{S_{1A}}(t_p) + \overline{S_{1B}}(t_p) = \\ &= \frac{\beta\rho_i}{A_o} \int_o dA \int_i dA' \psi^\delta(\alpha) I(\omega', t_p) + \\ &+ \frac{\beta\rho_i}{A_o} \sum_{l=0}^{N} \int_o dA \int_i dA' \delta T_l(\omega') \tilde{\psi}^h(\alpha, t_p - t_l). \end{aligned} \tag{3.39}$$

Similarly

$$\begin{aligned} \overline{S_2}(t_p) &\equiv \overline{S_{2A}}(t_p) + \overline{S_{2B}}(t_p) = \\ &= \frac{\beta\rho_o}{A_o} \int_o dA \int_o dA' \psi^\delta(\alpha) S(\omega', t_p) + \\ &+ \frac{\beta\rho_o}{A_o} \sum_{l=0}^{N} \int_o dA \int_o dA' \delta S_l(\omega') \tilde{\psi}^h(\alpha, t_p - t_l). \end{aligned} \tag{3.40}$$

Summary

To summarize this section, we report from above the complete expression of the SLE in its spatially–discretized form. We use (3.14), (3.33), (3.34), (3.37), (3.39), and (3.40), to obtain

$$S(\omega, t_p) = S_{1A} + S_{2A} + S_{1B} + S_{2B} + C(t_p), \tag{3.41}$$

with

$$S_{1A}(\omega, t_p) = \beta\rho_i \int_i dA' \psi^\delta(\alpha) I(\omega', t_p), \tag{3.42}$$

$$S_{2A}(\omega, t_p) = \beta\rho_o \int_o dA' \psi^\delta(\alpha) S(\omega', t_p), \tag{3.43}$$

$$S_{1B}(\omega, t_p) = \beta\rho_i \sum_{l=0}^{N} \int_i dA' \delta T_l(\omega') \tilde{\psi}^h(\alpha, t_p - t_l), \tag{3.44}$$

$$S_{2B}(\omega, t_p) = \beta\rho_i \sum_{l=0}^{N} \int_o dA' \delta S_l(\omega') \tilde{\psi}^h(\alpha, t_p - t_l), \tag{3.45}$$

and

$$C(t_p) = -\overline{S_{1A}} - \overline{S_{2A}} - \overline{S_{1B}} - \overline{S_{2B}} + S_E, \tag{3.46}$$

where

$$\overline{S_{1A}}(t_p) = \frac{\beta\rho_i}{A_o} \int_o dA \int_i dA' \psi^\delta(\alpha) I(\omega', t_p), \tag{3.47}$$

$$\overline{S_{2A}}(t_p) = \frac{\beta\rho_o}{A_o} \int_o dA \int_o dA' \psi^\delta(\alpha) S(\omega', t_p), \tag{3.48}$$

$$\overline{S_{1B}}(t_p) = \frac{\beta\rho_i}{A_o} \sum_{l=0}^{N} \int_o dA \int_i dA' \delta T_l(\omega') \tilde{\psi}^h(\alpha, t_p - t_l), \tag{3.49}$$

$$\overline{S_{2B}}(t_p) = \frac{\beta\rho_o}{A_o} \sum_{l=0}^{N} \int_o dA \int_o dA' \delta S_l(\omega') \tilde{\psi}^h(\alpha, t_p - t_l), \tag{3.50}$$

$$S_E(t_p) = -\frac{\rho_i}{\rho_o A_o} \int_i dA I(\omega, t_p). \tag{3.51}$$

3.2.2 Space

The spatial discretization of the SLE simply consists in replacing the integrals over the ice and the ocean domains with discrete sums, as illustrated by the general rule (3.12). The various terms that appear in (3.41) and (3.46) are separately considered in the two following paragraphs.

The terms S_{1A} and S_{2A}

According to (3.42), S_{1A} can be expressed by

$$S_{1A}(\omega_i, t_p) = \beta\rho_i \int_i dA' \psi^\delta(\omega_i, \omega') I(\omega', t_p), \tag{3.52}$$

where ω_i denotes[2] the coordinates of a given point on the Earth surface, and we have made explicit the dependence of ψ^δ from ω and ω' (see 1.60). In the above equation, the product $\psi^\delta(\omega_i, \omega')I(\omega', t_p)$ can be identified with the generic function $G(\omega', t)$ of equation (3.12), which represents the basic property of the discretization scheme that we are adopting here. It is now sufficient to introduce the two–dimensional arrays

$$I_{jp} = I(\omega_j, t_p), \tag{3.53}$$

and

$$\psi^\delta_{ij} = \psi^\delta(\omega_i, \omega_j), \tag{3.54}$$

in order to write

$$S_{1A}(\omega_i, t_p) \;\; = \;\; \xi\beta\rho_i \sum_{j\in\text{ice}} \psi^\delta_{ij} I_{jp}, \tag{3.55}$$

where the sum is over the active ice pixels, and ξ is the area of each pixel (see 3.13).

