C) LÉVY's PROBLEM - THE TRIANGULAR DAM

C.1) The SAINT-VENANT 's equations.

Differentiating a certain element of the strain tensor

$$\varepsilon_{ij} = \left(u_{i,j} + u_{j,i}\right)/2 \quad , \tag{c1}$$

it follows, for example, that

$$\begin{aligned} \varepsilon_{11,22} + \varepsilon_{22,11} &= \left(u_{1,1}\right)_{,22} + \left(u_{2,2}\right)_{,11} = u_{1,122} + u_{2,211} \\ &= \left(u_{1,2}\right)_{,12} + \left(u_{2,1}\right)_{,12} = \left(u_{1,2} + u_{2,1}\right)_{,12} = 2\varepsilon_{12,12} \end{aligned} \tag{c2}$$

Hence

$$\varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12} \tag{c3}$$

Also,

$$\varepsilon_{22,33} + \varepsilon_{33,22} = 2\varepsilon_{23,23} \tag{c4}$$

$$\varepsilon_{33,11} + \varepsilon_{11,33} = 2\varepsilon_{31,31} \tag{c5}$$

In a similar way, it follows that

$$\begin{aligned} \left(\varepsilon_{12,3} + \varepsilon_{23,1} - \varepsilon_{31,2}\right)_{,2} &= \varepsilon_{22,31} \\ \left(\varepsilon_{23,1} + \varepsilon_{31,2} - \varepsilon_{12,3}\right)_{,3} &= \varepsilon_{33,12} \\ \left(\varepsilon_{31,2} + \varepsilon_{12,3} - \varepsilon_{23,1}\right)_{,1} &= \varepsilon_{11,23} \end{aligned} \tag{c6)-(c8}$$

The above equations (c3)-(c8) represent the SAINT-VENANT's equations of compatibility.

C.2) The model . Simplifying hypothesis. The planar deformation state.

A horizontal dam of infinite length is considered. The cross-section is represented by a rectangular triangle OAB (Fig.C1). The length of the base is AB=l and the height is OA=h. On OA catheter is acting the hydrostatic pressure of a liquid (water) having the specific weight equal to γ. As a result, the dam is deformed. The dam is represented by an elastic homogeneous, isotropic material. Its specific weight is equal to Γ and its elastic constants are E and ν.

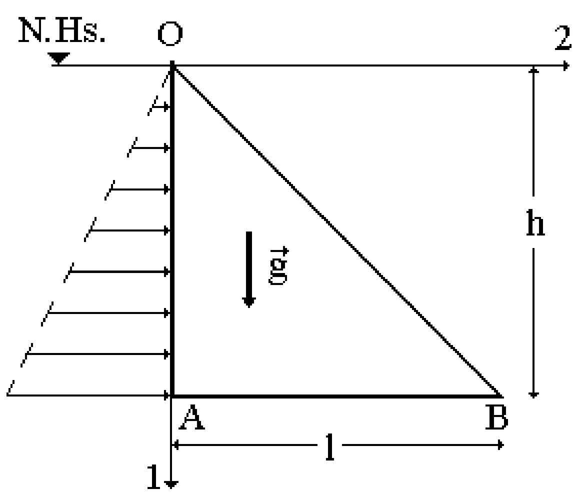

Fig.C1. A vertical cross section through the dam. N.Hs. is the free surface of the water, acting on OA side by a pressure linearly increasing with depth.

Because the shape of the dam, the displacement vector has the components like

$$\begin{cases} u_1 = u_1(x_1, x_2) \\ u_2 = u_2(x_1, x_2) \\ u_3 = 0 \end{cases} \tag{c9}$$

It follows the strain tensor components are like

$$\begin{aligned} &\varepsilon_{11} = u_{1,1} = \varepsilon_{11}(x_1, x_2) \\ &\varepsilon_{12} = \frac{1}{2}\left(u_{1,2} + u_{2,1}\right) = \varepsilon_{12}(x_1, x_2) \\ &\varepsilon_{13} = \frac{1}{2}\left(u_{1,3} + u_{3,1}\right) = 0 \\ &\varepsilon_{22} = u_{2,2} = \varepsilon_{22}(x_1, x_2) \\ &\varepsilon_{23} = \frac{1}{2}\left(u_{2,3} + u_{3,2}\right) = 0 \\ &\varepsilon_{33} = u_{3,3} = 0 \end{aligned} \tag{c10}$$

Hence the strain matrix is

$$\left[\varepsilon\right] = \left[\varepsilon\right](x_1, x_2) = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \tag{c11}$$

