F.15) Finite plates.

In the case of the finite plates, the boundary conditions on the contour of the plate are essential ones in order to obtain the values of the integration constants. Some particular, most common cases will analysed in detail. As a consequence, a previous examination of the mean values Σ_{ij} and M_{ij} , $i, j=1,2,3.$ it is necessary.

F.15 a) Significance of Σ_{ij} and M_{ij} for the bending state.

According to the definition of the stress tensor elements, σ_{ij} is the projection along j-axis of the surface force acting on the surface having the outer pointing normal equal to $\vec{n}_i$. Because the elements σ_{13} , σ_{23} are even functions, the mean values Σ_{13} , Σ_{23} are representing share forces, acting like in Fig.F3.

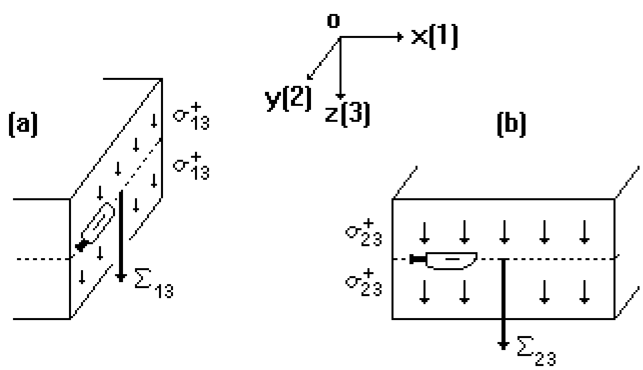

Fig.F3. The mean values Σ_{13} (a) and Σ_{23} (b). Both of them are share forces.

Similar considerations allow one to conclude that the mean values M_{11} and M_{22} are bending moments, while M_{12} , M_{21} are torsion moments (Fig.F4).

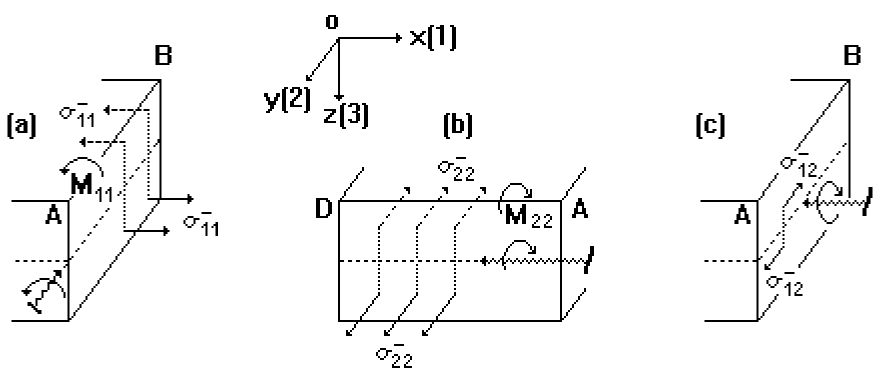

Fig.F4. The bending moments M_{11} (a) and M_{22} (b). The torsion moment M_{12} acting on the side having the outer pointing normal 1 is presented in Fig.4c. A similar torsion moment is acting on the side having the normal 2, but it is not presented in the figure.

F.15 b) The rectangular plate. Boundary conditions. LÉVY 's solution.

Let consider a rectangular plate having the sides equal to $2a$ and $2b$ respectively (Fig.F5). Consider, for example, the side AB, having the equation $x = a$, $y \in [-b, b]$). Among most commonly used boundary conditions are

-<u>the embedded side</u>: the flexure of the plate and the derivative of the flexure are both equal to zero

$$w(x, y) = 0 \,, \partial w/\partial x = 0 \;; \qquad \text{(f123)}$$

-<u>the rotating side</u>: the flexure of the plate and the bending moment are both equal to zero:

$$w(x, y) = 0 \,, M_{11} = 0 \quad , \qquad \text{(f124)}$$

i.e.:

$$w(x,y)=0\,,\lambda\,\Delta^{*}w+2\mu\frac{\partial^{2}w}{\partial x^{2}}=0\ . \qquad \text{(f125)}$$

-the free side: the share force, the bending moment and the torsion moment are all equal to zero.

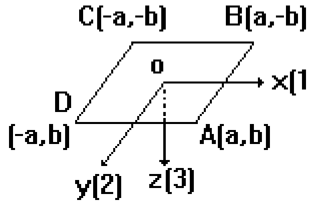

Fig.F5. The rectangular plate.

Similar conditions can be derived for plates of arbitrary shape.

