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ASYMPTOTIC FOURIER INTEGRAL OPERATOR METHODS

APPLIED TO THE INVERSE PROBLEMS OF GEOMETRIC ACOUSTICS

by

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A Dissertation

Presented to

the Faculty of Mathematical and Computer Sciences

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In Partial Fulfillment

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Doctor of Philosophy

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The velocity inverse problem of geometric acoustics for an inhomogeneous non-dispersive medium is studied. The elegant and powerful theory of pseudodifferential operators and Fourier integral operators is utilized to asymptotically solve a class of general asymptotic integral equations for which the important velocity inverse problem is a special case. It is demonstrated that explicit inversion algorithms can be generated for the three dimensional backscattered data configuration with constant reference velocity. In particular, the zero order and first order inversion algorithms are developed and extensively analyzed. It is shown that the first order inversion algorithm annihilates the linear error terms produced by the zero order algorithm. Consequently, a potentially valuable improvement over existing inversion algorithms is indicated. The results are shown to be consistent with exact closed form solutions obtained by the Cagniard-de Hoop method for stratified media. Furthermore, the contributions to the field are informative in demonstrating that asymptotic Fourier integral operator methods can be applied to the problems of geometric acoustics.

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CHAPTER 1

BACKGROUND THEORY

Over the last two decades, substantial and important progress has been made toward the solution of a class of significant problems known as inverse problems. Inverse problems are characterized by extensions of partial differential equations to the more general problem of the determination or approximation of unknown coefficient functions from various types of boundary data. The boundary data can be either time or spectral data.

The main topic of this dissertation concerns the velocity inverse problem arising from geometric acoustics theory. The velocity inverse problem is mathematically formulated in terms of determining velocity functions in a wave equation from boundary data measurements. This is the natural mathematical model for the seismic problem of imaging the geophysical discontinuities in the earth through the interpretation of reflected acoustical data generated by an impulsive wave source. In addition, the velocity inverse problem is relevant to non-destructive testing of materials and medical tomography. Furthermore, geometric acoustics phenomena have analogues with the electromagnetic propagation phenomena of geometric optics. In fact, the theory of geometric optics and geometric acoustics are essentially identical. Consequently, there is considerable potential for application of the inversion techniques to electromagnetic scattering phenomena.

In this introductory chapter, the necessary background theory is presented. A discussion of geometric acoustics methods and sound wave propagation phenomena in inhomogeneous non-dispersive media is required as a basis for later theory. The velocity inverse problem is described and formulated in terms of a fundamental integral equation for an index of refraction perturbation. This integral equation is then solved using Fourier integral operator techniques in subsequent chapters.

1.1 Historical Perspective

The velocity inverse problem was formulated as a perturbation problem by Cohen and Bleistein [7]. These researchers demonstrated that an index of refraction perturbation could be solved for in the case of a 2-dimensional backscattered data configuration using a constant reference velocity.

These techniques were later extended by Bleistein, Cohen, and Hagin [5] to include a more general integral equation utilizing a non-constant reference velocity. These authors pointed out the relationship of this perturbation formulation with the Born approximation found in theoretical physics. An algorithm was developed from the method of stationary phase and asymptotic analysis (see, for example, Bleistein and Handelsman [6], Erdélyi [13], Lebedev [23], or Olver [28]). This algorithm addressed the backscattered data problem with constant reference velocity.

Cohen and Hagin [8] then presented an algorithm for backscattered stacked seismic data in which the reference velocity varied with depth. This greatly enhanced the validity of the perturbation assumptions to more realistic inverse problems. Again, high frequency asymptotic and stationary phase methods were utilized to solve for the unknown index of refraction correction to the velocity. Moreover, it was noted that the integral equation was in the form of a generalized Fourier integral equation.

Along a parallel direction, several researchers have been using the powerful mathematical tools of Radon transforms, pseudodifferential operators, and Fourier integral operators. The Radon transform has become the fundamental tool in computerized tomography, the process of reconstructing images from recorded projections. The sources by Deans [11,12], Helgason [19,20], and Natterer [27] provide current treatments on Radon transform theory. Many researchers have been investigating the theory of pseudodifferential operators and their natural generalization to Fourier integral operators. The books by Treves [33] and Taylor [31] offer a modern perspective on the mathematical theory concerning these operators.

The connection between the theory of pseudodifferential and Fourier integral operators and the wave phenomena of geometric optics was discussed by Guillemin and Sternberg [17]. More recently, Beylkin [2,3] set the stage for very general inversion techniques for the problems of geometric acoustics. Beylkin made a significant contribution by realizing that the general mathematical approaches of Randon transforms, pseudodifferential operators and Fourier integral operators could be applied to the Born approximation integral equation of the velocity inverse problem. Beylkin then formulated inversion algorithms in terms of a generalized backprojection operator analogous to the Radon transform methods of computerized tomography. Beylkin also noted that this general setting includes all of the practical configurations of geophysics, tomography, and non-destructive testing.

Recently, the earlier stationary phase approaches were compared to Beylkin's methods by Cohen, Hagin, and Bleistein [9]. It was noticed that the problem of a developing a feasible algorithm for a particular seismic configuration is reduced to whether a certain Jacobian is computable. These authors then derived an expression for velocity inversion in a full 3-dimensional setting with a general data surface. The implementation of these algorithms into computationally feasible schemes was discussed.

1.2 Wave Propagation in Inhomogeneous Non-Dispersive Media

The problem of interest concerns acoustical wave propagation and the imaging of wave propagation velocity (or equivalently, index of refraction) discontinuities in an inhomogeneous medium. In the present section, the phenomenon of wave propagation is mathematically formulated in terms of an asymptotic series satisfying the wave equation. The method to be described follows Bleistein [4] and Lewis and Keller [24].

The inhomogeneous medium is modeled as an infinite half-space with wave velocity assumed to be a piecewise smooth function of only position. In particular, it is assumed that the medium shows an absence of significant dispersion (wave propagation velocity dependence on frequency) over the range of frequencies of interest. A right-handed coordinate system $\mathbf{x} = (x_1, x_2, x_3)$ is introduced where x_1 and x_2 are chosen to lie along the outer surface of the medium, and where x_3 is positive in the direction of depth into the medium.

We consider an impulse wave source emanating from a surface point $\xi = (\xi_1, \xi_2, 0)$. The source wave propagates through the medium and is reflected from internal surfaces of wave velocity discontinuity. It is assumed that the motion of the source wave $U(\mathbf{x},t)$ propagating through regions of smooth velocity is governed by the wave equation,

$$\nabla^2 U - \frac{1}{c^2(\mathbf{x})} \frac{\partial^2 U}{\partial t^2} = - \delta(\mathbf{x} - \xi)\, \delta(t) \;,$$

$$U(\mathbf{x},t) \equiv 0 \;, \quad t < 0 \;, \tag{1.1}$$

where ∇^2 is the spatial Laplacian operator, δ denotes the Dirac delta function, and $c(\mathbf{x})$ is the spatially dependent wave velocity function. On surfaces of velocity discontinuity, the wave bifurcates into reflected and transmitted waves weighted by the appropriate reflection and transmission coefficients.

We use the following convention for the n-dimensional Fourier transform and its inverse:

$$f(\mathbf{k}) = \mathfrak{F}_n[F(\mathbf{x})] = \int_{\mathbb{R}^n} F(\mathbf{x})\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} \,,$$

$$F(\mathbf{x}) = \mathfrak{F}_n^{-1}[f(\mathbf{k})] = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\mathbf{k})\, e^{-i\mathbf{x}\cdot\mathbf{k}}\, d\mathbf{k} \,. \tag{1.2}$$

The sign convention above is chosen to more closely reflect the form of the forthcoming integral equations. Taking the 1-dimensional Fourier transform $\mathfrak{F}_1$ of the wave equation with respect to time results in the inhomogeneous Helmholtz equation,

$$\nabla^2 u(\mathbf{x},\omega) + \frac{\omega^2}{c^2(\mathbf{x})}\, u(\mathbf{x},\omega) = -\,\delta(\mathbf{x} - \boldsymbol{\xi}) \,, \tag{1.3}$$

where $u(\mathbf{x},\omega)$ is the Fourier transform of $U(\mathbf{x},t)$.

For points $\mathbf{x}$ not at the source point $\boldsymbol{\xi}$, $u(\mathbf{x},\omega)$ satisfies the homogeneous Helmholtz equation. We assume a formal asymptotic series solution of the form

$$u(\mathbf{x},\omega) = \omega^{\beta}\, A(\mathbf{x},\omega)\, e^{i\omega\tau(\mathbf{x})} \,, \tag{1.4}$$

where $\tau(\mathbf{x})$ denotes travel time and where the amplitude function is formally written

$$A(\mathbf{x},\omega) = \sum_{j=0}^{\infty} \frac{A_j(\mathbf{x})}{(i\omega)^j} \,. \tag{1.5}$$

Note that the exponent β in equation (1.4) can not be determined from the homogeneous Helmholtz equation, but is determined by matching the solution asymptotically to prescribed data. $\beta = 0$, for the problems of interest. Substituting the asymptotic representation into the Helmholtz equation results in

$$\sum_{j=0}^{\infty} \left\{ \frac{A_j(\mathbf{x})}{(i\omega)^{j-2}} \left((\nabla\tau)^2 - \frac{1}{c^2(\mathbf{x})} \right) + \frac{1}{(i\omega)^{j-1}} \left(2\nabla\tau \cdot \nabla A_j + A_j(\mathbf{x})\, \nabla^2\tau \right) + \frac{\nabla^2 A_j}{(i\omega)^j} \right\} = 0 \ . \qquad (1.6)$$

Separately setting the coefficients equal to zero for each power of ω produces a system of equations for the travel time and amplitude functions. Travel time $\tau(\mathbf{x})$ satisfies a first order nonlinear partial differential equation known as the eikonal equation,

$$(\nabla\tau)^2 = \frac{1}{c^2(\mathbf{x})} \ . \qquad (1.7)$$

The first amplitude function $A_0(\mathbf{x})$ satisfies the first transport equation

$$2\nabla\tau \cdot \nabla A_0 + A_0(\mathbf{x})\, \nabla^2\tau = 0 \ . \qquad (1.8)$$

The higher order amplitude functions $A_j(\mathbf{x})$ satisfy the higher order transport equations

$$2\nabla\tau \cdot \nabla A_j + A_j(\mathbf{x})\, \nabla^2\tau = -\nabla^2 A_{j-1}(\mathbf{x}) \ , \quad (j = 1,2,3,\ldots) \ . \qquad (1.9)$$

The partial differential equations (1.7), (1.8), and (1.9) describe the wave propagation completely. However, for the problems of interest, the velocity function $c(\mathbf{x})$ is also unknown.

1.3 Derivation of the Backscattered Data Integral Equation

In this section, the problem of velocity discontinuity imaging is framed in the form of an integral equation. The method used follows the papers by Cohen and Bleistein [7], Bleistein, Cohen, and Hagin [5], and Beylkin [3]. We consider a source wave radiating from a point on the data surface $x_3 = 0$, $\boldsymbol{\xi} = (\xi_1, \xi_2, 0)$. The source wave propagates through the medium and is reflected from a point $\mathbf{x} = (x_1, x_2, x_3)$ at depth. If the point $\mathbf{x}$ lies on a surface of wave velocity discontinuity, the scattered wave has a non-zero amplitude at the receiver. The observed scattered field is measured with receivers on the data surface at the source point $\boldsymbol{\xi}$. The wave propagation geometry is depicted in Figure 1.

A known reference velocity $c_0(\mathbf{x})$ and an unknown index of refraction perturbation $\psi(\mathbf{x})$ are introduced in terms of the unknown velocity $c(\mathbf{x})$ by the equation

$$\frac{1}{c^2(\mathbf{x})} = \frac{1}{c_0^2(\mathbf{x})} (1 + \psi(\mathbf{x})) \ . \tag{1.10}$$

We extend the problem domain from the half-space $x_3 \geqq 0$ to the entire space $\mathbb{R}^3$ by imposing the condition that $\psi(\mathbf{x}) \equiv 0$ for $x_3 < 0$. This has the advantage of simplifying the notation and discussion.

The total field is denoted $u(\mathbf{x},\omega,\boldsymbol{\xi})$. The total field satisfies the Helmholtz equation with wave propagation velocity $c(\mathbf{x})$,

$$\nabla^2 u(\mathbf{x},\omega,\boldsymbol{\xi}) + \frac{\omega^2}{c^2(\mathbf{x})} u(\mathbf{x},\omega,\boldsymbol{\xi}) = -\,\delta(\mathbf{x} - \boldsymbol{\xi}) \ . \tag{1.11}$$

The total field is decomposed into an incident field $u_I(\mathbf{x},\omega,\boldsymbol{\xi})$ and a scattered field $u_S(\mathbf{x},\omega,\boldsymbol{\xi})$. Specifically,

$$u(\mathbf{x},\omega,\boldsymbol{\xi}) = u_I(\mathbf{x},\omega,\boldsymbol{\xi}) + u_S(\mathbf{x},\omega,\boldsymbol{\xi}) \ , \tag{1.12}$$

where the incident field satisfies the Helmholtz equation for an unperturbed medium with wave propagation velocity given by the

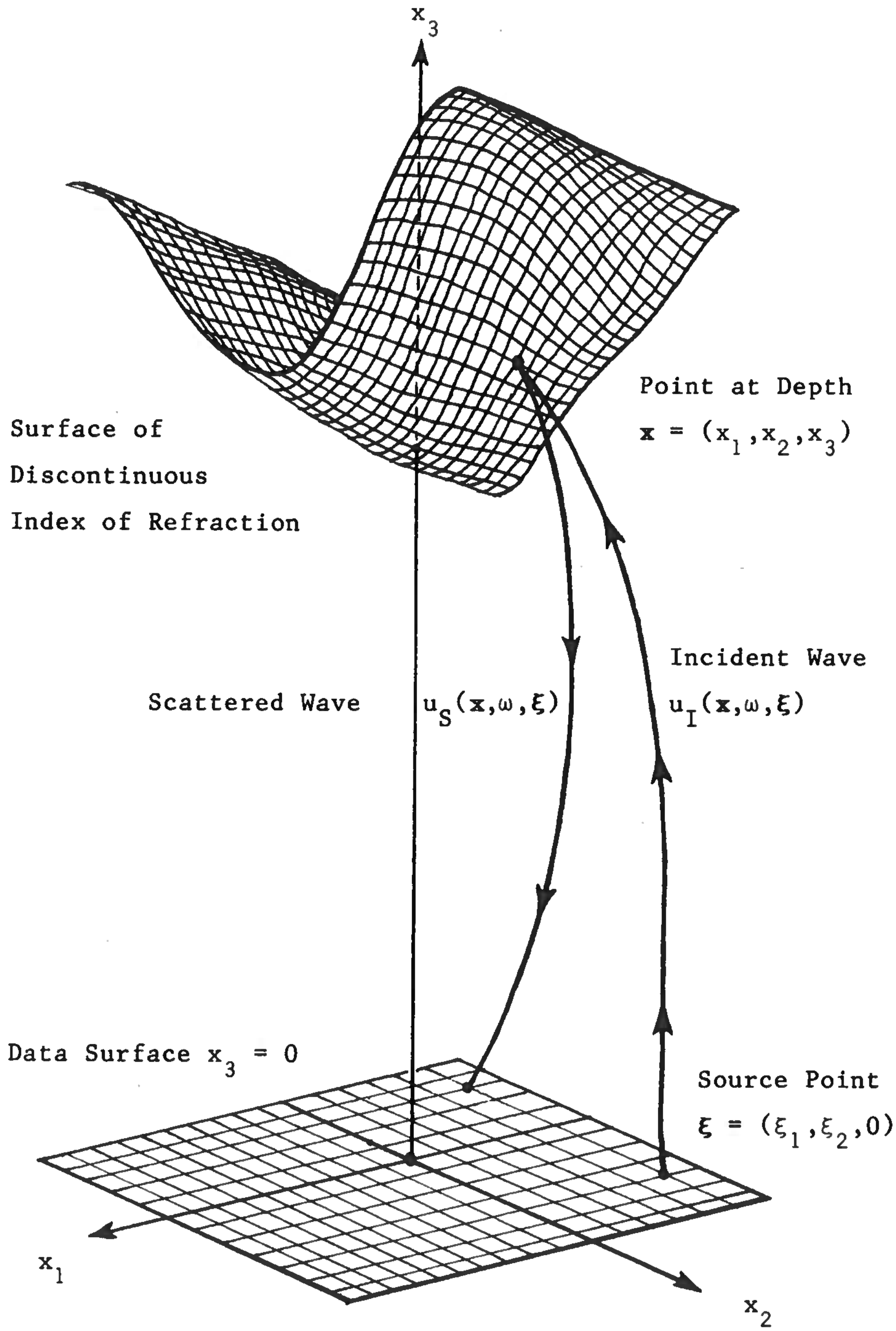

Figure 1. Wave Propagation Geometry

reference velocity $c_0(\mathbf{x})$,

$$\nabla^2 u_I(\mathbf{x},\omega,\boldsymbol{\xi}) + \frac{\omega^2}{c_0^2(\mathbf{x})} u_I(\mathbf{x},\omega,\boldsymbol{\xi}) = -\delta(\mathbf{x}-\boldsymbol{\xi}) , \tag{1.13}$$

and where the scattered field satisfies

$$\nabla^2 u_S(\mathbf{x},\omega,\boldsymbol{\xi}) + \frac{\omega^2}{c_0^2(\mathbf{x})} u_S(\mathbf{x},\omega,\boldsymbol{\xi})$$

$$= -\psi(\mathbf{x}) \frac{\omega^2}{c_0^2(\mathbf{x})} \left(u_I(\mathbf{x},\omega,\boldsymbol{\xi}) + u_S(\mathbf{x},\omega,\boldsymbol{\xi}) \right) . \tag{1.14}$$

Note that adding the equations (1.13) and (1.14) immediately results in the original Helmholtz equation (1.11).

It is to be observed that the incident field $u_I(\mathbf{x},\omega,\boldsymbol{\xi})$ is in fact the free-space Green's function for the unperturbed Helmholtz problem. The incident wave takes the following asymptotic form:

$$u_I(\mathbf{x},\omega,\boldsymbol{\xi}) = A(\mathbf{x},\omega,\boldsymbol{\xi})\, e^{i\omega\tau(\mathbf{x},\boldsymbol{\xi})} , \tag{1.15}$$

where

$$A(\mathbf{x},\omega,\boldsymbol{\xi}) = \sum_{j=0}^{\infty} \frac{A_j(\mathbf{x},\boldsymbol{\xi})}{(i\omega)^j} . \tag{1.16}$$

The phase function $\tau(\mathbf{x},\boldsymbol{\xi})$ is the travel time from the source point $\boldsymbol{\xi}$ to the point at depth $\mathbf{x}$ that satisfies the eikonal equation

$$(\nabla\tau)^2 = \frac{1}{c_0^2(\mathbf{x})} . \tag{1.17}$$

The first amplitude function $A_0(\mathbf{x},\boldsymbol{\xi})$ satisfies the transport equation

$$2\nabla\tau\cdot\nabla A_0 + A_0(\mathbf{x},\boldsymbol{\xi})\, \nabla^2\tau = 0 . \tag{1.18}$$

The higher order amplitude functions $A_j(\mathbf{x},\boldsymbol{\xi})$, $j = 1,2,3,\ldots$, satisfy the transport equations

$$2\nabla\tau \cdot \nabla A_j + A_j(\mathbf{x},\xi)\, \nabla^2\tau = -\nabla^2 A_{j-1} \ , \quad (j = 1,2,3,\ldots) \ . \tag{1.19}$$

Consider the linear differential operator $\mathcal{L}_0$ for the unperturbed Helmholtz problem,

$$\mathcal{L}_0 = \nabla^2 + \frac{\omega^2}{c_0^2(\mathbf{x})} \ . \tag{1.20}$$

We utilize Green's theorem for the Helmholtz operator $\mathcal{L}_0$ (see Bleistein [4] or Morse and Feshback [26]). If the two functions $u(\mathbf{x})$ and $v(\mathbf{x})$ have the necessary derivatives on a compact manifold Ω in $\mathbb{R}^3$ with a boundary $\partial\Omega$, then

$$\int_\Omega \left(u(\mathbf{x})\, \mathcal{L}_0\, v(\mathbf{x}) - v(\mathbf{x})\, \mathcal{L}_0\, u(\mathbf{x}) \right) d\mathbf{x} = \int_{\partial\Omega} \left(u(\mathbf{x}) \frac{\partial v}{\partial n} - v(\mathbf{x}) \frac{\partial u}{\partial n} \right) dS \ , \tag{1.21}$$

where $\frac{\partial}{\partial n}$ denotes normal derivative and dS denotes the surface differential form.