It corresponds to a *planar state of the strain* (the plane here being 1-2).
The components of the stress tensor are

$$\begin{aligned} &\sigma_{11} = \lambda\left(\varepsilon_{11} + \varepsilon_{22}\right) + 2\mu\,\varepsilon_{11} \\ &\sigma_{12} = 2\mu\,\varepsilon_{12} \\ &\sigma_{13} = 2\mu\,\varepsilon_{13} = 0 \\ &\sigma_{22} = \lambda\left(\varepsilon_{11} + \varepsilon_{22}\right) + 2\mu\,\varepsilon_{22} \\ &\sigma_{23} = 2\mu\,\varepsilon_{23} = 0 \\ &\sigma_{33} = \lambda\left(\varepsilon_{11} + \varepsilon_{22}\right) + 2\mu\,\varepsilon_{33} = \lambda\left(\varepsilon_{11} + \varepsilon_{22}\right) = \frac{\lambda}{2(\lambda + \mu)}\left(\sigma_{11} + \sigma_{22}\right) = \nu\left(\sigma_{11} + \sigma_{22}\right) \end{aligned} \tag{c12}$$

Hence the stress matrix is

$$\left[\sigma\right] = \left[\sigma\right](x_1, x_2) = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \nu\left(\sigma_{11} + \sigma_{22}\right) \end{pmatrix} \tag{c13}$$

Because the component 33 of the stress has a non-zero value, eq.(c13) shows that the stress state corresponding to a planar state of the strain is not generally a planar one too.

C.3) Equations of equilibrium. AIRY's potential.

The only body force acting on the dam is its weight. The equations of equilibrium are

$$\begin{cases} \sigma_{11,1} + \sigma_{12,2} + \Gamma = 0 \\ \sigma_{12,1} + \sigma_{22,2} = 0 \end{cases} \tag{c14}$$

Because the presence of Γ, eqs.(c14) represent a non-homogeneous system. In the beginning, the homogeneous system is solved, i.e.

$$\begin{cases}\Sigma_{11,1}+\Sigma_{12,2}=0\\ \Sigma_{12,1}+\Sigma_{22,2}=0\end{cases} \tag{c15}$$

Using an unknown function φ, the first equation of (c15) is verified for

$$\Sigma_{11}=\frac{\partial\varphi}{\partial x_2} \quad , \quad \Sigma_{12}=-\frac{\partial\varphi}{\partial x_1} \tag{c16}$$

In the same way, the second equation of (c15) is verified for

$$\Sigma_{12}=\frac{\partial\psi}{\partial x_2} \quad , \quad \Sigma_{12}=-\frac{\partial\psi}{\partial x_1} \tag{c17}$$

It follows that

$$\frac{\partial\varphi}{\partial x_1}+\frac{\partial\psi}{\partial x_2}=0 \tag{c18}$$

i.e. the unknown functions are

$$\varphi=\frac{\partial \mathcal{A}}{\partial x_2} \quad , \quad \psi=-\frac{\partial \mathcal{A}}{\partial x_1} \tag{c19}$$

The unknown function $\mathcal{A} = \mathcal{A}(x_1, x_2)$ represents the AIRY's potential. It allows one to obtain the next expressions for the components of the stress tensor when the body force are absent:

$$\Sigma_{11}=\mathcal{A}_{,22} \quad , \quad \Sigma_{12}=-\mathcal{A}_{,12} \quad , \quad \Sigma_{22}=\mathcal{A}_{,11} \tag{c20}$$

From (c13), the trace of the stress tensor can be written using LAPLACE's operator in 1-2 co-ordinates

$$\mathrm{tr}\,\Sigma = (1+\nu)\left(\Sigma_{11}+\Sigma_{22}\right)=(1+\nu)\Delta^*\mathcal{A} \tag{c21}$$

The components of the strain tensor are obtained using the reversed HOOKE's law

$$\varepsilon_{11}=\frac{1+\nu}{E}\left(\mathcal{A}_{,22}-\nu\Delta^*\mathcal{A}\right) \ ,\varepsilon_{22}=\frac{1+\nu}{E}\left(\mathcal{A}_{,11}-\nu\Delta^*\mathcal{A}\right) \ ,\varepsilon_{12}=-\frac{1+\nu}{E}\mathcal{A}_{,12} \ . \tag{c22}$$

Using (c22) and (c3) it follows

$$\mathcal{A}_{,2222}-\nu\left(\Delta^*\mathcal{A}\right)_{,22}+\mathcal{A}_{,1111}-\nu\left(\Delta^*\mathcal{A}\right)_{,11}=-2\mathcal{A}_{,1212} \quad , \tag{c23}$$

i.e.

$$(1-\nu)\Delta^*\Delta^*\mathcal{A}=0 \tag{c24}$$

Because $\nu < 0.5$, it follows that AIRY's potential is a solution of the bi-harmonic equation

$$\Delta^*\Delta^*\mathcal{A}=0 \tag{c25}$$

Because the trace of a tensor is an invariant, eq.(c25) holds too in the general case of the orthogonal curvilinear co-ordinates. However, eq.(c20) has to be modified.

