Answers To a Selection of Problems from

# Classical Electrodynamics

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# Contents

# Introduction

This is a collection of my answers to problems from a graduate course in electrodynamics. These problems are mainly from the book by Jackson [4], but appended are some practice problems. My answers are by no means guaranteed to be perfect, but I hope they will provide the reader with a guideline to understand the problems.

Throughout these notes I will refer to equations and pages of Jackson and Duffin [2]. The latter is a textbook in electricity and magnetism that I used as an undergraduate student. References to equations starting with a "D" are from the book by Duffin. Accordingly, equations starting with the letter "J" refer to Jackson.

In general, primed variables denote vectors or components of vectors related to the distance between source and origin. Unprimed coordinates refer to the location of the point of interest.

The text will be a work in progress. As time progresses, I will add more chapters.

# Chapter 1

# Introduction to Electrostatics

## 1.1 Electric Fields for a Hollow Conductor

### a. The Location of Free Charges in the Conductor

Gauss' law states that

$$\frac{\rho}{\epsilon_0} = \nabla \cdot \mathbf{E}, \tag{1.1}$$

where $\rho$ is the volume charge density and $\epsilon_0$ is the permittivity of free space. We know that conductors allow charges free to move within. So, when placed in an external static electric field charges move to the surface of the conductor, canceling the external field inside the conductor. Therefore, a conductor carrying only static charge can have no electric field within its material, which means the volume charge density is zero and excess charges lie on the surface of a conductor.

### b. The Electric Field inside a Hollow Conductor

When the free charge lies outside the cavity circumferenced by conducting material (see figure 1.1b), Gauss' law simplifies to Laplace's equation in the cavity. The conducting material forms a volume of equipotential, because the electric field in the conductor is zero and

$$\mathbf{E} = -\nabla\Phi \tag{1.2}$$

Since the potential is a continuous function across a charged boundary, the potential on the inner surface of the conductor has to be constant. This is now a problem satisfying Laplace's equation with Dirichlet boundary conditions. In section 1.9 of Jackson, it is shown that the solution for this problem is unique. The constant value of the potential on the outer surface of the cavity satisfies Laplace's equation and is therefore *the* solution. In other words, the hollow conductor acts like a electric field shield for the cavity.

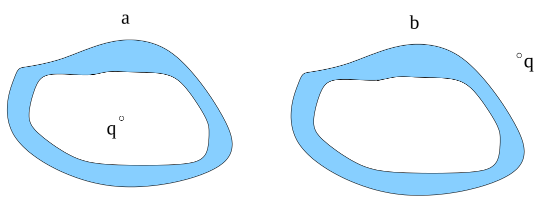


Figure 1.1: *a: point charge in the cavity of a hollow conductor. b: point charge outside the cavity of a hollow conductor.*

With a point charge $q$ *inside* the cavity (see figure 1.1a), we use the following representation of Gauss' law:

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\epsilon_0} \tag{1.3}$$

Therefore, the electric field inside the hollow conductor is non-zero. Note: the electric field outside the conductor due to a point source inside is influenced by the shape of the conductor, as you can see in part c.

## c. The Direction of the Electric Field outside a Conductor

An electrostatic field is conservative. Therefore, the circulation of $\mathbf{E}$ around any closed path is zero

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \tag{1.4}$$

This is called the circuital law for $\mathbf{E}$ (E4.14 or J1.21). I have drawn a closed path in four legs

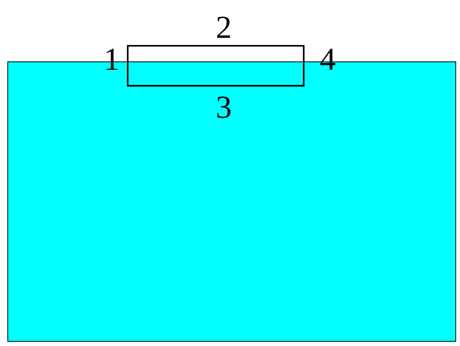


Figure 1.2: *Electric field near the surface of a charged spherical conductor. A closed path crossing the surface of the conductor is divided in four sections.*

through the surface of a rectangular conductor (figure 1.2). Sections 1 and 4 can be chosen negligible small. Also, we have seen earlier that the field in the conductor (section 3) is zero. For the total integral around the closed path to be zero, the tangential component (section 2) has to be zero. Therefore, the electric field is described by

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\,\hat{r} \tag{1.5}$$

where $\sigma$ is the surface charge density, since – as shown earlier – free charge in a conductor is located on the surface.

## 1.4 Charged Spheres

Here we have a conducting, a homogeneously charged and an in-homogeneously charged sphere. Their total charge is $Q$. Finding the electric field for each case in- and outside the sphere is an exercise in using Gauss' law

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\epsilon_0} \tag{1.6}$$

For all cases:

- Problem 1.1c showed that the electric field is directed radially outward from the center of the spheres.
- For $r > a$, $\mathbf{E}$ behaves as if caused by a point charge of magnitude of the total charge $Q$ of the sphere, at the origin.

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}}$$

As we have seen earlier, for a solid spherical conductor the electric field inside is zero (see figure 1.3). For a sphere with a homogeneous charge distribution the electric field at points inside the sphere increases with $r$. As the surface $S$ increases, the amount of charge surrounded increases (see equation 1.6):

$$\mathbf{E} = \frac{Qr}{4\pi\epsilon_0 a^3}\,\hat{\mathbf{r}} \tag{1.7}$$

For points inside a sphere with an inhomogeneous charge distribution, we use Gauss' law (once again)

$$\mathbf{E}4\pi r^2 = 1/\epsilon_0 \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho(\mathbf{r}') r'^2 \sin\theta' dr' d\phi' d\theta' \tag{1.8}$$

Implementing the volume charge distribution

$$\rho(r') = \rho_0 r'^n,$$

the integration over $r'$ for $n > -3$ is straightforward:

$$\mathbf{E} = \frac{\rho_0 r^{n+1}}{\epsilon_0(n+3)}\,\hat{\mathbf{r}}, \tag{1.9}$$

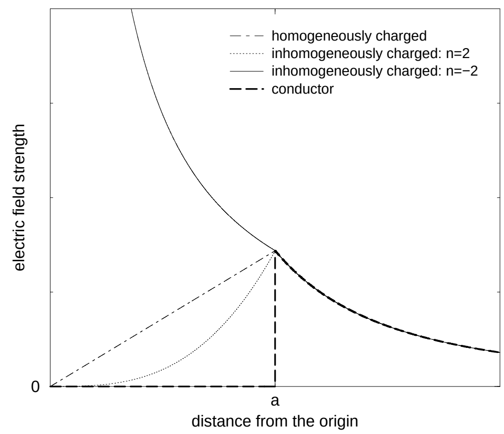


Figure 1.3: *Electric field for differently charged spheres of radius $a$. The electric field outside the spheres is the same for all, since the total charge is $Q$ in all cases.*

where

$$
\begin{aligned}
Q &= \int_0^a 4\pi\rho_0 r^{n+2} dr = \frac{4\pi\rho_0 a^{n+3}}{n+3} \quad \Leftrightarrow \\
\rho_0 &= \frac{Q(n+3)}{4\pi a^{n+3}}
\end{aligned} \tag{1.10}
$$

It can easily be verified that for $n = 0$, we have the case of the homogeneously charged sphere (equation 1.7). The electric field as a function of distance are plotted in figure 1.3 for the conductor, the homogeneously charged sphere and in-homogeneously charged spheres with $n = -2, 2$.

## 1.5 Charge Density for a Hydrogen Atom

The potential of a neutral hydrogen atom is

$$\Phi(r) = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right) \tag{1.11}$$

where $\alpha$ equals 2 divided by the Bohr radius. If we calculate the Laplacian, we obtain the volume charge density $\rho$, through Poisson's equation

$$\frac{\rho}{\epsilon_0} = \nabla^2\Phi$$

Using the Laplacian for spherical coordinates (see back-cover of Jackson), the result for $r > 0$ is

$$\rho(r) = -\frac{\alpha^3 q}{8\pi r^2} e^{-\alpha r} \tag{1.12}$$

For the case of $r \to 0$

$$\lim_{r\to 0} \Phi(r) = \lim_{r\to 0} \frac{q}{4\pi\epsilon_0 r} \tag{1.13}$$

From section 1.7 in Jackson we have (J1.31):

$$\nabla^2(1/r) = -4\pi\delta(r) \tag{1.14}$$

Combining (1.13), (1.14) and Poisson's equation, we get for $r \to 0$

$$\rho(r) = q\delta(r) \tag{1.15}$$

We can multiply the right side of equation (1.15) by $e^{-\alpha r}$ without consequences. This allows for a more elegant way of writing the discrete and the continuous parts together

$$\rho(r) = \left(\delta(r) - \frac{\alpha^3}{8\pi r^2}\right) q e^{-\alpha r} \tag{1.16}$$

The discrete part represents the stationary proton with charge $q$. Around the proton orbits an electron with charge $-q$. The continuous part of the charge density function is more a statistical distribution of the location of the electron.

## 1.7 Charged Cylindrical Conductors

Two very long cylindrical conductors, separated by a distance $d$, form a capacitor. Cylinder 1 has surface charge density $\lambda$ and radius $a_1$, and number 2 has surface charge density $-\lambda$ and radius $a_2$ (see figure 1.4). The electric field for each of the cylinders is radially directed outward

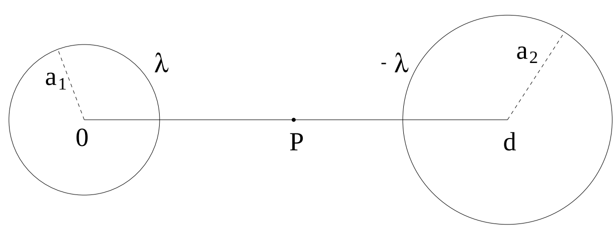


Figure 1.4: *Top view of two cylindrical conductors. The point $P$ is located in the plane connecting the axes of the cylinders.*

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0|\mathbf{r}|}\,\hat{\mathbf{r}}, \tag{1.17}$$

where $\hat{\mathbf{r}}$ is the radially directed outward unit vector. Taking a point $P$ on the plane connecting the axes of the cylinders, the electric field is constructed by superposition:

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{r} + \frac{1}{d-r}\right) \tag{1.18}$$

The potential difference between the two cylinders is

$$\begin{aligned} |V_{a_2} - V_{a_1}| &= \frac{\lambda}{2\pi\epsilon_0}\int_{d-a_2}^{a_1}\left(\frac{1}{r} + \frac{1}{d-r}\right)dr \\ &= \frac{\lambda}{2\pi\epsilon_0}\left[\ln r + \ln(d-r)\right]_{d-a_2}^{a_1} \\ &= \frac{\lambda}{2\pi\epsilon_0}\left[\ln a_1 + \ln(d-a_1) + \ln(d-a_2) - \ln a_2\right] \end{aligned} \tag{1.19}$$

If we average the radii of the cylinders to $a_1 = a_2 = a$ and assume $d \gg a$, then the potential difference is

$$|V_{a_2} - V_{a_1}| \approx \frac{\lambda}{\pi\epsilon_0}\left(\ln\frac{d}{a}\right) \tag{1.20}$$

The capacitance per unit length of the system of cylinders is given by

$$C = \frac{\lambda}{|V_{a_1} - V_{a_2}|} \approx \frac{\pi\epsilon_0}{\ln(d/a)} \tag{1.21}$$

From here, we can obtain the diameter $\delta$ of wire necessary to have a certain capacitance $C$ at a distance $d$:

$$\delta = 2a \approx 2d \cdot e^{-\frac{\pi\epsilon_0}{C}}, \tag{1.22}$$

where the permittivity in free space $\epsilon_0$ is $8.854 \cdot 10^{-12} F/m$. If $C = 1.2 \cdot 10^{-11} F/m$ and

- $d = 0.5$ cm, the diameter of the wire is 0.1 cm.
- $d = 1.5$ cm, the diameter of the wire is 0.3 cm.
- $d = 5$ cm, the diameter of the wire is 1 cm.

## 1.13 Green's Reciprocity Theorem

Two infinite grounded parallel conducting plates are separated by a distance $d$. What is the induced charge on the plates if there is a point charge $q$ in between the plates?

Split the problem up in two cases with the same geometry. The first is the situation as sketched by Jackson; two infinitely large grounded conducting plates, one at $x = 0$ and one at $x = d$ (see the right side of figure 1.5). In between the plates there is a point charge $q$ at $x = x_0$. We will apply Green's reciprocity theorem using a "mirror" set-up. In this geometry there is no point charge but the plates have a fixed potential $\psi_1$ and $\psi_2$, respectively (see the left side of figure 1.5).

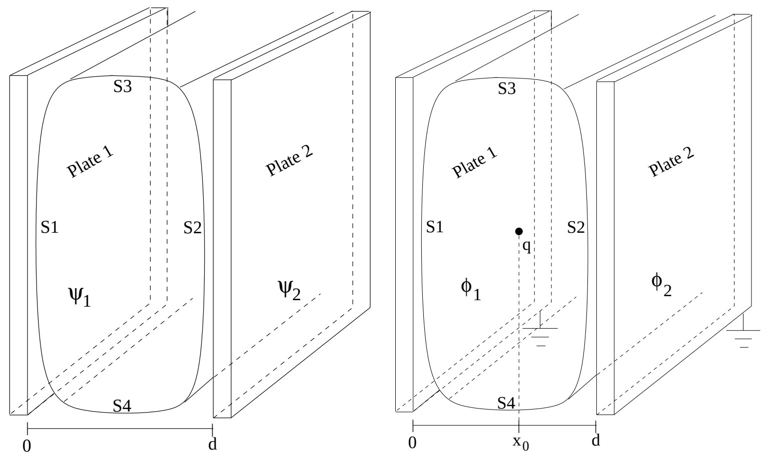


Figure 1.5: *Geometry of two conducting plates and a point-charge. $S$ is the surface bounding the volume between the plates. The right picture is the situation of the imposed problem with the point charge between two grounded plates. The left side is a problem with the same plate geometry, but we know the potential $\phi$ on the plates.*

Green's theorem (J1.35) states

$$\int_{\mathbf{V}} \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) d\mathbf{V} = \oint_{\mathbf{S}} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) d\mathbf{S} \tag{1.23}$$

The volume **V** is the space between the plates bounded by the surface **S**. **S1** and **S2** bound the plates and **S3** and **S4** run from plate 1 to plate 2 at + and - $\infty$, respectively. The normal derivative $\frac{\partial}{\partial n}$ at the surface **S** is directed outward from inside the volume **V**.

