H) ELEMENTS OF RHEOLOGY

H.1) Introduction.

Especially for geological processes at a large time scale and great values of the stress, the internal friction of the material cannot be ignored. Consequently, the HOOKE's law has to be replaced by assuming different constitutive equations (models). Their expressions are mainly depending on the time scale of the geophysical process to be modelled. In relation to seismic or seismological applications, for example, short periods and short stresses are required (usually, seconds up to minutes, with a maximum value around one hour for the fundamental mode of the free oscillations of the Earth). Here, the non-elasticity is related to the very short period irreversible changes in the crystal defect structures of the medium (e.g. opening/closing of pre-existing cracks) and/or to the energy lost by friction at the two sides of a crack or on the non-elastic boundary coupling grain particles to the adjacent material (Aki and Richards, 1980; Ranalli, 1987; Wahr, 1996).

With respect to the mathematical relation between stress and strain, there are two kinds of constitutive equations (models).

H.2) Linear models.

Simplified models involves a linear relation between stress (and its derivatives of various orders with respect to time) and strain (together with its time derivatives).

In the beginning, only the 1-D case will be discussed. More general examples follow to be presented in relation with the dynamic aspects of the flexure of a plate (shell) and to the accretion prism. For each constitutive equation, a mechanical analogue can be considered. The elastic part will be represented by a spring, while the inelastic (viscous) behaviour is associated to a dashpot. Both parts are supposed to be linear ones, i.e. a linear relation $\sigma = 2\mu\varepsilon$ is valid for the spring and a similar linear relation holds for the dashpot $\sigma = 2\eta\dot{\varepsilon}$.

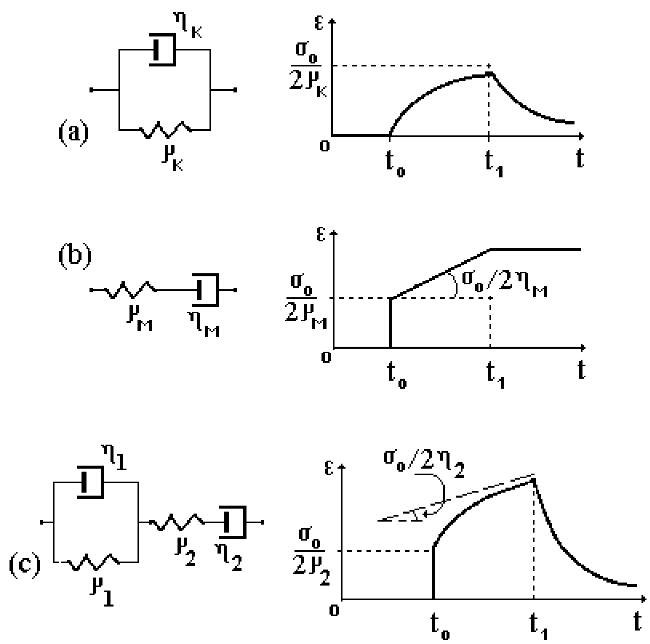

Fig.H1. (a) KELVIN-VOIGT model; (b) MAXWELL model; (c) BURGERS model

a) KELVIN-VOIGT (*strong viscous*) model.

The mechanical analogue of the first model to be considered is represented in Fig.H1a. The total stress is the sum between the stress in the spring and the stress in the dashpot, while the total deformation is equal to both the deformation of the spring and the deformation of the dashpot. It follows that KELVIN-VOIGT model has the next constitutive equation

$$\sigma = 2\mu_K \varepsilon + 2\eta_K \dot{\varepsilon} \ , \qquad \text{(h1)}$$

where the dot shows the (total, material) derivative with respect to time. Here, the second term is the inelastic one, η_K being the viscosity. Suppose now a constant stress equal to σ_0 is applied. Elementary computations show that the differential equation

$$\sigma_0 = 2\mu_K \varepsilon + 2\eta_K \frac{d\varepsilon}{dt} \tag{h2}$$

with the initial condition

$$\varepsilon\left(t = t_0\right) = 0 \tag{h3}$$

has the solution

$$\varepsilon = \frac{\sigma_0}{2\mu_K}\left[1 - \exp\left(-\frac{\mu_K}{\eta_K}(t - t_0)\right)\right] \quad , \tag{h4}$$