C.4) Boundary conditions. The final shape of the dam.

On the side OA of the dam is acting the hydrostatic pressure. It follows that

$$\sigma\left(-\vec{e}_2\right)=\gamma x_1\vec{e}_2 \tag{c26}$$

On the side OB of the dam is acting the negligible atmospheric pressure. It follows that

$$\sigma\,\vec{n}=\vec{0} \tag{c27}$$

where the outer pointing normal at the dam is

$$\vec{n}=-\sin\alpha\,\vec{e}_1+\cos\alpha\,\vec{e}_2 \tag{c28}$$

On the side OA, for $x_1 \in [0,h], x_2 = 0$, it follows that

$$\begin{cases}\sigma_{12} = 0, \\ \sigma_{22} = -\gamma x_1\end{cases} \tag{c29}$$

On the side OB it follows for $x_1 \in [0, h], x_2 = x_1 \tan\alpha$ that

$$\begin{cases}\sigma_{12} - \sigma_{11}\tan\alpha = 0, \\ \sigma_{22} - \sigma_{12}\tan\alpha = 0\end{cases} \tag{c30}$$

Eqs.(c29)-(c30) represent 4 boundary conditions, suggesting a solution of the bi-harmonic equation (c25) which depends on 4 unknown coefficients denoted by a, b, c, d, i.e.

$$A(x_1, x_2) = \frac{a}{6}x_1^3 + \frac{b}{2}x_1^2 x_2 + \frac{c}{2}x_1 x_2^2 + \frac{d}{6}x_2^3 \tag{c31}$$

Using (c20), the solution of the homogeneous system is

$$\begin{cases}\Sigma_{11} = cx_1 + dx_2 \\ \Sigma_{12} = -(bx_1 + cx_2) \\ \Sigma_{22} = ax_1 + bx_2\end{cases} \tag{c32}$$

A particular solution of the non-homogeneous system (c14) is

$$\begin{cases}\sigma_{11} = \sigma_{22} = 0 \\ \sigma_{12} = -\Gamma x_2\end{cases} \tag{c33}$$

It follows the general solution of (c14) is

$$\begin{cases}\sigma_{11} = cx_1 + dx_2 \\ \sigma_{12} = -(bx_1 + cx_2) - \Gamma x_2 \\ \sigma_{22} = ax_1 + bx_2\end{cases} . \tag{c34}$$

Replacing (c34) into (c29)-(c30) it follows that

$$\begin{cases}-(bx_1 + cx_2) - \Gamma x_2 = 0\text{, for } x_1 \in [0, h] \quad , x_2 = 0 \\ ax_1 + bx_2 = -\gamma x_1\text{, for } x_1 \in [0, h] \quad , x_2 = 0 \\ -(cx_1 + dx_2)\tan\alpha - (bx_1 + cx_2) - \Gamma x_2 = 0\text{, for } x_1 \in [0, h] \quad , x_2 = x_1\tan\alpha \\ (bx_1 + cx_2 + \Gamma x_2)\tan\alpha + ax_1 + bx_2 = 0\text{, for } x_1 \in [0, h] \quad , x_2 = x_1\tan\alpha\end{cases} \tag{c35}$$

It follows that

$$\begin{cases}a = -\gamma \\ b = 0 \\ c = -\Gamma + \gamma / \tan^2\alpha \\ d = \Gamma / \tan\alpha - 2\gamma / \tan^3\alpha\end{cases} \tag{c36}$$

and

$$\begin{cases}\sigma_{11} = Ax_1 + Bx_2 \\ \sigma_{12} = -Cx_2 \\ \sigma_{22} = -\gamma x_1\end{cases} \tag{c37}$$

where

$$A = \gamma h^2 / l^2 - \Gamma \quad , \quad B = \Gamma h / l - 2\gamma h^3 / l^3 \quad , \quad C = -\gamma h^2 / l^2 \tag{c38}$$