When the plates are grounded, the potential in the plates is zero. The potential is continuous across the boundary, so on $S1$ and $S2$ $\phi = 0$. Note that if the potential was not continuous the electric field ($\mathbf{E} = -\nabla\phi$) would go to infinity. At infinite distance from the point source, the potential is also zero:

$$\oint_{\mathbf{S}} \phi\frac{\partial\psi}{\partial n} d\mathbf{S} = 0 \tag{1.24}$$

The remaining part of the surface integral can be modified according to Jackson, page 36:

$$\frac{\partial\phi}{\partial n} = \nabla\phi\cdot\mathbf{n} = -\mathbf{E}\cdot\mathbf{n} \tag{1.25}$$

The electric field across a boundary with surface charge density $\sigma$ is (Jackson, equation 1.22):

$$\mathbf{n}\cdot(\mathbf{E}_{conductor} - \mathbf{E}_{void}) = \sigma/\epsilon_0 \tag{1.26}$$

However, inside the conductor $\mathbf{E} = 0$, therefore

$$\mathbf{n}\cdot\mathbf{E}_{void} = -\sigma/\epsilon_0, \tag{1.27}$$

for each of the plates. The total surface integral in equation 1.23 is then

$$\oint_{\mathbf{S}} \psi \frac{\partial \phi}{\partial n} d\mathbf{S} = \int_{\mathbf{S1}} \psi_1 \frac{\sigma_1}{\epsilon_0} d\mathbf{S} + \int_{\mathbf{S2}} \psi_2 \frac{\sigma_2}{\epsilon_0} d\mathbf{S} \tag{1.28}$$

In case of the plates of fixed potential $\psi$, the legs **S3** and **S4** have opposite potential and thus cancel. Using

$$\int_{\mathbf{S}} \sigma d\mathbf{S} = Q, \tag{1.29}$$

the surface integral of Greens theorem is

$$\oint_{\mathbf{S}} \psi \frac{\partial \phi}{\partial n} d\mathbf{S} = 1/\epsilon_0 (\psi_1 Q_{\mathbf{S1}} + \psi_2 Q_{\mathbf{S2}}) \tag{1.30}$$

In the volume integral in equation (1.23), the mirror case of the charged boundaries includes no free charges:

$$\nabla^2 \psi = -(\text{total charge})/\epsilon_0 = 0 \tag{1.31}$$

Applying Gauss' law to the case with the point charge gives us

$$\nabla^2 \phi = -(\text{total charge})/\epsilon_0 = -q/\epsilon_0 \delta(x - x_0), \tag{1.32}$$

where $x_0$ is the x-coordinate of the point source location. The volume integral will be

$$\int_{\mathbf{V}} -q/\epsilon_0 \delta(x - x_0) \psi(x) d\mathbf{V} = -q/\epsilon_0 \psi(x_0) \tag{1.33}$$

For two plates with fixed potentials, the potential in between is a linear function

$$\psi(x_0) = \psi_1 + (\frac{\psi_2 - \psi_1}{d})(1 - x_0)d = x_0\psi_1 + \psi_2(1 - x_0) \tag{1.34}$$

Green's theorem is now reduced to equation (1.30) and equation (1.34) in (1.33):

$$-q(x_0\psi_1 + (1 - x_0)\psi_2) = Q_{S1}\psi_1 + Q_{S2}\psi_2 \tag{1.35}$$

Since this equality must hold for all potentials, the charges on the plates must be

$$Q_{S1} = -qx_0 \text{ and } Q_{S2} = -q(1 - x_0) \tag{1.36}$$

# Chapter 2

# Boundary-Value Problems in Electrostatics: 1

## 2.2 The Method of Image Charges

### a. The Potential Inside the Sphere

This problem is similar to the example shown on pages 58, 59 and 60 of Jackson. The electric field due to a point charge $q$ inside a grounded conducting spherical shell can also be created by the point charge and an image charge $q'$ only. For reasons of symmetry it is evident that $q'$ is located on the line connecting the origin and $q$. The goal is to find the location and the magnitude of the image charge. The electric field can then be described by superposition of point charges:

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0|\mathbf{x}-\mathbf{y}|} + \frac{q'}{4\pi\epsilon_0|\mathbf{x}-\mathbf{y}'|} \tag{2.1}$$

In figure 2.1 you can see that $\mathbf{x}$ is the vector connecting origin and observation point. $\mathbf{y}$ connects the origin and the unit charge $q$. Finally, $\mathbf{y}'$ is the connection between the origin and the image charge $q'$. Next, we write the vectors in terms of a scalar times their unit vector and factor the scalars $y'$ and $x$ out of the denominators:

$$\Phi(\mathbf{x}) = \frac{q/4\pi\epsilon_0}{x|\mathbf{n} - \frac{y}{x}\mathbf{n}'|} + \frac{q'/4\pi\epsilon_0}{y'|\mathbf{n}' - \frac{x}{y'}\mathbf{n}|} \tag{2.2}$$

The potential for $x = a$ is zero, for all possible combinations of $\mathbf{n}\cdot\mathbf{n}'$. The magnitude of the image charge is

$$q' = -\frac{a}{y}q \tag{2.3}$$

at distance

$$y' = \frac{a^2}{y} \tag{2.4}$$

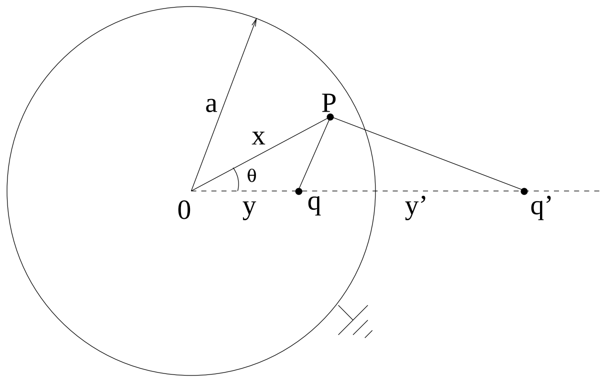


Figure 2.1: *A point charge $q$ in a grounded spherical conductor. $q'$ is the image charge.*

This is the same result as for the image charge inside the sphere and the point charge outside (like in the Jackson example). After implementing the amount of charge (2.3) and the location of the image (2.4) in (2.1), potential in polar coordinates is

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{y^2+r^2-2yr\cos\theta}} - \frac{\left(\frac{a}{y}\right)}{\sqrt{\left(\frac{a^2}{y}\right)^2 + r^2 - 2\left(\frac{a^2}{y}\right) r\cos\theta}}\right), \tag{2.5}$$

where $\theta$ is the angle between the line connecting the origin and the charges and the line connecting the origin and the point $P$ (see figure 2.1). $r$ is the length of the vector connecting the origin and observation point $P$.

## b. The Induced Surface Charge Density

The surface charge density on the sphere is

$$\sigma = \epsilon_0 \left(\frac{\partial \Phi}{\partial r}\right)_{r=a} \tag{2.6}$$

Differentiating equation (2.5) is left to the reader, but the result is

$$\sigma = -\frac{q}{4\pi a^2}\left(\frac{a}{y}\right)\frac{1-\left(\frac{a}{y}\right)^2}{\left(1+\left(\frac{a}{y}\right)^2 - 2\frac{a}{y}\cos\theta\right)^{3/2}} \tag{2.7}$$

### c. The Force on the Point Charge q

The force on the point charge $q$ by the field of the induced charges on the conductor is equal to the force on $q$ due to the field of the image charge:

$$\mathbf{F} = q\mathbf{E}' \tag{2.8}$$

The electric field at $y$ due to the image charge at $y'$ is directed towards the origin and of magnitude

$$|\mathbf{E}'| = \frac{q'}{4\pi\epsilon_0 (y'-y)^2} \tag{2.9}$$

We already computed the values for $y'$ and $q'$ in equation (2.4) and (2.3), respectively. The force is also directed towards the origin of magnitude

$$|\mathbf{F}| = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}\left(\frac{a}{y}\right)\left(1-\left(\frac{a}{y}\right)^2\right)^{-2} \tag{2.10}$$

### d. What If the Conductor Is Charged?

Keeping the sphere at a fixed non-zero potential requires net charge on the conducting shell. This can be imaged as an extra image charge at the center of the spherical shell. If we now compute the force on the conductor by means of the images, the result *will* differ from section c.

## 2.7 An Exercise in Green's Theorem

### a. The Green Function

The Green function for a half-space ($z > 0$) with Dirichlet boundary conditions can be found by the method of images. The potential field of a point source of unit magnitude at $z'$ from an infinitely large grounded plate in the x-y plane can be replaced by an image geometry with the unit charge $q$ and an additional (image) charge at $z = -z'$ of magnitude $q' = -q$. The situation is sketched in figure 2.2. The potential due to the two charges is the Green function $G_D$:

$$\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}} \tag{2.11}$$

### b. The Potential

The Green function as defined in equation (2.11) can serve as the "mirror set-up" required in Green's theorem:

$$\int_{\mathbf{V}} \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) d\mathbf{V} = \oint_{\mathbf{S}} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) d\mathbf{S} \tag{2.12}$$

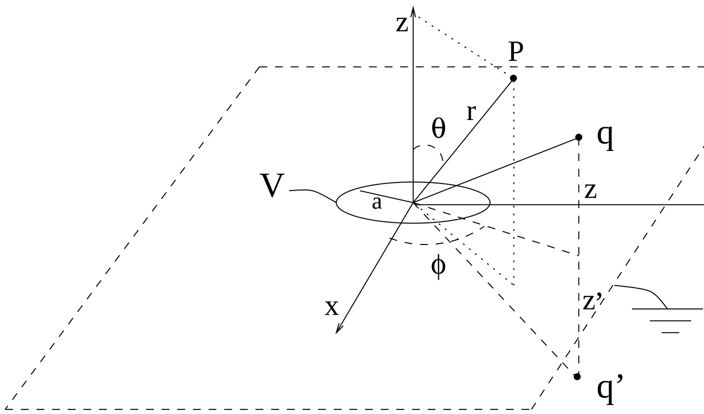


Figure 2.2: A very large grounded surface in which a circular shape is cut out and replaced by a conducting material of potential $V$.

with $G_D = \psi$ and $\Phi = \phi$. There are no charges the volume $z > 0$, so Laplace equation holds throughout the half-space $\mathbf{V}$:

$$\nabla^2 \Phi = 0 \tag{2.13}$$

Chapter one in Jackson (J1.39) showed

$$\nabla^2 G_D = -4\pi\delta(\rho - \rho') \tag{2.14}$$

leaving $\Phi(\rho')$ after performing the volume integration. The Green function on the surface $S$ ($G_D$) is constructed with the assumption that part of the surface (the base, if you will) is grounded, and the other parts stretch to infinity. Therefore

$$\oint_{\mathbf{S}} G_D d\mathbf{S} = 0 \tag{2.15}$$

The potential $\Phi = V$ in the circular area with radius $a$, but everywhere else $\Phi = 0$. Also:

$$\oint_{\mathbf{S}} \nabla G_D \cdot \hat{\mathbf{n}} d\mathbf{S} = -\oint_{\mathbf{S}} \frac{\partial G_D}{\partial z} d\mathbf{S}, \tag{2.16}$$

since the normal $\hat{\mathbf{n}}$ is in the negative z-direction $-\hat{\mathbf{k}}$. Thus we are left with the following remaining terms in Green's theorem (in cylindrical coordinates):

$$\Phi(\mathbf{r}') = \epsilon_0 V \int_0^a \int_0^{2\pi} \frac{\partial G_D}{\partial z} \rho d\rho d\phi \tag{2.17}$$

From here on we will exchange the primed and unprimed coordinates. This is OK, since the reciprocity theorem applies. Some algebra left to the reader leads to

$$\Phi(\mathbf{r}) = \frac{Vz}{2\pi} \int_0^a \int_0^{2\pi} \frac{\rho' d\rho' d\phi'}{(\rho^2 + \rho'^2 + z^2 - 2\rho\rho' \cos(\phi - \phi'))^{3/2}} \tag{2.18}$$

### c. The Potential on the z-axis

For $\rho = 0$, general solution (2.18) simplifies to

$$\begin{aligned}\Phi(\rho,\phi) &= \frac{Vz}{2\pi}\int_0^a\int_0^{2\pi}\frac{\rho' d\rho' d\phi'}{(\rho'^2+z^2)^{3/2}} \\ &= -Vz\left[\frac{1}{\sqrt{\rho'^2+z^2}}\right]_0^a \\ &= V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right) \qquad (2.19)\end{aligned}$$

### d. An Approximation

Slightly rewriting equation (2.18):

$$\Phi(\mathbf{r}) = \frac{Vz}{2\pi\left(\rho^2+z^2\right)^{3/2}}\int_0^a\int_0^{2\pi}\frac{\rho' d\rho' d\phi'}{\left(1+\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\right)^{3/2}} \qquad (2.20)$$

The denominator in the integral can be approximated by a binomial expansion. The first three terms of the approximation give

$$\Phi(\mathbf{r}) \approx \frac{Va^2z}{2\left(\rho^2+z^2\right)^{3/2}}\left(1-\frac{3a^2}{4\left(\rho^2+z^2\right)}+\frac{5a^4}{8\left(\rho^2+z^2\right)^2}+\frac{15a^2\rho^2}{8\left(\rho^2+z^2\right)^2}\right) \qquad (2.21)$$

Along the axis ($\rho = 0$) the expression simplifies to

$$\Phi(\phi,z) \approx \frac{Va^2}{2z^2}\left(1-\frac{3a^2}{4z^2}+\frac{5a^4}{8z^4}\right) \qquad (2.22)$$

This is the same result when we expand expression (2.19):

$$\begin{aligned}\Phi(\rho,\phi) &= V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right) \\ &= V\left(1-\left(1+\frac{a^2}{z^2}\right)^{-1/2}\right) \\ &\approx \frac{Va^2}{2z^2}\left(1-\frac{3a^2}{4z^2}+\frac{5a^4}{8z^4}\right) \qquad (2.23)\end{aligned}$$

## 2.9 Two Halves of a Conducting Spherical Shell

A conducting spherical shell consists of two halves. The cut plane is perpendicular to the homogeneous field (see figure 2.3). The goal is to investigate the force between the two halves introduced

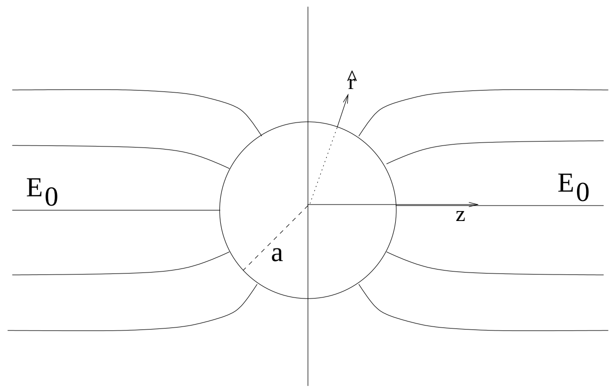


Figure 2.3: A spherical conducting shell in a homogeneous electric field directed in the z-direction.

by the induced charges.