In particular, for the unperturbed Helmholtz operator $\mathcal{L}_0$ acting on the incident and scattered field on a sphere Ω of large radius R, we obtain

$$\int_\Omega \left(u_I(\mathbf{x},\omega,\xi)\, \mathcal{L}_0\, u_S(\mathbf{x},\omega,\xi) - u_S(\mathbf{x},\omega,\xi)\, \mathcal{L}_0\, u_I(\mathbf{x},\omega,\xi) \right) d\mathbf{x} = \int_{\partial\Omega} \left(u_I(\mathbf{x},\omega,\xi) \frac{\partial u_S}{\partial n} - u_S(\mathbf{x},\omega,\xi) \frac{\partial u_I}{\partial n} \right) dS \ . \tag{1.22}$$

Moreover, by application of the radiation condition at infinity for $R \to \infty$, the surface integral vanishes. Substituting the unperturbed medium Helmholtz equations (1.13) and (1.14) for the incident and scattered field into the surviving volume integral and using the properties of the Dirac delta function leads to the integral equation

$$u_S(\xi,\omega,\xi) = \omega^2 \int_{\mathbb{R}^3} \frac{\psi(\mathbf{x})}{c_0^2(\mathbf{x})} u_I(\mathbf{x},\omega,\xi) \cdot \left(u_I(\mathbf{x},\omega,\xi) + u_S(\mathbf{x},\omega,\xi) \right) d\mathbf{x} . \tag{1.23}$$

The above expression is a nonlinear integral equation relating the unknown index of refraction perturbation $\psi(\mathbf{x})$ to the measurable data $u_S(\xi,\omega,\xi)$ of the scattered field at the surface $x_3 = 0$. However, the scattered field at depth $u_S(\mathbf{x},\omega,\xi)$ is also unknown.

Up to this point, there have been no approximations used in the formulation of the integral equation beyond the model of the medium and the appropriateness of the wave equation to describe the propagation of the wave. In this sense, the integral equation is an exact equation. We assume that the perturbation $\psi(\mathbf{x})$ is small. Consequently, the scattered field is also small relative to the incident field. In particular, the term $\psi(\mathbf{x})\ u_S(\mathbf{x},\omega,\xi)$ is essentially quadratic in the small parameter $\psi(\mathbf{x})$, whereas the term $\psi(\mathbf{x})\ u_I(\mathbf{x},\omega,\xi)$ is linear in $\psi(\mathbf{x})$. Hence, we are justified in making the approximation to a linear integral equation by dropping the product term $\psi(\mathbf{x})\ u_S(\mathbf{x},\omega,\xi)$. This technique is equivalent to the Born approximation or regular perturbation method of theoretical physics (see Merzbacher [25], Jackson [22], or Morse and Feshback [26], for example). Thus, the linearized integral equation is given by

$$D(\omega,\xi) = \omega^2 \int_{\mathbb{R}^3} \frac{\psi(\mathbf{x})}{c_0^2(\mathbf{x})} u_I^2(\mathbf{x},\omega,\xi)\ d\mathbf{x} , \tag{1.24}$$

where the data measurements are denoted

$$D(\omega,\xi) = u_S(\xi,\omega,\xi) . \tag{1.25}$$

In order to solve the linear integral equation (1.24), it is necessary to determine the incident wave explicitly. This involves solving the eikonal and transport equations. We now discuss the method by which these equations are solved.

1.4 Solution to the Eikonal and Transport Equations

In this section, the eikonal equation is solved using the method of characteristics. The method of characteristics is a general technique for solving first order nonlinear partial differential equations. The method is particularly suited to the eikonal equation. The solution is given in terms of parameterized trajectories referred to as rays. The method of characteristics originated with Cauchy. Details of the method can be found in several sources including Hartman [18], Spivak [29], Courant and Hilbert [10], and Garabedian [15]. Its application to wave phenomena can be found in Bleistein [4].

One key result of the present section is the reformulation of the eikonal equation into the ordinary differential equation (1.40) for the rays as a function of the reference velocity. In addition, the transport equations are solved through the exploitation of a ray Jacobian invariance property. This generates ordinary differential equations (1.48) and (1.57) for the amplitude functions along rays. These differential equations can be explicitly solved for particular reference velocity profiles. The solutions will be required in the forthcoming chapters.

Consider a general nonlinear first order partial differential equation of the form

$$F(\mathbf{x},u,\mathbf{p}) = 0 \ , \tag{1.26}$$

where $u(\mathbf{x})$ is the solution function in the variable $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ with gradient

$$\mathbf{p} = \left(\frac{\partial u}{\partial x_1} , \frac{\partial u}{\partial x_2} , \ldots , \frac{\partial u}{\partial x_n} \right) . \tag{1.27}$$

The differential equation is subject to the initial conditions

$$u(\mathbf{x}_0(\xi)) = u_0(\xi) \ , \tag{1.28}$$

on the (n-1)-dimensional submanifold $\mathfrak{M}$ in $\mathbb{R}^n$ represented parametrically by

$$\mathbf{x} = \mathbf{x}_0(\xi) \ , \quad \xi = (\xi_1, \xi_2, \ldots, \xi_{n-1}) \ . \tag{1.29}$$

If the function $F(\mathbf{x},u,\mathbf{p})$ is in the class $C^2(\mathbb{R}^{2n+1})$ and if the initial data is noncharacteristic on $\mathfrak{M}$, then there exists a unique solution $u = u(\mathbf{x})$ of class $C^2(\mathbb{R}^n)$ in a neighborhood of $\mathfrak{M}$. Moreover, the solution can be parameterized in the form

$$u = u(\mathbf{x}(\xi,\sigma)) \ , \tag{1.30}$$

such that

$$\mathbf{x}(\xi,\sigma_0) = \mathbf{x}_0(\xi) \ , \tag{1.31}$$

and such that the system of ordinary differential equations

$$\frac{d\mathbf{x}}{d\sigma} = \lambda(\mathbf{x}) \frac{\partial F}{\partial \mathbf{p}} \ , \tag{1.32}$$

$$\frac{d\mathbf{p}}{d\sigma} = - \lambda(\mathbf{x}) \left(\frac{\partial F}{\partial \mathbf{x}} + \mathbf{p} \frac{\partial F}{\partial u} \right) \ , \tag{1.33}$$

$$\frac{du}{d\sigma} = \lambda(\mathbf{x}) \sum_{j=1}^{n} p_j \frac{\partial F}{\partial p_j} \ , \tag{1.34}$$

are satisfied with an arbitrary non-vanishing function $\lambda(\mathbf{x})$. There is freedom in the choice of the function $\lambda(\mathbf{x})$ that corresponds to the freedom of choice in the parameterization σ in the solution curves $\mathbf{x}(\sigma)$.

Now the method of characteristics is specialized to the eikonal equation in $\mathbb{R}^3$ written in the form

$$F(\mathbf{x},\tau,\mathbf{p}) = \frac{p^2}{2} - \frac{1}{2c_0^2(\mathbf{x})} = 0 , \tag{1.35}$$

where p^2 denotes the square of the magnitude of the gradient of travel time τ,

$$p^2 = \mathbf{p}\cdot\mathbf{p} = \nabla\tau\cdot\nabla\tau . \tag{1.36}$$

The corresponding system of first order ordinary differential equations in terms of the general parameterization function $\lambda(\mathbf{x})$ is given by

$$\frac{d\mathbf{x}}{d\sigma} = \lambda(\mathbf{x})\ \mathbf{p} , \tag{1.37}$$

$$\frac{d\mathbf{p}}{d\sigma} = \frac{\lambda(\mathbf{x})}{2}\nabla\left(\frac{1}{c_0^2(\mathbf{x})}\right) = \frac{\lambda(\mathbf{x})}{c_0(\mathbf{x})}\nabla\left(\frac{1}{c_0(\mathbf{x})}\right) , \tag{1.38}$$

$$\frac{d\tau}{d\sigma} = \frac{\lambda(\mathbf{x})}{c_0^2(\mathbf{x})} . \tag{1.39}$$

Moreover, this system of first order differential equations can be expressed in terms of a second order differential equation for the ray trajectory $\mathbf{x}(\sigma)$,

$$\frac{d}{d\sigma}\left(\frac{1}{\lambda(\mathbf{x})}\frac{d\mathbf{x}}{d\sigma}\right) = \frac{\lambda(\mathbf{x})}{c_0(\mathbf{x})}\nabla\left(\frac{1}{c_0(\mathbf{x})}\right) . \tag{1.40}$$

We will be interested in curvilinear coordinate transformations of the general form

$$\begin{aligned} x_1 &= x_1(q_1,q_2,q_3) \\ x_2 &= x_2(q_1,q_2,q_3) \\ x_3 &= x_3(q_1,q_2,q_3) , \end{aligned} \tag{1.41}$$

where one of the transformation variables is chosen to be the ray parameter σ. For our purposes, $q_3 = \sigma$. The corresponding Jacobian of the curvilinear transformation is denoted

$$J = \det \frac{\partial(x_1, x_2, x_3)}{\partial(q_1, q_2, q_3)} = \det \frac{\partial(x_1, x_2, x_3)}{\partial(q_1, q_2, \sigma)} . \tag{1.42}$$

Furthermore, an application of the chain rule shows that the Jacobian satisfies the property:

$$\frac{dJ}{d\sigma} = J \,\nabla\cdot\frac{d\mathbf{x}}{d\sigma} = J \,\nabla\cdot(\lambda\nabla\tau) = J \,\nabla\lambda\cdot\nabla\tau + \lambda J \,\nabla^2\tau . \tag{1.43}$$

The above Jacobian property is utilized to solve the transport equation below.

Expressing the gradient of travel time $\mathbf{p} = \nabla\tau$ in terms of ray velocity by equation (1.37) and substituting the result into the first transport equation (1.18) results in an ordinary differential equation along the ray trajectory,

$$\frac{2}{\lambda} A_0 \,\nabla A_0\cdot\frac{d\mathbf{x}}{d\sigma} + A_0^2 \,\nabla^2\tau = 0 . \tag{1.44}$$

The Jacobian property in equation (1.43) and the first transport equation along rays (1.44) are used to derive a ray Jacobian invariance property. The ray Jacobian invariance property will permit the ready solution of the first transport differential equation for the wave amplitude function along rays. The initial step consists of calculating the derivative of $A_0^2 J/\lambda$ with respect to the ray parameter σ. We use the slight abuse of notation $A_0(\sigma) = A_0(\mathbf{x}(\sigma))$.

$$\frac{d}{d\sigma}\left(\frac{A_0^2(\sigma)\, J(\sigma)}{\lambda(\sigma)}\right) = \frac{2}{\lambda(\sigma)} A_0(\sigma)\, J(\sigma)\, \nabla A_0 \cdot \frac{d\mathbf{x}}{d\sigma} + \frac{A_0^2(\sigma)}{\lambda(\sigma)} \frac{dJ}{d\sigma} - \frac{A_0^2(\sigma)}{\lambda^2(\sigma)} J(\sigma)\, \nabla\lambda \cdot \frac{d\mathbf{x}}{d\sigma} . \tag{1.45}$$

The Jacobian property (1.43) provides an expression to substitute for the derivative of J with respect to σ.

$$\frac{d}{d\sigma}\left(\frac{A_0^2(\sigma)\ J(\sigma)}{\lambda(\sigma)}\right) = \frac{2}{\lambda(\sigma)} A_0(\sigma)\ J(\sigma)\ \nabla A_0 \cdot \frac{d\mathbf{x}}{d\sigma}$$
$$+ \frac{A_0^2(\sigma)}{\lambda(\sigma)} \left(J(\sigma)\ \nabla\lambda \cdot \nabla\tau + \lambda(\sigma)\ J(\sigma)\ \nabla^2\tau \right)$$
$$- \frac{A_0^2(\sigma)}{\lambda^2(\sigma)} J(\sigma)\ \nabla\lambda \cdot \frac{d\mathbf{x}}{d\sigma} \ . \tag{1.46}$$

A rearrangement of terms with an application of the differential equation (1.37) results in

$$\frac{d}{d\sigma}\left(\frac{A_0^2(\sigma)\ J(\sigma)}{\lambda(\sigma)}\right) = J(\sigma) \left(\frac{2}{\lambda(\sigma)} A_0(\sigma)\ \nabla A_0 \cdot \frac{d\mathbf{x}}{d\sigma} + A_0^2(\sigma)\ \nabla^2\tau \right)$$
$$+ \frac{A_0^2(\sigma)}{\lambda(\sigma)} J(\sigma)\ \nabla\lambda \cdot \nabla\tau$$
$$- \frac{A_0^2(\sigma)}{\lambda^2(\sigma)} J(\sigma)\ \nabla\lambda \cdot (\lambda(\sigma)\ \nabla\tau) \ . \tag{1.47}$$

Recalling the form of the first transport equation (1.44) along rays, the first term of the above expression immediately vanishes. Moreover, the second and third terms cancel each other. Thus, the desired ray Jacobian invariance property is obtained. Specifically,

$$\frac{d}{d\sigma}\left(\frac{A_0^2(\sigma)\ J(\sigma)}{\lambda(\sigma)}\right) = 0 \ . \tag{1.48}$$

It follows that $A_0^2 J/\lambda$ is constant along rays. Consequently, it becomes trivial to integrate the above differential equation and solve for the first amplitude function along rays. Thus,

$$A_0(\sigma) = A_0(\sigma_0) \sqrt{\frac{\lambda(\sigma)\ J(\sigma_0)}{\lambda(\sigma_0)\ J(\sigma)}} \ , \tag{1.49}$$

where σ_0 denotes the ray parameter value corresponding to the initial conditions. Hence, the solution of the first transport equation has been reduced to the calculation of a Jacobian J along rays.

A similar method can be exploited to solve the higher order transport equations along rays. Consider

$$\frac{d}{d\sigma}\left(A_j(\sigma)\sqrt{\frac{J(\sigma)}{\lambda(\sigma)}}\right) = \frac{d}{d\sigma}\left(\frac{A_j(\sigma)}{A_0(\sigma)}\, A_0(\sigma)\sqrt{\frac{J(\sigma)}{\lambda(\sigma)}}\right) . \tag{1.50}$$

Taking advantage of the ray Jacobian invariance property (1.49) results in the following.

$$\frac{d}{d\sigma}\left(A_j(\sigma)\sqrt{\frac{J(\sigma)}{\lambda(\sigma)}}\right) = A_0(\sigma_0)\sqrt{\frac{J(\sigma_0)}{\lambda(\sigma_0)}}\,\frac{d}{d\sigma}\left(\frac{A_j(\sigma)}{A_0(\sigma)}\right) . \tag{1.51}$$

Equivalently, by again applying equation (1.49),

$$\frac{d}{d\sigma}\left(A_j(\sigma)\sqrt{\frac{J(\sigma)}{\lambda(\sigma)}}\right) = A_0(\sigma)\sqrt{\frac{J(\sigma)}{\lambda(\sigma)}}\,\frac{d}{d\sigma}\left(\frac{A_j(\sigma)}{A_0(\sigma)}\right) . \tag{1.52}$$

Expanding, we obtain,

$$\frac{d}{d\sigma}\left(A_j(\sigma)\sqrt{\frac{J(\sigma)}{\lambda(\sigma)}}\right) = \sqrt{\frac{J(\sigma)}{\lambda(\sigma)}}\left(\frac{dA_j}{d\sigma} - \frac{A_j(\sigma)}{A_0(\sigma)}\,\frac{dA_0}{d\sigma}\right) . \tag{1.53}$$

Hence, by applying the ray equations,

$$\frac{d}{d\sigma}\left(A_j(\sigma)\sqrt{\frac{J(\sigma)}{\lambda(\sigma)}}\right) = \sqrt{\lambda(\sigma)\, J(\sigma)} \cdot \left(\nabla A_j \cdot \nabla\tau - \frac{A_j(\sigma)}{A_0(\sigma)}\, \nabla A_0 \cdot \nabla\tau \right) . \tag{1.54}$$

The transport equations (1.18) and (1.19) can be rewritten in the form:

$$\nabla A_0 \cdot \nabla\tau = -\frac{1}{2}\, A_0(\sigma)\, \nabla^2\tau , \tag{1.55}$$

$$\nabla A_j \cdot \nabla\tau = -\frac{1}{2}\, A_j(\sigma)\, \nabla^2\tau - \frac{1}{2}\, \nabla^2 A_{j-1} . \tag{1.56}$$

These expressions are substituted for the corresponding terms in equation (1.54). This substitution produces

$$\frac{d}{d\sigma}\left(A_j(\sigma)\sqrt{\frac{J(\sigma)}{\lambda(\sigma)}}\right) = -\frac{1}{2}\sqrt{\lambda(\sigma)\, J(\sigma)}\, \nabla^2 A_{j-1} \ . \tag{1.57}$$

The above equation is a simple inhomogeneous ordinary differential equation that can be solved recursively by direct integration.

$$A_j(\sigma)\sqrt{\frac{J(\sigma)}{\lambda(\sigma)}} - A_j(\sigma_0)\sqrt{\frac{J(\sigma_0)}{\lambda(\sigma_0)}} = -\frac{1}{2}\int_{\sigma_0}^{\sigma}\sqrt{\lambda(\sigma')\, J(\sigma')}\, \nabla^2 A_{j-1}(\sigma')\, d\sigma' \ . \tag{1.58}$$

Thus, the problem of solving the higher order transport equation has been again reduced to the calculation of the Jacobian J along rays. Hence, the amplitude is given by

$$A_j(\sigma) = A_j(\sigma_0)\sqrt{\frac{\lambda(\sigma)\, J(\sigma_0)}{\lambda(\sigma_0)\, J(\sigma)}} - \frac{1}{2}\sqrt{\frac{\lambda(\sigma)}{J(\sigma)}}\int_{\sigma_0}^{\sigma}\sqrt{\lambda(\sigma')\, J(\sigma')}\, \nabla^2 A_{j-1}(\sigma')\, d\sigma' \ . \tag{1.59}$$

Equations (1.40), (1.49), and (1.59) provide the fundamental equations required to specify the wave propagation in a particular medium. We now turn to the subject of pseudodifferential and Fourier integral operators.

1.5 Pseudodifferential and Fourier Integral Operators

In the forthcoming chapters, general integral operators appear often. In this concluding section, we provide definitions of the relevant integral operators. Specifically, we are concerned with two types of integral operators: the pseudodifferential operators and the Fourier integral operators. A rigorous treatment of these integral operators can be found in the sources by Treves [33] and Taylor [31].

A pseudodifferential operator $\mathbf{A}$ is represented by expressions of the following kind:

$$(\mathbf{A}u)(\mathbf{y}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\, a(\mathbf{y},\mathbf{k})\, u(\mathbf{x})\, d\mathbf{x}\, d\mathbf{k} \; . \tag{1.60}$$

The function $a(\mathbf{y},\mathbf{k})$ is known as the symbol of the pseudodifferential operator $\mathbf{A}$. The pseudodifferential operator (1.60) can also be expressed in terms of the Fourier transform of equation (1.2). Hence,

$$(\mathbf{A}u)(\mathbf{y}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\mathbf{y}\cdot\mathbf{k}}\, a(\mathbf{y},\mathbf{k})\, \hat{u}(\mathbf{k})\, d\mathbf{k} \; , \tag{1.61}$$

where $\hat{u}(\mathbf{k})$ denotes the n-dimensional Fourier transform of the function $u(\mathbf{x})$,

$$\hat{u}(\mathbf{k}) = \mathcal{F}_n[u(\mathbf{x})] \; . \tag{1.62}$$

The pseudodifferential operator (1.60) can be generalized to operators of the form

$$(\mathcal{F}u)(\mathbf{y}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\phi(\mathbf{x},\mathbf{y},\boldsymbol{\theta})}\, a(\mathbf{x},\mathbf{y},\boldsymbol{\theta})\, u(\mathbf{x})\, d\mathbf{x}\, d\boldsymbol{\theta} \; , \tag{1.63}$$

where the phase function $\phi(\mathbf{x},\mathbf{y},\boldsymbol{\theta})$ is subject to the conditions:

(1) $\phi \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n - \{0\})$,

(2) ϕ is positive-homogeneous with respect to θ of degree 1, and

(3) $\nabla_{\mathbf{x}}\phi$ and $\nabla_{\mathbf{y}}\phi$ do not vanish in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n - \{0\}$. (1.64)

An operator $\mathcal{F}$ satisfying (1.63) and (1.64) is called a standard Fourier integral operator.

The standard Fourier integral operators represent a class of operators that are easy to analyze from the point of view of functional analysis. However, for many important applications, one typically deals with phase functions $\phi(\mathbf{x},\mathbf{y},\theta)$ that violate the restrictions (1.64). Therefore, we define a general Fourier integral operator $\mathcal{F}$ as an operator of the form (1.63) but not necessarily satisfying the restrictions (1.64). Commonly, we simplify the terminology and speak of Fourier integral operators by dropping the word general. This concludes the background theory needed for the forthcoming chapters.