for $t \geq t_0$. Hence, for very great values of time, the strain approaches a limiting value equal to

$$\varepsilon_\infty = \frac{\sigma_0}{2\mu_K} \quad . \tag{h5}$$

The “flowage function”, denoted by

$$J(t) = \frac{1}{2\mu_K}\left[1 - \exp\left(-\frac{\mu_K}{\eta_K}(t - t_0)\right)\right] \quad , \tag{h6}$$

shows that for a constant stress (equal to unit here), there is a temporal variation of the strain.
Suppose now at a certain moment $t = t_1$, the constant stress σ_0 is removed, the corresponding strain at that moment being equal to ε_1. It follows now the corresponding solution decreases towards zero as

$$\varepsilon = \varepsilon_1 \exp\left(-\frac{\mu_K}{\eta_K}(t - t_1)\right) \quad . \tag{h7}$$

b) MAXWELL (*viscous-elastic*) model.

Consider the mechanical analogue represented in Fig.H1b. The total stress is equal to both the stress of the spring and to the stress of the dashpot, while the total deformation is the sum between the deformation of the spring and the deformation of the dashpot. It follows that MAXWELL model has the next constitutive equation

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{2\mu_M} + \frac{\sigma}{2\eta_M} \quad , \tag{h8}$$

with the initial condition represented by (h3).

Suppose again a constant stress equal to σ_0 is applied. The spring is instantly deformed to a value equal corresponding to the first term in the right hand of (h8), i.e.

$$\varepsilon_0 = \frac{\sigma_0}{2\mu_M} \quad . \tag{h9}$$

and the solution of (h8) (for a constant stress σ_0) with the initial condition (h9) is the straight line

$$\varepsilon = \frac{\sigma_0}{2\eta_M}\left(t - t_0\right) + \frac{\sigma_0}{2\mu_M} \quad , \tag{h10}$$

having the slope related to the stress σ_0 and to the viscosity of the dashpot. Suppose now at a certain moment $t = t_1$, the constant stress σ_0 is removed, the corresponding strain at that moment being equal to ε_1. It follows from (h8) that the strain remains constant.

c) BURGERS (*general linear*) model.

The third model to be considered has the mechanical analogue represented in Fig.H1c.
EXERCISE. Show that the corresponding differential equation is

$$2\eta_1 \ddot{\varepsilon} + 2\mu_1 \dot{\varepsilon} = \frac{\eta_1}{\mu_2}\ddot{\sigma} + \left(\frac{\eta_1}{\eta_2} + \frac{\mu_1}{\mu_2} + 1\right)\dot{\sigma} + \frac{\mu_1}{\eta_2}\sigma \quad . \tag{h11}$$

Consider now the same initial condition (h3) and suppose again a constant stress equal to σ_0 is applied. In a similar manner to MAXWELL model, the system is instantly deformed to a value equal to

$$\varepsilon_0 = \frac{\sigma_0}{2\mu_2} \tag{h12}$$

and the solution of (h11) (for a constant stress σ_0) with the initial condition (h12) is

$$\varepsilon = \frac{\sigma_0}{2\mu_2} + \frac{\sigma_0}{2\eta_2}(t - t_0) + C\left[1 - \exp\left(-\frac{\mu_1}{\eta_1}(t - t_0)\right)\right] \quad , \tag{h13}$$

where C is an unknown coefficient (because (h11) is a second order differential equation, two initial conditions are required to obtain the complete solution). However, differentiating (h13) it follows

$$\dot{\varepsilon} = \frac{\sigma_0}{2\eta_2} + C\frac{\mu_1}{\eta_1}\exp\left(-\frac{\mu_1}{\eta_1}(t - t_0)\right) \quad . \tag{h14}$$

Hence, for great values of time, the solution (h13) approaches asymptotically to a straight line having the slope equal to $\sigma_0 / 2\eta_2$. Suppose now at a certain moment $t = t_1$, the constant stress σ_0 is removed, the corresponding strain at that moment being equal to ε_1. It follows from (h13) that the strain decreases exponentially towards zero, i.e

$$\varepsilon = \varepsilon_1 \exp\left(-\frac{\mu_1}{\eta_1}(t - t_1)\right) \quad . \tag{h15}$$

EXERCISE. Show that two (or more) springs / dashpots connected in series (or parallel) sequence are equivalent to a single spring / dashpot. Justify that the BURGERS model is the general linear model.

d) Remarks on the linear models.

