G) THE SPHERICAL SHELL

G.1) The model. BERNOULLI's hypothesis. Displacement vector and strain tensor.

A spherical elastic, homogeneous shell having the elastic moduli equal to λ, μ is considered and the usual spherical co-ordinate system (r, θ, φ) is used. However, some of the derived results are also valid in the case of a more general, non-elastic spherical shell. In the initial state, the homogeneous density is denoted by ρ_0 and the median spherical surface of the shell has the equation

$$r = R \ , \tag{g1}$$

where R is the radius of the sphere. At a certain time during the deformation, the median surface will be

$$r = R - w(\theta, \varphi, t) \quad , \tag{g2}$$

where $w = w(\theta, \varphi, t)$ is the flexure of the shell, positive downward. Hence the unit vector normal to the median surface at a certain point of co-ordinates $(r = R - w, \theta, \varphi)$ is

$$\vec{n} = \left(\vec{e}_r + \frac{\partial w}{r \partial \theta} \vec{e}_\theta + \frac{1}{r \sin\theta} \frac{\partial w}{\partial \varphi} \vec{e}_\varphi \right) / \sqrt{1 + \frac{1}{r^2} \left[\left(\frac{\partial w}{\partial \theta} \right)^2 + \frac{1}{\sin^2\theta} \left(\frac{\partial w}{\partial \varphi} \right)^2 \right]} \ . \tag{g3}$$

Neglecting the quantities of the second order, it follows that

$$\vec{n} \cong \vec{e}_r + \frac{\partial w}{R \partial \theta} \vec{e}_\theta + \frac{1}{R \sin\theta} \frac{\partial w}{\partial \varphi} \vec{e}_\varphi \ . \tag{g4}$$

Consider the initial, non-deformed state of the shell and two points of co-ordinates $A_0(R, \theta, \varphi)$ and $B_0(R + h, \theta, \varphi)$, where H=2h is the thickness of the shell. It follows that the segment $\overrightarrow{A_0 B_0}$ has the unit vector equal to $\vec{e}_r$. At an arbitrary time, in the deformed state of the shell, the point A_0 is displaced to a point A having the position vector equal to

$$\vec{r}_A = R \vec{e}_r + \vec{u}(R, \theta, \varphi, t) \ , \tag{g5}$$

while the point B_0 is displaced to the point B having the position vector equal to

$$\vec{r}_B = (R + h) \vec{e}_r + \vec{u}(R + h, \theta, \varphi, t) \ . \tag{g6}$$

Here, $\vec{u}$ is the displacement vector at a point of certain spherical co-ordinates. Assuming the shell is thin, quantities of the order h^2 are neglected and the segment $\overrightarrow{AB}$ has the unit vector equal to

$$\left(\vec{r}_B - \vec{r}_A \right) / \left| \vec{r}_B - \vec{r}_A \right| \cong \left(1 + \frac{\partial u_r}{\partial r} \right) \vec{e}_r + \frac{\partial u_\theta}{\partial r} \vec{e}_\theta + \frac{\partial u_\varphi}{\partial r} \vec{e}_\varphi \ . \tag{g7}$$

The partial derivatives in (g7) are computed at the point (R, θ, φ). It is assumed that Bernoulli's hypothesis is valid for all time. It follows that a segment inside the shell, which is initially normal to the median spherical surface, will be always normal to the median surface during the deformation. From (g4) and (g7), it is supposed that the displacement vector has the elements

$$\begin{cases} u_r(r, \theta, \varphi, t) = -w(\theta, \varphi, t) \\ u_\theta(r, \theta, \varphi, t) = \dfrac{r - R}{R} \dfrac{\partial w}{\partial \theta} + x(\theta, \varphi, t) \\ u_\varphi(r, \theta, \varphi, t) = \dfrac{r - R}{R} \dfrac{1}{\sin\theta} \dfrac{\partial w}{\partial \varphi} + y(\theta, \varphi, t) \end{cases} \ , \tag{g8}$$

where x and y are two unknown functions representing the horizontal displacements of the points initially placed on the median sphere. A further hypothesis on x and y will be later considered. It follows the elements of the strain tensor are