The electric field due to the induced charges on the shell is (see [2], p. 51):

$$\mathbf{E}_{ind} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{r}} \tag{2.24}$$

You can see this as the resulting field in a capacitor with one of the plates at infinity. The electric field inside the conducting shell is zero. Therefore the external field has to be of the same magnitude (see figure 2.4).

The force of the external field on an elementary surface $dS$ of the conductor is:

$$d\mathbf{F} = \mathbf{E}_{ext}dq = \frac{\sigma^2 d\mathbf{S}}{2\epsilon_0}, \tag{2.25}$$

directed radially outward from the sphere's center. From the symmetry we can see that all forces cancel, except the component in the direction of the external field.[1]

## a. An Uncharged Shell

The derivation of the induced charge density on a conducting spherical shell in a homogeneous electrical field $E_0$ is given ([4], p. 64). The homogeneous field is portrayed by point charges of opposite magnitude at + and − infinity. Next, the location and magnitude of the image charges are computed. The result is

$$\sigma(\theta) = 3\epsilon_0 E_0 \cos\theta \tag{2.26}$$

[1] The external field is a superposition of the homogeneous electric field **plus** the electric field due to the induced charges excluding $dq$! This external field is perpendicular to the surface of the conductor with magnitude $\sigma/(2\epsilon_0)$. This is not addressed in Jackson.

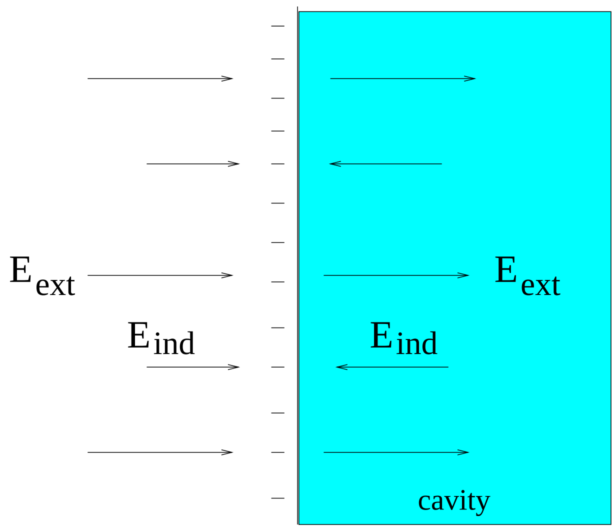


Figure 2.4: Zooming in on that small part of the conductor with induced charges, where the external field is at normal incidence.

When we plug this result into equation (2.25), we get for the horizontal component of the force ($d\mathbf{F}_z$) on an elementary surface:

$$d\mathbf{F}_z = d\mathbf{F}\cos\theta = \frac{9}{2}\epsilon_0 E_0^2 \cos^3\theta d\mathbf{S}. \tag{2.27}$$

Now we can integrate to get the total force on the sphere halves. From symmetry we can also see that the force on the left half is opposite of that on the right half (see figure 2.3). So we integrate over the right half and multiply by two to get the total net force:

$$\begin{aligned}
\mathbf{F_z} &= 2\int_0^{2\pi}\int_0^{\pi/2}\frac{9}{2}\epsilon_0 E_0^2\cos^3\theta a^2\sin\theta d\theta d\phi\ \hat{\mathbf{z}} \\
&= 9\pi a^2\epsilon_0 E_0^2\int_0^{\pi/2}\cos^3\theta\sin\theta d\theta\ \hat{\mathbf{z}} \\
&= 9\pi a^2\epsilon_0 E_0^2\left[-1/4\cos^4\theta\right]_0^{\pi/2}\ \hat{\mathbf{z}} \\
&= \frac{9}{4}\pi a^2\epsilon_0 E_0^2\hat{\mathbf{z}}
\end{aligned} \tag{2.28}$$

## b. A Shell with Total Charge $Q$

When the shell has a total charge $Q$ it changes the charge density of equation (2.26) to

$$\sigma(\theta) = 3\epsilon_0 E_0\cos\theta + \frac{Q}{4\pi a^2} \tag{2.29}$$

When we plug this expression into equation (2.25) and compute again the net (horizontal) component of the force, we find that

$$\mathbf{F_z} = \left(\frac{9}{4}\pi a^2 \epsilon_0 E_0^2 + \frac{Q^2}{32\pi\epsilon_0 a^2} + \frac{E_0 Q}{2}\right)\hat{\mathbf{z}} \tag{2.30}$$

The total force is bigger then for the uncharged case. This makes sense when we look at equation (2.25); when the shell is charged there is more charge per unit volume to, hence the force is bigger.

## 2.10 A Conducting Plate with a Boss

### a. $\sigma$ On the Boss

By inspection it can be seen that the system of images as proposed in figure (2.5) fits the geometry and the boundary conditions of our problem. We can write the potential as a function of these four point charges. This is done in Jackson (p. 63). It has to be noted that $R$ has to be chosen at infinity to apply to the homogeneous character of the field. The potential can then be described by expanding "the radicals after factoring out the $R^2$."

$$\begin{aligned}\Phi(r,\theta) &= \frac{Q}{4\pi\epsilon_0}\left(-\frac{2}{R^2}r\cos\theta + \frac{2a^3}{R^2 r^2}\cos\theta\right) + \ \ldots \\ &= -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta \end{aligned} \tag{2.31}$$

The surface charge density on the boss ($r = a$) is

$$\sigma = -\ \epsilon_0 \frac{\partial \Phi}{\partial r}\bigg|_{r=a} = 3\epsilon_0 E_0 \cos\theta \tag{2.32}$$

### b. The Total Charge on the Boss

The total charge $Q$ on the boss is merely an integration over half a sphere with radius $a$:

$$\begin{aligned} Q &= 3\epsilon_0 E_0 \int_0^{2\pi}\int_0^{\pi/2} \cos\theta a^2 \sin\theta d\theta d\phi \\ &= 3\epsilon_0 E_0 2\pi a^2 \left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2} \\ &= 3\epsilon_0 E_0 \pi a^2 \end{aligned} \tag{2.33}$$

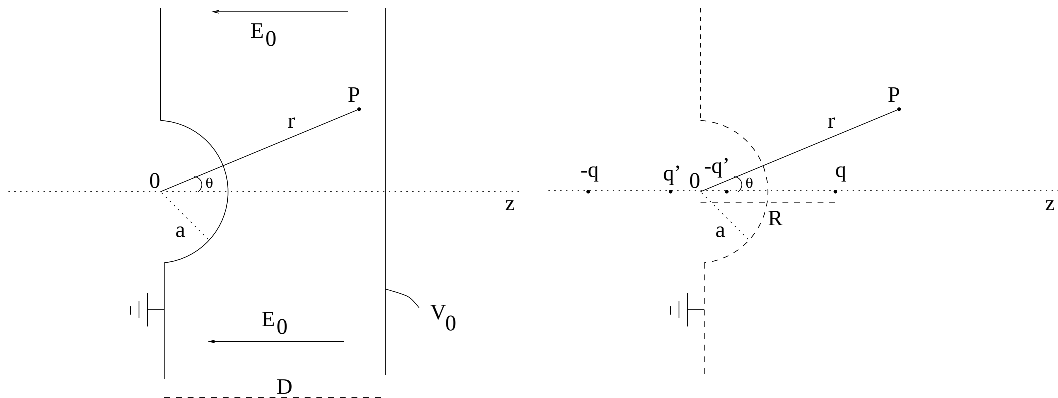


Figure 2.5: *On the left is the geometry of the problem: two conducting plates separated by a distance $D$. One of the plates has a hemispheric boss of radius $a$. The electric field between the two plates is $E_0$. On the right is the set of charges that image the field due to the conducting plates. In part a and b, $R \to \infty$ to image a homogeneous field. In c, $R = d$.*

## c. The Charge On the Boss Due To a Point Charge

Now we do not have a homogeneous field to image, but the result of a point charge on a grounded conducting plate with the boss. Again we use the method of images to replace the system with the plate by one entirely consisting of point charges. Checking the boundary conditions leads to the same set of four charges as drawn in figure 2.5. The only difference is that $R$ is not chosen at infinity to mimic the homogeneous field, but $R = d$. The potential is the superposition of the point charge $q$ at distance $d$ and its three image charges:

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(r^2+d^2+2rd\cos\theta)^{1/2}} - \frac{1}{(r^2+d^2-2rd\cos\theta)^{1/2}}\right.$$
$$\left. - \frac{a}{d\left(r^2+\frac{a^4}{d^2}+\frac{2a^2r}{d}\cos\theta\right)^{1/2}} + \frac{a}{d\left(r^2+\frac{a^4}{d^2}-\frac{2a^2r}{d}\cos\theta\right)^{1/2}}\right) \tag{2.34}$$

The charge density on the boss is

$$\sigma = -\epsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{r=a} \tag{2.35}$$

The total amount of charge is the surface charge density integrated over the surface of the boss:

$$Q = 2\pi a^2 \int_0^{\pi/2} \sigma \sin\theta d\theta \tag{2.36}$$

The differentiation of equation (2.34) to obtain the the surface charge density and the following integration in equation (2.36) are left to the reader. The resulting total charge is[2]

$$Q = -q\left[1 - \frac{d^2 - a^2}{d\sqrt{d^2 + a^2}}\right] \tag{2.37}$$

## 2.11 Line Charges and the Method of Images

### a. Magnitude and Position of the Image Charge(s)

Analog to the situation of point charges in previous image problems, one image charge of opposite magnitude at distance $\frac{b^2}{R}$ (see figure 2.6) satisfies the conditions the boundary conditions

$$\begin{aligned} \lim_{r\to\infty} \Phi(r,\phi) &= 0 \quad \text{and} \\ \Phi(b,\phi) &= V_0 \end{aligned}$$

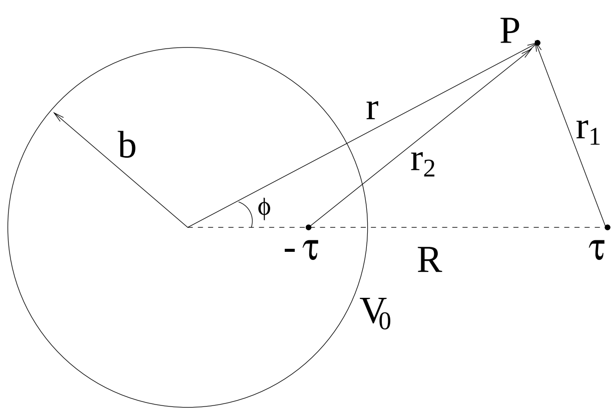


Figure 2.6: *A cross sectional view of a long cylinder at potential $V_0$ and a line charge $\tau$ at distance $R$, parallel to the axis of the cylinder. The image line charge $-\tau$ is placed at $b^2/R$ from the axis of the cylinder to realize a constant potential $V_0$ at radius $r = b$.*

[2] I have chosen to keep $Q$ as the symbol for the total charge. Jackson calls it $q'$. I find this confusing since the primed $q$ has been used for the image of $q$.

## b. The Potential

The potential in polar coordinates is simply a superposition of the line charge $\tau$ and the image line charge $\tau'$ with the conditions as proposed in section a. The result is

$$\Phi(r,\phi) = \frac{\tau}{4\pi\epsilon_0}\ln\left(\frac{(R^2r^2 + b^4 - 2rRb^2\cos\phi)}{R^2(r^2 + R^2 - 2Rr\cos\phi)}\right) \tag{2.38}$$

For the far field case ($r >> R$) we can factor out $(Rr)^2$. The $b^4$ and $R^2$ in equation (2.38) can be neglected:

$$\Phi(r,\phi) \approx \frac{\tau}{4\pi\epsilon_0}\ln\left(\frac{(Rr)^2(1 - \frac{2b^2}{Rr}\cos\phi)}{(Rr)^2(1 - \frac{2R}{r}\cos\phi)}\right) \tag{2.39}$$

The first order Taylor expansion is

$$\begin{aligned}
\Phi(r,\phi) &\approx \frac{\tau}{4\pi\epsilon_0}\ln\left((1 - \frac{2b^2}{Rr}\cos\phi)(1 + \frac{2R}{r}\cos\phi)\right) \\
&\approx \frac{\tau}{4\pi\epsilon_0}\ln\left((1 + \left(\frac{2R}{r}\cos\phi - \frac{2b^2}{Rr}\cos\phi\right)\right) \\
&\approx \frac{\tau}{2\pi\epsilon_0}\frac{(R^2 - b^2)}{Rr}\cos\phi \qquad \text{(using } \ln(1+x) \approx x\text{)}
\end{aligned} \tag{2.40}$$

## c. The Induced Surface Charge Density

$$\sigma(\phi) = -\epsilon_0\frac{\partial\Phi}{\partial r}\bigg|_{r=b} \tag{2.41}$$

Differentiation of equation (2.38) and substituting $r = b$:

$$\begin{aligned}
\sigma(\phi) &= -\frac{\tau}{4\pi}\left(\frac{2bR^2 - 2Rb^2\cos\phi}{R^2b^2 + R^2 + b^4 - 2b^3R\cos\phi} - \frac{R^2(2b - 2R\cos\phi}{R^2(b^2 + R^2 - 2Rb\cos\phi}\right) \\
&= -\frac{\tau}{2\pi}\left(\frac{(R/b)^2 - 1}{(R/b)^2 + 1 - 2(R/b)\cos\phi}\right)
\end{aligned} \tag{2.42}$$

When $R/b = 2$, the induced charge as a function of $\phi$ is

$$\sigma(\phi)|_{R=2b} = -\frac{\tau}{2\pi b}\left(\frac{3}{5 - 4\cos\phi}\right) \tag{2.43}$$

When the position of the line charge $\tau$ is four radii from the center of the cylinder, the surface charge density is

$$\sigma(\phi)|_{R=4b} = -\frac{\tau}{2\pi b}\left(\frac{15}{17 - 8\cos\phi}\right) \tag{2.44}$$

The graphs for either case are drawn in figure 2.7.