CHAPTER 2

GENERALIZED ASYMPTOTIC INTEGRAL EQUATION INVERSION

In the previous chapter, the nature of scattering data at an accessible surface due to reflections in inhomogeneous non-dispersive media was investigated. An integral equation for the backscattered data configuration was developed,

$$D(\omega,\xi) = \omega^2 \int_{\mathbb{R}^3} \frac{\psi(\mathbf{x})}{c_0^2(\mathbf{x})} u_I^2(\mathbf{x},\omega,\xi)\, d\mathbf{x} \;, \tag{2.1}$$

relating an unknown wave propagation index of refraction perturbation $\psi(\mathbf{x})$ (relative to a reference velocity $c_0(\mathbf{x})$) to the scattering data $D(\omega,\xi)$ at the surface. The incident field $u_I^2(\mathbf{x},\omega,\xi)$ is considered to emanate from the coincident source and receiver point ξ and possess the asymptotic representation

$$u_I^2(\mathbf{x},\omega,\xi) = A^2(\mathbf{x},\omega,\xi)\, e^{i\omega\phi(\mathbf{x},\xi)} \;, \tag{2.2}$$

where the amplitude has the expansion

$$A(\mathbf{x},\omega,\xi) = \sum_{j=0}^{\infty} \frac{A_j(\mathbf{x},\xi)}{(i\omega)^j} \;, \tag{2.3}$$

and where the phase is the sum of the travel time from the source to the point at depth and back,

$$\phi(\mathbf{x},\xi) = 2\tau(\mathbf{x},\xi) \;. \tag{2.4}$$

In the present chapter, the general inversion of a collection of integral equations is considered. The perturbation integral equation for backscattered data above can be viewed as a special case of this class of integral equations. The method involved follows the techniques supplied by the papers by Beylkin [2,3]. However, the method developed in this chapter deviates slightly from the Radon transform emphasis of Beylkin in that a purely Fourier integral operator approach is taken. Moreover, the procedure is extended to asymptotic integral equations of higher order from which explicit inversion algorithms are generated. A rigorous treatment of the theory of Fourier integral operators and the related pseudodifferential operators relevant to the techniques of the present chapter can be found in the books by Treves [33] and Taylor [31].

2.1 Generalized Asymptotic Integral Equation

In the present section, a class of asymptotic integral equations is explored. The main objective is to eventually invert the general class of integral equations. In particular, the inversion procedure will lead to explicit inversion algorithms for the backscattered data integral equation that is a special case of the more general integral equation.

The inversion methodology consists of operating on the general integral equation (that is similar to a Fourier transform in character) with a candidate inversion operator (that is similar to an inverse Fourier transform in character). The combined operator takes the form of a Fourier integral operator, or equivalently, a generalization of a pseudodifferential operator. It is then noted that making the Fourier integral operator asymptotically the identity operator is equivalent to selecting the unknown kernel functions in the inversion operator. The present section ends with an explicit asymptotic representation for the Fourier integral operator formulation. This representation is further manipulated in subsequent sections.

The general asymptotic integral equation to be considered relates an unknown function $\psi(\mathbf{x})$ to Fourier transformed data on a hyperplane $\mathbf{x} = \boldsymbol{\xi} = (\xi_1, \ldots, \xi_{n-1}, 0)$. The integral equation is of the form:

$$D(\omega,\boldsymbol{\xi}) = \omega^{n-1} \int_{\mathbb{R}^n} a(\mathbf{x},\omega,\boldsymbol{\xi})\, e^{i\omega\phi(\mathbf{x},\boldsymbol{\xi})}\, \psi(\mathbf{x})\, d\mathbf{x} \;, \tag{2.5}$$

where the phase function $\phi(\mathbf{x},\boldsymbol{\xi})$ and the amplitude $a(\mathbf{x},\omega,\boldsymbol{\xi})$ are C^∞. Furthermore, the amplitude function is assumed to have the asymptotic representation in the Fourier frequency variable ω,

$$a(\mathbf{x},\omega,\boldsymbol{\xi}) = \sum_{j=0}^{\infty} \frac{a_j(\mathbf{x},\boldsymbol{\xi})}{(i\omega)^j} \;. \tag{2.6}$$

We wish to invert the general integral equation for the unknown function $\psi(\mathbf{x})$. By noticing the Fourier-like nature of the integral kernel, a reasonable asymptotic form for the integral equation inverse operator can be conjectured:

$$\psi(\mathbf{y}) \sim \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} b(\mathbf{y},\omega,\boldsymbol{\xi})\, e^{-i\omega\phi(\mathbf{y},\boldsymbol{\xi})}\, D(\omega,\boldsymbol{\xi})\, d\omega\, d\boldsymbol{\xi} \;, \tag{2.7}$$

where the notation $d\boldsymbol{\xi}$ refers to the differential (n-1)-form $d\xi_1\, d\xi_2 \ldots d\xi_{n-1}$ and where the kernel function $b(\mathbf{y},\omega,\xi)$ is to be determined. We assume that the kernel function also has an asymptotic expansion,

$$b(\mathbf{y},\omega,\boldsymbol{\xi}) = \sum_{k=0}^{\infty} \frac{b_k(\mathbf{y},\boldsymbol{\xi})}{(i\omega)^k} \;. \tag{2.8}$$

Consequently, it is natural to combine the integral equation and the inverse operator to form the following Fourier integral operator $\mathcal{F}$ defined in terms of the unknown amplitude kernel function $b(\mathbf{y},\omega,\boldsymbol{\xi})$. Specifically,

$$(\mathcal{F}(\psi))(\mathbf{y}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} a(\mathbf{x},\omega,\boldsymbol{\xi})\, b(\mathbf{y},\omega,\boldsymbol{\xi}) \cdot e^{i\omega\Phi(\mathbf{x},\mathbf{y},\boldsymbol{\xi})}\, \psi(\mathbf{x})\, \omega^{n-1}\, d\mathbf{x}\, d\omega\, d\boldsymbol{\xi} \;, \tag{2.9}$$

where the phase is given by

$$\Phi(\mathbf{x},\mathbf{y},\xi) = \phi(\mathbf{x},\xi) - \phi(\mathbf{y},\xi) \ . \tag{2.10}$$

The strategy is to choose the kernel amplitude $b(\mathbf{y},\omega,\xi)$ in terms of the integral equation amplitude $a(\mathbf{x},\omega,\xi)$ and phase $\phi(\mathbf{x},\xi)$ such that the Fourier integral operator $\mathcal{F}$ is asymptotically the identity. This is precisely equivalent to the determination of the kernel function $b(\mathbf{y},\omega,\xi)$ in the integral equation inverse operator (2.7). This is also equivalent to inverting the original integral equation (2.5).

In order to make the Fourier integral operator $\mathcal{F}$ asymptotically the identity operator, it is necessary to expand the Fourier integral operator into a series of increasingly smooth integral operators. This procedure leads to very complicated and intricate expansions, but will eventually provide explicit expressions for the undetermined kernel functions.

To assist in the detailed expansions, the multi-index notation from the theory of partial differential operators is adopted (see Treves [32] or Folland [14]). Specifically, for the spatial variable $\mathbf{x} = (x_1,\ldots,x_n)$ and multi-index $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$ where each α_i is a nonnegative integer, the notation $\mathbf{x}^{\boldsymbol{\alpha}}$ refers to the multivariable polynomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots\ x_n^{\alpha_n}$. The notation $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}$ is a compact way to write the partial differential operator

$$\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} .$$

The norm of the multi-index $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\ldots,\alpha_n)$, denoted $|\boldsymbol{\alpha}|$, is the sum of the respective nonnegative integers, $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. The factorial symbol $\boldsymbol{\alpha}!$ refers to the product $\alpha_1!\ \alpha_2!\ \ldots\ \alpha_n!$.

If the Fourier integral operator $\mathcal{F}$ is made asymptotic to the identity operator, then the kernel of $\mathcal{F}$ is essentially an approximation to the Dirac delta function, $\delta(\mathbf{x} - \mathbf{y})$. Such an approximating function to $\delta(\mathbf{x}-\mathbf{y})$ will act predominantly in a neighborhood of $\mathbf{x}-\mathbf{y}=\mathbf{0}$. Consequently, it makes sense to expand the Fourier integral operator $\mathcal{F}$ for small $(\mathbf{x}-\mathbf{y})$. This requires expanding the various functions that define the Fourier integral operator into Taylor series. When the phase function $\Phi(\mathbf{x},\mathbf{y},\boldsymbol{\xi})$ is expanded in a Taylor series for small $(\mathbf{x}-\mathbf{y})$, we obtain the following:

$$\Phi(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) = \nabla_{\mathbf{y}}\phi(\mathbf{y},\boldsymbol{\xi})\cdot(\mathbf{x}-\mathbf{y}) + H(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) \ , \tag{2.11}$$

where

$$H(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) = \sum_{\alpha_1+\cdots+\alpha_n=2}^{\infty} \frac{1}{\alpha_1!\cdots\alpha_n!} \left(\frac{\partial}{\partial y_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial y_n}\right)^{\alpha_n} \phi(\mathbf{y},\boldsymbol{\xi}) \cdot (x_1-y_1)^{\alpha_1}\cdots(x_n-y_n)^{\alpha_n} \ , \tag{2.12}$$

or more succinctly in multi-index notation,

$$H(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}|=2}^{\infty} \frac{1}{\boldsymbol{\alpha}!} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \phi(\mathbf{y},\boldsymbol{\xi}) \ (\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}} \ . \tag{2.13}$$

The amplitude function $a(\mathbf{x},\omega,\boldsymbol{\xi})$ is also expanded in a Taylor series for small $(\mathbf{x}-\mathbf{y})$. This takes the following form in multi-index notation.

$$a(\mathbf{x},\omega,\boldsymbol{\xi}) = \sum_{j=0}^{\infty} \sum_{|\boldsymbol{\beta}|=0}^{\infty} \frac{1}{\boldsymbol{\beta}!} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} a_j(\mathbf{y},\boldsymbol{\xi}) \frac{(\mathbf{x}-\mathbf{y})^{\boldsymbol{\beta}}}{(i\omega)^j} \ . \tag{2.14}$$

The Taylor series relate the phase perturbation $H(\mathbf{x},\mathbf{y},\boldsymbol{\xi})$ and the amplitude function $a(\mathbf{x},\omega,\boldsymbol{\xi})$, that both depend on the integration variable $\mathbf{x}$, to the functions $\phi(\mathbf{y},\boldsymbol{\xi})$ and $a_j(\mathbf{y},\boldsymbol{\xi})$, that depend on the output variable $\mathbf{y}$. This makes it possible to integrate the series representations (2.11) and (2.14) with respect to $\mathbf{x}$ in forthcoming equations. In addition, the phase perturbation and amplitude series can be further expanded to yield the following multiple summation formulae.

$$H(\mathbf{x},\mathbf{y},\xi) = \frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\partial^2 \phi}{\partial y_p \partial y_q} (x_p - y_p)(x_q - y_q)$$
$$+ \frac{1}{6} \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{r=1}^{n} \frac{\partial^3 \phi}{\partial y_p \partial y_q \partial y_r} (x_p - y_p)(x_q - y_q)(x_r - x_r)$$
$$+ \cdots , \tag{2.15}$$

$$a(\mathbf{x},\omega,\xi) = \sum_{j=0}^{\infty} \left\{ a_j(\mathbf{y},\xi) + \sum_{p=1}^{n} \frac{\partial a_j}{\partial y_p} (x_p - y_p) \right.$$
$$\left. + \frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\partial^2 a_j}{\partial y_p \partial y_q} (x_p - y_p)(x_q - y_q) + \cdots \right\} . \tag{2.16}$$

The concept of a homotopy (see Hocking and Young [21]) is useful for the purpose of expanding the Fourier integral operator $\mathcal{F}$. Consider the homotopy $\mathfrak{h}_s$ with parameterization $s \varepsilon [0,1]$ of Fourier integral operators given by

$$(\mathfrak{h}_s(\psi))(\mathbf{y}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega\nabla_{\mathbf{y}}\phi(\mathbf{y},\xi)\cdot(\mathbf{x}-\mathbf{y})+i\omega s H(\mathbf{x},\mathbf{y},\xi)}$$
$$\cdot\ a(\mathbf{x},\omega,\xi)\ b(\mathbf{y},\omega,\xi)\ \psi(\mathbf{x})\ \omega^{n-1}\ d\mathbf{x}\ d\omega\ d\xi\ . \tag{2.17}$$

For the parameterization value $s = 0$, we obtain a simplified integral operator $\mathfrak{h}_0$ of the form

$$(\mathfrak{h}_0(\psi))(\mathbf{y}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega\nabla_{\mathbf{y}}\phi(\mathbf{y},\xi)\cdot(\mathbf{x}-\mathbf{y})}$$
$$\cdot \sum_{j=0}^{\infty} \frac{a_j(\mathbf{x},\xi)}{(i\omega)^j} \sum_{k=0}^{\infty} \frac{b_k(\mathbf{y},\xi)}{(i\omega)^k}\ \psi(\mathbf{x})\ \omega^{n-1}\ d\mathbf{x}\ d\omega\ d\xi\ . \tag{2.18}$$

Alternatively, for the parameterization value $s = 1$, we obtain the integral operator $\mathfrak{h}_1$

$$(\mathfrak{h}_1(\psi))(\mathbf{y}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega\nabla_{\mathbf{y}}\phi(\mathbf{y},\xi)\cdot(\mathbf{x}-\mathbf{y})+i\omega H(\mathbf{x},\mathbf{y},\xi)}$$
$$\cdot \sum_{j=0}^{\infty} \frac{a_j(\mathbf{x},\xi)}{(i\omega)^j} \sum_{k=0}^{\infty} \frac{b_k(\mathbf{y},\xi)}{(i\omega)^k}\ \psi(\mathbf{x})\ \omega^{n-1}\ d\mathbf{x}\ d\omega\ d\xi\ . \tag{2.19}$$

Moreover, the Fourier integral operators $\mathfrak{h}_1$ and $\mathcal{F}$ in equations (2.19) and (2.9), respectively, are in fact identical.

$$(\mathfrak{h}_1(\psi))(\mathbf{y}) = (\mathcal{F}(\psi))(\mathbf{y}) \ . \qquad (2.20)$$

The homotopy of Fourier integral operators $\mathfrak{h}_s$ suggests a method for relating the Fourier integral operator $\mathcal{F}$ to the simplified Fourier integral operator $\mathfrak{h}_0$ in equation (2.18) with phase functions of the form $\nabla_{\mathbf{y}}\phi\cdot(\mathbf{x}-\mathbf{y})$. The method involves the expansion of the operator $\mathcal{F}$ in a Taylor series of operators. This operator Taylor series is given formally by

$$\mathcal{F} = \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{d}{ds}\right)^m \mathfrak{h}_s \Big|_{s=0} \ . \qquad (2.21)$$

The m^{th} derivative in the above operator (2.21) is obtained from the differentiation of equation (2.17),

$$\left(\left(\frac{d}{ds}\right)^m \mathfrak{h}_s(\psi)\right)(\mathbf{y}) = \frac{i^m}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega\nabla_{\mathbf{y}}\phi(\mathbf{y},\xi)\cdot(\mathbf{x}-\mathbf{y})+i\omega sH(\mathbf{x},\mathbf{y},\xi)}$$

$$\cdot\ \omega^{m+n-1}\, H^m(\mathbf{x},\mathbf{y},\xi) \sum_{j=0}^{\infty} \frac{a_j(\mathbf{x},\xi)}{(i\omega)^j} \sum_{k=0}^{\infty} \frac{b_k(\mathbf{y},\xi)}{(i\omega)^k}\, \psi(\mathbf{x})\, d\mathbf{x}\, d\omega\, d\xi \ . \qquad (2.22)$$

where $H^m(\mathbf{x},\mathbf{y},\xi)$ denotes the m^{th} power of the phase perturbation $H(\mathbf{x},\mathbf{y},\xi)$,

$$H^m(\mathbf{x},\mathbf{y},\xi) = \Big(\sum_{|\alpha|=2}^{\infty} \frac{1}{\alpha!}\, \partial_{\mathbf{y}}^{\alpha}\, \phi(\mathbf{y},\xi)\ (\mathbf{x}-\mathbf{y})^{\alpha} \Big)^m \ . \qquad (2.23)$$

Combining the Taylor series (2.15) and (2.16) with the operator expansion (2.21) above, the main result of this section is produced. The result states that the Fourier integral operator $\mathcal{F}$ has an explicit asymptotic operator expansion of the form:

$$\mathcal{F} \sim \mathcal{F}_M = \sum_{m=0}^{M} \frac{1}{m!} \left(\frac{d}{ds}\right)^m \mathfrak{h}_s \Big|_{s=0} \ , \qquad (2.24)$$

where the truncated operator $\mathcal{F}_M$ for the nonnegative integer M is given by the expression

$$\big(\mathcal{F}_M(\psi)\big)(\mathbf{y}) = \sum_{m=0}^{M} \frac{i^m}{(2\pi)^n m!} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega \nabla_{\mathbf{y}}\phi(\mathbf{y},\xi)\cdot(\mathbf{x}-\mathbf{y})}$$

$$\cdot \sum_{k=0}^{\infty} \frac{b_k(\mathbf{y},\xi)}{(i\omega)^k} \Big\{ \frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\partial^2 \phi}{\partial y_p \partial y_q} (x_p - y_p)(x_q - y_q)$$

$$+ \frac{1}{6} \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{r=1}^{n} \frac{\partial^3 \phi}{\partial y_p \partial y_q \partial y_r} (x_p - y_p)(x_q - y_q)(x_r - y_r) + \cdots \Big\}^m$$

$$\cdot \sum_{j=0}^{\infty} \frac{1}{(i\omega)^j} \Big\{ a_j(\mathbf{y},\xi) + \sum_{p=1}^{n} \frac{\partial a_j}{\partial y_p} (x_p - y_p)$$

$$+ \frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\partial^2 a_j}{\partial y_p \partial y_q} (x_p - y_p)(x_q - y_q) + \cdots \Big\}$$

$$\cdot\ \omega^{m+n-1}\ \psi(\mathbf{x})\ d\mathbf{x}\ d\omega\ d\xi\ . \tag{2.25}$$

The asymptotic Fourier integral operator representation (2.25) is not in a form to perform an asymptotic analysis. This follows since it is not apparent at this point how the various derivatives and polynomials relate to frequency ω. This is investigated in the subsequent section.

2.2 Fourier Integral Operator Asymptotic Expansion

At this point in the discussion, the inversion of the general asymptotic integral equation (2.5) has been formulated in terms of the equivalent problem of making the Fourier integral operator (2.9) equal to the identity operator asymptotically. Through a succession of series expansions, an explicit asymptotic representation (2.25) is obtained.

In the present section, the manipulations of the asymptotic representation (2.25) are continued. The numerous terms of the lengthy expansion are collected into individual operators according to their corresponding order of frequency $(\frac{1}{\omega})^m$. However, it is not obvious initially that the individual operators of the decomposition possess the stated frequency dependence. This property becomes apparent only after the various manipulations and transformations have been made. In particular, the transformation $\mathbf{k} = \omega \, \nabla_{\mathbf{y}}\phi$ due to Beylkin [3] plays a key role. The Beylkin transformation reduces the Fourier integral operator to essentially a combination of a Fourier transform and Fourier inverse transform. Ultimately, explicit expressions for the first two terms of the asymptotic expansion of the Fourier integral operator (2.9) are developed in terms of combined Fourier transforms. The first two terms of this expansion are provided in equations (2.34) and (2.45). We now go through the details of this procedure.

Up to this point, the Fourier integral operator $\mathcal{F}$ defined in equation (2.9) has been expanded into the complicated asymptotic operator representation given in equations (2.24) and (2.25). We now decompose the asymptotic operator $\mathcal{F}_M$ into operators of increasing powers of $\frac{1}{\omega}$. It is not apparent that the operators subsequently defined actually possess the claimed order of ω. However, this result is deferred to the following section after considerable manipulations of the operator expressions. The result that is to be eventually explained is that the polynomial $(\mathbf{x} - \mathbf{y})^{\boldsymbol{\alpha}}$ in the expansions is exactly related to the power $(\frac{1}{\omega})^{|\boldsymbol{\alpha}|}$ for any multi-index $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$.