As an example, consider the rectangular plate with two opposite articulated sides. Neglecting the lateral forces and the gravity, it follows to solve the simplified equation of the flexure

$$\Delta\,\Delta\,w=q(x,y)\,/\,D\quad , \qquad \text{(f126)}$$

with the boundary conditions (f125) written for $x=\pm a$ and a 2-D LAPLACE operator. Consider a particular solution having the form

$$w(x,y)=\sum_{k=1}^{\infty}f_k(y)\sin(k\pi\,x\,/\,a)\,, \qquad \text{(f127)}$$

it follows that

$$w(\pm a,y)=0\,,$$
$$\frac{\partial^{2}w}{\partial x^{2}}=-\frac{\pi^{2}}{a^{2}}\sum_{k=1}^{\infty}k^{2}f_k\sin(k\pi\,x\,/\,a)\quad . \qquad \text{(f128)}$$

Hence

$$\partial^{2}w\,/\,\partial x^{2}=\partial^{2}w\,/\,\partial y^{2}=0\,,\text{for } x=\pm a\,. \qquad \text{(f129)}$$

It follows the conditions (f125) are fulfilled for the two opposite articulated sides. Substituting (f127) into (f126), it follows that

$$\sum_{k=1}^{\infty}\left[f_k''''(y)-2\left(\frac{k\pi}{a}\right)^{2}f_k''(y)+\left(\frac{k\pi}{a}\right)^{4}f_k(y)\right]\sin\frac{k\pi\,x}{a}=q(x,y)\,/\,D\ . \qquad \text{(f130)}$$

The function $q(x,y)$ is expanded in FOURIER series and the coefficients are identified. Both sides of eq.(f130) are multiplied by $\sin\frac{j\pi\,x}{a}$, $j=1,2,\ldots$ The result is integrated on the interval $[-a,a]$, taking into account that

$$\int_{-a}^{+a}\sin\frac{k\pi\,x}{a}\sin\frac{j\pi\,x}{a}dx=\begin{cases}0 & ,\text{pentru } j\neq k\\ a & ,\text{pentru } j=k\end{cases}\quad . \qquad \text{(f131)}$$

Hence

$$f_k''''(y)-2\left(\frac{k\pi}{a}\right)^{2}f_k''(y)+\left(\frac{k\pi}{a}\right)^{4}f_k(y)=\frac{1}{Da}\int_{-a}^{+a}q(x,y)\sin\frac{k\pi\,x}{a}dx\ ,k=1,2,\ldots\ . \qquad \text{(f132)}$$

In the beginning, the next homogeneous equation is solved, i.e.

$$F_k''''(y)-2\left(\frac{k\pi}{a}\right)^{2}F_k''(y)+\left(\frac{k\pi}{a}\right)^{4}F_k(y)=0\ ,k=1,2,\ldots\ . \qquad \text{(f133)}$$

The characteristic equation is

$$r^{4}-2\left(\frac{k\pi}{a}\right)^{2}r^{2}+\left(\frac{k\pi}{a}\right)^{4}=0\,,\text{deci}\quad r_1=r_2=-k\pi\,/\,a\quad ,r_3=r_4=k\pi\,/\,a\ , \qquad \text{(f134)}$$

hence the solution of the homogeneous equation is

$$F_k(y) = A_k \cosh\frac{k\pi y}{a} + B_k \sinh\frac{k\pi y}{a} + y\left(C_k \cosh\frac{k\pi y}{a} + D_k \sinh\frac{k\pi y}{a}\right) \quad , \qquad (f135)$$

where $A_k, B_k, C_k, D_k, \quad k = 1,2,\ldots$ are some constants following to be obtained from the boundary conditions on the other two (non-articulated) sides of the plate. The general solution of equation (f132) is the sum of (f135) and a particular solution. The last one can be obtained by the usual techniques (e.g. CAUCHY method). The above approach is due to LÉVY.

F.16) Vibrations of a plate laying on a viscous substratum.

In this chapter, the flexure is considered as a function of both spatial co-ordinates and time. The corresponding differential equation is derived, being solved in the case of the 2-dimensional plate.

a) The differential equation.

As usually, a co-ordinate system is used having the horizontal axes x and y. The z-axis is positive downward, having the unit vector denoted by $\vec{e}_z$. By applying the mean-value operator, the equations of motion for the bending state are

$$\partial M_{xx}/\partial x + \partial M_{xy}/\partial y + \left[q_x^d(x,y,t) - q_x^u(x,y,t)\right]/2 - \Sigma_{xz} = \rho\,\overline{z\,\ddot{u}_x} \quad , \qquad (f136)$$

$$\partial M_{xy}/\partial x + \partial M_{yy}/\partial y + \left[q_y^d(x,y,t) - q_y^u(x,y,t)\right]/2 - \Sigma_{yz} = \rho\,\overline{z\,\ddot{u}_y} \quad , \qquad (f137)$$

and

$$\partial \Sigma_{xz}/\partial x + \partial \Sigma_{yz}/\partial y + \frac{1}{2h}\left[q_z^d(x,y,t) + q_z^u(x,y,t)\right] + \rho g = \rho\,\overline{\ddot{u}_z} \quad . \qquad (f138)$$