The term S_{2A} is similar to S_{1A} (compare 3.43 with 3.42), but the ice variables are substituted by the ocean variables. Hence

$$S_{2A}(\omega_i, t_p) = \beta\rho_o \int_o dA' \psi^\delta(\omega_i, \omega') S(\omega', t_p). \tag{3.56}$$

Introducing the array

$$S_{jp} = S(\omega_j, t_p), \tag{3.57}$$

we simply have

$$S_{2A}(\omega_i, t_p) \;\; = \;\; \xi\beta\rho_o \sum_{j\in\text{oce}} \psi^\delta_{ij} S_{jp}. \tag{3.58}$$

[2]Here and in the following the index "i" should not be confused with the domain of integration.

The terms S_{1B} and S_{2B}

From (3.44) we obtain

$$S_{1B}(\omega_i, t_p) = \beta\rho_i \sum_{l=0}^{N} \int_i dA' \delta T_l(\omega') \tilde{\psi}^h(\omega_i, \omega', t_p - t_l). \tag{3.59}$$

Defining the arrays

$$\tilde{\psi}^h_{iplj} = \tilde{\psi}^h(\omega_i, \omega_j, t_p - t_l) \tag{3.60}$$

and

$$\delta I_{jl} = \delta T_l(\omega_j) \equiv I_{jl} - I_{j,l-1}, \tag{3.61}$$

we get the discretized form

$$S_{1B}(\omega_i, t_p) = \xi\beta\rho_i \sum_{l=0}^{N} \sum_{j\in\text{ice}} \tilde{\psi}_{iplj} \delta I_{jl}. \tag{3.62}$$

The term S_{2B} constitutes the oceanic counterpart of S_{1B} (see 3.45 and 3.44). Since

$$S_{2B}(\omega_i, t_p) = \beta\rho_o \sum_{l=0}^{N} \int_o dA' \delta S_l(\omega') \tilde{\psi}^h(\omega_i, \omega', t_p - t_l), \tag{3.63}$$

with the definition

$$\delta S_{jl} = \delta S_l(\omega_j) \equiv S_{jl} - S_{j,l-1}, \tag{3.64}$$

we obtain

$$S_{2B}(\omega_i, t_p) = \xi\beta\rho_i \sum_{l=0}^{N} \sum_{j\in\text{oce}} \tilde{\psi}_{iplj} \delta S_{jl}. \tag{3.65}$$

The terms $\overline{S_{1A}}$ and $\overline{S_{2A}}$

From (3.47):

$$\overline{S_{1A}}(t_p) = \frac{\beta\rho_i}{A_o} \int_o dA \int_i dA' \psi^\delta(\omega, \omega') I(\omega', t_p). \tag{3.66}$$

According to our discretization scheme, this double integral can be cast in the form

$$\overline{S_{1A}}(t_p) = \xi^2 \frac{\beta\rho_i}{A_o} \sum_{j\in\text{ice}} \sum_{k\in\text{oce}} I_{jp} \psi^\delta_{kj}. \tag{3.67}$$

Furthermore, since from (3.48) we have

$$\overline{S_{2A}}(t_p) = \frac{\beta\rho_o}{A_o}\int_o dA \int_o dA' \psi^\delta(\omega,\omega')S(\omega',t_p), \tag{3.68}$$

we obtain

$$\overline{S_{2A}}(t_p) = \xi^2 \frac{\beta\rho_o}{A_o}\sum_{j\in\text{oce}}\sum_{k\in\text{oce}} S_{jp}\psi^\delta_{kj}. \tag{3.69}$$