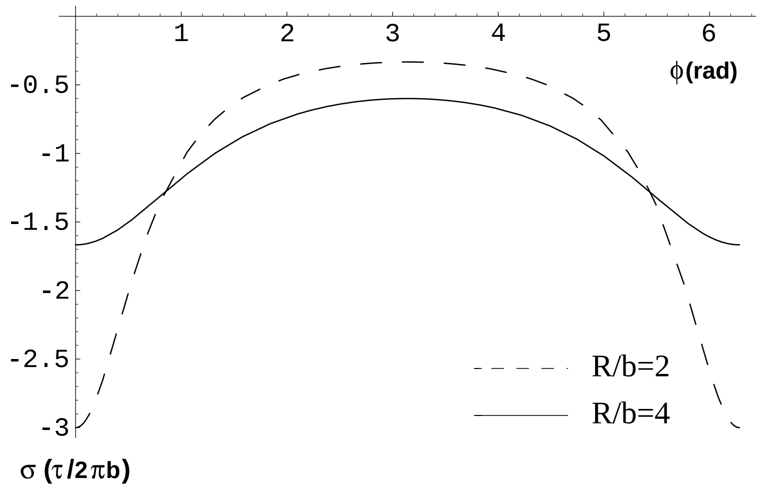


Figure 2.7: The behavior of the surface charge density $\sigma$ with angle for two different ratios between the radius of the cylinder and the distance to the line charge.

### d. The Force on the Line Charge

The force per meter on the line charge is Coulomb's law:

$$\mathbf{F} = \tau \mathbf{E}(R, 0) = -\frac{\tau^2}{2\pi\epsilon_0} \frac{1}{(R^2 - b^2)} \hat{\imath} \quad \text{per meter} \tag{2.45}$$

where $\hat{\imath}$ is the directed from the the axis of the cylinder to the line charge, perpendicular to the line charge and the cylinder axis.

## 2.13 Two Cylinder Halves at Constant Potentials

### a. The Potential inside the Cylinder

In this case (see figure 2.8) there are no free charges in the area of interest. Therefore the potential $\Phi$ inside the cylinder obeys Laplace's equation:

$$\nabla^2 \Phi = 0 \tag{2.46}$$

We can write the potential in cylindrical coordinates and separate the variables:

$$\Phi(\rho, \phi) = R(r) F(\phi) \tag{2.47}$$

The general solution is (see J2.71):

$$\Phi(\rho,\phi) \quad = \quad a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty}\left[a_n\rho^n \sin(n\phi+\alpha_n) + b_n\rho^{-n}\cos(n\phi+\alpha_n)\right] \tag{2.48}$$

From this geometry it is obvious that at the center $\rho = 0$ the solution may not blow up, so:

$$b_n = b_0 = 0 \tag{2.49}$$

This results in a potential

$$\Phi(\rho,\phi) \quad = \quad a_0 + \sum_{n=1}^{\infty} a_n\rho^n \sin(n\phi+\alpha_n) \tag{2.50}$$

The next step is to implement the boundary conditions

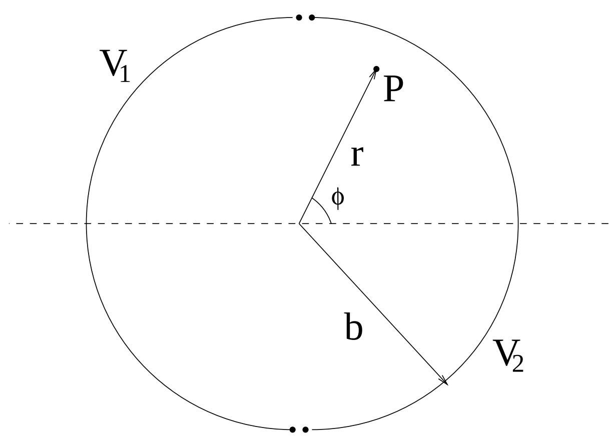


Figure 2.8: Cross-section of two cylinder halves with radius $b$ at constant potentials $V_1$ and $V_2$.

$$\Phi(b,\phi) = V_2 = a_0 + \sum_{n=1}^{\infty} a_n b^n \sin(n\phi+\alpha_n) \quad \text{for} \;\; (-\pi/2 < \phi < \pi/2)$$
$$\Phi(b,\phi) = V_1 = a_0 + \sum_{n=1}^{\infty} a_n b^n \sin(n\phi+\alpha_n) \quad \text{for} \;\; (\pi/2 < \phi < 3\pi/2) \tag{2.51}$$

## b. The Surface Charge Density

$$\sigma = -\,\epsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{r=b} \tag{2.52}$$

## 2.23 A Hollow Cubical Conductor

### a. The Potential inside the Cube

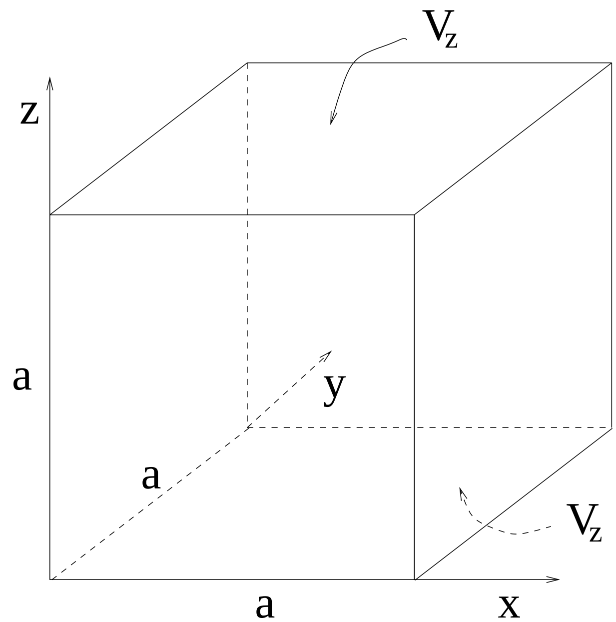


Figure 2.9: A hollow cube, with all sides but z=0 and z=a grounded.

$$\nabla^2\Phi = 0 \tag{2.53}$$

Separating the variables:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0. \tag{2.54}$$

$x$ and $y$ can vary independently so each term must be equal to a constant $-\alpha^2$:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \alpha^2 = 0 \quad \Rightarrow \quad X = A\text{cos}\alpha x + B\text{sin}\alpha x \tag{2.55}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} + \beta^2 = 0 \quad \Rightarrow \quad Y = C\text{cos}\beta y + D\text{sin}\beta y \tag{2.56}$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} + \gamma^2 = 0 \quad \Rightarrow \quad Z = E\text{sinh}(\gamma z) + F\text{cosh}(\gamma z), \tag{2.57}$$

where $\gamma^2 = \alpha^2 + \beta^2$. The boundary conditions determine the constants:

$$\Phi(0, y, z) = 0 \quad \Rightarrow \quad A = 0$$

$$\begin{aligned}
\Phi(a,y,z)=0 &\quad\Rightarrow\quad \alpha_n = n\pi/a \quad (n=1,2,3,...)\\
\Phi(x,0,z)=0 &\quad\Rightarrow\quad C=0\\
\Phi(x,a,z)=0 &\quad\Rightarrow\quad \beta_m = m\pi/a \quad (m=1,2,3,...)\\
&\quad\Rightarrow\quad \gamma_{nm} = \pi\sqrt{n^2+m^2}
\end{aligned}$$

The solution is thus reduced to

$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[A_{nm}\sinh\left(\frac{\gamma_{nm}z}{a}\right) + B_{nm}\cosh\left(\frac{\gamma_{nm}z}{a}\right)\right] \tag{2.58}$$

Now, let's use the last boundary conditions to find the coefficients $A_nm$ and $B_nm$. The top and bottom of the cube are held at a constant potential $V_z$, so

$$\Phi(x,y,0) = V_z = \sum_{n,m=1}^{\infty} B_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right) \tag{2.59}$$

This means that $B_{nm}$ are merely the coefficients of a double Fourier series (see for instance [1] on Fourier series):

$$B_{nm} = \frac{4V_z}{a^2}\int_0^a\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dxdy \tag{2.60}$$

It can be easily shown that the individual integrals in equation (2.60) are zero for even integer values and $\frac{2a}{n\pi}$ for $n$ is odd. Thus $B_nm$ is

$$B_{nm} = \frac{16V_z}{\pi^2 nm} \qquad \text{for odd } (n,m) \tag{2.61}$$

The top of the cube is also at constant potential $V_z$, so

$$\begin{aligned}
\Phi(x,y,0) &= V_z = \Phi(x,y,a) \Leftrightarrow\\
B_{nm} &= A_{nm}\sinh(\gamma_{nm}) + B_{nm}\cosh(\gamma_{nm}) \Leftrightarrow\\
A_{nm} &= B_{nm}\frac{1-\cosh(\gamma_{nm})}{\sinh(\gamma_{nm})}
\end{aligned} \tag{2.62}$$

Substituting the expressions for $A_{nm}$ and $B_{nm}$ into equation (2.58), gives us

$$\begin{aligned}
\Phi(x,y,z) = \frac{16V_z}{\pi^2}\sum_{n,m\ odd}^{\infty}\frac{1}{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)&\left[\frac{1-\cosh(\gamma_{nm})}{\sinh(\gamma_{nm})}\sinh\left(\frac{\gamma_{nm}z}{a}\right)\right.\\
&\left.+\cosh\left(\frac{\gamma_{nm}z}{a}\right)\right],
\end{aligned} \tag{2.63}$$

where $\gamma_{nm} = \pi\sqrt{n^2+m^2}$.

## b. The Potential at the Center of the Cube

The potential at the center of the cube is

$$\Phi(\frac{a}{2},\frac{a}{2},\frac{a}{2}) = \frac{16V_z}{\pi^2}\sum_{n,m\ odd}^{\infty}\frac{1}{nm}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{m\pi}{2}\right)\left[\frac{1-\cosh(\gamma_{nm})}{\sinh(\gamma_{nm})}\sinh\left(\frac{\gamma_{nm}}{2}\right) + \cosh\left(\frac{\gamma_{nm}}{2}\right)\right] \tag{2.64}$$

With just $n, m = 1$, the potential at the center is

$$\frac{16V_z}{\pi^2}\left[\frac{1-\cosh(\sqrt{2}\pi)}{\sinh(\sqrt{2}\pi)}\sinh\left(\frac{\pi}{\sqrt{2}}\right)+\cosh\left(\frac{\pi}{\sqrt{2}}\right)\right] \approx 0.347546V_z \tag{2.65}$$

When we add the two terms $(n = 3,\ m = 1)$ and $(n = 1,\ m = 3)$, the potential is $0.332498V_z$.

## c. The Surface Charge Density

The surface charge density on the top surface of the cube is given by

$$\sigma = -\epsilon_0\frac{\partial\Phi}{\partial z}\bigg|_{z=a} \tag{2.66}$$

In the appendix it is shown that the differentiation of the hyperbolic sine is the hyperbolic cosine. Furthermore

$$\frac{d\cosh(az)}{dz} = a\sinh(az) \tag{2.67}$$

Using this equality in differentiating the expression for the potential in equation (2.63), we get

$$\frac{\partial\Phi}{\partial z} = \frac{16V_z}{\pi^2}\sum_{n,m\ odd}^{\infty}\frac{\gamma_{nm}}{nma}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[\frac{1-\cosh(\gamma_{nm})}{\sinh(\gamma_{nm})}\cosh\left(\frac{\gamma_{nm}z}{a}\right) + \sinh\left(\frac{\gamma_{nm}z}{a}\right)\right], \tag{2.68}$$

where $\gamma_{nm} = \pi\sqrt{n^2+m^2}$. Now we evaluate this expression for $z = a$:

$$\sigma = -\epsilon_0\frac{\partial\Phi}{\partial z}\bigg|_{z=a} =$$

$$-\frac{16\epsilon_0 V_z}{\pi^2}\sum_{n,m\ odd}^{\infty}\frac{\gamma_{nm}}{nma}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[\frac{1-\cosh(\gamma_{nm})}{\sinh(\gamma_{nm})}\cosh(\gamma_{nm})+\sinh(\gamma_{nm})\right] =$$

$$-\frac{16\epsilon_0 V_z}{\pi^2}\sum_{n,m\ odd}^{\infty}\frac{\gamma_{nm}}{nma}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[(1-\cosh(\gamma_{nm}))\coth(\gamma_{nm})+\sinh(\gamma_{nm})\right] \tag{2.69}$$

Further simplification??