In the most general case, the linear relation between stress and strain can be written as

$$P(D)\sigma = Q(D)\varepsilon \quad , \tag{h16}$$

where

$$P(D) = A^0 + A^1 D + A^2 D^2 + \ldots + A^n D^n \tag{h17}$$

and

$$Q(D) = B^0 + B^1 D + B^2 D^2 + \ldots + B^m D^m \tag{h18}$$

are formal polynomials of the variable $D = \frac{d}{dt}$ representing the derivative with respect to time, applied to stress and strain respectively. Here, $A^0, A^1, \ldots, A^n, B^0, B^1, \ldots, B^m$ are fourth rank tensors. For example, with respect to the MAXWELL body having the constitutive equation (h8), it follows that

$$P(D) = \frac{1}{2\eta_M} + \frac{1}{2\mu_M} D \quad , \quad Q(D) = D \tag{h19}$$

A common way to solve (h16) is by using the LAPLACE transform (e.g. Sokolnikoff and Redheffer, 1958). Consider a certain function of one real variable $f(t)$, providing that

1. $f(t) = 0 \quad , \quad \text{for} \quad t < 0$;
2. $f(t)$ is piecewise continuous on every finite interval;
3. there are two constants $0 < M \quad , \quad a \le 0$ in order to have $|f(t)| \le M\exp(at)$, for an arbitrary t.

Under the above conditions, the LAPLACE transform of $f(t)$ is a new function of the variable p, defined by

$$L[f](p) = \int_0^\infty f(t)\exp(-pt)dt \tag{h20}$$

EXERCISE. Show that:

(a) The LAPLACE transform of the first derivative is

$$L\left[\frac{df}{dt}\right](p) = pL[f] - f(0+0) \tag{h21}$$

(b) $$L[\exp(-at)](p) = \frac{1}{p+a} \quad , \quad 0 \le a \tag{h22}$$

(c) The *convolution theorem*. Consider the functions f, g vanishing for negative values of their argument and a new function (also vanishing for negative values of the argument defined) by the *convolution product*

$$h(t) = (f * g)(t) = \int_{-\infty}^{+\infty} f(\tau) g(t-\tau) d\tau = \int_0^t f(\tau) g(t-\tau) d\tau \tag{h23}$$

Show that

$$L[f * g] = L[f] L[g] \tag{h24}$$

As an example, consider again the MAXWELL model by applying the LAPLACE transform to both sides of (h8). It follows that

$$pL[\boldsymbol{\varepsilon}] - \boldsymbol{\varepsilon}^0 = \frac{1}{2\mu_M}\left(pL[\boldsymbol{\sigma}] - \boldsymbol{\sigma}^0\right) + \frac{1}{2\eta_M} L[\boldsymbol{\sigma}] \quad , \tag{h25}$$

where $\boldsymbol{\sigma}^0 = \boldsymbol{\sigma}(0+0), \boldsymbol{\varepsilon}^0 = \boldsymbol{\varepsilon}(0+0)$ are the initial stress and strain respectively.

EXERCISE. By using the above properties of the LAPLACE transform, show that

$$\boldsymbol{\sigma}(t) = 2\mu_M \boldsymbol{\varepsilon}(t) - \frac{2\mu_M^2}{\eta_M} \int_0^t \boldsymbol{\varepsilon}(t) \exp\left[-\frac{\mu_M}{\eta_M}(t-\tau)\right] d\tau + \left(\boldsymbol{\sigma}^0 - 2\mu_M \boldsymbol{\varepsilon}^0\right) \exp\left(-\frac{\mu_M}{\eta_M} t\right) \tag{h26}$$

If the initial conditions are elastically coupled, i.e.