The decomposition of the truncated operator $\mathcal{F}_M$ that is asymptotic to the Fourier integral operator $\mathcal{F}$ takes the following form.

$$\mathcal{F} \sim \mathcal{F}_M = \mathfrak{T}_0 + \mathfrak{T}_1 + \mathfrak{T}_2 + \cdots + \mathfrak{T}_M \; . \tag{2.26}$$

The individual operators in the decomposition are explicitly obtained by carrying out the necessary multiplications and collection of terms in the complicated representation for $\mathcal{F}_M$ found in equation (2.25). This leads to the following definitions of the operators $\mathfrak{T}_0$, $\mathfrak{T}_1$, and $\mathfrak{T}_2$.

$$\big(\mathfrak{T}_0(\psi)\big)(\mathbf{y}) = + \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega\nabla_{\mathbf{y}}\phi\cdot(\mathbf{x}-\mathbf{y})} \cdot a_0(\mathbf{y},\xi)\, b_0(\mathbf{y},\xi)\, \omega^{n-1}\, \psi(\mathbf{x})\, d\mathbf{x}\, d\omega\, d\xi \; . \tag{2.27}$$

$$\begin{aligned}
\big(\mathfrak{T}_1(\psi)\big)(\mathbf{y}) = &- \frac{i}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega\nabla_{\mathbf{y}}\phi\cdot(\mathbf{x}-\mathbf{y})} \cdot a_0(\mathbf{y},\xi)\, b_1(\mathbf{y},\xi)\, \omega^{n-2}\, \psi(\mathbf{x})\, d\mathbf{x}\, d\omega\, d\xi \\
&- \frac{i}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega\nabla_{\mathbf{y}}\phi\cdot(\mathbf{x}-\mathbf{y})} \cdot a_1(\mathbf{y},\xi)\, b_0(\mathbf{y},\xi)\, \omega^{n-2}\, \psi(\mathbf{x})\, d\mathbf{x}\, d\omega\, d\xi \\
&+ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega\nabla_{\mathbf{y}}\phi\cdot(\mathbf{x}-\mathbf{y})}\, b_0(\mathbf{y},\xi) \cdot \Big(\sum_{p=1}^{n} \frac{\partial a_0}{\partial y_p} (x_p - y_p) \Big)\, \omega^{n-1}\, \psi(\mathbf{x})\, d\mathbf{x}\, d\omega\, d\xi \\
&+ \frac{i}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega\nabla_{\mathbf{y}}\phi\cdot(\mathbf{x}-\mathbf{y})}\, a_0(\mathbf{y},\xi)\, b_0(\mathbf{y},\xi)\, \omega^{n}\, \psi(\mathbf{x}) \cdot \Big(\frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\partial^2\phi}{\partial y_p \partial y_q} (x_p - y_p)(x_q - y_q) \Big)\, d\mathbf{x}\, d\omega\, d\xi \; .
\end{aligned} \tag{2.28}$$

$$(\mathfrak{C}_2(\psi))(\mathbf{y}) = -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega\nabla_{\mathbf{y}}\phi\cdot(\mathbf{x}-\mathbf{y})}$$

$$\cdot\; a_0(\mathbf{y},\xi)\; b_2(\mathbf{y},\xi)\; \omega^{n-3}\; \psi(\mathbf{x})\; d\mathbf{x}\; d\omega\; d\xi$$

$$-\frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega\nabla_{\mathbf{y}}\phi\cdot(\mathbf{x}-\mathbf{y})}$$

$$\cdot\; a_1(\mathbf{y},\xi)\; b_1(\mathbf{y},\xi)\; \omega^{n-3}\; \psi(\mathbf{x})\; d\mathbf{x}\; d\omega\; d\xi$$

$$-\frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega\nabla_{\mathbf{y}}\phi\cdot(\mathbf{x}-\mathbf{y})}$$

$$\cdot\; a_2(\mathbf{y},\xi)\; b_0(\mathbf{y},\xi)\; \omega^{n-3}\; \psi(\mathbf{x})\; d\mathbf{x}\; d\omega\; d\xi$$

$$-\frac{i}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega\nabla_{\mathbf{y}}\phi\cdot(\mathbf{x}-\mathbf{y})}\; b_1(\mathbf{y},\xi)$$

$$\cdot \left(\sum_{p=1}^{n} \frac{\partial a_0}{\partial y_p} (x_p - y_p) \right) \omega^{n-2}\; \psi(\mathbf{x})\; d\mathbf{x}\; d\omega\; d\xi$$

$$-\frac{i}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega\nabla_{\mathbf{y}}\phi\cdot(\mathbf{x}-\mathbf{y})}\; b_0(\mathbf{y},\xi)$$

$$\cdot \left(\sum_{p=1}^{n} \frac{\partial a_1}{\partial y_p} (x_p - y_p) \right) \omega^{n-2}\; \psi(\mathbf{x})\; d\mathbf{x}\; d\omega\; d\xi$$

$$+\frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega\nabla_{\mathbf{y}}\phi\cdot(\mathbf{x}-\mathbf{y})}\; a_0(\mathbf{y},\xi)\; b_1(\mathbf{y},\xi)\; \omega^{n-1}\; \psi(\mathbf{x})$$

$$\cdot \left(\frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\partial^2\phi}{\partial y_p \partial y_q} (x_p - y_p)(x_q - y_q) \right) d\mathbf{x}\; d\omega\; d\xi$$

$$+\frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega\nabla_{\mathbf{y}}\phi\cdot(\mathbf{x}-\mathbf{y})}\; a_1(\mathbf{y},\xi)\; b_0(\mathbf{y},\xi)\; \omega^{n-1}\; \psi(\mathbf{x})$$

$$\cdot \left(\frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\partial^2\phi}{\partial y_p \partial y_q} (x_p - y_p)(x_q - y_q) \right) d\mathbf{x}\; d\omega\; d\xi$$

$$+ \frac{i}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega \nabla_{\mathbf{y}}\phi \cdot (\mathbf{x}-\mathbf{y})}$$

$$\cdot \left(\frac{1}{6} \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{r=1}^{n} \frac{\partial^3 \phi}{\partial y_p \partial y_q \partial y_r} (x_p - y_p)(x_q - y_q)(x_r - y_r) \right)$$

$$\cdot\ a_0(\mathbf{y},\boldsymbol{\xi})\ b_0(\mathbf{y},\boldsymbol{\xi})\ \omega^n\ \psi(\mathbf{x})\ d\mathbf{x}\ d\omega\ d\boldsymbol{\xi}$$

$$+ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega \nabla_{\mathbf{y}}\phi \cdot (\mathbf{x}-\mathbf{y})}\ b_0(\mathbf{y},\boldsymbol{\xi})\ \omega^{n-1}\ \psi(\mathbf{x})$$

$$\cdot \left(\frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\partial^2 a_0}{\partial y_p \partial y_q} (x_p - y_p)(x_q - y_q) \right) d\mathbf{x}\ d\omega\ d\boldsymbol{\xi}$$

$$+ \frac{i}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega \nabla_{\mathbf{y}}\phi \cdot (\mathbf{x}-\mathbf{y})}$$

$$\cdot \left(\frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\partial^2 \phi}{\partial y_p \partial y_q} (x_p - y_p)(x_q - y_q) \right)$$

$$\cdot \left(\sum_{p=1}^{n} \frac{\partial a_0}{\partial y_p} (x_p - y_p) \right) b_0(\mathbf{y},\boldsymbol{\xi})\ \omega^n\ \psi(\mathbf{x})\ d\mathbf{x}\ d\omega\ d\boldsymbol{\xi}$$

$$- \frac{1}{2(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega \nabla_{\mathbf{y}}\phi \cdot (\mathbf{x}-\mathbf{y})}\ a_0(\mathbf{y},\boldsymbol{\xi})\ b_0(\mathbf{y},\boldsymbol{\xi})\ \omega^{n+1}\ \psi(\mathbf{x})$$

$$\cdot \left(\frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\partial^2 \phi}{\partial y_p \partial y_q} (x_p - y_p)(x_q - y_q) \right)^2 d\mathbf{x}\ d\omega\ d\boldsymbol{\xi}\ . \quad (2.29)$$

It is not obvious at this point what motivates the definitions of the operators $\mathfrak{C}_0$, $\mathfrak{C}_1$, and $\mathfrak{C}_2$ given in equations (2.27), (2.28) and the formidable (2.29) in the decomposition of $\mathfrak{F}$ in equation (2.26). The operator $\mathfrak{C}_0$ is defined by carrying out the multiplications in equation (2.25) and collecting the terms that are zero order in $\frac{1}{\omega}$. Similarly, the operators $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are defined by collecting terms that are first and second order in $\frac{1}{\omega}$, respectively. In general, the operator $\mathfrak{C}_m$ is constructed to be of order $\left(\frac{1}{\omega}\right)^m$, although the complexity of the form of the operator grows exponentially for $m > 2$. The main objective of the forthcoming discussion is to show the operators have the claimed order.

We make a change of variables following Beylkin [3] taking the coordinates $(\omega, \xi_1, \ldots, \xi_{n-1})$ into the coordinates $(k_1, k_2, \ldots, k_n)$ defined by

$$\mathbf{k} = \omega\, \nabla_{\mathbf{y}} \phi(\mathbf{y}, \boldsymbol{\xi}) \ . \tag{2.30}$$

The transformation has the corresponding matrix given by

$$\frac{\partial(k_1, k_2, \ldots, k_n)}{\partial(\omega, \xi_1, \ldots, \xi_{n-1})} = \begin{bmatrix} \dfrac{\partial\phi}{\partial y_1} & \cdots & \dfrac{\partial\phi}{\partial y_n} \\ \omega \dfrac{\partial^2\phi}{\partial y_1 \partial \xi_1} & \cdots & \omega \dfrac{\partial^2\phi}{\partial y_n \partial \xi_1} \\ \vdots & & \vdots \\ \omega \dfrac{\partial^2\phi}{\partial y_1 \partial \xi_{n-1}} & \cdots & \omega \dfrac{\partial^2\phi}{\partial y_n \partial \xi_{n-1}} \end{bmatrix} . \tag{2.31}$$

Moreover, the Beylkin change of variables can be written in terms of differential forms,

$$d\mathbf{k} = \omega^{n-1}\, h(\mathbf{y}, \boldsymbol{\xi})\, d\omega\, d\boldsymbol{\xi} \ , \tag{2.32}$$

where $h(\mathbf{y}, \boldsymbol{\xi})$ denotes the Jacobian determinant

$$h(\mathbf{y}, \boldsymbol{\xi}) = \det \begin{bmatrix} \dfrac{\partial\phi}{\partial y_1} & \cdots & \dfrac{\partial\phi}{\partial y_n} \\ \dfrac{\partial^2\phi}{\partial y_1 \partial \xi_1} & \cdots & \dfrac{\partial^2\phi}{\partial y_n \partial \xi_1} \\ \vdots & & \vdots \\ \dfrac{\partial^2\phi}{\partial y_1 \partial \xi_{n-1}} & \cdots & \dfrac{\partial^2\phi}{\partial y_n \partial \xi_{n-1}} \end{bmatrix} . \tag{2.33}$$

By performing the Beylkin transformation (2.30), the operators $\mathfrak{T}_0, \mathfrak{T}_1, \mathfrak{T}_2, \ldots, \mathfrak{T}_M$ are converted into the more recognizable combination of an n-dimensional Fourier transform and its inverse transform. The operator can not be directly integrated in this form as a result of the complicated dependence of ω and $\boldsymbol{\xi}$ on $\mathbf{k}$. The result for the operator $\mathfrak{T}_0$ is as follows. This operator is clearly of order zero in $\frac{1}{\omega}$.

$$\left(\mathfrak{T}_0(\psi)\right)(\mathbf{y})$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{a_0(\mathbf{y},\boldsymbol{\xi})}{h(\mathbf{y},\boldsymbol{\xi})} b_0(\mathbf{y},\boldsymbol{\xi})\ \psi(\mathbf{x})\ d\mathbf{x}\ d\mathbf{k}\ . \qquad (2.34)$$

The operator $\mathfrak{T}_1$ that is first order in $\frac{1}{\omega}$, as will be established below, takes the following form.

$$\left(\mathfrak{T}_1(\psi)\right)(\mathbf{y})$$

$$= -\frac{i}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{a_0(\mathbf{y},\boldsymbol{\xi})}{h(\mathbf{y},\boldsymbol{\xi})} b_1(\mathbf{y},\boldsymbol{\xi}) \frac{1}{\omega} \psi(\mathbf{x})\ d\mathbf{x}\ d\mathbf{k}$$

$$- \frac{i}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{a_1(\mathbf{y},\boldsymbol{\xi})}{h(\mathbf{y},\boldsymbol{\xi})} b_0(\mathbf{y},\boldsymbol{\xi}) \frac{1}{\omega} \psi(\mathbf{x})\ d\mathbf{x}\ d\mathbf{k}$$

$$+ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{h(\mathbf{y},\boldsymbol{\xi})} b_0(\mathbf{y},\boldsymbol{\xi})\ \psi(\mathbf{x})$$

$$\cdot \left(\sum_{p=1}^{n} \frac{\partial a_0}{\partial y_p} (x_p - y_p) \right) d\mathbf{x}\ d\mathbf{k}$$

$$+ \frac{i}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{a_0(\mathbf{y},\boldsymbol{\xi})}{h(\mathbf{y},\boldsymbol{\xi})} b_0(\mathbf{y},\boldsymbol{\xi})\ \omega\ \psi(\mathbf{x})$$

$$\cdot \left(\frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\partial^2 \phi}{\partial y_p \partial y_q} (x_p - y_p)(x_q - y_q) \right) d\mathbf{x}\ d\mathbf{k}\ . \qquad (2.35)$$

The operator $\mathfrak{C}_2$ that is second order in $\frac{1}{\omega}$ is given by the complicated expression

$$\big(\mathfrak{C}_2(\psi)\big)(\mathbf{y})$$

$$= -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{a_0(\mathbf{y},\xi)}{h(\mathbf{y},\xi)}\,b_2(\mathbf{y},\xi)\,\frac{1}{\omega^2}\,\psi(\mathbf{x})\,d\mathbf{x}\,d\mathbf{k}$$

$$-\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{a_1(\mathbf{y},\xi)}{h(\mathbf{y},\xi)}\,b_1(\mathbf{y},\xi)\,\frac{1}{\omega^2}\,\psi(\mathbf{x})\,d\mathbf{x}\,d\mathbf{k}$$

$$-\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{a_2(\mathbf{y},\xi)}{h(\mathbf{y},\xi)}\,b_0(\mathbf{y},\xi)\,\frac{1}{\omega^2}\,\psi(\mathbf{x})\,d\mathbf{x}\,d\mathbf{k}$$

$$-\frac{i}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{1}{h(\mathbf{y},\xi)}\,b_1(\mathbf{y},\xi)\,\frac{1}{\omega}\,\psi(\mathbf{x})$$

$$\cdot\Big(\sum_{p=1}^{n}\frac{\partial a_0}{\partial y_p}(x_p-y_p)\Big)\,d\mathbf{x}\,d\mathbf{k}$$

$$-\frac{i}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{1}{h(\mathbf{y},\xi)}\,b_0(\mathbf{y},\xi)\,\frac{1}{\omega}\,\psi(\mathbf{x})$$

$$\cdot\Big(\sum_{p=1}^{n}\frac{\partial a_1}{\partial y_p}(x_p-y_p)\Big)\,d\mathbf{x}\,d\mathbf{k}$$

$$+\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{a_0(\mathbf{y},\xi)}{h(\mathbf{y},\xi)}\,b_1(\mathbf{y},\xi)\,\psi(\mathbf{x})$$

$$\cdot\Big(\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{\partial^2\phi}{\partial y_p\partial y_q}(x_p-y_p)(x_q-y_q)\Big)\,d\mathbf{x}\,d\mathbf{k}$$

$$+\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{a_1(\mathbf{y},\xi)}{h(\mathbf{y},\xi)}\,b_0(\mathbf{y},\xi)\,\psi(\mathbf{x})$$

$$\cdot\Big(\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{\partial^2\phi}{\partial y_p\partial y_q}(x_p-y_p)(x_q-y_q)\Big)\,d\mathbf{x}\,d\mathbf{k}$$

$$+ \frac{i}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{a_0(\mathbf{y},\xi)}{h(\mathbf{y},\xi)}$$

$$\cdot \left(\frac{1}{6} \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{r=1}^{n} \frac{\partial^3 \phi}{\partial y_p \partial y_q \partial y_r} (x_p - y_p)(x_q - y_q)(x_r - y_r) \right)$$

$$\cdot\ b_0(\mathbf{y},\xi)\ \omega\ \psi(\mathbf{x})\ d\mathbf{x}\ d\mathbf{k}$$

$$+ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{h(\mathbf{y},\xi)} b_0(\mathbf{y},\xi)\ \psi(\mathbf{x})$$

$$\cdot \left(\frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\partial^2 a_0}{\partial y_p \partial y_q} (x_p - y_p)(x_q - y_q) \right) d\mathbf{x}\ d\mathbf{k}$$

$$+ \frac{i}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{h(\mathbf{y},\xi)}$$

$$\cdot \left(\sum_{p=1}^{n} \frac{\partial a_0}{\partial y_p} (x_p - y_p) \right) b_0(\mathbf{y},\xi)\ \omega\ \psi(\mathbf{x})$$

$$\cdot \left(\frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\partial^2 \phi}{\partial y_p \partial y_q} (x_p - y_p)(x_q - y_q) \right) d\mathbf{x}\ d\mathbf{k}$$

$$- \frac{1}{2(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{a_0(\mathbf{y},\xi)}{h(\mathbf{y},\xi)} b_0(\mathbf{y},\xi)\ \omega^2\ \psi(\mathbf{x})$$

$$\cdot \left(\frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\partial^2 \phi}{\partial y_p \partial y_q} (x_p - y_p)(x_q - y_q) \right)^2 d\mathbf{x}\ d\mathbf{k}\ . \qquad (2.36)$$

For obvious reasons, we consider only the operators $\mathfrak{C}_0$ and $\mathfrak{C}_1$ in detail. However, the forthcoming techniques can be equally applied to $\mathfrak{C}_2$ and higher order operators, despite the extreme complexity of their expressions.

In equation (2.35) for the operator $\mathfrak{C}_1$, it is now desirable to replace the polynomials $(\mathbf{x} - \mathbf{y})^{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$ is an arbitrary multi-index, with derivatives $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}$ with respect to the transformation $\mathbf{k} = \omega\ \nabla_{\mathbf{y}}\phi$. This replacement is accomplished through a combination of integration by parts and the nature of differentiation of the Fourier kernel. If u and v are functions belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ (space of C^∞ functions on $\mathbb{R}^n$ rapidly decaying at infinity, see Treves [33] or Helgason [20]), then integration by parts can be expressed in terms of multi-index notation by the formula

$$\int_{\mathbb{R}^n} u \, \partial_{\mathbf{k}}^{\alpha} v \, d\mathbf{k} = (-1)^{|\alpha|} \int_{\mathbb{R}^n} v \, \partial_{\mathbf{k}}^{\alpha} u \, d\mathbf{k} . \tag{2.37}$$

The relevant property of the Fourier kernel is given by

$$\frac{1}{i^{|\alpha|}} \partial_{\mathbf{k}}^{\alpha} \left(e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \right) = (\mathbf{x}-\mathbf{y})^{\alpha} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} . \tag{2.38}$$

The Fourier kernel property (2.38) is applied to the expression (2.35) for the $\mathfrak{C}_1$ operator to replace powers of $(\mathbf{x}-\mathbf{y})$ with the corresponding derivatives with respect to $\mathbf{k}$ on the exponents. We then integrate by parts using equation (2.37) in order to remove the derivative with respect to $\mathbf{k}$ from the exponents to the remaining functions in the terms of equation (2.35). The amplitude functions in (2.35) decay rapidly at infinity as a result of the radiation condition. Applying the above Fourier kernel property, integration by parts, radiation condition, and freely interchanging the order of integration results in another relation for the $\mathfrak{C}_1$ operator.

$$\left(\mathfrak{T}_1(\psi)\right)(\mathbf{y})$$

$$= -\frac{i}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{a_0(\mathbf{y},\boldsymbol{\xi})}{h(\mathbf{y},\boldsymbol{\xi})}\,b_1(\mathbf{y},\boldsymbol{\xi})\,\frac{1}{\omega}\,\psi(\mathbf{x})\,d\mathbf{x}\,d\mathbf{k}$$

$$-\frac{i}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{a_1(\mathbf{y},\boldsymbol{\xi})}{h(\mathbf{y},\boldsymbol{\xi})}\,b_0(\mathbf{y},\boldsymbol{\xi})\,\frac{1}{\omega}\,\psi(\mathbf{x})\,d\mathbf{x}\,d\mathbf{k}$$

$$+\frac{i}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$$

$$\cdot\left\{\sum_{p=1}^{n}\frac{\partial}{\partial k_p}\left(\frac{\partial a_0}{\partial y_p}\,\frac{b_0(\mathbf{y},\boldsymbol{\xi})}{h(\mathbf{y},\boldsymbol{\xi})}\right)\right\}\psi(\mathbf{x})\,d\mathbf{x}\,d\mathbf{k}$$

$$-\frac{i}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\psi(\mathbf{x})$$

$$\cdot\left\{\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{\partial^2}{\partial k_p\partial k_q}\left(\frac{a_0(\mathbf{y},\boldsymbol{\xi})}{h(\mathbf{y},\boldsymbol{\xi})}\,\frac{\partial^2\phi}{\partial y_p\partial y_q}\,b_0(\mathbf{y},\boldsymbol{\xi})\,\omega\right)\right\}d\mathbf{x}\,d\mathbf{k}$$

(2.39)

It is necessary to replace the derivatives with respect to $\mathbf{k}$ by the equivalent differential operators with respect to ω and $\boldsymbol{\xi}$. This is desirable since ω and $\boldsymbol{\xi}$ are the natural variables for the phase and amplitude functions. In order to determine explicit expressions for the differential operators $\partial/\partial k_p$, we calculate the inverse of the change of variables transformation matrix of equation (2.31). Specifically, we are interested in calculating the matrix

$$\frac{\partial(\omega,\xi_1,\ldots,\xi_{n-1})}{\partial(k_1,k_2,\ldots,k_n)} = \left[\frac{\partial(k_1,k_2,\ldots,k_n)}{\partial(\omega,\xi_1,\ldots,\xi_{n-1})}\right]^{-1} . \qquad (2.40)$$