It will be assumed that Bernoulli's hypothesis is valid for all time. Hence the displacement vector has the elements

$$u_x(x,y,z,t) = -z\frac{\partial w}{\partial x} \quad , \quad u_y(x,y,z,t) = -z\frac{\partial w}{\partial y} \quad , \quad u_z(x,y,z,t) = w(x,y,t) \quad . \qquad (f139)$$

Substituting (f139) into (f136)-(f138) it follows after elementary computations that

$$D\Delta^*\Delta^* w = \rho g H - \rho H \ddot{w} + \frac{\rho}{12}H^3\Delta^*\ddot{w} + q_z^l(x,y,t) + q_z^u(x,y,t) + \frac{H}{2}\left\{\partial\left[q_x^l(x,y,t) - q_x^u(x,y,t)\right]\Big/\partial x + \partial\left[q_y^l(x,y,t) - q_y^u(x,y,t)\right]\Big/\partial y\right\} \quad , \qquad (f140)$$

As usually, the horizontal loads for the upper face of the plate are neglected, the surface forces being assumed to be

$$q_x^u = 0 \quad , \quad q_y^u = 0 \quad , \quad q_z^u = \rho^F g w + P(1 - \ddot{w}/g) \quad , \qquad (f141)$$

where ρ^F is the density of the filling sediments and P is the load. The last term in (f141) is an inertial one. For the material below the plate, the next constitutive equation is assumed

$$\sigma = [p_0 - \rho^M g(H + w)]\mathbf{1} + \lambda^M \mathrm{tr}\,\varepsilon\,\mathbf{1} + 2\mu^M \varepsilon + 2\eta^M \dot{\varepsilon} \qquad (f142)$$

where p_0 is a reference pressure and ρ^M is the density of the material below the plate. The LAMÉ elastic coefficients are λ^M, μ^M and η^M is the viscosity. Hence the loads on the lower face of the plate are

$$q_x^l = \mu^M \frac{\partial w}{\partial x} + \eta^M \frac{\partial \dot{w}}{\partial x} \quad , \quad q_y^l = \mu^M \frac{\partial w}{\partial y} + \eta^M \frac{\partial \dot{w}}{\partial x} \quad , \qquad . \qquad \text{(f143)}$$
$$q_z^d = p_0 - \frac{H}{2}\lambda^M \Delta^* w + \rho^M g(H + w)$$

Again, a correction due to the compressive horizontal stresses σ_x^c , σ_y^c acting along the x- and y-axes respectively at the ends of the plate follows to be considered further. The reference pressure p_0 is selected in order the flexure w to vanish in the absence of the load P . Finally, a generalisation of the Sophie GERMAIN equation for a time dependent flexure is obtained as

$$D\Delta^*\Delta^* w + \frac{H}{2}\left(\lambda^M - \mu^M\right)\Delta^* w + H\left(\sigma_x^c \frac{\partial^2 w}{\partial x^2} + \sigma_y^c \frac{\partial^2 w}{\partial y^2}\right) + (\rho^M - \rho^F)gw$$
$$= P + \eta^M \frac{H}{2}\Delta^* \dot{w} - (\rho H + P/g)\ddot{w} + \frac{\rho}{12}H^3\Delta^* \ddot{w} \qquad \text{(f144)}$$

b) The rectangular plate with 3 welded sides.

For usual materials $\lambda^M = \mu^M$, hence the second term on the left side of (f144) vanishes. Because the load is mainly represented by the relief, having the elevations much smaller then the thickness of the plate, the inertial term is usually negligible on the right side of (15). Let $w^e = w^e(x, y)$ be the equation of the flexure corresponding to the state of equilibrium in the presence of the load, i.e.

$$D\Delta^*\Delta^* w^e + H\left(\sigma_x^c \partial^2 w^e / \partial x^2 + \sigma_y^c \partial^2 w^e / \partial y^2\right) + (\rho^M - \rho^F)gw^e = P \qquad \text{(f145)}$$