$$\boldsymbol{\sigma}^0 = 2\mu_M \boldsymbol{\varepsilon}^0 \tag{h27}$$

it follows an integral representation of the stress which is independent on the initial conditions

$$\boldsymbol{\sigma}(t) = 2\mu_M \boldsymbol{\varepsilon}(t) - \frac{2\mu_M^2}{\eta_M} \int_0^t \boldsymbol{\varepsilon}(t) \exp\left[-\frac{\mu_M}{\eta_M}(t-\tau)\right] d\tau \tag{h28}$$

Hence, the actual value of the stress is related both to the actual value of the deformation and to the previous values of the strain (i.e. the stress depends on the "history" of the deformation). If a constant strain $\boldsymbol{\varepsilon}(t) = \boldsymbol{\varepsilon}^0$, for $t \ge 0$ is applied to the MAXWELL body, it follows from (h28) that the stress decreases as

$$\boldsymbol{\sigma}(t) = 2\mu_M \boldsymbol{\varepsilon}^0 \exp\left(-\frac{\mu_M}{\eta_M} t\right) \quad , \tag{h29}$$

representing a "relaxation phenomenon" (stress decreases in time if a constant deformation is present). Here, the function

$$G(t) = 2\mu_M \exp(-t/\tau_M) \tag{h30}$$

is the "relaxation" kernel and the parameter $\tau_M = \eta_M / \mu_M$ is the relaxation time.

A very similar approach (e.g. Wahr 1996) is based on the use of FOURIER transform (see Chapter F.13). Formally, the results derived by using FOURIER transform are derived from the same results obtained with LAPLACE transform by performing the substitution $p = iu$.

The above 1-D models can be generalised for the 3-D case. For example, consider again the MAXWELL body. There is a strong experimental evidence that the MAXWELL Rheology applies only to the dissipation of the shear energy, i.e. the stress and strain tensors in (h8) are the deviatoric tensors

$$\boldsymbol{\sigma} \leftrightarrow \boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}\,\boldsymbol{\sigma}\,\mathbf{1} \quad , \quad \boldsymbol{\varepsilon} \leftrightarrow \boldsymbol{\varepsilon} - \frac{1}{3}\mathrm{tr}\,\boldsymbol{\varepsilon}\,\mathbf{1} \tag{h31}$$

It will be further assumed that there is no dissipation of the compressional energy, i.e. a proportionality like

$$\mathrm{tr}\,\boldsymbol{\sigma} = (3\lambda_M + 2\mu_M)\,\mathrm{tr}\,\boldsymbol{\varepsilon} \quad , \tag{h32}$$

is valid. By differentiating with respect to time

$$\mathrm{tr}\,\dot{\boldsymbol{\sigma}} = (3\lambda_M + 2\mu_M)\,\mathrm{tr}\,\dot{\boldsymbol{\varepsilon}} \tag{h33}$$

Substituting (h31) and (h33) in (h8) it follows

$$\dot{\boldsymbol{\varepsilon}} - \frac{1}{3}\mathrm{tr}\,\dot{\boldsymbol{\varepsilon}}\mathbf{1} = \frac{1}{2\mu_M}\left(\dot{\boldsymbol{\sigma}} - \frac{3\lambda_M + 2\mu_M}{3}\mathrm{tr}\,\dot{\boldsymbol{\varepsilon}}\mathbf{1}\right) + \frac{1}{2\eta_M}\left(\boldsymbol{\sigma} - \frac{3\lambda_M + 2\mu_M}{3}\mathrm{tr}\,\boldsymbol{\varepsilon}\mathbf{1}\right) \tag{h34}$$

Hence

$$\dot{\sigma}+\frac{\mu_M}{\eta_M}\sigma = 2\mu_M\dot{\varepsilon}+\lambda_M \operatorname{tr}\dot{\varepsilon}1+\frac{\mu_M}{\eta_M}\frac{3\lambda_M+2\mu_M}{3}\operatorname{tr}\varepsilon 1 \qquad \text{(h35)}$$