The desired transformation matrix is obtained by matrix inversion and the result is given by the following.

$$\frac{\partial(\omega,\xi_1,\ldots,\xi_{n-1})}{\partial(k_1,k_2,\ldots,k_n)} = \begin{bmatrix} D_1(\mathbf{y},\boldsymbol{\xi}) & \frac{1}{\omega}E_{11}(\mathbf{y},\boldsymbol{\xi}) & \cdots & \frac{1}{\omega}E_{1\;n-1}(\mathbf{y},\boldsymbol{\xi}) \\ D_2(\mathbf{y},\boldsymbol{\xi}) & \frac{1}{\omega}E_{21}(\mathbf{y},\boldsymbol{\xi}) & \cdots & \frac{1}{\omega}E_{2\;n-1}(\mathbf{y},\boldsymbol{\xi}) \\ \vdots & \vdots & & \vdots \\ D_n(\mathbf{y},\boldsymbol{\xi}) & \frac{1}{\omega}E_{n1}(\mathbf{y},\boldsymbol{\xi}) & \cdots & \frac{1}{\omega}E_{n\;n-1}(\mathbf{y},\boldsymbol{\xi}) \end{bmatrix}, \tag{2.41}$$

where $h(\mathbf{y},\boldsymbol{\xi})$ is the determinant found in equation (2.33). The cofactors $D_p(\mathbf{y},\boldsymbol{\xi})$, $(p = 1,\ldots,n)$, are explicitly given by

$$D_p = \frac{(-1)^{p+1}}{h(\mathbf{y},\boldsymbol{\xi})}\det\begin{bmatrix} \frac{\partial^2\phi}{\partial y_1\partial\xi_1} & \cdots & \frac{\partial^2\phi}{\partial y_{p-1}\partial\xi_1} & \frac{\partial^2\phi}{\partial y_{p+1}\partial\xi_1} & \cdots & \frac{\partial^2\phi}{\partial y_n\partial\xi_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial^2\phi}{\partial y_1\partial\xi_{n-1}} & \cdots & \frac{\partial^2\phi}{\partial y_{p-1}\partial\xi_{n-1}} & \frac{\partial^2\phi}{\partial y_{p+1}\partial\xi_{n-1}} & \cdots & \frac{\partial^2\phi}{\partial y_n\partial\xi_{n-1}} \end{bmatrix}. \tag{2.42}$$

The determinants $E_{pq}(\mathbf{y},\boldsymbol{\xi})$, $(p = 1,\ldots,n;\ q = 1,\ldots,n)$, are given by the following.

$$E_{pq} = \frac{(-1)^{p+q+1}}{h(\mathbf{y},\boldsymbol{\xi})} \det \begin{bmatrix} \frac{\partial\phi}{\partial y_1} & \cdots & \frac{\partial\phi}{\partial y_{p-1}} & \frac{\partial\phi}{\partial y_{p+1}} & \cdots & \frac{\partial\phi}{\partial y_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial^2\phi}{\partial y_1 \partial\xi_{q-2}} & \cdots & \frac{\partial^2\phi}{\partial y_{p-1}\partial\xi_{q-2}} & \frac{\partial^2\phi}{\partial y_{p+1}\partial\xi_{q-2}} & \cdots & \frac{\partial^2\phi}{\partial y_n\partial\xi_{q-2}} \\ \frac{\partial^2\phi}{\partial y_1 \partial\xi_q} & \cdots & \frac{\partial^2\phi}{\partial y_{p-1}\partial\xi_q} & \frac{\partial^2\phi}{\partial y_{p+1}\partial\xi_q} & \cdots & \frac{\partial^2\phi}{\partial y_n\partial\xi_q} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial^2\phi}{\partial y_1 \partial\xi_{n-1}} & \cdots & \frac{\partial^2\phi}{\partial y_{p-1}\partial\xi_{n-1}} & \frac{\partial^2\phi}{\partial y_{p+1}\partial\xi_{n-1}} & \cdots & \frac{\partial^2\phi}{\partial y_n\partial\xi_{n-1}} \end{bmatrix} . \tag{2.43}$$

Thus, the nature of $\partial/\partial k_p$ as a differential operator acting on C^∞ functions is established through an application of the chain rule,

$$\frac{\partial}{\partial k_p} = D_p(\mathbf{y},\boldsymbol{\xi}) \frac{\partial}{\partial\omega} + \frac{1}{\omega} E_{p1}(\mathbf{y},\boldsymbol{\xi}) \frac{\partial}{\partial\xi_1} + \cdots + \frac{1}{\omega} E_{p\,n-1}(\mathbf{y},\boldsymbol{\xi}) \frac{\partial}{\partial\xi_{n-1}} . \tag{2.44}$$

Replacing the differential operators $\partial/\partial k_p$ in equation (2.39) with the above expression (2.44) produces the desired form for the $\mathfrak{C}_1$ operator,

$$\begin{aligned}
&\left(\mathfrak{C}_1(\psi)\right)(\mathbf{y}) \\
&= -\frac{i}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{a_0(\mathbf{y},\xi)}{h(\mathbf{y},\xi)} b_1(\mathbf{y},\xi) \frac{1}{\omega} \psi(\mathbf{x})\, d\mathbf{x}\, d\mathbf{k} \\
&\quad - \frac{i}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{a_1(\mathbf{y},\xi)}{h(\mathbf{y},\xi)} b_0(\mathbf{y},\xi) \frac{1}{\omega} \psi(\mathbf{x})\, d\mathbf{x}\, d\mathbf{k} \\
&\quad + \frac{i}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \\
&\qquad \cdot \left\{ \sum_{p=1}^{n} \sum_{q=1}^{n-1} E_{pq} \frac{\partial}{\partial \xi_q} \left(\frac{\partial a_0}{\partial y_p} \frac{b_0(\mathbf{y},\xi)}{h(\mathbf{y},\xi)} \right) \right\} \frac{1}{\omega} \psi(\mathbf{x})\, d\mathbf{x}\, d\mathbf{k} \\
&\quad - \frac{i}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \\
&\qquad \cdot \left\{ \frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{r=1}^{n-1} E_{pr} \frac{\partial}{\partial \xi_r} \left\{ D_q + \sum_{s=1}^{n-1} E_{qs} \frac{\partial}{\partial \xi_s} \right\} \right. \\
&\qquad \cdot \left. \left(\frac{a_0(\mathbf{y},\xi)}{h(\mathbf{y},\xi)} \frac{\partial^2 \phi}{\partial y_p \partial y_q} b_0(\mathbf{y},\xi) \right) \right\} \frac{1}{\omega} \psi(\mathbf{x})\, d\mathbf{x}\, d\mathbf{k}\,. \qquad (2.45)
\end{aligned}$$

We have previously noticed that the operator $\mathfrak{C}_0$ found in equation (2.34) is explicitly of order zero in $\frac{1}{\omega}$. Furthermore, it is also clear that the operator $\mathfrak{C}_1$ in equation (2.45) above is explicitly first order in $\frac{1}{\omega}$. If a similar examination of the complicated expressions found in equations (2.29) and (2.36) is performed, it can be shown that the operator $\mathfrak{C}_2$ is explicitly second order in $\frac{1}{\omega}$. Although this claim will not be demonstrated, it is a result of the discussion in the following section.

It is also worth noticing that the operators $\mathfrak{C}_0$ and $\mathfrak{C}_1$ found in equations (2.34) and (2.45) are explicitly pseudodifferential operators of the form of equation (1.60). This follows since we can view $\xi = \xi(\mathbf{k})$ and $\omega = \omega(\mathbf{k})$ as functions of $\mathbf{k}$.

2.3 Relationship between Taylor Polynomials and Asymptotics

The key result that the operator $\mathfrak{C}_m$ is of order $\left(\frac{1}{\omega}\right)^m$ follows as a direct result of the construction of the operator in the decomposition of the Fourier integral operator $\mathfrak{F}$ in equation (2.26). However, the reasons for the collection of certain terms in the operators $\mathfrak{C}_0$, $\mathfrak{C}_1$, and $\mathfrak{C}_2$ in equations (2.27), (2.28), and (2.29), respectively, become apparent only after the extensive algebraic manipulations of the previous section. In particular, it is to be noticed that the polynomial $(\mathbf{x} - \mathbf{y})^{\boldsymbol{\alpha}}$ occurring in the operators is exactly related to $\left(\frac{1}{\omega}\right)^{|\boldsymbol{\alpha}|}$ after the lengthy calculations. This fundamental phenomenon of the Fourier integral operator expansion is now illustrated.

The amplitude functions appearing in the $\mathfrak{C}_m$ operators are always of the form $\omega^{\beta}\, \mu(\mathbf{y},\boldsymbol{\xi})$ where β is an integer. Consider the multiplication of this amplitude function by the polynomial $(\mathbf{x} - \mathbf{y})^{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$ is any multi-index, in the generic integral

$$J = \int_{\mathbb{R}^n} \omega^{\beta}\, \mu(\mathbf{y},\boldsymbol{\xi})\, (\mathbf{x} - \mathbf{y})^{\boldsymbol{\alpha}}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\, d\mathbf{k} \;. \tag{2.46}$$

The integration by parts formula of equation (2.37) and Fourier kernel property of equation (2.38) lead to the equivalent integral

$$J = (-1)^{|\boldsymbol{\alpha}|} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\, \partial_{\mathbf{k}}^{\boldsymbol{\alpha}} \left(\omega^{\beta}\, \mu(\mathbf{y},\boldsymbol{\xi})\right) d\mathbf{k} \;. \tag{2.47}$$

Replacement of the differential operator $\partial_{\mathbf{k}}^{\boldsymbol{\alpha}}$ by the expression (2.44) results in the following explicit integral.

$$J = (-1)^{|\boldsymbol{\alpha}|} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \cdot \prod_{p=1}^{n} \left\{ D_p(\mathbf{y},\boldsymbol{\xi}) \frac{\partial}{\partial\omega} + \frac{1}{\omega} \sum_{q=1}^{n-1} E_{pq}(\mathbf{y},\boldsymbol{\xi}) \frac{\partial}{\partial\xi_p} \right\}^{\alpha_p} \left(\omega^{\beta}\, \mu(\mathbf{y},\boldsymbol{\xi})\right) d\mathbf{k} \;. \tag{2.48}$$

Applying the partial differential operator appearing in equation (2.48) to the function in parentheses produces

$$J = (-1)^{|\boldsymbol{\alpha}|} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{\omega^{|\boldsymbol{\alpha}|}}$$

$$\cdot \prod_{\substack{p=1 \\ (\alpha_p > 0)}}^{n} \prod_{r=1}^{\alpha_p} \left\{ (\beta - r)\, D_p(\mathbf{y},\boldsymbol{\xi}) + \sum_{q=1} \frac{\partial}{\partial \xi_q} E_{pq}(\mathbf{y},\boldsymbol{\xi}) \frac{\partial}{\partial \xi_q} \right\}$$

$$\cdot \left(\omega^{\beta}\, \mu(\mathbf{y},\boldsymbol{\xi})\right) d\mathbf{k} \,, \qquad (2.49)$$

or equivalently, by lumping the functions of $\mathbf{y}$ and $\boldsymbol{\xi}$ into a single function $T(\mathbf{y},\boldsymbol{\xi})$,

$$J = \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{\omega^{|\boldsymbol{\alpha}|}} \omega^{\beta}\, T(\mathbf{y},\boldsymbol{\xi})\, d\mathbf{k} \,. \qquad (2.50)$$

A comparison between equation (2.46) and (2.50) above immediately shows that the Taylor polynomial $(\mathbf{x} - \mathbf{y})^{\boldsymbol{\alpha}}$ is exactly related to the frequency order $\left(\frac{1}{\omega}\right)^{|\boldsymbol{\alpha}|}$. This indicates that the assumption of small $(\mathbf{x} - \mathbf{y})$ used in the Taylor expansions is equivalent to the assumption of large frequency ω.

Moreover, the generic integral J in equation (2.46) has been expressed explicitly in the form of a pseudodifferential operator (1.60) in equation (2.50). The symbol of the pseudodifferential operator is given by the function $\omega^{\beta-|\boldsymbol{\alpha}|}\, T(\mathbf{y},\boldsymbol{\xi})$ when viewed as a function of $\mathbf{y}$ and $\mathbf{k}$.

2.4 General Asymptotic Integral Equation Inversion Algorithm

We now summarize some key results of the earlier sections and develop an explicit inversion algorithm for the general asymptotic integral equation. The integral equation to be inverted is of the asymptotic form

$$D(\omega,\boldsymbol{\xi}) = \omega^{n-1} \int_{\mathbb{R}^n} \sum_{j=0}^{\infty} \frac{a_j(\mathbf{x},\boldsymbol{\xi})}{(i\omega)^j} e^{i\omega\phi(\mathbf{x},\boldsymbol{\xi})} \psi(\mathbf{x})\, d\mathbf{x} . \tag{2.51}$$

The objective of the inversion is to obtain the unknown function $\psi(\mathbf{x})$ in terms of an asymptotic inverse operator acting on the hyperplane data $D(\omega,\boldsymbol{\xi})$ given by

$$\psi(\mathbf{y}) \sim \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \sum_{k=0}^{\infty} \frac{b_k(\mathbf{y},\boldsymbol{\xi})}{(i\omega)^k} e^{-i\omega\phi(\mathbf{y},\boldsymbol{\xi})} D(\omega,\boldsymbol{\xi})\, d\omega\, d\boldsymbol{\xi} . \tag{2.52}$$

Combining the two integral equations (2.51) and (2.52) above leads to the consideration of the Fourier integral operator $\mathcal{F}$ with unknown amplitude kernel functions $b_k(\mathbf{y},\boldsymbol{\xi})$. Specifically,

$$\big(\mathcal{F}(\psi)\big)(\mathbf{y}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\omega\phi(\mathbf{x},\boldsymbol{\xi}) - i\omega\phi(\mathbf{y},\boldsymbol{\xi})}$$

$$\cdot \sum_{j=0}^{\infty} \frac{a_j(\mathbf{x},\boldsymbol{\xi})}{(i\omega)^j} \sum_{k=0}^{\infty} \frac{b_k(\mathbf{y},\boldsymbol{\xi})}{(i\omega)^k} \psi(\mathbf{x})\, \omega^{n-1}\, d\mathbf{x}\, d\omega\, d\boldsymbol{\xi} . \tag{2.53}$$

Through the series of manipulations presented in the previous discussion and based on the Beylkin transformation,

$$\mathbf{k} = \omega\, \nabla_{\mathbf{y}}\phi(\mathbf{y},\boldsymbol{\xi}) , \tag{2.54}$$

it is concluded that the Fourier integral operator $\mathcal{F}$ has an asymptotic representation,

$$\mathcal{F} \sim \mathcal{F}_M = \mathfrak{C}_0 + \mathfrak{C}_1 + \cdots + \mathfrak{C}_M , \tag{2.55}$$

where $\mathfrak{C}_m$ is exactly of order $\left(\frac{1}{\omega}\right)^m$. The individual operators $\mathfrak{C}_m$ in the

Fourier integral operator decomposition are given by the expressions

$$\left(\mathbb{C}_m(\psi)\right)(\mathbf{y}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} K_m(\mathbf{y},\xi) \left(\frac{1}{\omega}\right)^m \psi(\mathbf{x})\, d\mathbf{x}\, d\mathbf{k} \,, \tag{2.56}$$

where $K_m(\mathbf{y},\xi)$ are the kernel functions produced by the procedures of the previous sections. As is seen in equation (2.34), the kernel function $K_0(\mathbf{y},\xi)$ is explicitly calculated to be

$$K_0(\mathbf{y},\xi) = \frac{a_0(\mathbf{y},\xi)}{h(\mathbf{y},\xi)}\, b_0(\mathbf{y},\xi) \,, \tag{2.57}$$

where $h(\mathbf{y},\xi)$ is the Jacobian of equation (2.33). Similarly, by observing equation (2.45), the kernel function $K_1(\mathbf{y},\xi)$ is found to be the more complex expression,

$$\begin{aligned} K_1(\mathbf{y},\xi) = &- i\, \frac{a_0(\mathbf{y},\xi)}{h(\mathbf{y},\xi)}\, b_1(\mathbf{y},\xi) - i\, \frac{a_1(\mathbf{y},\xi)}{h(\mathbf{y},\xi)}\, b_0(\mathbf{y},\xi) \\ &+ i \left\{ \sum_{p=1}^{n} \sum_{q=1}^{n-1} E_{pq}(\mathbf{y},\xi) \frac{\partial}{\partial \xi_q} \left(\frac{\partial a_0}{\partial y_p} \frac{b_0(\mathbf{y},\xi)}{h(\mathbf{y},\xi)} \right) \right\} \\ &- i \left\{ \frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{r=1}^{n-1} E_{pr}(\mathbf{y},\xi) \frac{\partial}{\partial \xi_r} \right. \\ &\quad \cdot \left\{ D_q(\mathbf{y},\xi) + \sum_{s=1}^{n-1} E_{qs}(\mathbf{y},\xi) \frac{\partial}{\partial \xi_s} \right\} \\ &\quad \left. \cdot \left(\frac{a_0(\mathbf{y},\xi)}{h(\mathbf{y},\xi)} \frac{\partial^2 \phi}{\partial y_p \partial y_q}\, b_0(\mathbf{y},\xi) \right) \right\} \,, \end{aligned} \tag{2.58}$$

where the determinant functions $D_p(\mathbf{y},\xi)$ and $E_{pq}(\mathbf{y},\xi)$ are defined in equations (2.42) and (2.43).

In order to solve the generalized asymptotic integral equation (2.51), the inversion operator kernel functions $b_0(\mathbf{y},\xi)$, $b_1(\mathbf{y},\xi),\ldots,$ $b_K(\mathbf{y},\xi)$ are chosen such that the decomposition operator kernels

$K_0(\mathbf{y},\xi)$, $K_1(\mathbf{y},\xi),\ldots,$ $K_K(\mathbf{y},\xi)$ satisfy the relations

$$K_0(\mathbf{y},\xi) = 1 \ ,$$

$$K_1(\mathbf{y},\xi) = 0 \ ,$$

$$\vdots$$

$$K_K(\mathbf{y},\xi) = 0 \ . \tag{2.59}$$

This choice of kernel functions forces the Fourier integral operator $\mathcal{F}$ to be asymptotically the identity operator up to a smoothing operator involving $(\frac{1}{\omega})^{K+1}$, specifically,

$$\mathcal{F} \sim \mathbf{I} + \mathbf{0} + \cdots + \mathbf{0}$$

$$+ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} K_{K+1}(\mathbf{y},\xi) \left(\frac{1}{\omega}\right)^{K+1} \psi(\mathbf{x}) \ d\mathbf{x} \ d\mathbf{k}$$

$$+ \cdots \ . \tag{2.60}$$

Solving the equations (2.59) results in the desired expressions for the inverse operator kernel functions $b_0(\mathbf{y},\xi)$, $b_1(\mathbf{y},\xi),\ldots,$ $b_K(\mathbf{y},\xi)$ that define the inverse algorithm of equation (2.52). The first kernel function is obtained by solving the equation $K_0(\mathbf{y},\xi) = 1$. This produces

$$b_0(\mathbf{y},\xi) = \frac{h(\mathbf{y},\xi)}{a_0(\mathbf{y},\xi)} \ . \tag{2.61}$$

The kernel function $b_1(\mathbf{y},\xi)$ is obtained by solving the second equation in (2.59), $K_1(\mathbf{y},\xi) = 0$. Hence, it follows from equation (2.58) that $b_1(\mathbf{y},\xi)$ is given by

$$b_1(\mathbf{y},\xi) = \frac{h(\mathbf{y},\xi)}{a_0(\mathbf{y},\xi)} \left\{ - \frac{a_1(\mathbf{y},\xi)}{a_0(\mathbf{y},\xi)} \right.$$

$$+ \left\{ \sum_{p=1}^{n} \sum_{q=1}^{n-1} E_{pq}(\mathbf{y},\xi) \frac{\partial}{\partial \xi_q} \left(\frac{1}{a_0(\mathbf{y},\xi)} \frac{\partial a_0}{\partial y_p} \right) \right\}$$

$$- \left[\frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{r=1}^{n-1} E_{pr}(\mathbf{y},\xi) \frac{\partial}{\partial \xi_r} \right.$$

$$\left. \left. \cdot \left\{ D_q(\mathbf{y},\xi) + \sum_{s=1}^{n-1} E_{qs}(\mathbf{y},\xi) \frac{\partial}{\partial \xi_s} \right\} \left(\frac{\partial^2 \phi}{\partial y_p \partial y_q} \right) \right] \right\} . \qquad (2.62)$$

Thus, the order zero inversion algorithm to the generalized asymptotic integral equation (2.51) is explicitly given by

$$\psi(\mathbf{y}) \sim \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \frac{h(\mathbf{y},\xi)}{a_0(\mathbf{y},\xi)} D(\omega,\xi)\, e^{-i\omega\phi(\mathbf{y},\xi)}\, d\omega\, d\xi . \qquad (2.63)$$

The first order inversion algorithm to the generalized asymptotic integral equation is given by

$$\psi(\mathbf{y}) \sim \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \frac{h(\mathbf{y},\xi)}{a_0(\mathbf{y},\xi)} \left\{ 1 + \frac{1}{i\omega} \left[- \frac{a_1(\mathbf{y},\xi)}{a_0(\mathbf{y},\xi)} \right. \right.$$

$$+ \sum_{p=1}^{n} \sum_{q=1}^{n-1} E_{pq}(\mathbf{y},\xi) \frac{\partial}{\partial \xi_q} \left(\frac{1}{a_0(\mathbf{y},\xi)} \frac{\partial a_0}{\partial y_p} \right)$$

$$- \frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{r=1}^{n-1} E_{pr}(\mathbf{y},\xi) \frac{\partial}{\partial \xi_r}$$

$$\left. \left. \cdot \left\{ D_q(\mathbf{y},\xi) \frac{\partial^2 \phi}{\partial y_p \partial y_q} + \sum_{s=1}^{n-1} E_{qs}(\mathbf{y},\xi) \frac{\partial}{\partial \xi_s} \left(\frac{\partial^2 \phi}{\partial y_p \partial y_q} \right) \right\} \right] \right\}$$

$$\cdot D(\omega,\xi)\, e^{-i\omega\phi(\mathbf{y},\xi)}\, d\omega\, d\xi . \qquad (2.63)$$

This concludes the discussion of the inversion of the generalized asymptotic integral equation. We now specialize this theory to sound wave propagation in 3-dimensions for constant reference velocity. It will be shown for this special case that equation (2.63) simplifies greatly.