The terms $\overline{S_{1B}}$ and $\overline{S_{2B}}$

From (3.39):

$$\overline{S_{1B}}(t_p) = \frac{\beta\rho_i}{A_o}\sum_{l=0}^{N}\int_o dA \int_i dA' \delta T_l(\omega')\tilde{\psi}^h(\omega,\omega',t_p-t_l), \tag{3.70}$$

that can be discretized as

$$\overline{S_{1B}}(t_p) = \xi^2 \frac{\beta\rho_i}{A_o}\sum_{l=0}^{N}\sum_{j\in\text{ice}}\sum_{k\in\text{oce}} \delta I_{jl}\tilde{\psi}^{hv}_{kplj}, \tag{3.71}$$

with δI_{jl} and ψ^{hv}_{iplj} given by (3.61) and (3.60), respectively. Since (3.50) is the oceanic counterpart of (3.49), we have

$$\overline{S_{2B}}(t_p) = \frac{\beta\rho_o}{A_o}\sum_{l=0}^{N}\int_o dA \int_o dA' \delta S_l(\omega')\tilde{\psi}^h(\omega,\omega',t_p-t_l), \tag{3.72}$$

that can be immediately discretized as

$$\overline{S_{2B}}(t_p) = \xi^2 \frac{\beta\rho_o}{A_o}\sum_{l=0}^{N}\sum_{j\in\text{oce}}\sum_{k\in\text{oce}} \delta S_{jl}\tilde{\psi}^{hv}_{kplj}, \tag{3.73}$$

with δS_{jl} given by (3.64).

The term S_E

According to (3.51), the eustatic term is

$$S_E(t_p) = -\frac{\rho_i}{\rho_o A_o}\int_i dA I(\omega,t_p). \tag{3.74}$$

Hence

$$S_E(t_p) = -\xi \frac{\rho_i}{\rho_o A_o}\sum_{j\in\text{ice}} I_{jp}, \tag{3.75}$$

with I_{jp} given by (3.53).

Summary

It is now possible to write the SLE in its final discretized form. With

$$S_{ip} = S(\omega_i, t_p), \tag{3.76}$$

from the results of this section we have

$$\begin{aligned} S_{ip} &= c_1 \sum_{j\in\text{ice}} \psi^{\delta}_{ij} I_{jp} + c_2 \sum_{j\in\text{oce}} \psi^{\delta}_{ij} S_{jp} \\ &+ c_1 \sum_{l=0}^{N} \sum_{j\in\text{ice}} \tilde{\psi}^{h}_{iplj} \delta I_{jl} + c_2 \sum_{l=0}^{N} \sum_{j\in\text{oce}} \tilde{\psi}^{h}_{iplj} \delta S_{jl} \\ &- d_1 \sum_{j\in\text{ice}} \sum_{k\in\text{oce}} I_{jp} \psi^{\delta}_{kj} - d_2 \sum_{j\in\text{oce}} \sum_{k\in\text{oce}} S_{jp} \psi^{\delta}_{kj} \\ &- d_1 \sum_{l=0}^{N} \sum_{j\in\text{ice}} \sum_{k\in\text{oce}} \delta I_{jl} \tilde{\psi}^{h}_{kplj} - d_2 \sum_{l=0}^{N} \sum_{j\in\text{oce}} \sum_{k\in\text{oce}} \delta S_{jl} \tilde{\psi}^{h}_{kplj} \\ &- q \sum_{j\in\text{ice}} I_{jp}, \end{aligned} \tag{3.77}$$

with

$$\begin{aligned} c_1 &= \xi\beta\rho_i = \frac{3}{N_p}\frac{\rho_i}{\bar{\rho}_e} \\ c_2 &= \xi\beta\rho_o = \frac{3}{N_p}\frac{\rho_o}{\bar{\rho}_e} \\ d_1 &= \xi^2\beta\frac{\rho_i}{A_o} = \frac{3}{N_p^2}\frac{A_e}{A_o}\frac{\rho_i}{\bar{\rho}_e} \\ d_2 &= \xi^2\beta\frac{\rho_o}{A_o} = \frac{3}{N_p^2}\frac{A_e}{A_o}\frac{\rho_o}{\bar{\rho}_e} \\ q &= \xi\frac{\rho_i}{\rho_o A_o} = \frac{1}{N_p}\frac{A_e}{A_o}\frac{\rho_i}{\rho_o}, \end{aligned} \tag{3.78}$$

where we have used (1.71), (3.13), and (3.92), and $A_e = 4\pi a^2$ is the area of the surface of the Earth.