By applying the FOURIER transform to both sides of (h35), it follows

$$\Phi\left[\sigma\right]=\tilde{\lambda}\operatorname{tr}\Phi\left[\varepsilon\right]1+2\tilde{\mu}\Phi\left[\varepsilon\right] \quad , \qquad \text{(h36)}$$

i.e. a relation similar to HOOKE's law is valid between the FOURIER (or LAPLACE) transforms of stress and strain. Here, the coefficients similar to LAMÉ parameters are

$$\tilde{\lambda}=\lambda_M\frac{iu+\frac{\mu_M}{\eta_M}\left(1+\frac{2}{3}\frac{\mu_M}{\lambda_M}\right)}{iu+\frac{\mu_M}{\eta_M}} \quad , \quad \tilde{\mu}=\mu_M\frac{iu}{iu+\frac{\mu_M}{\eta_M}} \qquad \text{(h37)}$$

At short periods T correspond high values of the pulsation $u=2\pi/T$. From (h37) it follows

$$\tilde{\lambda}=\lambda_M \quad , \quad \tilde{\mu}=\mu_M \qquad \text{(h38)}$$

and the behaviour of the MAXWELL body is an elastic one.

At long periods T correspond low values of the pulsation and

$$\tilde{\lambda}=\frac{3\lambda_M+2\mu_M}{3}=K_M \quad , \quad \tilde{\mu}=0 \qquad \text{(h39)}$$

and the MAXWELL body is a fluid having the compressional coefficient denoted by K_M.

H.3) Non-linear models.

For tectonic applications, large stresses and periods of thousands to millions of years are appropriate. Here, the non-elastic behaviour is probably related to the diffusion or dislocation creep of the molecules, a major factor being the high temperatures.

There are great difficulties to consider constitutive equations with non-linear relations between stress and strain, but some attempts have been made. A very common non-linear model is the work-hardening plasticity (e.g. Ranalli, 1994)

$$\varepsilon_{ij}=\frac{3}{2}\left(\frac{\sigma_E^*}{\sigma_0}\right)^{n-1}\frac{\sigma_{ij}^*}{\sigma_0} \quad , \qquad \text{(h40)}$$

where σ_{ij}^* is the component of the deviatoric stress, $\sigma_E^*=\sqrt{\frac{3}{2}\sigma_{ij}^*\sigma_{ij}^*}$ is related to the second invariant of the deviatoric stress, and σ_0, n are material parameters.

H.4) Brittle. Creep. Empirical criteria.

The usual materials are reacting in an elastic manner only for small values of the (deviatoric) stress, i.e. for stress values smaller than a limiting value representing the yield strength (or the yield stress), denoted by σ_Y. The yield stress is a function of the nature of the material, of the temperature, pressure, the chemical composition of the adjacent rocks and, finally, of the history of the deformation (i.e. the intermediate steps followed to attend the yield value). When the yield stress is attended, there are two possibilities of behaviour of the material:

- a rupture deformation of the rock, when the continuity of the deformation is lost, usually along a fault surface; this is the case of the **brittle materials**. The process is illustrated in Fig.H2a and b.
- a plastic, irreversibly flow of the material (**creep**), when, apparently, the continuity holds. The phenomenon is quite similar to the usual viscous flow, but it can be observed only when the yield stress is attended. This is the case of the **ductile materials**. The process is illustrated in Fig.H2c. An usual non-linear constitutive equation is the BYERLEE power-law creep

$$\dot{\varepsilon}=A\sigma^n\exp(-H/RT) \quad , \qquad \text{(h41)}$$

where A, n are material parameters, H is the activation enthalpy, R is the gas constant and T is the absolute temperature. It should be noted that the same material can act as a brittle or a ductile one according to the external conditions.

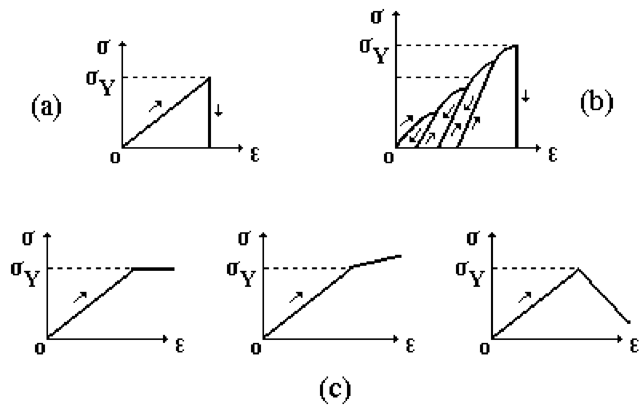

Fig.H2. (a) Faulting of a brittle material; (b) Increasing of the yield stress due to the history of deformation; (c) Creep of a ductile material.