# Chapter 8

# Waveguides, Resonant Cavities and Optical Fibers

## 8.1 Time Averaged Forces Per Unit Area on a Conductor

### a. A Good Conductor

The relationship between force $\mathbf{F}$ on one side and current density $\mathbf{J}$ and magnetic field $\mathbf{H}$ on the other is

$$\mathbf{F} = \mu_c \int_{\mathbf{V}} \mathbf{J} \times \mathbf{H} \, d\mathbf{V}. \tag{8.1}$$

The force per unit area (or pressure) would then be

$$\frac{d\mathbf{F}}{d\mathbf{A}} = -\mu_c \int_{\xi} \mathbf{J} \times \mathbf{H} \, d\xi, \tag{8.2}$$

where $\xi$ is the normal pointing into the conductor. This is opposite of the "normal" $\hat{n}$, resulting in the change of sign. The time average of this quantity $\mathbf{f}$ is

$$\mathbf{f} = \langle \frac{d\mathbf{F}}{d\mathbf{A}} \rangle = -\frac{1}{2}\mu_c \int_{\xi} \Re \left[ \mathbf{J} \times \mathbf{H}^* \right] \, d\xi. \tag{8.3}$$

See section 6.9 in Jackson for the time-averaging. When we insert the following properties (see J8.9):

$$\mathbf{H} = \mathbf{H}_{\parallel} e^{\xi/\delta(1-\imath)} \quad \text{and} \quad \mathbf{J} = \sigma \mathbf{E}, \tag{8.4}$$

and the approximation for a good conductor that

$$\mathbf{E} \approx \sqrt{\frac{\mu_c \omega}{2\sigma}} (1-\imath) \left( \hat{n} \times \mathbf{H}_{\parallel} \right) e^{\xi/\delta(\imath-1)}, \tag{8.5}$$

into equation 8.3, we are left with a simple integration of the exponential

$$\int_0^\infty e^{-2\xi/\delta}\, d\xi = \frac{-\delta}{2} \tag{8.6}$$

and a series of cross products (for which we have the equality as defined in the appendix and in the inside cover of Jackson)

$$\left(\hat{n} \times \mathbf{H}_\parallel\right) \times \mathbf{H}_\parallel^* = -\mathbf{H}_\parallel^* \times \left(\hat{n} \times \mathbf{H}_\parallel\right) = -\left(\mathbf{H}_\parallel^* \cdot \mathbf{H}_\parallel\right)\hat{n} + \left(\mathbf{H}_\parallel \cdot \hat{n}\right)\mathbf{H}_\parallel. \tag{8.7}$$

The first dot-product is the square of the magnetic field magnitude and the last dot-product is obviously zero, since the magnetic field is perpendicular to the normal. Putting all this together, gives us the time average force per unit area

$$\mathbf{f} = -\hat{n}\frac{\mu_c}{4}\left|\mathbf{H}_\parallel\right|^2. \tag{8.8}$$

## b. A Perfect Conductor

$$d\mathbf{F} = I d\mathbf{l} \times \mathbf{B}. \tag{8.9}$$

$$d\mathbf{K} = \frac{I}{d\mathbf{l}} \tag{8.10}$$

Therefore

$$\mathbf{f} = \langle \frac{d\mathbf{F}}{d\mathbf{A}} \rangle = \frac{1}{2}\mathbf{K} \times \mathbf{B}^*. \tag{8.11}$$

Using

$$\mathbf{B}^* = \mu\mathbf{H}^* \quad \text{and} \quad \mathbf{K} = \hat{n} \times \mathbf{H}, \tag{8.12}$$

we get in a perfect conductor for the time averaged force per unit area

$$\mathbf{f} = -\frac{\mu}{2}\hat{n}|\mathbf{H}_\parallel|^2. \tag{8.13}$$

This is twice as large as the force found in a good conductor in part *a*. This has to do with the discontinuity between the fields inside and outside the conductor. For an explaination, see [3].

## c. A Superposition of Different Frequencies

As in section 6.8 in Jackson, time averaging leads to

$$\langle |\mathbf{H}|^2 \rangle = \langle \Re[\mathbf{H}] \cdot \Re[\mathbf{H}] \rangle = 1/2\mathbf{H} \cdot \mathbf{H}^*, \tag{8.14}$$

where $\mathbf{H} = \sum_k \mathbf{H}_\parallel e^{\imath\omega_k t}$. Therefore, the dot-product of the leads to an exponential dependence, but due to averaging the individual frequencies over time, they all cancel:

$$\langle |\mathbf{H}|^2 \rangle = \frac{1}{2}\sum_k \sum_l \mathbf{H}_\parallel \cdot \mathbf{H}_\parallel^*\, e^{\imath(\omega_k - \omega_l)t} = \frac{1}{2}\mathbf{H}_\parallel \cdot \mathbf{H}_\parallel^* = \frac{1}{2}|\mathbf{H}_\parallel|^2. \tag{8.15}$$

The rest of the derivation stays the same, so for the result to stay the same we have to replace $|\mathbf{H}_\parallel|^2$ by $2\langle|\mathbf{H}|^2\rangle$ in equation 8.8.

## 8.2 TEM Waves in a Medium of Two Concentric Cylinders

### a. Time Averaged Power Flow Along the Guide

For TEM waves, $E_z$ and $H_z$ are by definition zero. This simplifies equations J8.23 and J8.25 to

$$\nabla_t \cdot \mathbf{E}_{TEM} = 0 \quad \text{and} \quad \nabla_t \times \mathbf{E}_{TEM} = 0 \tag{8.16}$$

This implies that our problem is reduced to an electrostatic potential $\Phi$ that obeys Laplace's equation in the cavity between the two cylindrical surfaces:

$$\mathbf{E}_{TEM} = -\nabla_t \Phi \quad \text{and} \quad \nabla_t^2 \Phi = 0. \tag{8.17}$$

When we solve this equation in cylindrical coordinates and use the symmetry of the problem, the solution is only a function of $\rho$:

$$\Phi(\rho) = A \ln \rho + B \tag{8.18}$$

where $A$ and $B$ are arbitrary constants which will be defined by the boundary conditions. The electric field $\mathbf{E}_{TEM}$ is

$$\mathbf{E}_{TEM}(\rho) = -\nabla_t \Phi = -\frac{A\hat{\rho}}{\rho}. \tag{8.19}$$

According to equation J8.28, the magnetic field is

$$\mathbf{H}_{TEM}(\rho) = \pm\sqrt{\frac{\epsilon}{\mu}}\hat{\mathbf{z}} \times \mathbf{E}_{TEM} = \mp\frac{\sqrt{\frac{\epsilon}{\mu}}A\hat{\rho} \times \hat{\mathbf{z}}}{\rho}. \tag{8.20}$$

The boundary condition for this problem is that at $\rho = a$, $H = H_0$. This means the constant is defined is

$$A = aH_0\sqrt{\frac{\mu}{\epsilon}}. \tag{8.21}$$

Next, we compute the Poynting vector

$$\mathbf{S} = \frac{1}{2}\left(\mathbf{E} \times \mathbf{H}^*\right) = \frac{a^2\sqrt{\frac{\mu}{\epsilon}}|H_0|^2\left(\hat{\rho} \times (\hat{\mathbf{z}} \times \hat{\rho})\right)}{2\rho^2}. \tag{8.22}$$

Using the vector identity from the appendix and realizing that $\hat{\rho} \cdot \hat{\mathbf{z}} = 0$, we are left with

$$\mathbf{S} = \frac{a^2\sqrt{\frac{\mu}{\epsilon}}|H_0|^2}{2\rho^2}\hat{\mathbf{z}}. \tag{8.23}$$

Equation J8.49 gives us the power flow

$$P = \int_A \mathbf{S} \cdot \hat{\mathbf{z}} da = \int_0^{2\pi}\int_a^b\int_0^{\pi} d\phi d\rho d\theta \frac{a^2\sqrt{\frac{\mu}{\epsilon}}|H_0|^2}{2\rho^2}\rho \sin\theta = \sqrt{\frac{\mu}{\epsilon}}\pi a^2 |H_0|^2 \ln\left(\frac{b}{a}\right). \tag{8.24}$$

## b. Attenuation of the Transmitted Power

Attenuation is due to power lost through the walls of the conductor. According to equations J8.56, J8.57 and J8.58, the power flow is

$$P(z) = P_0 e^{\frac{1}{P}\frac{dP}{dz}}, \tag{8.25}$$

The goal is therefore to find $P$, which is related to the Poynting vector as defined in equation 8.22. $S$ follows from equation 8.22. All that is left is to determine the values for $\mathbf{H}$ and $\mathbf{E}$. The magnetic field is continuous across the boundary and thus is the same as in part a:

$$\mathbf{H}_{\|} = \frac{aH_0\hat{\phi}}{\rho} \tag{8.26}$$

The electric field can then be approximated as before in equation 8.5, evaluated at the surface $\xi = 0$:

$$\mathbf{E}_{\|} \approx \sqrt{\frac{\mu_c\omega}{2\sigma}}(1-\imath)\left(\hat{\mathbf{n}} \times \mathbf{H}_{\|}\right). \tag{8.27}$$

The Poynting vector is

$$\mathbf{S} = \frac{1}{2}\left(\mathbf{E}_{\|} \times \mathbf{H}_{\|}^*\right) = -\frac{1}{2}\sqrt{\frac{\mu_c\omega}{2\sigma}}\left(\frac{a}{\rho}\right)^2 |H_0|^2\hat{\rho}. \tag{8.28}$$

This Poynting vector is directed outward of the guide walls. This is the direction of the power loss. This leaves us with the contributions of the two cylinders to the power loss $P$:

$$P = P_a + P_b = -\pi\sqrt{\frac{\mu_c\omega}{2\sigma}}a^2|H_0|^2\left(\int_{\rho=a}\frac{1}{\rho}dz + \int_{\rho=b}\frac{1}{\rho}dz\right) \tag{8.29}$$

Therefore

$$\frac{dP}{dz} = -\pi\sqrt{\frac{\mu_c\omega}{2\sigma}}a^2|H_0|^2\left(\frac{1}{b}+\frac{1}{a}\right) \tag{8.30}$$

Using $P$ from equation 8.24, we can write $|H_0|^2$ as function of $P$ and substitute into equation 8.30:

$$\frac{dP}{dz} = -2\pi\sqrt{\frac{\mu_c\omega}{2\sigma}}\sqrt{\frac{\epsilon}{\mu}}\frac{P}{\pi}\left(\frac{1/a+1/b}{\ln\left(\frac{b}{a}\right)}\right) = -2\gamma P. \tag{8.31}$$

So $P$ is indeed $P_0e^{-2\gamma z}$ where

$$\gamma = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{1/a+1/b}{\ln\left(\frac{b}{a}\right)}\right) \tag{8.32}$$

## c. The Characteristic Impedance

The voltage difference between the cylinders is

$$\Delta V = -\int_b^a \mathbf{E}\cdot d\mathbf{l} = \int_a^b \mathbf{E}\cdot d\rho\hat{\rho}. \tag{8.33}$$

Using the electric field that we computed in part $a$ (equation 8.19), we get

$$\Delta V = aH_0\sqrt{\frac{\mu}{\epsilon}}\ln\left(\frac{b}{a}\right). \tag{8.34}$$

By Ampere's Law, the current is related to the magnetic field as computed in equation 8.20.

$$I = \int \mathbf{H}\cdot d\mathbf{l} = \int \frac{aH_0}{\rho}\hat{\phi}\cdot\rho d\phi\hat{\phi} = 2\pi aH_0. \tag{8.35}$$

## d. Series Resistance and Inductance

To find the resistance per unit length, we use that

$$\frac{dP}{dz} = \frac{1}{2}I^2R_l, \tag{8.36}$$

where $R_l$ is the resistance per unit length. In part $c$, we found that

$$I = 2\pi aH_0 \tag{8.37}$$

and $\frac{dP}{dz}$ we obtained in equation8.30. Plugging these results into equation 8.36, gives

$$R_l = \frac{1}{2\pi\delta\sigma}\left(\frac{1}{a}+\frac{1}{b}\right), \tag{8.38}$$

where we used that $\sigma\delta = \sqrt{\frac{2\sigma}{\mu_c\omega}}$.

Finally, we'll compute the inductance $L$. The inductance is related to the power loss as follows:

$$\frac{dP}{dz} = \frac{1}{2}\left(\frac{\Phi}{L}\right)^2 R_l, \tag{8.39}$$

where $\Phi$ is the flux thorugh the walls of the conductor and medium, as

$$\Phi = \int \mathbf{B}_{cond}\cdot d\mathbf{A} + \int \mathbf{B}_{medium}\cdot d\mathbf{A} \tag{8.40}$$

Using the resistance $R_l$ as computed above, we get

$$L^2 = \frac{R_l}{2\frac{dP}{dz}}\left(\int \mathbf{B}_{cond}\cdot d\mathbf{A} + \int \mathbf{B}_{medium}\cdot d\mathbf{A}\right)^2 \tag{8.41}$$

## 8.3 TEM Waves Between Metal Strips

### a. Two Identical Thin Strips

#### The Power

In the previous problem we saw that for TEM waves, we can define a potential that satisfies Laplace's equation. Since $b >> a$, we consider only variations in electric field in the $x$-direction. The Laplace equation simplifies to

$$\frac{\partial \Phi^2}{\partial^2 x} = 0. \tag{8.42}$$

The solutions are a linearly varying field and a constant. When we compute the electric field by taking the gradient, we find that

$$\mathbf{E}_{TEM} = E_0 \hat{x}. \tag{8.43}$$

From equation J8.28, we know the relation between the magnetic and electric field for TEM waves to be

$$\mathbf{H}_{TEM} = \pm\sqrt{\frac{\epsilon}{\mu}}\hat{\mathbf{z}} \times \mathbf{E}_{TEM} = \pm\sqrt{\frac{\epsilon}{\mu}} E_0 \hat{y}. \tag{8.44}$$

If $H_0$ is the (peak) amplitude for the magnetic field, then $H_0 = \sqrt{\frac{\epsilon}{\mu}} E_0$. The Poynting vector is defined as

$$\mathbf{S} = \frac{1}{2}\left(\mathbf{E} \times \mathbf{H}^*\right) = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}|H_0|^2 \hat{z}. \tag{8.45}$$

$$P = \int_A \mathbf{S} \cdot \hat{\mathbf{z}} da = \int_0^a \int_0^b \mathbf{S} dx dy = \frac{ab}{2}\sqrt{\frac{\mu}{\epsilon}}|H_0|^2. \tag{8.46}$$

#### The Exponential Decay Factor

$$\gamma = -\frac{1}{2P}\frac{dP}{dz}, \tag{8.47}$$

where $P$ was calculated in the first part of this exercise and $\frac{dP}{dz}$ is

$$\frac{dP}{dz} = -\frac{1}{2\sigma\delta}\oint_C |\hat{n} \times \mathbf{H}|^2 dl. \tag{8.48}$$

The magnetic field is oriented in the z-direction and the normal $\hat{n}$ is in the x-direction. Now we integrate around the plates individually to get the total power. Only between the plates there is a magnetic field. Therefore the closed line integral reduces to

$$\frac{dP}{dz} = -\frac{1}{2\sigma\delta}\oint_C |\hat{n} \times \mathbf{H}|^2 dl = \int_0^b |\hat{x} \times H_0 \hat{z} dl = \frac{|H_0|^2 b}{2\sigma\delta}. \tag{8.49}$$

This is the ohmic power loss per plate. The exponential decay factor is thus

$$\gamma = -\frac{1}{2P}\frac{dP}{dz} = \sqrt{\frac{\epsilon}{\mu}}\frac{1}{a\sigma\delta}. \tag{8.50}$$

### The Characteristic Impedance

THe characteristic impedance was defined in problem 8.2 as the voltage frop between the plates divided by the current flowing along one of the plates. The voltage difference between the cylinders is

$$\Delta V = -\int_b^a \mathbf{E} \cdot d\mathbf{l} = \int_0^a E_0 dx = \int_0^a \sqrt{\frac{\mu}{\epsilon}} H_0 = a\sqrt{\frac{\mu}{\epsilon}} H_0. \tag{8.51}$$