Hence

$$u_{1,1} = \varepsilon_{11} = C_1 x_1 + C_2 x_2 \text{ ,} \tag{c39}$$

where

$$C_1 = (1+\nu)[A - \nu(A-\gamma)]/E \quad , \quad C_2 = (1-\nu^2)B/E \qquad \text{(c40)}$$

It follows that

$$u_1 = C_1 x_1^2/2 + C_2 x_1 x_2 + f_1(x_2) \qquad \text{(c41)}$$

where the unknown function f_1 follows to be found. In the same manner,

$$u_2 = C_3 x_1 x_2 + C_4 x_2^2/2 + f_2(x_1) \qquad \text{(c42)}$$

But

$$\varepsilon_{12} = \frac{1}{2}\left(u_{1,2} + u_{2,1}\right) = \frac{1}{2}(C_2 x_1 + f_1'(x_2) + C_3 x_2 + f_2'(x_1)) = \frac{1+\nu}{E}\sigma_{12} = -\frac{1+\nu}{E}Cx_2 \qquad \text{(c43)}$$

Hence

$$\begin{cases} C_2 x_1 + f_2'(x_1) = K \\ C_3 x_2 + f_1'(x_2) = -K \end{cases} , \qquad \text{(c44)}$$

where K is an arbitrary constant. It follows

$$f_1(x_2) = -C_3 x_2^2/2 - Kx_2 + K_1 \quad , \quad f_2(x_1) = -C_2 x_1^2/2 + Kx_1 + K_2 \qquad \text{(c45)}$$

Hence, the displacement field is

$$\begin{cases} u_1 = C_1 x_1^2/2 + C_2 x_1 x_2 - [C_3 + 2(1+\nu)C/E]x_2^2/2 - Kx_2 + K_1 \\ u_2 = -C_2 x_1^2/2 + C_3 x_1 x_2 + C_4 x_2^2/2 + Kx_1 + K_2 \end{cases} \qquad \text{(c46)}$$

The last terms into (c46) represent a rigid roto-translation.
It should be outlined that the above boundary conditions on stress values on the sides OA and OB are not complete ones. As a result, the unknown constants C_3, C_4 are present in (c46). Boundary conditions on stress values (or displacements) on the side AB are required in order to obtain an unique solution of the problem
For example, consider the case when the points A and B are fixed ones. It follows

$$\begin{cases} u_1 = C_1\left(x_1^2 - h^2\right)/2 + C_2\left(x_1 - h\right)x_2 - [C_3 + 2(1+\nu)C/E]x_2\left(l - x_2\right)/2 \\ u_2 = C_2\left(h^2 - x_1^2\right)/2 + C_3 x_2\left(x_1 - x_2 h/l\right) + \left\{C_2 h - \left[C_3 + 2(1+\nu)C/E\right]l/2\right\}\left(x_1 - h\right) \end{cases} \qquad \text{(c47)}$$

An arbitrary point placed initially on the side AB has the initial co-ordinates $\left(X_1 = h; X_2\right)$. Its final position is

$$\begin{cases} x_1 = X_1 + u_1(X_1, X_2) = h + [C_3 + 2(1+\nu)C/E]X_2(l - X_2)/2 \\ x_2 = X_2 + u_2(X_1, X_2) = X_2 + C_3 h X_2 (l - X_2)/l \end{cases} \qquad \text{(c48)}$$

Elementary computations show that

$$C_3 + \frac{2(1+\nu)}{E}C = -\frac{1+\nu}{E}\left[\gamma(1-\nu) - \nu\Gamma - (2-\nu)\gamma\frac{h^2}{l^2}\right] \qquad \text{(c49)}$$

If

$$C_3 + \frac{2(1+\nu)}{E}C < 0 \quad , \qquad \text{(c50)}$$

the final shape of the side AB is a concave parabolic segment. Because the possibility of the water to flow below the dam, that situation is not recommended in real cases. Therefore, it is asked to

$$\gamma(1-\nu) - \nu\Gamma - (2-\nu)\gamma(h/l)^2 \le 0 \quad , \qquad \text{(c51)}$$

i.e.

$$h/l \ge \sqrt{\left[\gamma(1-\nu) - \nu\Gamma\right]/(2-\nu)/\gamma} \qquad \text{(c52)}$$

For example, assuming that $\gamma = 1000\,\text{Kgs}/m^3$, $\Gamma = 2400\,\text{Kgs}/m^3$, $\nu = 0.25$ it follows that $h \ge 0.29\,l$.
EXERCISE. Obtain the final shape of the dam in the above hypothesis.