H.5) Empirical criteria for shear-faulting. TRESCA criterion. COULOMB-NAVIER criterion.

A first criterion also (valid for plasticity) is due to TRESCA. It assumes that faulting (for brittle materials) or creep (for ductile ones) is attended at that points of the material where the maximum value of the shear (tangential) stress is equal to a yield value denoted by σ_Y. Consider now a homogeneous (constant) stress field inside the material. Such a case can be obtained either by considering an infinitesimal volume of material or by taking into account a prismatic body with very large (infinite) sides. Consider the eigen-values $\sigma_1, \sigma_2, \sigma_3$ of the stress tensor. It will be assumed that they are denoted in order to have $\sigma_3 < \sigma_2 < \sigma_1$, with the remark that ,in real life, stress is assumed to be positive for compression. Hence, with respect to Fig.H3, let $\sigma_{11} = -\sigma_1$, $\sigma_{22} = -\sigma_3$ and $\sigma_{12} = 0$ in eqs. (d11) and (d13). It follows that

$$\begin{aligned} \sigma &= -\sigma_{rr} = \frac{\sigma_1 + \sigma_3}{2} - \frac{\sigma_1 - \sigma_3}{2}\cos 2\psi \\ \tau &= -\sigma_{r\theta} = \frac{\sigma_1 - \sigma_3}{2}\sin 2\psi \end{aligned} , \qquad \text{(h42)}$$

where σ , τ are so-called “normal stress” and “tangential (shear) stress” respectively, acting on a plane inside the material. The plane is at an angle $\psi = \theta - \frac{\pi}{2}$ with the direction of the maximum compressive stress. The outer-pointing normal at that plane makes an angle θ with the direction of the maximum compressive stress. With respect to a $\sigma - \tau$ reference system, eqs.(h42) are the parametric equations of the MOHR circle (see Section D4), plotted in Fig. H3. In the most general case, stress field is varying from point to point inside the material. Hence both the eigen-vectors of the stress tensor (i.e. the local directions of the maximum / minimum compressive stress) and the eigen-values of that tensor (i.e. the magnitudes of the maximum / minimum compressive stress) are also changing from point to point. For a fixed point inside the material, both normal stress and shear (tangential) stress are varying with the angle between the plane (with respect to normal and tangential stresses are defined) and the direction of the local maximum compressive stress. Consider a certain point inside the material and imagine various planes passing through that point. Hence, according to TRESCA empirical criterion, failure of the material is produced here if

$$\max_{\psi} |\tau| = \sigma_Y \qquad \text{(h43)}$$

Using eq.(h42b), it follows

$$\max_{\psi} \left| \frac{\sigma_1 - \sigma_3}{2}\sin 2\psi \right| = \frac{\sigma_1 - \sigma_3}{2}\max_{\psi}|\sin 2\psi| = \sigma_Y \qquad \text{(h44)}$$

Consider now a homogeneous stressed material subject to progressive increasing values of the difference $\sigma_1 - \sigma_3$. The material is characterised by a yield value denoted by σ_Y. Eq.(h44) shows that:

- if $\left(\sigma_1 - \sigma_3\right)/2 < \sigma_Y$, there is no failure inside the material;
- when the equality

$$\sigma_1 - \sigma_3 = 2\sigma_Y \tag{h45}$$

is attended, a failure is produced along the planes at angles $\psi = \pm\frac{\pi}{4}$ with the direction of the maximum compressive stress.