The other leg is the current along one of the plates. To find this, we use

$$I = \oint_C H_0 \hat{y} \cdot dl = \int_0^b H_0 dy = H_0 b. \tag{8.52}$$

The characteristic impedance is therefore

$$Z_0 = \frac{dV}{I} = \sqrt{\frac{\mu}{\epsilon}} \frac{a}{b}. \tag{8.53}$$

### The Series Resistance

$$R_l = \frac{2}{I_0^2} \left| \frac{dP}{dz} \right| = \frac{2}{(H_0 b)^2} \frac{b|H_0|^2}{\sigma\delta} = \frac{2}{b\sigma\delta}. \tag{8.54}$$

### The Series Inductance

The inductance is

$$L = \frac{\Phi}{I}, \tag{8.55}$$

where $\Phi = \int \mathbf{B} \cdot \hat{n} da$ and we know $I = H_0 b$. In the conducting plate (and we are looking at only one plate, like we did to compute the current, too), the flux is

$$\Phi_c = \mu_c H_0 \int_0^\delta \int_0 \infty \hat{y} \cdot dx dz \hat{y}. \tag{8.56}$$

Therefore, the flux per unit length in the conductor is

$$\Phi_{l,c} = \mu_c H_0 \delta. \tag{8.57}$$

For the medium, we integrate from zero to $a$, and get

$$\Phi_{l,0} = \mu_0 H_0 a. \tag{8.58}$$

The total inductance per unit length is the addition of the two, divided by the current:

$$L_l = \frac{\mu_0 a + \mu_c \delta}{b}. \tag{8.59}$$

## 8.4 TE and TM Waves along a Brass Cylinder

### 8.4.1 a. Cutoff Frequencies

A brass cylinder with radius $R$ is oriented with its axis along the z-direction. TE or TM waves propagate along the cylinder. Both TE and TM waves have a z-component that satisfies

$$\left(\nabla_t + \gamma^2\right)\psi = 0, \tag{8.60}$$

where $\gamma^2 = \mu\epsilon\omega^2 - k^2$ and $\psi$ is either $E_z$ for TM waves or $B_z$ for TE waves. Solving this equation in cylindrical coordinates (and recognizing that there is no z-dependence), we get solutions

$$\psi = e^{\imath m\phi} J_m(\gamma\rho) e^{\imath(kz-wt)}. \tag{8.61}$$

We already discarded the other solution to the Bessel equation, because it has a singularity at the origin. Lets apply the rest of the boundary conditions for the appropriate wave. For TM waves we know that

$$\psi|_s = \psi|_{\rho=R} = 0. \tag{8.62}$$

Hence the Bessel function is zero when $\rho = R$:

$$\gamma_{mn} R = x_{mn}, \tag{8.63}$$

where $x_{mn}$ is the n-th root of the m-th order Bessel function. The cutoff frequency is the lowest frequency for which the wave does not become inhomogeneous or evanescent. In other words, the wave number should be real. The cutoff frequency is therefore

$$\omega_{mn} = \frac{\gamma_{mn}}{\sqrt{\mu\epsilon}} = \frac{x_{mn}}{R\sqrt{\mu\epsilon}} \tag{8.64}$$

All that's left to do is look up the four smallest roots. From page 114 in Jackson or any book with tables of Bessel functions it is obvious that $\omega_{11}$ is the dominant frequency. The next three are $\omega_{21}, \omega_{02}, \omega_{12}$, respectively.

For TE waves, the boundary condition is

$$\left.\frac{\partial\psi}{\partial n}\right|_{\rho=R} = 0, \tag{8.65}$$

where the normal is $\hat{n} = \hat{\rho}$. We'll call

$$\left.\frac{\partial\psi}{\partial\rho}\right|_{\rho=R} = \left.\frac{\partial J_m(\gamma\rho)}{\partial\rho}\right|_{\rho=R} = J_m'(\gamma R), \tag{8.66}$$

which is a function conveniently tabulated on page 370 of Jackson. The dominant mode is

$$\omega_{11} = \frac{1.841}{R\sqrt{\mu\epsilon}}. \tag{8.67}$$

The next four higher modes have ratios:

$$\frac{\omega_{21}}{\omega_{11}} = 1.656,$$
$$\frac{\omega_{01}}{\omega_{11}} = 2.081,$$
$$\frac{\omega_{31}}{\omega_{11}} = 2.282,$$
$$\frac{\omega_{12}}{\omega_{11}} = 2.896.$$

# Chapter 10

# Scattering and Diffraction

## 10.3 Scattering Due to a Solid Uniform Conducting Sphere

### a. The Magnetic Field around the Sphere

Since there are no currents present, we find that

$$\nabla \times \mathbf{H} = \mathbf{J} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{H} = 0. \tag{10.1}$$

Therefore the magnetic field can be described by a magnetic scalar potential $\Phi_m$ that satisfies Laplace's equation:

$$\begin{aligned} \mathbf{H} &= -\nabla \Phi_m \\ \nabla^2 \Phi_m &= 0. \end{aligned} \tag{10.2}$$

The general solution to Laplace's equation, simplifies for our boundary conditions to (J5.117):

$$\Phi_m = -H_{inc} r \cos\theta + \sum_{l=0}^{\infty} \frac{\alpha_l}{r^{l+1}} P_l(\cos\theta). \tag{10.3}$$

where $\alpha_l$ is defined by (J5.121). With $a = 0$ (our shell is a solid sphere), $\mu = 0$, because we are dealing with a perfectly conducting sphere, $\alpha_l = 0$ for $l \neq 1$ and $\alpha_1 = -\frac{b^3 H_{inc}}{2}$. This means that the magnetic scalar potential is

$$\Phi_m = -H_{inc} \cos\theta \left( r + \frac{R^3}{2r^2} \right) \tag{10.4}$$

and

$$\mathbf{H} = H_r + H_\theta = -\frac{1}{r}\frac{\partial \Phi_m}{\partial \theta}\hat{\theta} = -H_{inc} \sin\theta \left( 1 + \frac{R^3}{2r^3} \right) \hat{\theta}. \tag{10.5}$$

$H_r$ is zero inside the sphere $\mathbf{H} = 0$ and the boundary conditions say that $H_r$ is continuous across the boundary. Just outside the sphere, the magnetic field is

$$\mathbf{H}|_{r=R} = -\frac{3}{2} H_{inc} \sin\theta \, \hat{\theta} \tag{10.6}$$

### b. The Absorption Cross Section

According to (J8.12), the power loss per unit area is

$$\frac{dP_{loss}}{da} = \frac{1}{4}\mu_c\omega\delta \left|H_{\|}\right|^2 = \frac{9}{16}\mu_c\omega\delta H_{inc}^2 \sin^2\theta. \tag{10.7}$$

The loss of power is then just a simple integration over area:

$$P_{loss} = \int_S \frac{dP_{loss}}{da} da = \frac{9}{16}\mu_c\omega\delta H_{inc}^2 2\pi R^2 \int_0^\pi \sin^2\theta \sin\theta d\theta = \frac{3}{2}\pi R^2 \mu_c\omega\delta H_{inc}^2. \tag{10.8}$$

According to the experts (I could not find this anywhere), the absorption cross section is defined as

$$\sigma_{abs} = \frac{P_{loss}}{dP_{inc}/da}, \tag{10.9}$$

where the denominator is the total incidence power per unit area. For plane waves, Griffiths writes:

$$\frac{dP_{inc}}{da} = \frac{1}{2}\Re\left[\hat{n} \cdot \mathbf{E}_{inc} \times \mathbf{H}_{inc}^*\right] = \frac{\left|\mathbf{H}_{inc}\right|^2}{2c}. \tag{10.10}$$

Plugging this and the expression for the loss of power into equation 10.9 shows that the absorption cross section is proportional to the square root of frequency:

$$\sigma_{abs} = \frac{\frac{3}{2}\pi R^2 \mu_c\omega\delta H_{inc}^2}{\frac{|\mathbf{H}_{inc}|^2}{2c}} = \frac{3\pi R^2 \mu_c\omega\delta}{c} = \frac{3\pi R^2 \mu_c\omega\sqrt{2}}{c\sqrt{\mu_c\omega\sigma}} = \frac{3\pi R^2}{c}\sqrt{\frac{2\mu_c}{\sigma}}\sqrt{\omega}. \tag{10.11}$$

## 10.14 Diffraction from a Rectangular Opening

This problem involves an infinite conducting sheet in the x-y plane with diffraction from a rectangular aperture as depicted in figure 10.1.

### a. The Smythe-Kirchhoff Relation

According to the vector Smythe-Kirchoff relation, the electric field at an arbitrary point $O$ at location $\mathbf{x}$ due to diffraction from an aperture $S1$ is

$$\mathbf{E}(\mathbf{x}) = \frac{\imath e^{\imath kr}}{2\pi r}\mathbf{k} \times \int_{S1} \mathbf{n} \times \mathbf{E}(\mathbf{x}') e^{-\imath \mathbf{k}\cdot\mathbf{x}'} da', \tag{10.12}$$

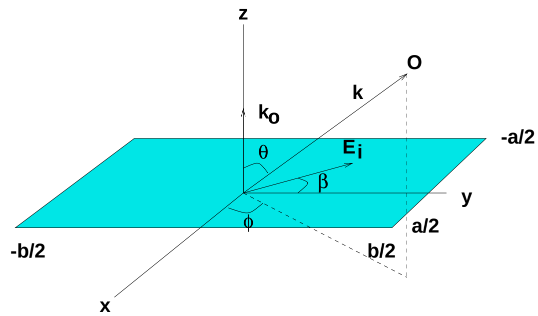


Figure 10.1: *A rectangular aperture in a conducting sheet in the x-y plane.*

where the electric field inside the aperture is assumed to be that of the incoming one:

$$\mathbf{E}(\mathbf{x}') = E_0 e^{-\imath \mathbf{k}\cdot\mathbf{x}'} \left(\sin\beta\hat{x} + \cos\beta\hat{y}\right). \qquad (10.13)$$

Since $\mathbf{n} = \hat{z}$,

$$\hat{z}\times\mathbf{E}(\mathbf{x}') = E_0 e^{-\imath \mathbf{k}\cdot\mathbf{x}'} \left(\sin\beta\hat{y} - \cos\beta\hat{x}\right). \qquad (10.14)$$

The integration is now straightforward:

$$\begin{aligned}\int_{S1} \mathbf{n}\times\mathbf{E}(\mathbf{x}')e^{-\imath\mathbf{k}\cdot\mathbf{x}'}da' &= E_0\left(\sin\beta\hat{y} - \cos\beta\hat{x}\right)\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} e^{-\imath\left(k_x x' + k_y y'\right)}dx'dy' \\ &= E_0\left(\sin\beta\hat{y} - \cos\beta\hat{x}\right)\frac{4}{k_x k_y}\sin\left(\frac{k_x a}{2}\right)\sin\left(\frac{k_y b}{2}\right). \qquad (10.15)\end{aligned}$$

With $\mathbf{k} = k(\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z})$, the total expression for the diffracted electric field is

$$\begin{aligned}\mathbf{E}(\mathbf{x}) &= \frac{2\imath E_0 e^{\imath kr}}{\pi r k_x k_y}\sin\left(\frac{k_x a}{2}\right)\sin\left(\frac{k_y b}{2}\right)\mathbf{k}\times\left(\sin\beta\hat{y} - \cos\beta\hat{x}\right) \\ &= \frac{2\imath E_0 e^{\imath kr}}{\pi r k_x k_y}\sin\left(\frac{k_x a}{2}\right)\sin\left(\frac{k_y b}{2}\right)\left(\hat{z}\left(k_x\sin\beta + k_y\cos\beta\right) - \hat{x}k_z\sin\beta - \hat{y}k_z\cos\beta\right) \\ &= \frac{2\imath E_0 e^{\imath kr}}{\pi r k\sin^2\theta\cos\phi\sin\phi}\sin\left(\frac{ka\sin\theta\cos\phi}{2}\right)\sin\left(\frac{kb\sin\theta\sin\phi}{2}\right)\times \\ &\quad\times\left(\hat{z}\left(\sin\theta\cos\phi\sin\beta + \sin\theta\sin\phi\cos\beta\right) - \hat{x}\cos\theta\sin\beta - \hat{y}\cos\theta\cos\beta\right) \\ &= \frac{2\imath E_0 e^{\imath kr}}{\pi r k\sin^2\theta\cos\phi\sin\phi}\sin\left(\frac{ka\sin\theta\cos\phi}{2}\right)\sin\left(\frac{kb\sin\theta\sin\phi}{2}\right)\times \\ &\quad\times\left(\hat{z}\sin\theta\sin(\phi-\beta) - \hat{x}\cos\theta\sin\beta - \hat{y}\cos\theta\cos\beta\right) \qquad (10.16)\end{aligned}$$

The magnetic field for plane waves has a relation to the electric field that can easily be understood by applying the far field (where waves are "plane") approximation of $\nabla = \imath\mathbf{k}$:

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \quad\Leftrightarrow\quad \imath k\hat{k}\times\mathbf{E} = \imath\omega\mathbf{B} \quad\Leftrightarrow$$

$$\begin{aligned}\mathbf{B}(\mathbf{x}) &= \frac{\mathbf{k}\times\mathbf{E}}{c} = \frac{2\imath E_0 e^{\imath kr}}{c\pi r\sin^2\theta\cos\phi\sin\phi}\sin\left(\frac{ka\sin\theta\cos\phi}{2}\right)\sin\left(\frac{kb\sin\theta\sin\phi}{2}\right)\times \\ &\times \Big(\hat{z}\left(\sin^2\theta\cos\phi(\sin(\phi-\beta)) - \sin\theta\sin\phi(\cos\theta\sin\beta)\right) - \\ &\quad \hat{x}\left(\sin^2\theta\sin\phi(\sin(\phi-\beta)) - \cos\theta\cos\theta\cos\beta\right) - \\ &\quad \hat{y}\left(\sin^2\theta\cos\phi(\sin(\phi-\beta)) - \cos\theta(\cos\theta\sin\beta)\right)\Big)\,. \end{aligned} \tag{10.17}$$

The power per unit angle is defined as

$$\begin{aligned}\frac{dP}{d\omega} &= \frac{r^2}{2Z_0}|\mathbf{E}|^2 = \\ &\quad \frac{2E_0^2}{Z_0k^2\pi^2\sin^4\theta\cos^2\phi\sin^2\phi}\sin^2\left(\frac{ka\sin\theta\cos\phi}{2}\right)\sin^2\left(\frac{kb\sin\theta\sin\phi}{2}\right)\times \\ &\times \left(\sin^2\theta\sin^2(\phi-\beta) + \cos^2\theta\right). \end{aligned} \tag{10.18}$$