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = 0 \ , \tag{g9}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) = -\frac{x}{2r} \ , \tag{g10}$$

$$\varepsilon_{r\varphi} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \varphi} - \frac{u_\varphi}{r} + \frac{\partial u_\varphi}{\partial r}\right) = -\frac{y}{2r} \ , \tag{g11}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) = \frac{r-R}{rR}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\left(\frac{\partial x}{\partial \theta} - w\right) \ , \tag{g12}$$

$$\varepsilon_{\theta\varphi} = \frac{1}{2r}\left(\frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial \varphi} - \frac{u_\varphi}{\tan\theta} + \frac{\partial u_\varphi}{\partial \theta}\right) = \frac{r-R}{rR}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial w}{\partial \varphi}\right) + \frac{1}{2r}\left(\frac{1}{\sin\theta}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial \theta} - \frac{y}{\tan\theta}\right) \ , \tag{g13}$$

and

$$\varepsilon_{\varphi\varphi} = \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + u_r + \frac{u_\theta}{\tan\theta}\right) = \frac{r-R}{rR}\left(\frac{1}{\sin^2\theta}\frac{\partial^2 w}{\partial \varphi^2} + \frac{1}{\tan\theta}\frac{\partial w}{\partial \theta}\right) + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial y}{\partial \varphi} - w + \frac{x}{\tan\theta}\right) \tag{g14}$$

Hence the trace of the strain tensor is

$$\mathrm{tr}\,\varepsilon = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\varphi\varphi} = \frac{R(r-R)}{r}\Delta^* w + \frac{1}{r\sin\theta}\left[-2w\sin\theta + \frac{\partial}{\partial \theta}(x\sin\theta) + \frac{\partial y}{\partial \varphi}\right] \ , \tag{g15}$$

where the two-dimensional LAPLACE operator is

$$\Delta^* w = \frac{1}{R^2\sin\theta}\left[\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial w}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 w}{\partial \varphi^2}\right] \ . \tag{g16}$$

G.2) Quasi-mean values. Equations of motion.

Consider the stress tensor in spherical co-ordinates. For each element of the tensor, the corresponding quasi-mean value is defined, for example, by

$$\Sigma_{rr} = \overline{\sigma_{rr}} = \frac{1}{H}\int_{R-h}^{R+h} \frac{r}{R}\sigma_{rr}\,dr \ . \tag{g17}$$

In the same way, the corresponding quasi-moment is defined by

$$M_{rr} = \frac{1}{H}\int_{R-h}^{R+h} \frac{r}{R}(r-R)\sigma_{rr}\,dr \ . \tag{g18}$$

The equations of motion in spherical co-ordinates are

$$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + 2\frac{\sigma_{rr}}{r} + \frac{\partial \sigma_{r\theta}}{r\partial \theta} + \frac{\sigma_{r\theta}}{r\tan\theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{r\varphi}}{\partial \varphi} - \frac{\sigma_{\theta\theta} + \sigma_{\varphi\varphi}}{r} - \rho g = \rho\frac{\partial^2 u_r}{\partial t^2} \\
&\frac{\partial \sigma_{r\theta}}{\partial r} + 3\frac{\sigma_{r\theta}}{r} + \frac{\partial \sigma_{\theta\theta}}{r\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\theta\varphi}}{\partial \varphi} + \frac{\sigma_{\theta\theta} - \sigma_{\varphi\varphi}}{r\tan\theta} = \rho\frac{\partial^2 u_\theta}{\partial t^2} \\
&\frac{\partial \sigma_{r\varphi}}{\partial r} + 3\frac{\sigma_{r\varphi}}{r} + \frac{\partial \sigma_{\theta\varphi}}{r\partial \theta} + \frac{2\sigma_{\theta\varphi}}{r\tan\theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} = \rho\frac{\partial^2 u_\varphi}{\partial t^2}
\end{aligned} \ . \tag{g19}$$

In the above equations, ρ denotes the density of the shell in the deformed state. Assumed to be a negligible second order effect, that density will be replaced by the initial density ρ_0 at the right side of (g19).

The first equation in (g19) will be multiplied by r^2 and the quasi-mean operator defined by (g17) will be applied. The next two equations in (g19) will be multiplied by r and the quasi-moment operator defined by (g18) will be applied. In order to do that, some intermediary results are necessary.

G.3) Integrals of the stress elements. Quasi-moments.

Let the stress values on the upper/lower faces of the shell be denoted as

$$\sigma^U = \sigma \ (R+h, \theta, \varphi, t) \tag{g20}$$

and, respectively, by

$$\sigma^L = \sigma \ (R-h, \theta, \varphi, t) \quad . \tag{g21}$$

Elementary computations show that

$$\frac{1}{H}\frac{1}{R}\int_{R-h}^{R+h}\left(r^2\frac{\partial \sigma_{rr}}{\partial r}+2r\sigma_{rr}\right)dr=\frac{R}{H}\left[\left(1+\frac{h}{R}\right)^2\sigma_{rr}^U-\left(1-\frac{h}{R}\right)^2\sigma_{rr}^L\right] \quad . \tag{g22}$$