CHAPTER 3

INVERSION FOR A CONSTANT REFERENCE VELOCITY MEDIUM

Thus far, the general Fourier integral operator techniques for the inversion of asymptotic integral equations have been explored. In the present chapter, the inversion techniques are specialized to the backscattered data integral equation developed in Chapter 1. In particular, the wave propagation phenomena investigated are restricted to 3-dimensional space. In addition, the reference velocity of the medium is chosen to be constant and hence independent of position. The resulting backscattered data integral equation constitutes the simplest (but non-trivial and physically meaningful) example to which the general theory of the previous chapter can be applied. Although it is beyond the scope of the present discussion, it is worth mentioning that the constant reference velocity backscattered data inversion problem also encompasses a wide spectrum of significant applications including the seismic imaging of geophysical discontinuities, the non-destructive testing of materials, and medical tomography.

The present chapter starts with a review of wave propagation in a 3-dimensional medium and how it specializes to the constant reference velocity case. The backscattered data integral equation is then cast in this special setting and the Fourier integral operator formulation of the problem is made. This chapter concludes with the development of both the zero and the first order inversion algorithms following the elegant Fourier integral operator methodology of Chapter 2. Moreover, it is discovered that the first order inversion algorithm given in equation (2.63) reduces to the surprisingly simple algorithm (3.54). This result occurs despite the apparent complexity of equation (2.63).

3.1 Wave Propagation with Constant Reference Velocity

In the present section, we consider the wave propagation in a 3-dimensional medium. Hence, the theory is specialized for n=3. The reference velocity is taken to be a constant independent of spatial location. We then solve the eikonal and transport equations with the constant reference velocity. The ray solutions will be shown to follow straight line trajectories. Furthermore, by imposing a certain choice of asymptotic initial conditions, the wave propagation amplitude can be modeled as a spherical wave front amplitude.

For the present discussion, the reference velocity $c_0(\mathbf{x})$ is chosen to be a constant,

$$c_0(\mathbf{x}) = c_0 \ . \tag{3.1}$$

We consider a source wave emanating from the source point $\boldsymbol{\xi} = (\xi_1, \xi_2, 0)$ on the data surface $x_3 = 0$ given by the asymptotic representation

$$u_I(\mathbf{x},\omega,\boldsymbol{\xi}) = A(\mathbf{x},\omega,\boldsymbol{\xi})\, e^{i\omega\tau(\mathbf{x},\boldsymbol{\xi})} \ , \tag{3.2}$$

where $\tau(\mathbf{x},\boldsymbol{\xi})$ is the travel time from the source point $\boldsymbol{\xi}$ to the point at depth $\mathbf{x}$ and where $A(\mathbf{x},\omega,\boldsymbol{\xi})$ is the amplitude function possessing the asymptotic expansion in frequency ω given by

$$A(\mathbf{x},\omega,\boldsymbol{\xi}) = \sum_{j=0}^{\infty} \frac{A_j(\mathbf{x},\boldsymbol{\xi})}{(i\omega)^j} \ . \tag{3.3}$$

Travel time satisfies the eikonal equation for constant reference velocity,

$$\nabla_{\mathbf{x}}\tau \cdot \nabla_{\mathbf{x}}\tau = \frac{1}{c_0^2} \ . \tag{3.4}$$

We make use of the notation $\mathbf{p}$ for the gradient of travel time with respect to $\mathbf{x}$ (sometimes referred to as the slowness vector since its dimensions are inverse velocity),

$$\mathbf{p} = \nabla_{\mathbf{x}}\tau \ . \tag{3.5}$$

From the above, the system of ordinary differential equations generated by the method of characteristics can be written down. The corresponding system of first order equations for the constant reference velocity case are given by

$$\frac{d\mathbf{x}}{d\sigma} = \lambda(\mathbf{x})\ \mathbf{p}\ ,$$

$$\frac{d\mathbf{p}}{d\sigma} = \frac{\lambda(\mathbf{x})}{2}\ \nabla\left(\frac{1}{c_0^2}\right) = 0\ ,$$

$$\frac{d\tau}{d\sigma} = \frac{\lambda(\mathbf{x})}{c_0^2}\ . \tag{3.6}$$

The ray parameter σ corresponds to the arbitrary non-vanishing parameterization function $\lambda(\mathbf{x})$.

For simplicitly in the discussion, we choose the parameterization most common to analytical purposes, specifically,

$$\lambda(\mathbf{x}) \equiv 1\ . \tag{3.7}$$

Other choices of parameterization functions giving arc length or travel time could have been used equally well. With the selected parameterization, the differential equations are integrated to obtain the ray paths. In the constant reference velocity case, the rays are straight lines emanating from the source point $\boldsymbol{\xi}$,

$$x_1 = \xi_1 + \kappa_1\sigma$$

$$x_2 = \xi_2 + \kappa_2\sigma$$

$$x_3 = \sigma\sqrt{\frac{1}{c_0^2} - \kappa_1^2 - \kappa_2^2}\ , \tag{3.8}$$

where κ_1 and κ_2 are constants corresponding to the initial conditions for p_1 and p_2 at $\sigma = 0$. Note that we can solve the system of equations (3.8) for the parameters κ_1, κ_2, and σ from specified source point $\boldsymbol{\xi}$ and depth point $\mathbf{x}$, provided $\mathbf{x} \neq \boldsymbol{\xi}$. Hence, there is a one-to-one correspondence between points at depth $\mathbf{x}$ and the parameters κ_1, κ_2, and σ. We are motivated to consider the curvilinear transformation of the coordinates (q_1, q_2, q_3) to the coordinates (x_1, x_2, x_3) defined by

$$q_1 = \kappa_1 \;, \quad q_2 = \kappa_2 \;, \quad q_3 = \sigma \;. \tag{3.9}$$

The matrix of partial derivatives for this transformation is given by

$$\frac{\partial(x_1, x_2, x_3)}{\partial(q_1, q_2, q_3)} = \begin{bmatrix} q_3 & 0 & q_1 \\ 0 & q_3 & q_2 \\ -\dfrac{q_1 q_3}{\sqrt{\dfrac{1}{c_0^2} - q_1^2 - q_2^2}} & -\dfrac{q_2 q_3}{\sqrt{\dfrac{1}{c_0^2} - q_1^2 - q_2^2}} & \sqrt{\dfrac{1}{c_0^2} - q_1^2 - q_2^2} \end{bmatrix} . \tag{3.10}$$

The determinant J of the above matrix is referred to as the conoidal ray Jacobian,

$$J = \det \frac{\partial(x_1, x_2, x_3)}{\partial(q_1, q_2, q_3)} = \frac{q_3^2}{c_0^2 \sqrt{\dfrac{1}{c_0^2} - q_1^2 - q_2^2}} \;. \tag{3.11}$$

The conoidal ray Jacobian $J(q_3) = J(\sigma)$ plays an important role in the solution of the transport equations.

It has been established in Chapter 1 that the amplitude functions $A_j(\mathbf{x}, \boldsymbol{\xi})$ satisfy the transports equations (1.18) and (1.19). It has also been shown that the transport equations can be reduced to the ordinary differential equations (1.48) and (1.57). Some initial conditions need to be prescribed in order to solve the differential equations. However, the amplitude functions become singular at the initial ray parameter value $\sigma = 0$. Hence, initial conditions are imposed in an asymptotic sense for $\sigma \to 0$. This process essentially involves the determination of the far-field wave propagation from specified near-field conditions. We impose the condition that the impulsive source wave is asymptotically a spherical wave in the near-field. Specifically,

$$A(\mathbf{x},\omega,\xi) \to \frac{Q_0}{4\pi\sigma} , \tag{3.12}$$

as $\sigma \to 0$. The implications of other choices of initial conditions are postponed to the next chapter.

For the constant reference velocity case with the choice of parameterization (3.7), the ordinary differential equation for the first amplitude function reduces to the ray Jacobian invariance property (1.48). The invariance property states that the product of the conoidal ray Jacobian J and the square of the first amplitude function is constant along rays. That is,

$$\frac{d}{d\sigma} \left(J(\sigma)\ A_0(\sigma) \right) = 0 , \tag{3.13}$$

where the notation $A_0(\sigma) = A_0(\mathbf{x}(\sigma),\xi)$ is implied. Substituting the conoidal ray Jacobian (3.11) into the invariance property above shows that $A_0(\sigma)$ is proportional to the reciprocal of σ. Hence, it is then immediately concluded that the first amplitude function is given by

$$A_0(\sigma) = \frac{Q_0}{4\pi\sigma} . \tag{3.14}$$

This is precisely the wave propagation of a spherical wave front in a homogeneous medium.

The ordinary differential equation for the higher order transport equation along rays reduces to

$$\frac{d}{d\sigma}\left(A_j(\sigma)\sqrt{J(\sigma)} \right) = -\frac{1}{2}\sqrt{J(\sigma)}\ \nabla^2 A_{j-1}(\sigma) \ . \tag{3.15}$$

By writing the Laplacian ∇^2 in spherical coordinates, it is quickly observed that the Laplacian of $A_0(\sigma)$ is identically zero. Consequently, $A_1(\sigma)$ satisfies exactly the same differential equation as $A_0(\sigma)$. By examining the imposed initial conditions (3.12) and continuing the argument inductively, it is found that all of the higher order amplitude functions vanish,

$$A_j(\sigma) = 0 \ , \quad (j = 1,2,3,\ldots) \ . \tag{3.16}$$

It is to be noted that the above functions do not necessarily vanish for other types of imposed initial conditions.

3.2 Backscattered Data Integral Equation

In this section, the backscattered data integral equation is specialized to the constant reference velocity example. A candidate inversion algorithm is proposed to invert the backscattered data integral equation. The discussion concludes with many of the calculations necessary for the development of an explicit representation of an inversion algorithm. The details of the inversion algorithm are deferred to the following section.

In Chapter 1, an integral equation (1.24) for the backscattered data configuration is developed relating the index of refraction perturbation $\psi(\mathbf{x})$ to data measurements $D(\omega,\xi)$ at the accessible surface $x_3 = 0$. For the constant reference velocity example, the integral equation simplifies to

$$D(\omega,\xi) = \frac{\omega^2}{c_0^2}\int_{\mathbb{R}^3} a(\mathbf{x},\omega,\xi)\ e^{i\omega\phi(\mathbf{x},\xi)}\ \psi(\mathbf{x})\ d\mathbf{x} \ , \tag{3.17}$$

where the phase is given by

$$\phi(\mathbf{x},\boldsymbol{\xi}) = 2\tau(\mathbf{x},\boldsymbol{\xi}) = \frac{2|\mathbf{x} - \boldsymbol{\xi}|}{c_0} , \tag{3.18}$$

and where the amplitude is found to be

$$a(\mathbf{x},\omega,\boldsymbol{\xi}) = \frac{Q_0^2}{16\pi^2\sigma^2} = \frac{Q_0^2}{16\pi^2 c_0^2|\mathbf{x} - \boldsymbol{\xi}|^2} . \tag{3.19}$$

The above equation for the amplitude is a result of the form of the integral equation (1.24) and the imposed initial conditions (3.12).

The methods developed in the previous chapter are used to invert the integral equation (3.17). As before, a candidate inversion operator is proposed.

$$\psi(\mathbf{y}) \sim \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}} b(\mathbf{y},\omega,\boldsymbol{\xi})\, e^{-i\omega\phi(\mathbf{y},\boldsymbol{\xi})}\, D(\omega,\boldsymbol{\xi})\, d\omega\, d\boldsymbol{\xi} , \tag{3.20}$$

where $d\boldsymbol{\xi}$ denotes the differential form $d\xi_1\ d\xi_2$ and where the unknown kernel function $b(\mathbf{y},\omega,\boldsymbol{\xi})$ has the asymptotic expansion in frequency ω,

$$b(\mathbf{y},\omega,\boldsymbol{\xi}) = \sum_{k=0}^{\infty} \frac{b_k(\mathbf{y},\boldsymbol{\xi})}{(i\omega)^k} . \tag{3.21}$$

An integral operator $\mathcal{F}$ is defined by substituting the integral equation (3.17) into the candidate inversion operator (3.20). The resulting operator is a Fourier integral operator acting on the unknown index of refraction perturbation $\psi(\mathbf{x})$. Specifically,

$$\big(\mathcal{F}(\psi)\big)(\mathbf{y}) = \frac{1}{8\pi^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}^3} b(\mathbf{y},\omega,\boldsymbol{\xi}) \frac{Q_0^2\ \omega^2}{16\pi^2 c_0^4|\mathbf{x} - \boldsymbol{\xi}|^2}$$

$$\cdot\ e^{i\omega\Phi(\mathbf{x},\mathbf{y},\boldsymbol{\xi})}\ \psi(\mathbf{x})\ d\mathbf{x}\ d\omega\ d\boldsymbol{\xi} , \tag{3.22}$$

where the phase function takes the form

$$\Phi(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) = \phi(\mathbf{x},\boldsymbol{\xi}) - \phi(\mathbf{y},\boldsymbol{\xi}) = \frac{2}{c_0}\left(|\mathbf{x} - \boldsymbol{\xi}| - |\mathbf{y} - \boldsymbol{\xi}|\right) . \tag{3.23}$$

The Fourier integral operator (3.22) is made asymptotic to the identity operator by proper choice of the kernel function $b_k(\mathbf{y},\boldsymbol{\xi})$ in accordance with the techniques of Chapter 2.

It is first necessary to expand the Fourier integral operator phase $\Phi(\mathbf{x},\mathbf{y},\boldsymbol{\xi})$ in a Taylor series for small $(\mathbf{x} - \mathbf{y})$ away from the singular point $\boldsymbol{\xi}$,

$$\Phi(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) = \nabla_{\mathbf{y}}\phi(\mathbf{y},\boldsymbol{\xi})\cdot(\mathbf{x} - \mathbf{y}) + \sum_{|\boldsymbol{\alpha}|=2}^{\infty} \frac{1}{\boldsymbol{\alpha}!}\,\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\phi(\mathbf{y},\boldsymbol{\xi})\;(\mathbf{x} - \mathbf{y})^{\boldsymbol{\alpha}}\;. \tag{3.24}$$

We use the Beylkin change of variables,

$$\mathbf{k} = \omega\,\nabla_{\mathbf{y}}\phi(\mathbf{y},\boldsymbol{\xi})\;. \tag{3.25}$$

This change of variables has the following equivalent representation in terms of differential forms:

$$d\mathbf{k} = \omega^2\,h(\mathbf{y},\boldsymbol{\xi})\;d\omega\;d\boldsymbol{\xi}\;, \tag{3.26}$$

where $h(\mathbf{y},\boldsymbol{\xi})$ is the fundamental determinant given by

$$h(\mathbf{y},\boldsymbol{\xi}) = \det\begin{bmatrix} \dfrac{\partial\phi}{\partial y_1} & \dfrac{\partial\phi}{\partial y_2} & \dfrac{\partial\phi}{\partial y_3} \\[2ex] \dfrac{\partial^2\phi}{\partial y_1\partial\xi_1} & \dfrac{\partial^2\phi}{\partial y_2\partial\xi_1} & \dfrac{\partial^2\phi}{\partial y_3\partial\xi_1} \\[2ex] \dfrac{\partial^2\phi}{\partial y_1\partial\xi_2} & \dfrac{\partial^2\phi}{\partial y_2\partial\xi_2} & \dfrac{\partial^2\phi}{\partial y_3\partial\xi_2} \end{bmatrix}. \tag{3.27}$$

In the explicit calculations of the derivatives in the equation above, the notation is greatly simplified by defining r to be the distance from the source point $\boldsymbol{\xi}$ to the output point $\mathbf{y}$.

$$r = |\mathbf{y} - \boldsymbol{\xi}| = \sqrt{(y_1 - \xi_1)^2 + (y_2 - \xi_2)^2 + (y_3 - \xi_3)^2}\;. \tag{3.28}$$

Then the phase $\phi(\mathbf{y},\boldsymbol{\xi})$ is simply

$$\phi(\mathbf{y},\boldsymbol{\xi}) = \frac{2r}{c_0} \; . \tag{3.29}$$

Moreover, the gradient $\nabla_{\mathbf{y}}\phi$ has the components given by

$$\frac{\partial\phi}{\partial y_1} = \frac{2}{c_0 r}(y_1 - \xi_1) \; , \quad \frac{\partial\phi}{\partial y_2} = \frac{2}{c_0 r}(y_2 - \xi_2) \; , \quad \frac{\partial\phi}{\partial y_3} = \frac{2}{c_0 r} y_3 \; . \tag{3.30}$$

By differentiating with respect to $\mathbf{y}$, the following derivatives are obtained.

$$\begin{aligned}
\frac{\partial^2\phi}{\partial y_1^2} &= \frac{2}{c_0 r^3}[(y_2 - \xi_2)^2 + y_3^2] \; , \\
\frac{\partial^2\phi}{\partial y_3^2} &= \frac{2}{c_0 r^3}[(y_1 - \xi_1)^2 + y_3^2] \; , \\
\frac{\partial^2\phi}{\partial y_3^2} &= \frac{2}{c_0 r^3}[(y_1 - \xi_1)^2 + (y_2 - \xi_2)^2] \; , \\
\frac{\partial^2\phi}{\partial y_1 \partial y_2} &= -\frac{2}{c_0 r^3}(y_1 - \xi_1)(y_2 - \xi_2) \; , \\
\frac{\partial^2\phi}{\partial y_1 \partial y_3} &= -\frac{2}{c_0 r^3}(y_1 - \xi_1)\, y_3 \; , \\
\frac{\partial^2\phi}{\partial y_2 \partial y_3} &= -\frac{2}{c_0 r^3}(y_2 - \xi_2)\, y_3 \; .
\end{aligned} \tag{3.31}$$

Similarly, derivatives with respect to $\boldsymbol{\xi}$ are immediately obtained from the expressions above since

$$\frac{\partial^2\phi}{\partial y_p \partial \xi_q} = -\frac{\partial^2\phi}{\partial y_p \partial y_q} \; . \tag{3.32}$$

From the explicit derivatives given in equations (3.30), (3.31), and (3.32) above, several additional quantities can be expressly written out. In particular, the change of variables to $\mathbf{k}$ takes the form

$$\mathbf{k} = \frac{2\omega}{c_0 r}(\mathbf{y} - \boldsymbol{\xi}) \; . \tag{3.33}$$

The determinant $h(\mathbf{y},\boldsymbol{\xi})$ can be derived through a series of straightforward calculations using equations (3.30), (3.31), and (3.32).

$$h(\mathbf{y},\boldsymbol{\xi}) = \frac{8y_3}{c_0^3 r^3} . \tag{3.34}$$

The transformation matrix relating the variables (k_1, k_2, k_3) to the variables (ω, ξ_1, ξ_2) is particularly important.

$$\frac{\partial(k_1, k_2, k_3)}{\partial(\omega, \xi_1, \xi_2)} = \begin{bmatrix} \dfrac{\partial\phi}{\partial y_1} & \dfrac{\partial\phi}{\partial y_2} & \dfrac{\partial\phi}{\partial y_3} \\ \omega \dfrac{\partial^2\phi}{\partial y_1 \partial\xi_1} & \omega \dfrac{\partial^2\phi}{\partial y_2 \partial\xi_1} & \omega \dfrac{\partial^2\phi}{\partial y_3 \partial\xi_1} \\ \omega \dfrac{\partial^2\phi}{\partial y_1 \partial\xi_2} & \omega \dfrac{\partial^2\phi}{\partial y_2 \partial\xi_2} & \omega \dfrac{\partial^2\phi}{\partial y_3 \partial\xi_2} \end{bmatrix} . \tag{3.35}$$