## b. The Scalar Kirchhoff Approximation

We can get a scalar approximation from the far field version of the Kirchhoff integral (J10.79), which is (J10.108):

$$\psi(\mathbf{x}) = \frac{e^{\imath kr}}{4\pi r}\int_{S1} e^{-\imath\mathbf{k}\cdot\mathbf{x}'}\left(\mathbf{n}\cdot\nabla'\psi(\mathbf{x}') + \imath\mathbf{k}\cdot\mathbf{n}\psi(\mathbf{x}')\right)da', \tag{10.19}$$

where the assumption is that the scalar $\psi(\mathbf{x}')$ in the aperture is the magnitude of the incoming field:

$$\psi(\mathbf{x}') = E_0 e^{\imath\mathbf{k_0}\cdot\mathbf{x}'} = E_0 e^{\imath kz'}. \tag{10.20}$$

Therefore

$$\mathbf{n}\cdot\nabla'\psi(\mathbf{x}')\big|_{z'=0} = \imath kE_0. \tag{10.21}$$

And

$$\imath\mathbf{k}\cdot\mathbf{n}\psi(\mathbf{x}')\big|_{z'=0} = \imath k\cos\theta E_0. \tag{10.22}$$

So what we are left with is an integral we solved in part a:

$$\begin{aligned}\psi(\mathbf{x}) &= \frac{\imath kE_0 e^{\imath kr}\left(1+\cos\theta\right)}{4\pi r}\int_{S1} e^{-\imath\mathbf{k}\cdot\mathbf{x}'}da' \\ &= \frac{\imath E_0 e^{\imath kr}\left(1+\cos\theta\right)}{\pi rk\sin^2\theta\cos\phi\sin\phi}\sin\left(\frac{ka\sin\theta\cos\phi}{2}\right)\sin\left(\frac{kb\sin\theta\sin\phi}{2}\right). \end{aligned} \tag{10.23}$$

The power per unit angle in the scalar approximation is

$$\frac{dP}{d\omega} = \frac{E_0^2\left(1+\cos\theta\right)^2}{2Z_0\pi^2k^2\sin^4\theta\cos^2\phi\sin^2\phi}\sin^2\left(\frac{ka\sin\theta\cos\phi}{2}\right)\sin^2\left(\frac{kb\sin\theta\sin\phi}{2}\right). \tag{10.24}$$

## c. A Comparison

For the case: $b = a$, $\beta = \pi/4$, $\phi \to 0$ and $ka/2 = kb/2 = 2\pi$ the power per unit angle for the vector and the scalar approximation is respectively

$$\begin{aligned}
\frac{dP}{d\omega} &= \frac{2E_0^2}{k^2\pi^2 \sin^4\theta \cos^2\phi \sin^2\phi} \sin^2\left(\frac{ka\sin\theta\cos\phi}{2}\right) \sin^2\left(\frac{kb\sin\theta\sin\phi}{2}\right) \times \\
&\times \left(\sin^2\theta \sin^2(\phi-\beta) + \cos^2\theta\right) \\
&= \frac{2E_0^2}{k^2\pi^2\sin^4\theta\phi^2} \sin^2(2\pi\sin\theta)\sin^2(2\pi\sin\theta\phi)\left(\frac{\sin^2\theta}{2} + \cos^2\theta\right). \qquad (10.25)
\end{aligned}$$

Using that

$$\lim_{\phi\to 0} \frac{\sin^2(2\pi\sin\theta\phi)}{\phi^2} \approx (2\pi\sin\theta)^2 = 4\pi^2\sin^2\theta, \qquad (10.26)$$

we get for the vector approximation of the power per unit angle that

$$\lim_{\phi\to 0}\frac{dP}{d\omega} = \frac{8E_0^2}{k^2\sin^2\theta}\sin^2(2\pi\sin\theta)\left(\frac{1}{2} + \frac{\cos^2\theta}{2}\right) = \frac{4E_0^2\sin^2(2\pi\sin\theta)\left(1+\cos^2\theta\right)}{k^2\sin^2\theta}. \qquad (10.27)$$

And for the scalar approximation:

$$\lim_{\phi\to 0}\frac{dP}{d\omega} = \frac{E_0^2\left(1+\cos\theta\right)^2}{2\pi^2k^2\sin^4\theta\phi^2}\sin^2\left(2\pi\sin\theta\right)\sin^2\left(2\pi\sin\theta\phi\right) = \frac{2E_0^2\sin^2\left(2\pi\sin\theta\right)\left(1+\cos\theta\right)^2}{k^2\sin^2\theta}. \qquad (10.28)$$

The results are plotted (with $\frac{E_0}{k^2} = 1$) in figure 10.2. Note that for small scattering angle $\theta$ the results are identical.

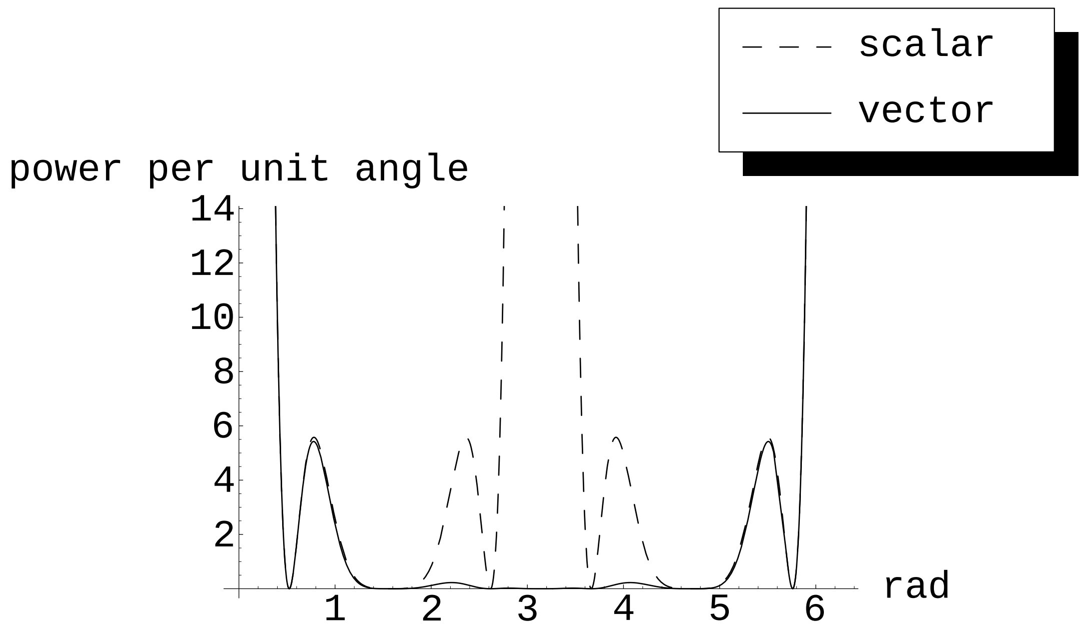


Figure 10.2: *Scalar and vector approximation of the radiated power per unit angle. For $\theta$ close to zero (i.e. close to the direction of $\mathbf{k_0}$, the approximations lead to the same result.*

# Chapter 11

# Special Theory of Relativity

## 11.3 The Parallel-velocity Addition Law

Two reference frames $K'$,$K''$ move in the same direction from $K$ with different velocities $v_1$ and $v_2$, respectively. The frames can be oriented such that the direction of propagation with respect to $K$ is $\hat{x}$. The Lorentz transform can then be written in matrix form $\mathbf{x}' = A_{K\to K'}\mathbf{x}$

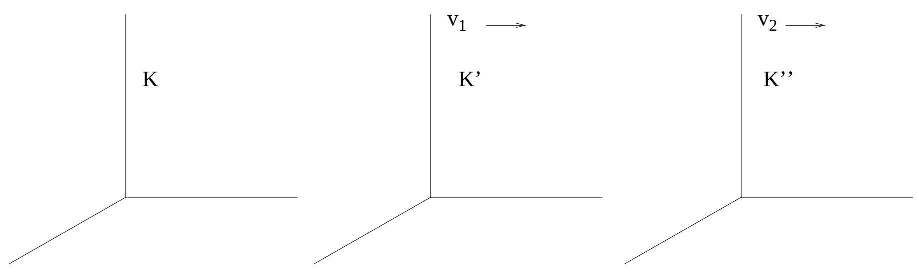


Figure 11.1: *Three reference frames.* $K'$ *and* $K''$ *move in the same direction from* $K$ *with velocity* $v_1$ *and* $v_2$*, respectively.*

$$\begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{11.1}$$

where

$$\beta = \frac{v}{c} \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{11.2}$$

with $c$ equal to the speed of light. The transform from $K$ to $K''$ can then be written as the multiplication of the transform matrix $A_{K\to K'}$ and $A_{K'\to K''}$. Here we will show that that is the same as a direct Lorentz transform $A_{K\to K''}$.

$$\begin{pmatrix} \gamma_1 & -\gamma_1\beta_1 & 0 & 0 \\ -\gamma_1\beta_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_2 & -\gamma_2\beta_2 & 0 & 0 \\ -\gamma_2\beta_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \gamma_1\gamma_2 + \gamma_1\gamma_2\beta_1\beta_2 & -\gamma_1\gamma_2\beta_2 - \gamma_1\gamma_2\beta_1 & 0 & 0 \\ -\gamma_2\gamma_1\beta_1 - \gamma_2\gamma_1\beta_2 & \gamma_1\gamma_2\beta_1\beta_2 + \gamma_1\gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11.3}$$

We can define $\gamma$ and $\beta$ in the total transformation $A_{K\to K''}$ and find the total transformation velocity:

$$\gamma = \gamma_1\gamma_2 + \gamma_1\gamma_2\beta_1\beta_2 \quad \text{and} \quad \beta\gamma = \gamma_1\gamma_2\beta_2 - \gamma_1\gamma_2\beta_1. \tag{11.4}$$

Therefore

$$\begin{aligned} \beta &= \frac{\gamma_1\gamma_2\beta_2 - \gamma_1\gamma_2\beta_1}{\gamma} = \frac{\gamma_1\gamma_2\beta_2 - \gamma_1\gamma_2\beta_1}{\gamma_1\gamma_2 + \gamma_1\gamma_2\beta_1\beta_2} = \frac{\beta_2 + \beta_1}{1 + \beta_1\beta_2} = \frac{\frac{v_1}{c} + \frac{v_2}{c}}{1 + \frac{v_1 v_2}{c^2}} = \frac{v}{c} \Leftrightarrow \\ v &= \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}. \end{aligned} \tag{11.5}$$

## 11.5 The Lorentz Transformation Law for Acceleration

Reference frame $K'$ moves from $K$ with velocity $\mathbf{v}$. We arrange our reference frame that $\mathbf{v} = v\hat{x}$. To find the acceleration transform we first find how velocity $\mathbf{u}$ and time $t$ transform, so that we can define $\mathbf{a} = \frac{d\mathbf{u}}{dt}$. From equation (J11.18) it is clear that

$$dt = \frac{dx_0}{c} = \frac{\gamma}{c}\left(dx_0' + \beta dx_1'\right) = \gamma\left(dt' + \frac{v}{c^2}dx'\right) \tag{11.6}$$

The derivative of the transformed displacement with respect to time (11.6) gives us (J11.31) which states

$$u_\parallel = \frac{u_\parallel' + v}{1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}} \quad \text{and} \quad \mathbf{u}_\perp = \frac{\mathbf{u}_\perp'}{\gamma\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)}. \tag{11.7}$$

The acceleration parallel to the relative velocity between the reference frames is

$$a_\parallel = \frac{du_\parallel}{dt} = \frac{\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)du_\parallel' - \left(u_\parallel' + v\right)\left(\frac{v}{c^2}\right)du_\parallel'}{\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^2 \gamma\left(dt' + \frac{v}{c^2}dx'\right)}. \tag{11.8}$$

Use $dx'_1 = \frac{u'_1}{c} dx'_0$ (Jackson page 531) and $a'_\parallel = du'_\parallel / dt'$ to conclude that

$$a_\parallel = \frac{a'_\parallel \left(1 + \frac{v^2}{c^2}\right)}{\gamma \left(1 + \frac{vu'_\parallel}{c^2}\right)^3} = \frac{a'_\parallel \left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 + \frac{vu'_\parallel}{c^2}\right)^3}. \tag{11.9}$$

The acceleration perpendicular to the direction of propagation is

$$\mathbf{a}_\perp = \frac{d\mathbf{u}_\perp}{dt} = \frac{d\left(\frac{\mathbf{u}'_\perp}{\gamma\left(1+\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)}\right)}{\gamma\left(dt' + \frac{v}{c^2}dx'\right)}. \tag{11.10}$$

After computing the differential of the numerator with the chain rule, we obtain

$$d\mathbf{u}_\perp = \frac{d\mathbf{u}'_\perp}{\gamma\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)} - \frac{\mathbf{u}_\perp{}' \frac{\mathbf{v}\cdot d\mathbf{u}'}{c^2}}{\gamma\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^2}. \tag{11.11}$$

Use $\mathbf{v} \cdot d\mathbf{u}' = v du'_\parallel$ and we have

$$\mathbf{a}_\perp = \frac{\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right) d\mathbf{u}'_\perp - \frac{v}{c^2}\mathbf{u}_\perp{}' du'_\parallel}{\gamma^2 dt' \left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}. \tag{11.12}$$

Realizing that $\frac{d\mathbf{u}_\perp}{dt'} = a'_\perp$, $\frac{d\mathbf{u}_\parallel}{dt'} = a'_\parallel$ and $1/\gamma^2 = 1 - v^2/c^2$ simplifies equation 11.12 to

$$\mathbf{a}_\perp = \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3} \left(\mathbf{a}'_\perp + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\mathbf{a}'_\perp - \frac{\mathbf{v}\cdot\mathbf{a}'}{c^2}\mathbf{u}_\perp{}'\right). \tag{11.13}$$

Writing out the outer products $\mathbf{v} \times (\mathbf{a}' \times \mathbf{u}')$ shows that

$$\mathbf{a}_\perp = \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3} \left(\mathbf{a}'_\perp + \frac{\mathbf{v}}{c^2} \times (\mathbf{a}' \times \mathbf{u}')\right). \tag{11.14}$$