In the case of the elastic shell, the HOOKE's law leads to

$$\sigma_{r\theta}=2\mu\,\varepsilon_{r\theta}=-\mu x/r \quad , \quad \Sigma_{r\theta}=-\mu x/R \quad . \tag{g23}$$

Integrating by parts, it follows that

$$\begin{aligned}\frac{1}{H}\int_{R-h}^{R+h}\frac{r^2}{R}(r-R)\frac{\partial \sigma_{r\theta}}{\partial r}dr &= \frac{1}{HR}\left\{\left[r^2(r-R)\sigma_{r\theta}\right]_{R-h}^{R+h}+\mu x\int_{R-h}^{R+h}(3r-2R)dr\right\} \\ &=\frac{R}{2}\left[\left(1+\frac{h}{R}\right)^2\sigma_{r\theta}^U+\left(1-\frac{h}{R}\right)^2\sigma_{r\theta}^L\right]-R\,\Sigma_{r\theta}\end{aligned} \quad . \tag{g24}$$

Also,

$$\frac{1}{H}\int_{R-h}^{R+h}\frac{r^2}{R}(r-R)\frac{3\sigma_{r\theta}}{r}dr=0 \quad . \tag{g25}$$

Similar relations are derived for $\sigma_{r\varphi}$.

Also,

$$M_{\theta\theta}=\left(\lambda\Delta^* w+\frac{2\mu}{R^2}\frac{\partial^2 w}{\partial\theta^2}\right)\frac{1}{H}\int_{R-h}^{R+h}(r-R)^2\,dr=\frac{h^2}{3}\left(\lambda\Delta^* w+\frac{2\mu}{R^2}\frac{\partial^2 w}{\partial\theta^2}\right) \quad . \tag{g26}$$

In the same way, it follows that

$$M_{\theta\varphi}=\frac{h^2}{3}\frac{2\mu}{R^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial w}{\partial\varphi}\right) \quad , \tag{g27}$$

and

$$M_{\varphi\varphi}=\frac{h^2}{3}\left[(\lambda+2\mu)\Delta^* w-\frac{2\mu}{R^2}\frac{\partial^2 w}{\partial\theta^2}\right] \quad . \tag{g28}$$

G.4) Integrals of the displacement vector.

Using (g8), it follows

$$\frac{1}{H}\int_{R-h}^{R+h}\frac{r^2}{R}\rho_0\frac{\partial^2 u_r}{\partial t^2}dr=-\rho_0 R\left[1+\frac{1}{3}\left(\frac{h}{R}\right)^2\right]\frac{\partial^2 w}{\partial t^2} \quad . \tag{g29}$$

Also,

$$\frac{1}{H}\int_{R-h}^{R+h}\frac{r^2(r-R)}{R}\rho_0\frac{\partial^2 u_\theta}{\partial t^2}dr=\rho_0 R^2\frac{\partial^2}{\partial t^2}\left[\left(\frac{h^2}{3R^2}+\frac{h^4}{5R^4}\right)\frac{\partial w}{\partial\theta}+\frac{2x}{3}\left(\frac{h}{R}\right)^2\right] , \qquad (g30)$$

and

$$\frac{1}{H}\int_{R-h}^{R+h}\frac{r^2(r-R)}{R}\rho_0\frac{\partial^2 u_\varphi}{\partial t^2}dr=\rho_0 R^2\frac{\partial^2}{\partial t^2}\left[\left(\frac{h^2}{3R^2}+\frac{h^4}{5R^4}\right)\frac{1}{\sin\theta}\frac{\partial w}{\partial\varphi}+\frac{2y}{3}\left(\frac{h}{R}\right)^2\right] . \qquad (g31)$$

G.5) Equations of motion in quasi-mean values.

Let the first equation in (g19) be multiplied by r^2. By applying the quasi-mean operator defined by (g17), it follows

$$\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,\Sigma_{r\theta}\right)+\frac{\partial\Sigma_{r\varphi}}{\partial\varphi}\right]-\left(\Sigma_{\theta\theta}+\Sigma_{\varphi\varphi}\right)-\frac{1}{RH}\int_{R-h}^{R+h}\rho r^2 g(r)dr$$
$$+\frac{R}{H}\left[(1+h/R)^2\sigma_{rr}^U-(1-h/R)^2\sigma_{rr}^L\right]=-\rho_0 R\left[1+\frac{1}{3}\left(\frac{h}{R}\right)^2\right]\frac{\partial^2 w}{\partial t^2} . \qquad (g32)$$