Substituting the expressions for the derivatives results in an explicit representation for the transformation matrix:

$$\begin{bmatrix} \dfrac{2}{c_0 r}(y_1 - \xi_1) & \dfrac{2}{c_0 r}(y_2 - \xi_2) & \dfrac{2}{c_0 r} y_3 \\ -\dfrac{2\omega}{c_0 r^3}[(y_2 - \xi_2)^2 + y_3^2] & \dfrac{2\omega}{c_0 r^3}(y_1 - \xi_1)(y_2 - \xi_2) & \dfrac{2\omega}{c_0 r^3}(y_1 - \xi_1) y_3 \\ \dfrac{2\omega}{c_0 r^3}(y_1 - \xi_1)(y_2 - \xi_2) & -\dfrac{2\omega}{c_0 r^3}[(y_1 - \xi_1)^2 + y_3^2] & \dfrac{2\omega}{c_0 r^3}(y_2 - \xi_2) y_3 \end{bmatrix} . \tag{3.36}$$

The above transformation matrix can be inverted to yield the transformation matrix relating the variables (ω, ξ_1, ξ_2) to the variables (k_1, k_2, k_3). The inverse transformation matrix is obtained by a series of algebraic manipulations and has the form

$$\frac{\partial(\omega,\xi_1,\xi_2)}{\partial(k_1,k_2,k_3)} = \begin{bmatrix} D_1(\mathbf{y},\boldsymbol{\xi}) & \frac{1}{\omega} E_{11}(\mathbf{y},\boldsymbol{\xi}) & \frac{1}{\omega} E_{12}(\mathbf{y},\boldsymbol{\xi}) \\ D_2(\mathbf{y},\boldsymbol{\xi}) & \frac{1}{\omega} E_{21}(\mathbf{y},\boldsymbol{\xi}) & \frac{1}{\omega} E_{22}(\mathbf{y},\boldsymbol{\xi}) \\ D_3(\mathbf{y},\boldsymbol{\xi}) & \frac{1}{\omega} E_{31}(\mathbf{y},\boldsymbol{\xi}) & \frac{1}{\omega} E_{32}(\mathbf{y},\boldsymbol{\xi}) \end{bmatrix} . \tag{3.37}$$

The individual terms of the inverse matrix are given by

$$D_1(\mathbf{y},\boldsymbol{\xi}) = \frac{c_0}{2r}(y_1 - \xi_1) ,$$

$$D_2(\mathbf{y},\boldsymbol{\xi}) = \frac{c_0}{2r}(y_2 - \xi_2) ,$$

$$D_3(\mathbf{y},\boldsymbol{\xi}) = \frac{c_0}{2r} y_3 ,$$

$$E_{11}(\mathbf{y},\boldsymbol{\xi}) = -\frac{c_0 r}{2} , \qquad E_{12}(\mathbf{y},\boldsymbol{\xi}) = 0 ,$$

$$E_{21}(\mathbf{y},\boldsymbol{\xi}) = 0 , \qquad E_{22}(\mathbf{y},\boldsymbol{\xi}) = -\frac{c_0 r}{2} ,$$

$$E_{31}(\mathbf{y},\boldsymbol{\xi}) = \frac{c_0 r}{2y_3}(y_1 - \xi_1) , \quad E_{32}(\mathbf{y},\boldsymbol{\xi}) = \frac{c_0 r}{2y_3}(y_2 - \xi_2) . \tag{3.38}$$

The amplitude function $a(\mathbf{x},\omega,\boldsymbol{\xi})$ needs to also be expanded in a Taylor series for small $(\mathbf{x} - \mathbf{y})$. Since the terms $a_j(\mathbf{y},\boldsymbol{\xi})$ vanish for all $j \geqq 1$, the series takes the following form:

$$a(\mathbf{x},\omega,\boldsymbol{\xi}) = \sum_{|\boldsymbol{\beta}|=0}^{\infty} \frac{1}{\boldsymbol{\beta}!} \partial_{\mathbf{y}}^{\boldsymbol{\beta}} a_0(\mathbf{y},\boldsymbol{\xi}) (\mathbf{x} - \mathbf{y})^{\boldsymbol{\beta}} . \tag{3.39}$$

The first amplitude function $a_0(\mathbf{x},\boldsymbol{\xi})$ in the constant reference velocity case is given by

$$a_0(\mathbf{y},\boldsymbol{\xi}) = \frac{Q_0^2}{16\pi^2 c_0^2 |\mathbf{y} - \boldsymbol{\xi}|^2} = \frac{Q_0^2}{16\pi^2 c_0^2 r^2} . \tag{3.40}$$

The gradient of the first amplitude function with respect to $\mathbf{y}$ is simply

$$\nabla_{\mathbf{y}} a_0(\mathbf{y},\boldsymbol{\xi}) = -\frac{Q_0^2}{8\pi^2 c_0^2 r^4} (\mathbf{y} - \boldsymbol{\xi}) . \tag{3.41}$$

Moreover, the logarithmic derivative of the first amplitude function is calculated by combining equations (3.40) and (3.41).

$$\frac{1}{a_0(\mathbf{y},\boldsymbol{\xi})} \nabla_{\mathbf{y}} a_0(\mathbf{y},\boldsymbol{\xi}) = -\frac{2}{r^2} (\mathbf{y} - \boldsymbol{\xi}) . \tag{3.42}$$

This provides all of the preliminary calculations required for the development of an inversion algorithm for the backscattered data integral equation (3.17). We now turn to the formulation of an explicit inversion algorithm from these calculations.

3.3 Backscattered Data Inversion Algorithm

We now produce the main result of the present chapter. The numerous calculations of the previous section are utilized to derive the zero and first order inversion algorithms to the backscattered data integral equation (3.17). Explicit calculations of the complicated inversion operator amplitude functions $b_0(\mathbf{y},\boldsymbol{\xi})$ and $b_1(\mathbf{y},\boldsymbol{\xi})$ found in equations (2.61) and (2.62) are performed. It is demonstrated that after considerable algebraic manipulation that the amplitude kernel functions $b_0(\mathbf{y},\boldsymbol{\xi})$ and $b_1(\mathbf{y},\boldsymbol{\xi})$ greatly simplify yielding the desired inversion algorithms of equations (3.45) and (3.54).

We begin by determining the zero order kernel function $b_0(\mathbf{y},\boldsymbol{\xi})$. The zero order kernel function can readily be calculated from the expressions (3.34) and (3.40),

$$b_0(\mathbf{y},\boldsymbol{\xi}) = \frac{h(\mathbf{y},\boldsymbol{\xi})}{a_0(\mathbf{y},\boldsymbol{\xi})} = \frac{128\pi^2 y_3}{Q_0^2 c_0 r} . \tag{3.43}$$

Substituting the above kernel function into the inversion operator,

$$\psi(\mathbf{y}) \sim \frac{1}{8\pi^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}} b_0(\mathbf{y},\xi)\, e^{-i\omega\phi(\mathbf{y},\xi)}\, D(\omega,\xi)\, d\omega\, d\xi , \tag{3.44}$$

leads to the zero order inversion algorithm,

$$\psi(\mathbf{y}) \sim \int_{\mathbb{R}^2} \int_{\mathbb{R}} \frac{16y_3}{\pi Q_0^2 c_0 r}\, D(\omega,\xi)\, e^{-i\left(\frac{2r\omega}{c_0}\right)}\, d\omega\, d\xi . \tag{3.45}$$

The inversion algorithm (3.45) obtained using the methods of Beylkin [2,3] can be shown to agree with the inversion algorithm obtained by stationary phase methods of Cohen, Hagin, and Bleistein [9].

The first order kernel function $b_1(\mathbf{y},\xi)$ is considerably more difficult to compute. This function is given by the following formidable expression obtained in Chapter 2:

$$\begin{aligned} b_1(\mathbf{y},\xi) &= \frac{h(\mathbf{y},\xi)}{a_0(\mathbf{y},\xi)} \\ &\cdot \left\{ -\frac{a_1(\mathbf{y},\xi)}{a_0(\mathbf{y},\xi)} + \left\{ \sum_{p=1}^{3} \sum_{q=1}^{2} E_{pq}(\mathbf{y},\xi) \frac{\partial}{\partial \xi_q} \left(\frac{1}{a_0(\mathbf{y},\xi)} \frac{\partial a_0}{\partial y_p} \right) \right\} \right. \\ &\quad - \left[\frac{1}{2} \sum_{p=1}^{3} \sum_{q=1}^{3} \sum_{r=1}^{2} E_{pr}(\mathbf{y},\xi) \frac{\partial}{\partial \xi_r} \right. \\ &\quad \left.\left. \cdot \left\{ D_q(\mathbf{y},\xi) + \sum_{s=1}^{2} E_{qs}(\mathbf{y},\xi) \frac{\partial}{\partial \xi_s} \right\} \left(\frac{\partial^2 \phi}{\partial y_p \partial y_q} \right) \right] \right\} . \end{aligned} \tag{3.46}$$

At this point, all of the individual coefficient functions $D_p(\mathbf{y},\xi)$ and $E_{pq}(\mathbf{y},\xi)$ have been determined along with the partial derivatives of $a_0(\mathbf{y},\xi)$ and $\phi(\mathbf{y},\xi)$ in the previous section.

Only the second and third term contribute in equation (3.46) since the amplitude $a_1(\mathbf{y},\xi)$ vanishes for the selected asymptotic initial conditions (3.12). From equations (3.38) and (3.42), the second term in the first order kernel function $b_1(\mathbf{y},\xi)$ is produced,

$$\sum_{p=1}^{3} \sum_{q=1}^{2} E_{pq}(\mathbf{y},\xi) \frac{\partial}{\partial \xi_q} \left(\frac{1}{a_0(\mathbf{y},\xi)} \frac{\partial a_0}{\partial y_p} \right) = - \frac{2c_0}{r} . \qquad (3.47)$$

It is to be noted that despite the complexity involved in the inversion of the coordinate transformation and the various algebraic manipulations in the summations, there is considerable simplification and cancellation in the final form of equation (3.47). This cancellation is a direct result of the simplicity of the constant reference velocity wave propagation phenomena. Consequently, such simplifications are to be expected.

The calculation of the third term in equation (3.46) is substantially more difficult and tedious to perform, but the method is clear and straightforward. A minimum of details are provided. However, the following intermediate step is included for reference. Define a matrix C with elements C_{pq} by

$$C_{pq} = \sum_{r=1}^{2} E_{pr}(\mathbf{y},\xi) \frac{\partial}{\partial \xi_r}$$

$$\cdot \left\{ D_q(\mathbf{y},\xi) \frac{\partial^2 \phi}{\partial y_p \partial y_q} + \sum_{s=1}^{2} E_{qs}(\mathbf{y},\xi) \frac{\partial}{\partial \xi_s} \left(\frac{\partial^2 \phi}{\partial y_p \partial y_q} \right) \right\} . \qquad (3.48)$$

Then, it is found by a group of lengthy calculations that the matrix elements are given by the following. We use the notation $Y_1 = y_1 - \xi_1$, $Y_2 = y_2 - \xi_2$, and $Y_3 = y_3$.

$$C_{11} = - \frac{c_0}{r^5} [- 3Y_1^2 + Y_2^2 + Y_3^2][Y_2^2 + Y_3^2] ,$$

$$C_{12} = - \frac{c_0}{2r^5} [- Y_1^4 + 6Y_1^2 Y_2^2 - Y_2^4 + Y_3^4] ,$$

$$C_{13} = - \frac{c_0}{r^5} [- Y_1^4 + 3y_1^2 Y_3^2 + Y_2^4 + Y_2^2 Y_3^2] ,$$

$$C_{21} = - \frac{c_0}{2r^5} [- Y_1^4 + 6Y_1^2 Y_2^2 - Y_2^4 + Y_3^4] ,$$

$$C_{22} = - \frac{c_0}{r^5} [Y_1^2 - 3Y_2^2 + Y_3^2][Y_1^2 + Y_3^2] ,$$

$$C_{23} = -\frac{c_0}{r^5} [\ Y_1^4 + Y_1^2 Y_3^2 - Y_2^4 + 3Y_2^2 Y_3^2\ ] ,$$

$$C_{31} = -\frac{c_0}{r^5} Y_1^2 [\ - Y_1^2 + 3Y_2^2 + 3Y_3^2\ ] + \frac{c_0}{r^5} Y_2^2 [\ 3Y_1^2 - Y_2^2 - Y_3^2\ ] ,$$

$$C_{32} = \frac{c_0}{r^5} Y_1^2 [\ - Y_1^2 + 3Y_2^2 - Y_3^2\ ] - \frac{c_0}{r^5} Y_2^2 [\ 3Y_1^2 - Y_2^2 + 3Y_3^2\ ] ,$$

$$C_{33} = -\frac{c_0}{r^5} [\ Y_1^2 + Y_2^2\ ][\ 3Y_1^2 + 3Y_2^2 - Y_3^2\ ] . \tag{3.49}$$

Thus, the third term in the expression (3.46) for the first order kernel function $b_1(\mathbf{y},\xi)$ is composed essentially from the summation over the matrix elements C_{pq}. After a series of calculations, it is found that the sum of the matrix elements is given by

$$\sum_{p=1}^{3} \sum_{q=1}^{3} C_{pq} = -\frac{3c_0}{r} . \tag{3.50}$$

Hence the third term in equation (3.46) for the first order kernel function produces the startling result,

$$-\frac{1}{2} \sum_{p=1}^{3} \sum_{q=1}^{3} \sum_{r=1}^{2} E_{pr}(\mathbf{y},\xi) \frac{\partial}{\partial \xi_r} \cdot \{\ D_q(\mathbf{y},\xi) + \sum_{s=1}^{2} E_{qs}(\mathbf{y},\xi) \frac{\partial}{\partial \xi_s}\ \} \left(\frac{\partial^2 \phi}{\partial y_p \partial y_q} \right) = \frac{3c_0}{2r} . \tag{3.51}$$

This surprising simplicity of the form of the end result is again a consequence of the simplicity of the constant reference velocity wave propagation phenomena.

Thus, the first order kernel function is produced by combining the results of equations (3.43), (3.47), and (3.51). Hence,

$$b_1(\mathbf{y},\xi) = \frac{128\pi^2 y_3}{Q_0^2 c_0 r} \left(-\frac{c_0}{2r} \right) . \tag{3.52}$$

Substituting the above kernel function into the inversion operator,

$$\psi(\mathbf{y}) \sim \frac{1}{8\pi^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \left(b_0(\mathbf{y},\xi) + \frac{1}{i\omega} b_1(\mathbf{y},\xi) \right)$$

$$\cdot\, e^{-i\omega\phi(\mathbf{y},\xi)} D(\omega,\xi)\, d\omega\, d\xi \ , \tag{3.53}$$

generates the first order inversion algorithm for the backscattered data integral equation (3.17). Specifically,

$$\psi(\mathbf{y}) \sim \int_{\mathbb{R}^2} \int_{\mathbb{R}} \frac{16y_3}{\pi Q_0^2 c_0 r} \left(1 - \frac{c_0}{2i\omega r} \right) D(\omega,\xi)\, e^{-i\left(\frac{2r\omega}{c_0}\right)} d\omega\, d\xi \ . \tag{3.54}$$

It is worth observing that the inversion algorithm (3.54) for the backscattered data problem involves a $\frac{1}{\omega}$ term and consequently represents a higher order version of the inversion algorithm (3.45). In fact, by dropping the $\frac{1}{\omega}$ term, the first order inversion algorithm (3.54) reduces to the zero order algorithm (3.45). However, at this point, it is not clear whether the first order algorithm is an improvement over the zero order algorithm. This fundamental question will be discussed in detail in the next chapter.

In principle, higher order algorithms of arbitrary order $\left(\frac{1}{\omega}\right)^m$ can be derived by application of the same Fourier integral operator techniques discussed in the present chapter. However, in practice, the complexity of the necessary calculations grows exponentially with increasing order $m \geqq 2$. On the other hand, it is anticipated that significant simplification and cancellation of terms will occur in the final form of the inverse algorithm. This is a possible area for future research.

CHAPTER 4

ANALYSIS OF THE INVERSION ALGORITHM

The discussion so far has led to the zero and first order inversion algorithms for the backscattered data integral equation subject to constant reference velocity restrictions. The first order inversion algorithm involves an additional $\frac{1}{\omega}$ term. It is suspected that this is an improvement over the zero order algorithm. The main objective of the present chapter is to explore in detail the nature of the improvement that the $\frac{1}{\omega}$ term provides.

We proceed by representing the output of the inversion algorithm approximations in terms of Fourier integral operators. After a series of manipulations, it is recognized that the zero order inversion algorithm approximation generates an error term involving a $\frac{1}{\omega}$ term that is exactly cancelled by the first order inversion algorithm. Thus, the first order inversion algorithm is indeed an improvement over the zero order algorithm. We conclude this chapter with some miscellaneous topics concerning the constant reference velocity example. The correction term of the first order inversion algorithm is specialized for stratified media where the index of refraction perturbation depends on depth only. The correction term is shown to be a essentially a linear ramp correction for a step discontinuity in the index of refraction perturbation. More precisely, the first order inversion algorithm is further verified using Cagniard-de Hoop data for a single reflector. We end this chapter with a discussion of the extension of the first order inversion algorithm for more general impulsive wave sources.

4.1 Fourier Integral Operator Representation of the Approximations

Recall that in the preceding chapter, the backscattered data integral equation for constant reference velocity,

$$D(\omega,\boldsymbol{\xi}) = \frac{\omega^2}{c_0^2} \int_{\mathbb{R}^3} \left(\frac{Q_0}{4\pi|\mathbf{x} - \boldsymbol{\xi}|} \right)^2 e^{i\omega\left(\frac{2|\mathbf{x} - \boldsymbol{\xi}|}{c_0}\right)} \psi(\mathbf{x})\, d\mathbf{x} \,, \tag{4.1}$$

has been solved explicitly for the zero order and first order inversion algorithms. The zero order inversion algorithm is given by

$$\tilde{\psi}_0(\mathbf{y}) = \int_{\mathbb{R}^2} \int_{\mathbb{R}} \frac{16 y_3}{\pi Q_0^2 c_0 |\mathbf{y} - \boldsymbol{\xi}|} D(\omega,\boldsymbol{\xi})\, e^{-i\omega\left(\frac{2|\mathbf{y} - \boldsymbol{\xi}|}{c_0}\right)} d\omega\, d\boldsymbol{\xi} \,, \tag{4.2}$$

where the notation $\tilde{\psi}_0(\mathbf{y})$ is used to indicate the zero order approximation to the unknown exact index of refraction perturbation $\psi(\mathbf{y})$. The zero order inversion algorithm has previously been described by Cohen, Hagin, and Bleistein [9].

The first order inversion algorithm derived in the previous chapter by Fourier integral operator techniques is given by

$$\tilde{\psi}_1(\mathbf{y}) = \int_{\mathbb{R}^2} \int_{\mathbb{R}} \frac{16 y_3}{\pi Q_0^2 c_0 |\mathbf{y} - \boldsymbol{\xi}|} \left(1 - \frac{c_0}{2i\omega|\mathbf{y} - \boldsymbol{\xi}|} \right) \cdot D(\omega,\boldsymbol{\xi})\, e^{-i\omega\left(\frac{2|\mathbf{y} - \boldsymbol{\xi}|}{c_0}\right)} d\omega\, d\boldsymbol{\xi} \,. \tag{4.3}$$

The inverse operator (4.3) has potential for being a significant improvement over the inverse operator (4.2).

In order to investigate the nature in which equation (4.3) is a refinement of equation (4.2), the action of the additional $\frac{1}{\omega}$ term contributes to the accuracy of the inversion is explored. A Fourier integral operator approach is again taken. We rewrite the inversion integral in terms of Fourier integral operators by substituting the original integral equation (4.1) into the inverse operators (4.2) and (4.3).