## 11.6 The Rocket Ship

### a. How long are they gone?

There a four legs to the trip. We'll do the calculations on the first and then use symmetry to find the total answer. The first leg is five years to the people in the rocket ship: $t'_1 = 5$ years. They experience an acceleration of $a'_1 = g$ in the direction which I will call $\hat{x}$. What we want to know is how long are those five years in the rocket ship to the observers (the other part of the twins) on earth. In other words, what is $t$? They are related by

$$t = \gamma\left(t' + \frac{\beta}{c}x'\right), \tag{11.15}$$

where

$$\beta = \frac{v}{c} \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{11.16}$$

The twin in the rocket ship feels a force but does not move in its reference frame: $x' = 0$, so

$$dt = \gamma dt'. \tag{11.17}$$

I wrote this in differentials because $\gamma$ is function of velocity and the velocity of the space ship as seen from earth increases with time. We'll find this velocity via the acceleration $a$. We know $a' = g$ and that

$$\begin{aligned}
\mathbf{a} = a_\parallel \hat{x} &= \frac{a'_\parallel \left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 + \frac{vu'_\parallel}{c^2}\right)^3}\hat{x} = g\left(1 - \frac{v^2}{c^2}\right)^{3/2}\hat{x} \Leftrightarrow \\
\frac{dv}{dt} &= g\left(1 - \frac{v^2}{c^2}\right)^{3/2}\hat{x} \Leftrightarrow \\
gdt &= \left(1 - \frac{v^2}{c^2}\right)^{-3/2} dv \Leftrightarrow \\
gt &= \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}.
\end{aligned} \tag{11.18}$$

Rewriting this result as a function of velocity leads to

$$v(t) = \frac{gt}{\sqrt{1 + \left(\frac{gt}{c}\right)^2}} \tag{11.19}$$

Plugging this result into equation 11.16 gives

$$dt' = dt/\gamma = \frac{dt}{\sqrt{1 + \left(\frac{gt}{c}\right)^2}}. \tag{11.20}$$

This we integrate to

$$\tau = \frac{c}{g}\sinh^{-1}\left(\frac{gt}{c}\right). \tag{11.21}$$

The time on the clock in the space ship records $\tau = 5$ years for the first leg. Equation 11.21 written as a function of time on earth $t$ is

$$t = \frac{c}{g}\sinh\left(\frac{g\tau}{c}\right) = 83.7612 \text{ years}. \tag{11.22}$$

Since all four stages are symmetric the total time spent away from earth is $4 \cdot 83.7612 = 335.05$ years, while 20 years passed on the rocket. The space ship left in 2100, so the year of return is 2435.05 AD.

## b. How far did the rocket ship get?

From Earth $dx = vdt$, where the velocity for leg one is calculated in equation 11.19. Therefore the distance traveled is merely an integration over time from zero to the $t_1$ seconds from part a (83.7612 years):

$$d_1 = \int_0^{t_1} v(t)dt = \int_0^{t_1} \frac{gt}{\sqrt{1 + \left(\frac{gt}{c}\right)^2}} dt = fracc^2 g \left( \sqrt{1 + \left(\frac{gt_1}{c}\right)^2} - 1 \right) = 7.83 \times 10^{17} \text{ m} = 82.77 \text{ light years.} \tag{11.23}$$

The total is distance is just twice the first leg distance: $d = 2d_1 = 165.55$ light years.

# Chapter 12

# Practice Problems

## 12.1 Angle between Two Coplanar Dipoles

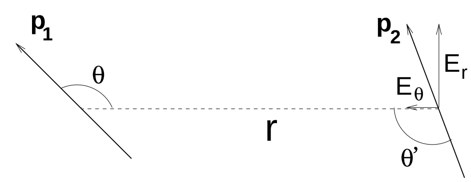


Figure 12.1: *Coplanar dipoles separated by a distance $r$.*

Two dipoles are separated by a distance $r$. Dipole $\mathbf{p_1}$ is fixed at an angle $\theta$ as defined in figure 12.1, while $\mathbf{p_2}$ is free to rotate. The orientation of the latter is defined by the angle $\theta'$ as defined in figure 12.1. Let us calculate the angular dependence between the two dipoles in equilibrium.

The electric field due to a dipole can be decomposed into a radial and a tangential component (D3.39):

$$\mathbf{E}(r,\theta) = \frac{2p\mathrm{cos}\theta}{4\pi\epsilon_0 r^3}\hat{\mathbf{r}} + \frac{p\mathrm{sin}\theta}{4\pi\epsilon_0 r^3}\hat{\boldsymbol{\theta}}, \tag{12.1}$$

where $\epsilon_0$ is the dielectric permittivity. Writing $\mathbf{p_2}$ in terms of $r$ and $\theta$:

$$\mathbf{p}_2(r,\theta) = (\mathbf{p}_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} + (\mathbf{p}_2 \cdot \hat{\boldsymbol{\theta}} = p_2\mathrm{cos}\theta'\hat{\mathbf{r}} + p_2\mathrm{sin}\theta'\hat{\boldsymbol{\theta}} \tag{12.2}$$

The potential energy of the second dipole in the electric field due to the first dipole is

$$U_2(r,\theta,\theta') = -\mathbf{E}_1 \cdot \mathbf{p}_2 = -\frac{p_1 p_2}{4\pi\epsilon_0 r^3}(2\mathrm{cos}\theta\mathrm{cos}\theta' - \mathrm{sin}\theta\mathrm{sin}\theta') \tag{12.3}$$

The second dipole will rotate to minimize its potential energy, defining the angular dependence between the two dipoles:

$$\begin{aligned}
\frac{\partial U_2}{\partial \theta'} = 0 \quad &\Leftrightarrow \\
\frac{p_1 p_2}{4\pi\epsilon_0 r^3}(2\cos\theta\sin\theta' + \sin\theta\cos\theta') = 0 \quad &\Leftrightarrow \\
2\cos\theta\sin\theta' = -\sin\theta\cos\theta' \quad &\Leftrightarrow \\
\tan\theta' = -(\tan\theta)/2. &
\end{aligned} \tag{12.4}$$

## 12.2 The Potential in Multipole Moments

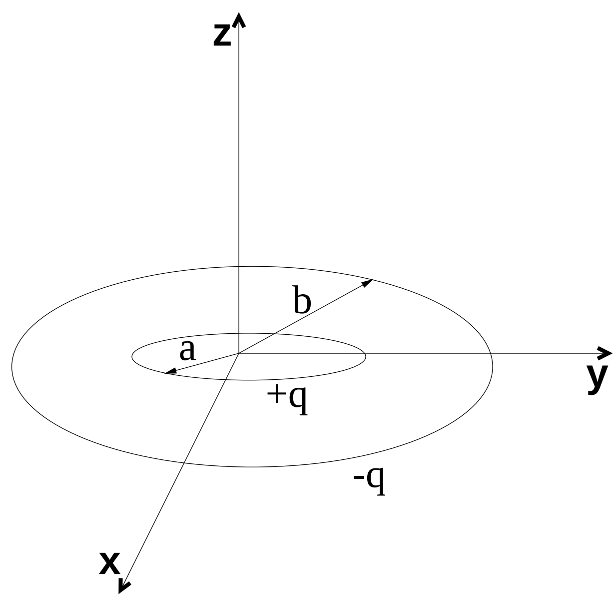


Figure 12.2: *Concentric rings of radii $a$ and $b$. Their charge is $q$ and $-q$, respectively.*

This exercise is how to find the potential due to two charged concentric rings (see figure 12.2) in terms of the monopole dipole and quadrupole moments. Discarding higher order moments (J4.10):

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r} + \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\epsilon_0 r^3} + \frac{1}{8\pi\epsilon_0 r^5}\sum_{i,j} x_i x_j Q_{ij} + .... \tag{12.5}$$

The monopole moment is zero since there is no net charge. The dipole moment is (J4.8):

$$\mathbf{p} = \int_{\mathbf{V}} \mathbf{r}\rho(\mathbf{r})d\mathbf{V}, \tag{12.6}$$

where the volume charge density for our case can be written in cylindrical coordinates:

$$\rho(\mathbf{r}) = \frac{q}{2\pi a}\delta(r-a)\delta(z) - \frac{q}{2\pi b}\delta(r-b)\delta(z) \tag{12.7}$$

After the x-component of the dipole moment is also written in cylindrical coordinates, the integral can be easily evaluated:

$$p_x = \int_{\mathbf{V}} x\rho(r,z)d\mathbf{V} = \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} \cos\phi\rho(r,z)r^2 dr d\phi dz = 0, \tag{12.8}$$

because the $\cos\phi$ integrated over one period is zero. The y-component is zero, because the integration involves a sinusoid over one period. Finally, the z-component is zero, because

$$p_z \propto \int z\delta(z)dz = 0 \tag{12.9}$$

if the value 0 is within the integration limits.

The quadrupole moment is a tensor:

$$Q_{ij} = \int_{\mathbf{V}} (3x_i x_j - \mathbf{r}^2)\rho(\mathbf{r})d\mathbf{V} \tag{12.10}$$

We use the following properties of the $\delta$-function:

$$\begin{aligned} &\int \delta(z)dz = 1 \quad \text{and} \\ &\int r\delta(r-a)dr = a, \end{aligned} \tag{12.11}$$

where 0 and $a$ are within the respective integral limits. Knowing this, the calculations for each element of $Q$ is left to the reader. After some algebra, the quadrupole moment turns out to be

$$Q_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \frac{q}{2}\left(a^2 - b^2\right) \tag{12.12}$$

This means the potential of the two charged concentric rings is approximately

$$\Phi(\mathbf{r}) \approx \frac{(a^2-b^2)}{16\pi\epsilon_0 r^5}(x^2+y^2-2z^2) \tag{12.13}$$

## 12.3 Potential by Taylor Expansion

Here, I will show that the electrostatic potential $\Phi(x, y, z)$ can be approximated by the average of the potentials at the positions perturbed by a small quantity $+/-a$ by doing a Taylor expansion.

This expansion is correct to the third order. First, the Taylor expansion around the $\Phi(x, y, z)$ perturbed in the positive x-component

$$\Phi(x+a, y, z) = \Phi(x, y, z) + \frac{\partial\Phi(x, y, z)}{\partial x}a + \frac{\partial^2\Phi(x, y, z)}{\partial x^2}a^2 + \frac{\partial^3\Phi(x, y, z)}{\partial x^3}a^3 + O(a^4) \tag{12.14}$$

Next, the same expansion around $\Phi(x - a, y, z)$:

$$\Phi(x-a, y, z) = \Phi(x, y, z) - \frac{\partial\Phi(x, y, z)}{\partial x}a + \frac{\partial^2\Phi(x, y, z)}{\partial x^2}a^2 - \frac{\partial^3\Phi(x, y, z)}{\partial x^3}a^3 + O(a^4) \tag{12.15}$$

When we add up these two equations, the odd powers of $a$ cancel. This is the same for the y- and z-component. The $a^2$-term adds up to the Laplacian $\nabla^2$. Assuming there is no charge within the radius $a$ of $(x, y, z)$, Laplace's equation holds:

$$\nabla^2\Phi = 0 \tag{12.16}$$

and thus the $a^2$ term is zero, too. Therefore, the potential at $(x, y, z)$ can be given by:

$$\Phi(x, y, z) = 1/6(\Phi(x+a, y, z) + \Phi(x-a, y, z) + \Phi(x, y+a, z) + \Phi(x, y-a, z) + \Phi(x, y, z+a) + \Phi(x, y, z-a)) + O(a^4) \tag{12.17}$$

# Appendix A

# Mathematical Tools

## A.1 Partial integration

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx \tag{A.1}$$

## A.2 Vector analysis

### Stokes' theorem

$$\oint_L \mathbf{M} \cdot d\mathbf{l} = \int_{\mathbf{S}} \mathrm{curl}\mathbf{M} \cdot d\mathbf{S} \tag{A.2}$$

### Gauss' theorem

$$\oint_S \mathbf{M} \cdot d\mathbf{S} = \int_{\mathbf{V}} \mathrm{div}\mathbf{M} \cdot d\mathbf{V} \tag{A.3}$$

### Computation of the curl

$$\mathrm{curl}\mathbf{M} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_i \hat{i} & h_2 \hat{j} & h_3 \hat{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_2} \\ h_1 M_1 & h_2 M_2 & h_3 M_3 \end{vmatrix} \tag{A.4}$$

$h_i$ are the geometrical components that depend on the coordinate system. For a Cartesian coordinate system they are one. For the cylindrical system:

$$h_1 = 1$$
$$h_2 = r$$
$$h_3 = 1$$

For the spherical coordinate system:

$$h_1 = 1$$
$$h_2 = r$$
$$h_3 = r \sin\theta$$

### Computation of the divergence

$$\mathrm{div}\mathbf{M} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial x_1}(h_2 h_3 M_1) + \frac{\partial}{\partial x_2}(h_1 h_3 M_2) + \frac{\partial}{\partial x_3}(h_1 h_2 M_3)\right) \tag{A.5}$$

With the same factors $h_i$ depending on the coordinate system.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}. \tag{A.6}$$

### Relations between grad and div

$$\nabla \times \nabla \times \mathbf{M} = \nabla\nabla \cdot \mathbf{M} - \nabla^2 \mathbf{M} \tag{A.7}$$

$$\mathbf{M} = \nabla\Phi + \nabla \times \mathbf{A} \tag{A.8}$$

## A.3 Expansions

### Taylor series

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2 f''(a)}{2!} + .... + \frac{x^n}{n!}(x-a)^n f^{(n)}(a) \tag{A.9}$$

## A.4 Euler Formula

$$e^{i\phi} = \cos\phi + i\sin\phi \tag{A.10}$$

$$\begin{aligned}
\sin\theta &= \frac{e^{\imath\theta} - e^{-\imath\theta}}{2i} && \text{(A.11)}\\
\cos\theta &= \frac{e^{\imath\theta} + e^{-\imath\theta}}{2} && \text{(A.12)}\\
\sinh\theta &= \frac{e^{\theta} - e^{-\theta}}{2} && \text{(A.13)}\\
\cosh\theta &= \frac{e^{\theta} + e^{-\theta}}{2} && \text{(A.14)}
\end{aligned}$$

## A.5 Trigonometry

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2} \tag{A.15}$$

$$\begin{aligned}
\cos(\phi - \theta) &= Re\left[e^{\imath(\phi-\theta)}\right] = Re\left[e^{\imath\phi}e^{-\imath\theta}\right]\\
&= Re\left[\cos\phi\cos\theta + \sin\phi\sin\theta + i(..)\right]\\
&= \cos\phi\cos\theta + \sin\phi\sin\theta && \text{(A.16)}
\end{aligned}$$

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