The next two equations in (g19) are multiplied by r and the quasi-moment operator defined by (g18) is applied. Hence

$$\frac{1}{R}\left(\frac{\partial M_{\theta\theta}}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial M_{\theta\varphi}}{\partial\varphi}\right)-\Sigma_{r\theta}+\frac{M_{\theta\theta}-M_{\varphi\varphi}}{R\tan\theta}+\frac{1}{2}\left[(1+h/R)^2\sigma_{r\theta}^U+(1-h/R)^2\sigma_{r\theta}^L\right]$$
$$=\rho_0 R\frac{\partial^2}{\partial t^2}\left\{\left[\frac{1}{3}\left(\frac{h}{R}\right)^2+\frac{1}{5}\left(\frac{h}{R}\right)^4\right]\frac{\partial w}{\partial\theta}+\frac{2x}{3}\left(\frac{h}{R}\right)^2\right\} \qquad (g33)$$

and

$$\frac{1}{R}\left(\frac{\partial M_{\theta\varphi}}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial M_{\varphi\varphi}}{\partial\varphi}\right)-\Sigma_{r\varphi}+\frac{2M_{\theta\varphi}}{R\tan\theta}+\frac{1}{2}\left[(1+h/R)^2\sigma_{r\varphi}^U+(1-h/R)^2\sigma_{r\varphi}^L\right]$$
$$=\rho_0 R\frac{\partial^2}{\partial t^2}\left\{\left[\frac{1}{3}\left(\frac{h}{R}\right)^2+\frac{1}{5}\left(\frac{h}{R}\right)^4\right]\frac{1}{\sin\theta}\frac{\partial w}{\partial\varphi}+\frac{2y}{3}\left(\frac{h}{R}\right)^2\right\} . \qquad (g34)$$

G.6) Quasi-mean value of the shell density. The differential equation.

According to (g17), the quasi-mean of a constant is equal to that constant. Applying the quasi-mean operator to the mass balance equation in the linear approximation gives

$$\bar{\rho}=\rho_0(1-\overline{\mathrm{tr}\,\varepsilon}) . \qquad (g35)$$

It will be further assumed that the quasi-mean of the density is constant and equal to its initial value. From (g35) it follows that

$$\overline{\mathrm{tr}\,\varepsilon}=0 , \qquad (g36)$$

i.e., using (g15),

$$\frac{\partial}{\partial\theta}(x\sin\theta)+\frac{\partial y}{\partial\varphi}=2w\sin\theta . \qquad (g37)$$

For the elastic shell,

$$\sigma=\lambda\,\mathrm{tr}\,\varepsilon\;\mathbf{1}+2\mu\,\varepsilon . \qquad (g38)$$

By using (g9), it follows

$$\sigma_{\theta\theta} + \sigma_{\varphi\varphi} = 2(\lambda+\mu)\mathrm{tr}\,\varepsilon \quad , \tag{g39}$$

or, using (g36),

$$\Sigma_{\theta\theta} + \Sigma_{\varphi\varphi} = 0 \quad . \tag{g40}$$

The term $(h/R)^4$ will be neglected. Using (g40) and substituting $\Sigma_{r\theta}$ and $\Sigma_{r\varphi}$ between (g32)-(g34) it follows

$$\begin{aligned}&\frac{1}{R^2}\left[\frac{\partial^2(\sin\theta M_{\theta\theta})}{\partial\theta^2} + \frac{2}{\sin\theta}\frac{\partial^2(\sin\theta M_{\theta\varphi})}{\partial\theta\partial\varphi} + \frac{1}{\sin\theta}\frac{\partial^2 M_{\varphi\varphi}}{\partial\varphi^2} - \frac{\partial}{\partial\theta}(\cos\theta M_{\varphi\varphi})\right] \\ &+\frac{\sin\theta}{H}\left[(1+h/R)^2\sigma^U_{rr} - (1-h/R)^2\sigma^L_{rr}\right] + \frac{1}{2R}\frac{\partial}{\partial\theta}\left\{\sin\theta\left[(1+h/R)^2\sigma^U_{r\theta} + (1-h/R)^2\sigma^L_{r\theta}\right]\right\} \\ &+\frac{1}{2R}\frac{\partial}{\partial\varphi}\left[(1+h/R)^2\sigma^U_{r\varphi} + (1-h/R)^2\sigma^L_{r\varphi}\right] - \frac{1}{HR}\int_{R-h}^{R+h}\rho r^2 g dr \\ &\qquad = \rho_0\sin\theta\left\{\frac{h^2}{3}\Delta^*\left(\frac{\partial^2 w}{\partial t^2}\right) - \left[1-\left(\frac{h}{R}\right)^2\right]\frac{\partial^2 w}{\partial t^2}\right\}\end{aligned} \tag{g41}$$

A correction due to the compressive horizontal stresses σ^{NS}, σ^{WE} acting at the ends of the shell, approximately along the $\theta-$ and $\varphi-$ axes respectively, follows to be further considered.