Consider the difference

$$\delta = \delta(x, y, t) = w(x, y, t) - w^e(x, y) \qquad \text{(f146)}$$

It follows that

$$\Delta^*\Delta^*\delta + \frac{12}{\rho a^2}\frac{1}{H^2}\left(\sigma_x^c \frac{\partial^2 \delta}{\partial x^2} + \sigma_y^c \frac{\partial^2 \delta}{\partial y^2}\right) + 12\frac{\rho^M - \rho^F}{\rho}\frac{g}{H^3 a^2}\delta = \frac{1}{a^2}\left(\Delta^* \ddot{\delta} + \frac{6\eta^M}{\rho H^2}\Delta^* \dot{\delta} - \frac{12}{H^2}\ddot{\delta}\right) \qquad \text{(f147)}$$

where a denotes the velocity of P-waves through the plate.
Consider the lengths of the sides are L_x , L_y respectively. Suppose the plate is welded according to

$$\begin{aligned} &\delta(x,0,t) = 0, \quad \text{with} \quad 0 \le x \le L_x \\ &\delta(0,y,t) = 0, \quad \text{with} \quad 0 \le y \le L_y \\ &\delta(L_x, y, t) = 0, \quad \text{with} \quad 0 \le y \le L_y \end{aligned} \quad , \qquad \text{(f148)}$$

at any time. A particular solution satisfying (f148) is

$$\delta_{mn}(x, y, t) = \sin\left(m\pi x / L_x\right)\sin[(n - 0.5)\pi y / L_y]\,\tau_n(t) \quad , \quad m, n = 1,2,\ldots \qquad \text{(f149)}$$

Substituting (f149) in (f147) it follows the modes are damped harmonic, i.e.

$$\ddot{\tau} + 2\zeta_{mn}\dot{\tau} + \frac{4\pi^2}{T_{mn}^2}\tau = 0 \quad , \qquad \text{(f150)}$$

For $m, n = 1,2,\ldots$ the periods are

$$T_{mn} = 2\pi \frac{H}{a}\sqrt{\frac{A_{mn} + 12}{A_{mn}^2 - 12\pi^2[\sigma_x^c m^2 H_x^2 + \sigma_y^c (n - 0.5)^2 H_y^2]/(\rho a^2) + 12\rho^* gH / a^2}} \, , \qquad \text{(f151)}$$

and the decay constants are

$$\zeta_{mn} = \frac{3\eta^M}{\rho H^2} \frac{A_{mn}}{A_{mn} + 12} \quad , \tag{f152}$$

where

$$H_x = H / L_x \quad , \quad H_y = H / L_y \quad , \quad \rho^* = (\rho^M - \rho^F) / \rho \tag{f153}$$

and

$$A_{mn} = \pi^2 \left[m^2 H_x^2 + (n - 0.5)^2 H_y^2 \right] \quad . \tag{f154}$$

According to (f139), the modes are both toroidal and spheroidal.
For each mode, a critical viscosity can be found from

$$\zeta_{mn}^{cr} = 2\pi / T_{mn} \quad , \tag{f155}$$

i.e.

$$\eta_{mn}^{cr} = \frac{2\pi\rho H^2}{3T_{mn}} \frac{A_{mn} + 12}{A_{mn}} \quad . \tag{f156}$$

For values of viscosity less than the critical value (f156), the motion of the plate is represented by a sum of damped oscillations. The decay constants are obtained from (f152) and the periods are

$$T_{mn}^* = T_{mn} / \sqrt{1 - \left(\zeta_{mn} / \zeta_{mn}^{cr} \right)^2} \tag{f157}$$

For values of viscosity greater than the critical value, the motion of the plate is aperiodic and the characteristic roots of (21) are $-\left(\zeta_{mn} + \sqrt{\zeta_{mn}^2 - 4\pi^2 / T_{mn}^2} \right)$ and $-\left(\zeta_{mn} - \sqrt{\zeta_{mn}^2 - 4\pi^2 / T_{mn}^2} \right)$. It follows that for large scale of times, the solution of (f150) behaves like $\exp(-Ct)$, where the decay constant is

$$C = \zeta_{11} - \sqrt{\zeta_{11}^2 - 4\pi^2 / T_{11}^2} \quad , \tag{f158}$$

all the other terms being faster attenuated. In many real applications, an approximate value of (f159) is

$$C = \frac{4\pi^2 / T_{11}^2}{\zeta_{11} + \sqrt{\zeta_{11}^2 - 4\pi^2 / T_{11}^2}} \cong \frac{2\pi^2}{\zeta_{11} T_{11}^2} \tag{f159}$$

With regard to (f149), the decay constant (f158) or (f159) can be obtained as the ratio between the amplitude of the velocity $\partial\delta / \partial t$ and the amplitude of δ. By using eq.(f152), that ratio can be used to estimate the mean viscosity of the material below the plate

$$\eta^M = \frac{\rho H^2}{3C} \frac{2\pi^2}{T_{11}^2} \frac{A_{mn} + 12}{A_{mn}} \tag{f160}$$

A numerical application with respect to the Moesian Platform, a micro-plate bounded by Carpathians and Balkan Mountains and by the Black Sea is presented in detail by (Ivan 1997a,b).