The elastic and viscous matrix elements that appear in (3.76) can be easily made explicit. From (3.19) and (3.54) we have

$$\psi^{\delta}_{ij} = \sum_{n=0}^{\infty}(1 + k^e_n - h^e_n)P_{nij}, \tag{3.79}$$

while from (3.20) and (3.60) after simple algebra we obtain

$$\tilde{\psi}^h_{iplj} = \begin{cases} 0 & p < l \\ -\sum_{n=0}^{\infty} \left[\sum_{m=1}^{M} \frac{k_{nm} - h_{nm}}{s_{nm}} \left(1 - e^{s_{nm}(p-l)\Delta} \right) \right] P_{nij}, & p \geq l, \end{cases} \tag{3.80}$$

where

$$P_{nij} = P_n(\cos \alpha_{ij}) \tag{3.81}$$

with

$$\cos \alpha_{ij} = \cos \theta_i \cos \theta_j + \sin \theta_i \sin \theta_j \cos(\lambda_i - \lambda_j). \tag{3.82}$$

3.3 Iterative solution of the SLE

Here we briefly describe the iterative scheme that is used to solve the discretized form of the SLE, and we present the discretized forms of the by–products of the SLE considered in §2.5. The formulas presented here are directly implemented in the various programs that compose the package SELEN (see Part II).

From the results of the last section (see in particular 3.77), we can formally write the discretized version of the SLE as

$$S_{ip} = H_{ip} + Z_{ip}(S_{kl}), \tag{3.83}$$

where in the array H_{ip} we have collected the terms that only involve the ice thickness variations, whereas Z_{ip} depends on the sealevel S_{kl}.

As mentioned in §2.3, the SLE is solved by an iterative technique that provides successive approximations to the solution of the equation. Using the form (3.83), we can express the iterative scheme as

$$\begin{cases} S^{(0)}_{ip} = S^E_p, & n = 0 \\ S^{(n)}_{ip} = H_{ip} + Z_{ip}(S^{(n-1)}_{kl}), & n \geq 1, \end{cases} \tag{3.84}$$

where n denotes the step number, and $S^E_p = S_E(t_p)$ represents the eustatic term of the SLE (see 3.75). A moderate number of steps is normally necessary to obtain convergence, as the reader can verify using the code SELEN.

Once the array

$$S_{ip} = \lim_{\text{n large enough}} S_{ip}^{(n)} \tag{3.85}$$

has been obtained from (3.84), it can be used to express in discretized form all of the by–products of the SLE that we have dealt with in §2.5. These discretized forms are directly implemented in the code SELEN.

From the definition (2.65), the relative sealevel at time k kyrs BP can be expressed as

$$RSL_{jk} = S_{j,N-k} - S_{j,N}, \tag{3.86}$$

where j is the label of the pixel containing the RSL site of coordinates ω_j and N is the number of time–steps of the time–discretization (see 3.1).

The rate of sealevel change at present time, defined by (2.66), takes the discretized form

$$\dot{S}_j = S_{j,N} - S_{j,N-1}, \tag{3.87}$$

where j labels the pixels containing a tide–gauge station, and the rate has been computed over a time interval $\delta t = 1$ kyr.

The discretized solution for the vertical displacement U can be retrieved in a few steps. From (2.5.2) we recall that

$$U(\omega, t) = \rho_i G_u \otimes_i I + \rho_o G_u \otimes_o S, \tag{3.88}$$

where, in analogy with what we have done with the sealevel GF (see 3.18), it is convenient to write the vertical displacement GF as

$$G_u = \beta[u^\delta(\alpha)\delta(t) + u^h(\alpha, t)H(t)], \tag{3.89}$$

with

$$u^\delta(\alpha) = \sum_{n=0}^{\infty} h_n^e P_n(\cos\alpha), \tag{3.90}$$

$$u^h(\alpha, t) = \sum_{n=0}^{\infty} \left(\sum_{m=1}^{M} h_{nm} e^{s_{nm} t} \right) P_n(\cos\alpha), \tag{3.91}$$

and

$$\beta = \frac{a}{m_e}. \tag{3.92}$$

It now suffices to observe that the two terms on the right–hand side of (3.88) have the same form of the terms S_1 and S_2 of the SLE (see 3.14), provided that the ratio G_s/γ_o is substituted with G_u. Hence, from (3.77), we have immediately

$$U_{ip} = c_1 \sum_{j\in\text{ice}} u^{\delta}_{ij} I_{jp} + c_2 \sum_{j\in\text{oce}} u^{\delta}_{ij} S_{jp} + c_1 \sum_{l=0}^{N} \sum_{j\in\text{ice}} \tilde{u}^{h}_{iplj} \delta I_{jl} + c_2 \sum_{l=0}^{N} \sum_{j\in\text{oce}} \tilde{u}^{h}_{iplj} \delta S_{jl}, \quad (3.93)$$