G.7) The buckling of a spherical shell.

Consider the element ABCD of the mean surface of the deformed shell from Fig.G1. It is centred at the point M, where the local unit vectors are $\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi$. The centres of the lateral sides are denoted by M_k , $k = 1,2,3,4$. On the meridian cross section M_4MM_2 is acting a normal compressive stress σ^{WE}, having the approximate direction from West to East, and a tangential stress τ^{NS}, having the approximate direction from North to South. On the parallel cross section M_1MM_3 is acting a normal compressive stress σ^{NS}, having the approximate direction from North to South, and a tangential stress τ^{WE}, having the approximate direction from West to East. Let Φ be the angle between the normal vector to the meridian section M_4MM_2 and the unit vector $\vec{e}_\varphi$. Also, let Θ be the angle between the normal vector to the parallel section M_1MM_3 and the unit vector $\vec{e}_\theta$. The concentrated force acting at the point M_1 is

$$\begin{aligned}\vec{\Sigma}_1 = &\left\{\left(\sigma^{WE} - \frac{\partial\sigma^{WE}}{\partial\varphi}\frac{d\varphi}{2}\right)\left[\cos\left(\Phi - \frac{\partial\Phi}{\partial\varphi}\frac{d\varphi}{2}\right)\vec{e}_\varphi + \sin\left(\Phi - \frac{\partial\Phi}{\partial\varphi}\frac{d\varphi}{2}\right)\vec{e}_r\right]\right. \\ &\left.+\left(\tau^{NS} - \frac{\partial\tau^{NS}}{\partial\varphi}\frac{d\varphi}{2}\right)\left[\cos\left(\Theta - \frac{\partial\Theta}{\partial\varphi}\frac{d\varphi}{2}\right)\vec{e}_\theta + \sin\left(\Theta - \frac{\partial\Theta}{\partial\varphi}\frac{d\varphi}{2}\right)\vec{e}_r\right]\right\}\times\left(R - w + \frac{\partial w}{\partial\varphi}\frac{d\varphi}{2}\right)Hd\theta\end{aligned} \tag{g42}$$

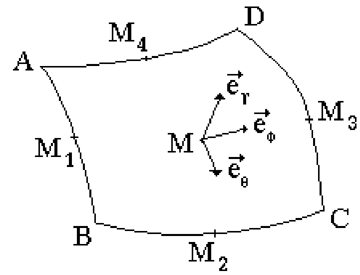

Fig.G1. A rectangular element of the mean surface of the deformed shell

In the same way, the concentrated force acting at the point M_3 is equal to

$$\vec{\Sigma}_3 = -\left\{\left(\sigma^{WE} + \frac{\partial \sigma^{WE}}{\partial \varphi}\frac{d\varphi}{2}\right)\left[\cos\left(\Phi + \frac{\partial \Phi}{\partial \varphi}\frac{d\varphi}{2}\right)\vec{e}_\varphi + \sin\left(\Phi + \frac{\partial \Phi}{\partial \varphi}\frac{d\varphi}{2}\right)\vec{e}_r\right]\right.$$
$$\left.+\left(\tau^{NS} + \frac{\partial \tau^{NS}}{\partial \varphi}\frac{d\varphi}{2}\right)\left[\cos\left(\Theta + \frac{\partial \Theta}{\partial \varphi}\frac{d\varphi}{2}\right)\vec{e}_\theta + \sin\left(\Theta + \frac{\partial \Theta}{\partial \varphi}\frac{d\varphi}{2}\right)\vec{e}_r\right]\right\} \times \left(R - w - \frac{\partial w}{\partial \varphi}\frac{d\varphi}{2}\right) H d\theta \tag{g43}$$

The concentrated forces acting at the points M_2 and M_4 are respectively equal to

$$\vec{\Sigma}_{2,4} = \mp\left\{\left(\sigma^{NS} \pm \frac{\partial \sigma^{NS}}{\partial \theta}\frac{d\theta}{2}\right)\left[\cos\left(\Theta \pm \frac{\partial \Theta}{\partial \theta}\frac{d\theta}{2}\right)\vec{e}_\theta + \sin\left(\Theta \pm \frac{\partial \Theta}{\partial \theta}\frac{d\theta}{2}\right)\vec{e}_r\right]\right.$$
$$\left.+\left(\tau^{WE} \pm \frac{\partial \tau^{WE}}{\partial \theta}\frac{d\theta}{2}\right)\left[\cos\left(\Phi \pm \frac{\partial \Phi}{\partial \theta}\frac{d\theta}{2}\right)\vec{e}_\varphi + \sin\left(\Phi \pm \frac{\partial \Phi}{\partial \theta}\frac{d\theta}{2}\right)\vec{e}_r\right]\right\} \tag{g44}$$
$$\times \left(R - w \mp \frac{\partial w}{\partial \theta}\frac{d\theta}{2}\right)\sin(\theta \pm d\theta) H d\varphi$$