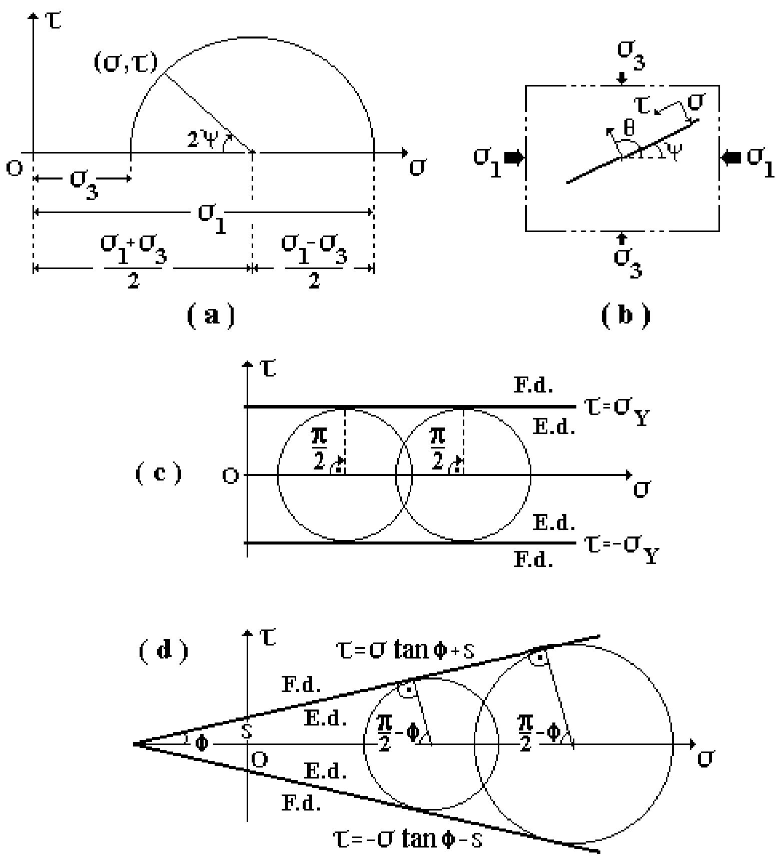

Fig.H3.(a) The MOHR circle;
(b) A plane inside the material, at an angle ψ with the direction of the maximum compressive stress;
(c) TRESCA criterion. E.d and F.D. denote the elastic domain and the failure domain respectively;
(d) COULOMB - NAVIER criterion.

Eq.(h45) represents the TRESCA criterion in terms of the eigenvalues of the stress tensor. In the case of a material subject to a non-homogeneous stress field, eq.(h45) is a local condition. Here, the eigen-values are obtained (see Section A). For a compressive stress, that values are expected to be negative ones. They have to be denoted by $-\sigma_1, -\sigma_2, -\sigma_3$, where $\sigma_3 < \sigma_2 < \sigma_1$.

A second criterion is due to COULOMB and NAVIER. It can be used to describe only the shear fracture. According to it, a material is characterised by the cohesive strength denoted by S and by the coefficient of friction, denoted by $\mu^* = \tan\phi$. Here, ϕ is the angle of internal friction ($\phi = 30^\circ$ in most rocks). According to COULOMB - NAVIER empirical criterion, shear failure of the material is produced at its points where

$$\max_{\psi}\left(\left|\ \tau\ \right| - \mu^* \sigma\right) = S \tag{h46}$$

Using eqs.(h42), it follows

$$\max_{\psi}\left[\frac{\sigma_1-\sigma_3}{2}\left|\sin 2\psi\right| - \tan\phi\left(\frac{\sigma_1+\sigma_3}{2} - \frac{\sigma_1-\sigma_3}{2}\cos 2\psi\right)\right] = S \qquad \text{(h47)}$$

If ψ is a solution of (h47), $\pi-\psi$ (or just $-\psi$) is a solution too. Hence, without loss of generality, the values of the angle ψ will be limited to the first quadrant, where eq.(h47) is

$$\frac{\sigma_1-\sigma_3}{2}\frac{1}{\cos\phi}\max_{\psi}\left[\sin(2\psi+\phi)\right] = S + \tan\phi\frac{\sigma_1+\sigma_3}{2} \qquad \text{(h48)}$$

Eq.(h48) shows that:

- if $\frac{\sigma_1-\sigma_3}{2}\frac{1}{\cos\phi} - \tan\phi\frac{\sigma_1+\sigma_3}{2} < S$, there is no failure of the material;
- when the equality

$$\frac{\sigma_1-\sigma_3}{2}\frac{1}{\cos\phi} - \tan\phi\frac{\sigma_1+\sigma_3}{2} = S \qquad \text{(h49)}$$

is attended, a share fracture is produced along the planes at angles $\psi = \pm\left(\frac{\pi}{4}-\frac{\phi}{2}\right)$ with the direction of the maximum compressive stress. Eq.(h49) represents the COULOMB - NAVIER criterion in terms of the eigenvalues of the stress tensor. Taking into account that $\mu^* = \tan\phi$, eq.(h49) can be written as

$$\sigma_1\left[\sqrt{\left(\mu^*\right)^2+1}-\mu^*\right] - \sigma_3\left[\sqrt{\left(\mu^*\right)^2+1}+\mu^*\right] = 2S \quad , \qquad \text{(h50)}$$

outlining that COULOMB - NAVIER criterion is a generalisation of TRESCA criterion for a non-zero internal friction.

H.6) Von MISES-HENCKY criterion for ductile flow (plasticity).

Because the ductile (plastic) flow is independent of the co-ordinate system used, it depends only on the invariants of the stress tensor (Section A, eq.(a22)). Hence an equation like

$$f\left(I_1, I_2, I_3\right) = 0 \quad , \qquad \text{(h51)}$$

will be valid. There are strong experimental evidence that the plastic flow does not depend on the hydrostatic pressure, being also similar for compressive and tensile states of stress. It follows the function f in (h51) depends only on the second invariant of the stress deviator (see eq.(a23)). Hence, ductile flow occurs only at those points of material where the second invariant of the deviator stress reaches a certain value, depending on the nature of the material. Using eq.(a24), the criterion of Von MISES - HENCKY assumes that ductile flow occurs at those points of the material where

$$\left(\sigma_{11}-\sigma_{22}\right)^2 + \left(\sigma_{22}-\sigma_{33}\right)^2 + \left(\sigma_{33}-\sigma_{11}\right)^2 + 6\left(\sigma_{12}^2+\sigma_{23}^2+\sigma_{13}^2\right) = 6k^2 \qquad \text{(h52)}$$

In terms of the principal stresses, eq.(h52) is

$$\left(\sigma_1-\sigma_2\right)^2 + \left(\sigma_2-\sigma_3\right)^2 + \left(\sigma_3-\sigma_1\right)^2 = 6k^2 \qquad \text{(h53)}$$

Hence the criterion of Von MISES - HENCKY can be regarded too as a generalisation of TRESCA criterion, by taking into account the presence of the intermediate stress.

H.7) Rheological models.

a) SAINT-VENANT body (elastic-plastic material).

The behaviour of that material is characterised by linear elasticity for stress values below the yield strength. When the yield stress is attended, the body exhibits a pure plasticity. Its constitutive equation (using deviator tensors) has the symbolic form

$$\begin{cases} \sigma^* = 2\mu_S \varepsilon^* & , \quad \sigma < \sigma_Y \\ f(I_2^*, I_3^*) = \sigma_Y & , \quad \sigma = \sigma_Y \end{cases} \tag{h54}$$

The above material has the mechanical analogue presented in Fig. (h4a), being referred as a SAINT-VENANT body.

b) BINGHAM body (visco-plastic material).

Similar to the SAINT-VENANT body, that material exhibits linear elasticity for stress values lower than the yield strength., but flows linearly above that value. The strain rate is proportional to the difference between the deviatoric stress and the yield strength. Its constitutive equation is

$$\begin{cases} \sigma^* = 2\mu_B \varepsilon^* & , \quad \sigma < \sigma_Y \\ \sigma^* = \sigma_Y + 2\eta_B \dfrac{d\varepsilon^*}{dt} & , \quad \sigma \geq \sigma_Y \end{cases} \tag{h55}$$

The above material has the mechanical analogue presented in Fig. (h4b), being referred as a BINGHAM body.

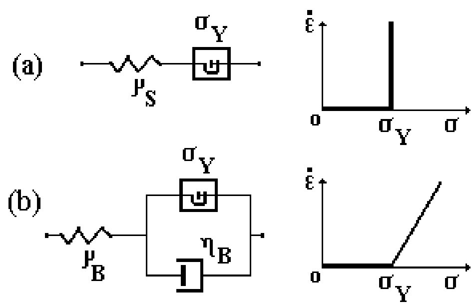

Fig.H4. (a) The SAINT-VENANT body; (b) The BINGHAM body.