where δS_{jl} is expressed by (3.64) in terms of S_{jl} and

$$U_{ip} = U(\omega_i, t_p). \quad (3.94)$$

Similarly to (3.79) we also have

$$u^{\delta}_{ij} = \sum_{n=0}^{\infty} h^{e}_{n} P_{nij}, \quad (3.95)$$

and

$$\tilde{u}^{h}_{iplj} = \begin{cases} 0, & p < l \\ -\sum_{n=0}^{\infty} \left[\sum_{m=1}^{M} \frac{h_{nm}}{s_{nm}} \left(1 - e^{s_{nm}(p-l)\Delta} \right) \right] P_{nij}, & p \geq l, \end{cases} \quad (3.96)$$

with P_{nij} given by (3.81).

The discretized form of the geoid height change can be easily obtained using (2.4), (3.77) and (3.93) as

$$N_{ip} = S_{ip} + U_{ip}. \quad (3.97)$$

Following (3.87), the present–day rate of vertical uplift is

$$\dot{U}_i = U_{i,N} - U_{i,N-1} \quad (3.98)$$

while the rate of change of the geoid height computed at present time can be expressed as

$$\dot{N}_i = \dot{S}_i + \dot{U}_i. \quad (3.99)$$

Finally, from (2.76) and (3.12), it is straightforward to verify that the discretized form of time–derivative of the variations of the Stokes coefficients reads

$$\left\{ \begin{array}{c} \dot{\delta c} \\ \dot{\delta s} \end{array} \right\}_{lm} (t_p) = \frac{4\pi}{N_p} \frac{(2-\delta_{0m})\mu^2_{lm}}{a} \sum_{i=1}^{N_p} \left\{ \begin{array}{c} \cos m\lambda_i \\ \sin m\lambda_i \end{array} \right\} \dot{N}_i P_{lm}(\cos\theta_i), \quad (3.100)$$

where $(l \geq 2, m = 0, 1, ..., l)$, t_p is present time, N_p is the total number of pixels, and $\dot{N}_i$ is given by (3.99).

Chapter 4

Appendices

4.1 Spherical Harmonics

Here we provide the basic definitions and conventions concerning the spherical harmonics functions. We adopt the same conventions of Messiah [6].

We first define the associated Legendre function of degree l ($l = 0, 1, 2, \ldots$) and order m ($m = 0, 1, 2, \ldots l$) as

$$P_{lm}(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_l(x), \tag{4.1}$$

where $x = \cos\theta$, θ is colatitude, and the Legendre polynomials of degree l are defined by the Rodriguez formula

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l. \tag{4.2}$$

With the above definitions, the complex spherical harmonics are

$$Y_{lm}(\theta, \lambda) = \mu_{lm} P_{lm}(\cos\theta)\ \mathrm{e}^{im\lambda}, \tag{4.3}$$

where $i = \sqrt{-1}$. The normalization constant

$$\mu_{lm} = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \tag{4.4}$$

ensures that the following orthogonality relationship holds

$$\int_\Omega Y^*_{l'm'}(\theta, \lambda) Y_{lm}(\theta, \lambda) d\Omega = \delta_{ll'} \delta_{mm'}, \tag{4.5}$$

where the asterisk denotes complex conjugation, δ_{ij} is the Kronecker delta, and

$$\int_{\Omega} (\cdot) d\Omega \equiv \int_0^{2\pi} \int_0^{\pi} (\cdot) \sin\theta d\theta d\lambda, \tag{4.6}$$

where $(\cdot)$ is any scalar function. The spherical harmonics with negative order are obtained by

$$Y_{l-m}(\theta, \lambda) \equiv (-)^m Y^*_{lm}(\theta, \lambda). \tag{4.7}$$

Let (θ, λ) and (θ', λ') the polar spherical coordinates of two points on the surface of a sphere, and let Θ be the colatitude of the second relative to the first, such that

$$\cos\Theta = \frac{\vec{r}' \cdot \vec{r}}{rr'}, \tag{4.8}$$

with $r' = \|\vec{r}'\|$ and $r = \|\vec{r}\|$. The addition theorem states that

$$P_l(\cos\Theta) = \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} Y^*_{lm}(\theta', \lambda') Y_{lm}(\theta, \lambda). \tag{4.9}$$

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