But

$$\cos\Phi \cong 1 \quad , \quad \cos\Theta \cong 1 \quad , \quad \sin\Phi \cong \Phi \cong \frac{1}{R\sin\theta}\frac{\partial w}{\partial \varphi} \quad , \quad \sin\Theta \cong \Theta \cong \frac{1}{R}\frac{\partial w}{\partial \theta} \quad . \tag{g45}$$

Let $p = p(\theta, \varphi)$ a surface density of forces normal to the element of the shell, having the same mechanical effect as the presence of the compressive stress. The force due to that density is equal to

$$\vec{\Sigma}_p = p(\theta, \varphi)(R - w)^2 \sin\theta d\theta d\varphi \, \vec{e}_r \quad . \tag{g46}$$

It follows the deformed element of the shell is into an equilibrium state due to the action of the lateral stress and to the opposite force $-\vec{\Sigma}_p$, i.e.

$$\vec{\Sigma}_1 + \vec{\Sigma}_2 + \vec{\Sigma}_3 + \vec{\Sigma}_4 - \vec{\Sigma}_p = \vec{0} \quad . \tag{g47}$$

Hence the next three equations of equilibrium are obtained:

$$\left(2\cos\theta-\frac{\sin\theta}{R}\frac{\partial w}{\partial\theta}\right)\left(\sigma^{NS}\frac{\partial w}{\partial\theta}+\frac{\tau^{WE}}{\sin\theta}\frac{\partial w}{\partial\varphi}\right)$$
$$+\sin\theta\frac{\partial}{\partial\theta}\left(\sigma^{NS}\frac{\partial w}{\partial\theta}+\frac{\tau^{WE}}{\sin\theta}\frac{\partial w}{\partial\varphi}\right)+\frac{\partial}{\partial\varphi}\left(\frac{\sigma^{WE}}{\sin\theta}\frac{\partial w}{\partial\varphi}+\tau^{NS}\frac{\partial w}{\partial\theta}\right)=-p(\theta,\varphi)\frac{R(R-w)}{H}\sin\theta \quad , \tag{g48}$$

$$\sigma^{NS}\left(2\cos\theta-\frac{\sin\theta}{R}\frac{\partial w}{\partial\theta}\right)+\sin\theta\frac{\partial\sigma^{NS}}{\partial\theta}+\frac{\partial\tau^{NS}}{\partial\varphi}=0 \quad , \tag{g49}$$

and

$$\frac{\partial\sigma^{WE}}{\partial\varphi}+\tau^{WE}\left(2\cos\theta-\frac{\sin\theta}{R}\frac{\partial w}{\partial\theta}\right)+\sin\theta\frac{\partial\tau^{WE}}{\partial\theta}=0 \quad . \tag{g50}$$

For the particular case when σ^{WE} is a constant and $\tau^{NS}=\tau^{WE}=0$, eqs.(g48)-(g50) show that the presence of the lateral compressive stress is equivalent to a supplemental load placed on the upper face of the shell, having the value

$$p(\theta,\varphi)=-\frac{H}{R^2}\left(\sigma^{NS}\frac{\partial^2 w}{\partial\theta^2}+\sigma^{WE}\frac{1}{\sin^2\theta}\frac{\partial^2 w}{\partial\varphi^2}\right) \quad , \tag{g51}$$

a result similar to the case of the plane plate (Timoshenko and Woinowsky-Krieger 1959; Nowacki 1961). Quantities of the second order, like $(\partial w/\partial\theta)^2$ have been neglected again.

G.8) Load on the upper face. Stress on the lower surface of the shell.

Consider now the differential equation (g41). Usually, the horizontal loads for the upper face of the shell are neglected, and it is assumed that

$$\sigma^U_{rr}=-\rho^F gw-P\left(1-\frac{1}{g}\frac{\partial^2 w}{\partial t^2}\right) \quad , \quad \sigma^U_{r\theta}=0 \quad , \quad \sigma^U_{r\varphi}=0 \quad , \tag{g52}$$

where ρ^F is the density of the filling sediments, P is the load and an inertial term is considered. For the material below the shell, the next constitutive equation is assumed

$$\sigma^M=\left[p_0-\rho^M g(R-h-w)\right]\mathbf{1}+\lambda^M \mathrm{tr}\,\varepsilon^M\mathbf{1}+2\mu^M\varepsilon^M+2\eta^M\frac{\partial\varepsilon^M}{\partial t} \quad , \tag{g53}$$

where p_0 is a reference pressure, ρ^M is the density of that material and $\mathbf{1}$ is the unit tensor. The elastic coefficients are λ^M,μ^M and the viscosity is denoted by η^M. The strain tensor inside the material is ε^M and the strain rate here is $\partial\varepsilon^M/\partial t$.

The first boundary condition assumed on the lower face of the shell, having the equation $r=R-h$, is the continuity of the displacement vector. Hence the elements of the displacement vector inside the material placed immediately below the lower face of the shell are assumed to be equal to the same elements at the points of the shell placed on the lower face. By using eqs. (g8), it follows

$$\begin{cases} u_r(r,\theta,\varphi,t)=-w(\theta,\varphi,t) \\ u_\theta(r,\theta,\varphi,t)=\dfrac{-h}{R}\dfrac{\partial w}{\partial\theta}+x(\theta,\varphi,t) \\ u_\varphi(r,\theta,\varphi,t)=\dfrac{-h}{R}\dfrac{1}{\sin\theta}\dfrac{\partial w}{\partial\varphi}+y(\theta,\varphi,t) \end{cases} \quad . \tag{g54}$$

Hence the next values for the strain tensor immediately below the shell are obtained

$$\varepsilon_{rr}^{M} = 0 \quad , \quad \varepsilon_{r\theta}^{M} = -\frac{1}{2R}\frac{\partial w}{\partial \theta} - \frac{x}{2(R-h)} \quad , \quad \varepsilon_{r\varphi}^{M} = -\frac{1}{2R}\frac{1}{\sin\theta}\frac{\partial w}{\partial \varphi} - \frac{y}{2(R-h)} \quad . \qquad \text{(g55)}$$

The normal vector at the lower face of the shell is $\vec{e}_r$. From Newton's third low, it follows a relation between the stress σ^S inside the shell and the stress σ^M inside the material below the shell, i.e.

$$\sigma^S\left(-\vec{e}_r\right) = -\sigma^M\left(\vec{e}_r\right) \quad , \qquad \text{(g56)}$$

on the lower surface of the shell. Here, the next elements of the stress are obtained after some elementary computations

$$\sigma_{rr}^{L} = p_0 - \rho^M g(R-h-w) - \frac{Rh}{R-h}\lambda^M \Delta^* w \quad , \qquad \text{(g57)}$$

$$\sigma_{r\theta}^{L} = -\mu^M\left(\frac{1}{R}\frac{\partial w}{\partial \theta} + \frac{x}{R-h}\right) - \eta^M\left[\frac{1}{R}\frac{\partial}{\partial \theta}\left(\frac{\partial w}{\partial t}\right) + \frac{1}{R-h}\frac{\partial x}{\partial t}\right] \quad , \qquad \text{(g58)}$$

and

$$\sigma_{r\varphi}^{L} = -\mu^M\left(\frac{1}{R\sin\theta}\frac{\partial w}{\partial \varphi} + \frac{y}{R-h}\right) - \eta^M\left[\frac{1}{R\sin\theta}\frac{\partial}{\partial \varphi}\left(\frac{\partial w}{\partial t}\right) + \frac{1}{R-h}\frac{\partial y}{\partial t}\right] \quad . \qquad \text{(g59)}$$

G.9) The differential equation of time dependent flexure.

The reference pressure p_0 in (g53) is selected in order the flexure of the shell to vanish in the absence of the load. Substituting the loads on the upper/lower faces of the shell in eq.(s41) and taking into account the presence of the lateral gstress, a generalisation of the Sophie GERMAIN plain plate static equation in the case of a time dependent flexure of a spherical elastic shell is obtained as

$$\begin{aligned}
&\frac{H^3}{12}\left[(\lambda+2\mu)\Delta^*\Delta^* w + 2\mu\frac{\Delta^* w}{R^2}\right] - \left(1+\frac{h}{R}\right)^2\left[\rho^F g w + P\left(1-\frac{1}{g}\frac{\partial^2 w}{\partial t^2}\right)\right] \\
&+\left(1-\frac{h}{R}\right)^2\left(\rho^M g w + \frac{Rh}{R-h}\lambda^M\Delta^* w\right) - \frac{H}{2R}\left(1-\frac{h}{R}\right)^2\mu^M\left(R\Delta^* w + \frac{2w}{R-h}\right) \\
&-\frac{H}{2R}\left(1-\frac{h}{R}\right)^2\eta^M\left[R\Delta^*\left(\frac{\partial w}{\partial t}\right) + \frac{2}{R-h}\frac{\partial w}{\partial t}\right] + \frac{H}{R^2}\left(\sigma^{NS}\frac{\partial^2 w}{\partial \theta^2} + \sigma^{WE}\frac{1}{\sin^2\theta}\frac{\partial^2 w}{\partial \varphi^2}\right) \quad . \qquad \text{(g60)} \\
&\quad = \rho_0 H\left\{\frac{H^2}{12}\Delta^*\left(\frac{\partial^2 w}{\partial t^2}\right) - \left[1-\left(\frac{h}{R}\right)^2\right]\frac{\partial^2 w}{\partial t^2}\right\}
\end{aligned}$$

If quantities of the order $(h/R)^2$ are neglected with respect to unit, it follows finally that

$$\begin{aligned}
&D\Delta^*\Delta^* w + \frac{H}{2}\left(\lambda^M - \mu^M\right)\Delta^* w + \left(\rho^M - \rho^F - \rho^*\right)g w \\
&+H\left[\left(\sigma^{NS} + \sigma^*\right)\frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2} + \left(\sigma^{WE} + \sigma^*\right)\frac{1}{R^2\sin^2\theta}\frac{\partial^2 w}{\partial \varphi^2}\right] \quad , \qquad \text{(g61)} \\
&= P^* + \frac{H}{2}\eta^M\frac{\partial}{\partial t}\left(\Delta^* w + \frac{2w}{R^2}\right) - \left(\rho H + \frac{P^*}{g}\right)\frac{\partial^2 w}{\partial t^2} + \frac{\rho}{12}H^3\frac{\partial^2}{\partial t^2}(\Delta^* w)
\end{aligned}$$

where

$$D = \frac{(\lambda + 2\mu)H^3}{12} = \rho a^2 \frac{H^3}{12} \qquad \text{(g62)}$$

is the flexural rigidity and a is the velocity of the P-wave through the shell. Also,

$$\rho^* = \left(\rho^M + \rho^F + \frac{\mu^M}{gR}\right)\frac{H}{R} \quad , \quad \sigma^* = \frac{\mu}{6}\frac{H^2}{R^2} \quad , \quad P^* = P\left(1 + \frac{H}{R}\right) \quad . \qquad \text{(g63)}$$

G.10) Spherical effects with respect to the plane plate.

For the usual materials, $\lambda^M = \mu^M$. In that case, the corresponding equation for the plane plate (Ivan 1997) is.

$$D\Delta^*\Delta^* w + H\left(\sigma_x^c \frac{\partial^2 w}{\partial x^2} + \sigma_y^c \frac{\partial^2 w}{\partial y^2}\right) + \left(\rho^M - \rho^F\right) gw$$
$$= P + \frac{H}{2}\eta^M \frac{\partial}{\partial t}\left(\Delta^* w\right) - \left(\rho H + \frac{P}{g}\right)\frac{\partial^2 w}{\partial t^2} + \frac{\rho}{12}H^3 \frac{\partial^2}{\partial t^2}(\Delta^* w) \quad . \qquad \text{(g64)}$$

With respect to (g64), a change of the density difference $\rho^M - \rho^F$ according to (g63a) and a substitution of the real lateral stresses σ^{NS}, σ^{WE} by their apparent values $\sigma^{NS} + \sigma^*, \sigma^{WE} + \sigma^*$ can be observed in eq.(s61). A supplemental load is present according to (g63c). For usual values (e.g. Ivan 1997a) like $H / R \propto 1/100$, $\mu, \mu^M \propto 10^{11}\,\text{Pa}, \rho^M, \rho^F \propto 3000\,\text{kgs} / \text{m}^3, \sigma^{NS}, \sigma^{WE} \propto 30\,\text{MPa}$, all these effects are usually negligible and difficult to be observed in real life. To compare $2w / R^2$ to $\Delta^* w$ in the left side of (g61), the case of a rectangular plate having the sides equal to L_x, L_y is considered. Here, the flexure is proportional to a product of sins (cosines) functions, i.e. $w \propto \sin(m\pi x / L_x)\sin(n\pi y / L_y)$. It follows the LAPLACE operator of the flexure is proportional to $\Delta^* w \propto \pi^2\left(m^2 / L_x^2 + n^2 / L_y^2\right)\sin(m\pi x / L_x)\sin(n\pi y / L_y)$. For the fundamental mode ($m = n = 1$) it follows that

$$\frac{2w / R^2}{\Delta^* w} = \frac{1}{\pi^2 R^2 (1 / L_x^2 + 1 / L_y^2)} \quad . \qquad \text{(g65)}$$

That ratio is negligible too in the usual cases.

It can be concluded that in the usual cases, the sphericity of the crustal plates can be ignored and the equation derived for the flat plate can be used.