The zero order Fourier integral operator $\mathcal{Z}_0$ obtained by this substitution is given by the following operator acting on the unknown index of refraction perturbation $\psi(\mathbf{x})$.

$$\tilde{\psi}_0(\mathbf{y}) = \big(\mathcal{Z}_0(\psi)\big)(\mathbf{y})$$

$$= \frac{1}{\pi^3 c_0^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{\omega^2 y_3}{|\mathbf{y}-\boldsymbol{\xi}|\;|\mathbf{x}-\boldsymbol{\xi}|^2} \cdot e^{\frac{2i\omega}{c_0}(|\mathbf{x}-\boldsymbol{\xi}| - |\mathbf{y}-\boldsymbol{\xi}|)} \psi(\mathbf{x})\, d\mathbf{x}\, d\omega\, d\boldsymbol{\xi} \,. \tag{4.4}$$

The first order operator $\mathcal{Z}_1$ is given by the expression

$$\tilde{\psi}_1(\mathbf{y}) = \big(\mathcal{Z}_1(\psi)\big)(\mathbf{y}) = \big(\mathcal{Z}_0(\psi)\big)(\mathbf{y})$$

$$- \frac{1}{2i\pi^3 c_0^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{\omega y_3}{|\mathbf{y}-\boldsymbol{\xi}|^2 |\mathbf{x}-\boldsymbol{\xi}|^2} \cdot e^{\frac{2i\omega}{c_0}(|\mathbf{x}-\boldsymbol{\xi}| - |\mathbf{y}-\boldsymbol{\xi}|)} \psi(\mathbf{x})\, d\mathbf{x}\, d\omega\, d\boldsymbol{\xi} \,. \tag{4.5}$$

For convenience, we denote the common phase function by $\Phi(\mathbf{x},\mathbf{y},\boldsymbol{\xi})$ as before,

$$\Phi(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) = \frac{2}{c_0}(|\mathbf{x}-\boldsymbol{\xi}| - |\mathbf{y}-\boldsymbol{\xi}|) \,. \tag{4.6}$$

Using the Beylkin change of variables,

$$\mathbf{k} = \omega\, \nabla_{\mathbf{x}} \Phi \big|_{\mathbf{x}=\mathbf{y}} = \frac{2\omega}{c_0} \frac{(\mathbf{y}-\boldsymbol{\xi})}{|\mathbf{y}-\boldsymbol{\xi}|} \,, \tag{4.7}$$

and the corresponding change of differential forms found in equations (3.26) and (3.34),

$$d\mathbf{k} = \frac{\partial(k_1,k_2,k_3)}{\partial(\omega,\xi_1,\xi_2)}\, d\omega\, d\boldsymbol{\xi} = \frac{8\omega^2 y_3}{c_0^3 |\mathbf{y}-\boldsymbol{\xi}|^3}\, d\omega\, d\boldsymbol{\xi} \,, \tag{4.8}$$

we can write the integral operator of equation (4.4) in terms of the variable **k**. This produces

$$\tilde{\psi}_0(\mathbf{y}) = \big(\mathcal{Z}_0(\psi)\big)(\mathbf{y})$$

$$= \frac{1}{8\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\mathbf{y} - \xi|^2}{|\mathbf{x} - \xi|^2} \, \psi(\mathbf{x}) \, e^{i\omega\Phi(\mathbf{x},\mathbf{y},\xi)} \, d\mathbf{x} \, d\mathbf{k} \, . \tag{4.9}$$

Similarly, the integral operator of equation (4.5) can also be converted to the variable **k**,

$$\tilde{\psi}_1(\mathbf{y}) = \big(\mathcal{Z}_1(\psi)\big)(\mathbf{y}) = \big(\mathcal{Z}_0(\psi)\big)(\mathbf{y})$$

$$- \frac{c_0}{8\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\mathbf{y} - \xi|^2}{|\mathbf{x} - \xi|^2} \left(\frac{c_0}{2|\mathbf{y} - \xi|} \right) \frac{1}{i\omega}$$

$$\cdot \, \psi(\mathbf{x}) \, e^{i\omega\Phi(\mathbf{x},\mathbf{y},\xi)} \, d\mathbf{x} \, d\mathbf{k} \, . \tag{4.10}$$

We now expand some of the terms directly into Taylor series with small $(\mathbf{x} - \mathbf{y})$. The phase function has the expansion

$$i\omega\Phi(\mathbf{x},\mathbf{y},\xi) = i(\mathbf{x} - \mathbf{y})\cdot\mathbf{k}$$

$$+ \frac{i\omega}{2} \sum_{p=1}^{3} \sum_{q=1}^{3} (x_p - y_p)(x_q - y_q) \left. \frac{\partial^2\Phi}{\partial x_p \partial x_q} \right|_{\mathbf{x}=\mathbf{y}} + \cdots \, . \tag{4.11}$$

It is also desirable to expand the following function in a Taylor series.

$$\frac{|\mathbf{y} - \xi|^2}{|\mathbf{x} - \xi|^2} = 1 + (\mathbf{x} - \mathbf{y})\cdot\nabla_{\mathbf{x}} \left(\frac{|\mathbf{y} - \xi|^2}{|\mathbf{x} - \xi|^2} \right) \bigg|_{\mathbf{x}=\mathbf{y}} + \cdots$$

$$= 1 - \frac{2(\mathbf{x} - \mathbf{y})\cdot(\mathbf{y} - \xi)}{|\mathbf{y} - \xi|^2} + \cdots \, . \tag{4.12}$$

We use the following property of the exponential function.

$$e^{A+B} = e^{A} e^{B} = e^{A} (1 + B + \cdots) . \tag{4.13}$$

Substituting the series (4.11) into the exponential function $e^{i\omega\Phi}$ produces one additional series based on the property (4.13).

$$e^{i\omega\Phi(\mathbf{x},\mathbf{y},\xi)}$$

$$= e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}} \Big(1 + \frac{i\omega}{2} \sum_{p=1}^{3} \sum_{q=1}^{3} (x_p - y_p)(x_q - y_q) \frac{\partial^2\Phi}{\partial x_p \partial x_q}\Big|_{\mathbf{x}=\mathbf{y}} + \cdots \Big) . \tag{4.14}$$

The preceding series expansions of equations (4.11), (4.12), and (4.14) are formally substituted into the expressions (4.9) and (4.10) for the Fourier integral operators $\mathcal{Z}_0$ and $\mathcal{Z}_1$. The resulting equation for the zero order integral operator $\mathcal{Z}_0$ is given by

$$\tilde{\psi}_0(\mathbf{y}) = \big(\mathcal{Z}_0(\psi)\big)(\mathbf{y})$$

$$= \frac{1}{8\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Big(1 - \frac{2(\mathbf{x} - \mathbf{y})\cdot(\mathbf{y} - \xi)}{|\mathbf{y} - \xi|^2} + \cdots \Big)$$

$$\cdot \Big(1 + \frac{i\omega}{2} \sum_{p=1}^{3} \sum_{q=1}^{3} (x_p - y_p)(x_q - y_q) \frac{\partial^2\Phi}{\partial x_p \partial x_q}\Big|_{\mathbf{x}=\mathbf{y}} + \cdots \Big)$$

$$\cdot\ \psi(\mathbf{x})\, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\, d\mathbf{x}\, d\mathbf{k} . \tag{4.15}$$

Similarly, the equation for the first order integral operator $\mathcal{Z}_1$ is given by

$$\tilde{\psi}_1(\mathbf{y}) = \big(\mathcal{Z}_1(\psi)\big)(\mathbf{y}) = \big(\mathcal{Z}_0(\psi)\big)(\mathbf{y})$$

$$+ \frac{1}{8\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left(1 - \frac{2(\mathbf{x}-\xi)\cdot(\mathbf{y}-\xi)}{|\mathbf{y}-\xi|^2} + \cdots \right)$$

$$\cdot \left(1 + \frac{i\omega}{2} \sum_{p=1}^{3} \sum_{q=1}^{3} (x_p - y_p)(x_q - y_q) \frac{\partial^2 \Phi}{\partial x_p \partial x_q}\bigg|_{\mathbf{x}=\mathbf{y}} + \cdots \right)$$

$$\cdot \left(\frac{c_0}{2|\mathbf{y}-\xi|} \right) \frac{1}{i\omega}\, \psi(\mathbf{x})\, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\, d\mathbf{x}\, d\mathbf{k} \,. \tag{4.16}$$

Recall that in the previous chapter an important relationship was established between the Taylor polynomial $(\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}}$ for small $(\mathbf{x}-\mathbf{y})$ and the asymptotic order $\left(\frac{1}{\omega}\right)^{|\boldsymbol{\alpha}|}$ for large frequency ω. These apparently unrelated quantities represent the same order of asymptotics in a Fourier integral operator sense. Thus, we consider the asymptotic terms of smaller order than $\left(\frac{1}{\omega}\right)^2$ in the operators $\mathcal{Z}_0$ and $\mathcal{Z}_1$. Hence, it is observed that the integral operator $\mathcal{Z}_0$ has three significant terms (excluding $\left(\frac{1}{\omega}\right)^2$ and higher order terms). The expansion for the integral operator $\mathcal{Z}_1$ involves only one additional term. Specifically,

$$\tilde{\psi}_0(\mathbf{y}) = \big(\mathcal{Z}_0(\psi)\big)(\mathbf{y}) = \frac{1}{8\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi(\mathbf{x})\, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\, d\mathbf{x}\, d\mathbf{k}$$

$$- \frac{1}{4\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(\mathbf{x}-\mathbf{y})\cdot(\mathbf{y}-\xi)}{|\mathbf{y}-\xi|^2}\, \psi(\mathbf{x})\, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\, d\mathbf{x}\, d\mathbf{k}$$

$$+ \frac{i}{16\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \omega \sum_{p=1}^{3} \sum_{q=1}^{3} (x_p - y_p)(x_q - y_q) \frac{\partial^2 \Phi}{\partial x_p \partial x_q}\bigg|_{\mathbf{x}=\mathbf{y}}$$

$$\cdot\, \psi(\mathbf{x})\, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\, d\mathbf{x}\, d\mathbf{k} + \cdots \,. \tag{4.17}$$

$$\tilde{\psi}_1(\mathbf{y}) = \big(\mathcal{Z}_1(\psi)\big)(\mathbf{y}) = \big(\mathcal{Z}_0(\psi)\big)(\mathbf{y})$$

$$- \frac{1}{8\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left(\frac{c_0}{2|\mathbf{y}-\xi|} \right) \frac{1}{i\omega}\, \psi(\mathbf{x})\, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\, d\mathbf{x}\, d\mathbf{k} + \cdots \,. \tag{4.18}$$

4.2 Alternate Forms for the Integral Operator Terms

In the preceding section, the zero order and first order inverse algorithm approximations to the backscattered data integral equation were formulated in terms of the Fourier integral operators $\mathcal{Z}_0$ and $\mathcal{Z}_1$. Through a series of manipulations, equations (4.17) and (4.18) were derived providing explicit expressions for the two integral operators.

It is to be noticed that the integral operator $\mathcal{Z}_0$ of equation (4.17) consists of three individual Fourier integral operators of order less than $(\frac{1}{\omega})^2$ plus additional higher order operators. The integral operator $\mathcal{Z}_1$ of equation (4.18) consists of an additional operator of order $\frac{1}{\omega}$ plus additional higher order operators. These individual terms are investigated individually in this section.

The first term in the Fourier integral operator expression (4.17) is easily recognized as the identity operator. This follows since the action of the Fourier kernel $e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}$ is precisely equivalent to the Dirac delta function $\delta(\mathbf{x}-\mathbf{y})$. Specifically,

$$\frac{1}{8\pi^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\psi(\mathbf{x})\, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\, d\mathbf{x}\, d\mathbf{k} = \int_{\mathbb{R}^3}\psi(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y})\, d\mathbf{x} = \psi(\mathbf{y}) \ . \tag{4.19}$$

If the above operator were the only term involved in equation (4.17) for the operator $\mathcal{Z}_0$, then $\mathcal{Z}_0$ would be in fact the identity operator. Consequently, the inversion algorithm (4.2) would be exact. However, the other terms in equation (4.17) comprise errors that degrade the zero order inversion algorithm.

Consider the second Fourier integral operator term found in equation (4.17). We wish to write this expression in an alternate form by utilizing the same Fourier kernel property and integration by parts used in Chapter 2. The Fourier kernel property (2.38) is specialized to a 3-dimensional gradient,

$$(\mathbf{x} - \mathbf{y})\, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}} = \frac{1}{i} \nabla_{\mathbf{k}} \left(e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}} \right) . \tag{4.20}$$

The integration by parts that is needed is a 3-dimensional version of equation (2.37) and is written simply as

$$\int_{\mathbb{R}^3} u\, \nabla_{\mathbf{k}}\, v\, d\mathbf{k} = - \int_{\mathbb{R}^3} v\, \nabla_{\mathbf{k}}\, u\, d\mathbf{k} . \tag{4.21}$$

Hence, using a combination of integration by parts (4.21) and the Fourier kernel property (4.19), the second term of equation (4.17) can be rewritten as follows.

$$- \frac{1}{4\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(\mathbf{x} - \mathbf{y})\cdot(\mathbf{y} - \boldsymbol{\xi})}{|\mathbf{y} - \boldsymbol{\xi}|^2} \psi(\mathbf{x})\, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\, d\mathbf{x}\, d\mathbf{k}$$

$$= - \frac{1}{4\pi^3 i} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\, \psi(\mathbf{x})\, \nabla_{\mathbf{k}}\cdot\left(\frac{(\mathbf{y} - \boldsymbol{\xi})}{|\mathbf{y} - \boldsymbol{\xi}|^2} \right) d\mathbf{x}\, d\mathbf{k} . \tag{4.22}$$

Recall that for the constant reference velocity case, the Beylkin transformation is given by

$$\mathbf{k} = \omega\, \nabla_{\mathbf{x}} \Phi \big|_{\mathbf{x}=\mathbf{y}} = \frac{2\omega}{c_0} \frac{(\mathbf{y} - \boldsymbol{\xi})}{|\mathbf{y} - \boldsymbol{\xi}|} . \tag{4.23}$$

This transformation can be explicitly inverted to yield the variables (ω, ξ_1, ξ_2) as functions of the variables (k_1, k_2, k_3) and take the form

$$\omega = \pm \frac{c_0}{2} |\mathbf{k}| , \quad \xi_1 = y_1 - y_3\left(\frac{k_1}{k_3} \right) , \quad \xi_2 = y_2 - y_3\left(\frac{k_2}{k_3} \right) . \tag{4.24}$$

Viewing $\boldsymbol{\xi}$ as a function of $\mathbf{k}$, it immediately follows that

$$|\mathbf{y} - \boldsymbol{\xi}|^2 = \frac{y_3^2}{k_3^2} |\mathbf{k}|^2 , \tag{4.25}$$

and also that the following holds,

$$\frac{(\mathbf{y} - \boldsymbol{\xi})}{|\mathbf{y} - \boldsymbol{\xi}|^2} = \frac{k_3}{y_3 |\mathbf{k}|^2} \mathbf{k} . \tag{4.26}$$

From equation (4.26), the divergence of the unit vector function $(\mathbf{y} - \xi)/|\mathbf{y} - \xi|^2$ with respect to $\mathbf{k}$ can be obtained through a series of straightforward calculations. The divergence is given by

$$\nabla_{\mathbf{k}} \cdot \left(\frac{(\mathbf{y} - \xi)}{|\mathbf{y} - \xi|^2} \right) = \frac{2k_3}{y_3 |\mathbf{k}|^2} . \tag{4.27}$$

Replacing the divergence in equation (4.22) by the expression (4.27) above results in

$$- \frac{1}{4\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{y} - \xi)}{|\mathbf{y} - \xi|^2} \psi(\mathbf{x}) \, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}} \, d\mathbf{x} \, d\mathbf{k}$$

$$= \frac{1}{2\pi^3 i} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}} \, \psi(\mathbf{x}) \frac{k_3}{y_3 |\mathbf{k}|^2} \, d\mathbf{x} \, d\mathbf{k} . \tag{4.28}$$

This is the desired alternate form for the second Fourier integral operator term in the explicit expansion (4.17) for $\mathcal{Z}_0$.

We now turn to the third term and most complicated term of the expansion (4.17) of the operator $\mathcal{Z}_0$. Again the Fourier kernel property (2.38) is specialized to second order partial derivatives,

$$(x_p - y_p)(x_q - y_q) \, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}} = - \frac{\partial^2}{\partial k_p \partial k_q} \left(e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}} \right) . \tag{4.29}$$

We also use the double integration by parts that reduces equation (2.37) to the form

$$\int_{\mathbb{R}^3} u \frac{\partial^2}{\partial k_p \partial k_q} v \, d\mathbf{k} = \int_{\mathbb{R}^3} v \frac{\partial^2}{\partial k_p \partial k_q} u \, d\mathbf{k} . \tag{4.30}$$

Thus, by using the Fourier kernel property (4.29) and integration by parts (4.30) above, the third term of the expansion (4.17) for $\mathcal{Z}_0$ becomes

$$\frac{i}{16\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \omega \sum_{p=1}^{3} \sum_{q=1}^{3} (x_p - y_p)(x_q - y_q) \frac{\partial^2 \Phi}{\partial x_p \partial x_q}\Big|_{\mathbf{x}=\mathbf{y}}$$

$$\cdot\ \psi(\mathbf{x})\ e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\ d\mathbf{x}\ d\mathbf{k}$$

$$= -\frac{i}{16\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{p=1}^{3} \sum_{q=1}^{3} \frac{\partial^2}{\partial k_p \partial k_q} \left(\omega \frac{\partial^2 \Phi}{\partial x_p \partial x_q}\Big|_{\mathbf{x}=\mathbf{y}} \right)$$

$$\cdot\ \psi(\mathbf{x})\ e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\ d\mathbf{x}\ d\mathbf{k}\ . \qquad (4.31)$$

We explicitly evaluate the derivatives above in equation (4.31) through the use of the inverse transformation between (k_1, k_2, k_3) and (ω, ξ_1, ξ_2) given in equation (4.24). However, there remains an ambiguity in the sign of ω in equation (4.24) that needs to be resolved. By examining the Beylkin transformation (4.23), it is observed that the frequency ω can be represented unambiguously by

$$\omega = \frac{c_0 k_3 |\mathbf{y} - \boldsymbol{\xi}|}{2y_3}\ . \qquad (4.32)$$

From the transformation relations (4.24) and (4.32) along with the explicit form of the phase function $\Phi(\mathbf{x},\mathbf{y},\boldsymbol{\xi})$ given in equation (4.6), the following explicit derivatives can be generated.

$$\omega \frac{\partial^2 \Phi}{\partial x_1^2}\Big|_{\mathbf{x}=\mathbf{y}} = \left(\frac{c_0 k_3 |\mathbf{y} - \boldsymbol{\xi}|}{2y_3} \right) \left(\frac{2}{c_0} \frac{[(y_2 - \xi_2)^2 + y_3^2]}{|\mathbf{y} - \boldsymbol{\xi}|^3} \right)$$

$$= \frac{k_3}{y_3 |\mathbf{k}|^2} (k_2^2 + k_3^2)\ ,$$

$$\omega \frac{\partial^2 \Phi}{\partial x_2^2}\Big|_{\mathbf{x}=\mathbf{y}} = \left(\frac{c_0 k_3 |\mathbf{y} - \boldsymbol{\xi}|}{2y_3} \right) \left(\frac{2}{c_0} \frac{[(y_1 - \xi_1)^2 + y_3^2]}{|\mathbf{y} - \boldsymbol{\xi}|^3} \right)$$

$$= \frac{k_3}{y_3 |\mathbf{k}|^2} (k_1^2 + k_3^2)\ ,$$

$$\omega \frac{\partial^2 \Phi}{\partial x_3^2}\Big|_{\mathbf{x}=\mathbf{y}} = \left(\frac{c_0 k_3 |\mathbf{y} - \boldsymbol{\xi}|}{2y_3} \right) \left(\frac{2}{c_0} \frac{[(y_1 - \xi_1)^2 + (y_2 - \xi_2)^2]}{|\mathbf{y} - \boldsymbol{\xi}|^3} \right)$$

$$= \frac{k_3}{y_3 |\mathbf{k}|^2} (k_1^2 + k_2^2)\ ,$$

$$\omega \frac{\partial^2 \Phi}{\partial x_1 \partial x_2}\Big|_{\mathbf{x}=\mathbf{y}} = \left(\frac{c_0 k_3 |\mathbf{y} - \boldsymbol{\xi}|}{2y_3} \right) \left(-\frac{2}{c_0} \frac{(y_1 - \xi_1)(y_2 - \xi_2)}{|\mathbf{y} - \boldsymbol{\xi}|^3} \right)$$
$$= -\frac{k_1 k_2 k_3}{y_3 |\mathbf{k}|^2} ,$$
$$\omega \frac{\partial^2 \Phi}{\partial x_1 \partial x_3}\Big|_{\mathbf{x}=\mathbf{y}} = \left(\frac{c_0 k_3 |\mathbf{y} - \boldsymbol{\xi}|}{2y_3} \right) \left(-\frac{2}{c_0} \frac{(y_1 - \xi_1) y_3}{|\mathbf{y} - \boldsymbol{\xi}|^3} \right)$$
$$= -\frac{k_1 k_3^2}{y_3 |\mathbf{k}|^2} ,$$
$$\omega \frac{\partial^2 \Phi}{\partial x_2 \partial x_3}\Big|_{\mathbf{x}=\mathbf{y}} = \left(\frac{c_0 k_3 |\mathbf{y} - \boldsymbol{\xi}|}{2y_3} \right) \left(-\frac{2}{c_0} \frac{(y_2 - \xi_2) y_3}{|\mathbf{y} - \boldsymbol{\xi}|^3} \right)$$
$$= -\frac{k_2 k_3^2}{y_3 |\mathbf{k}|^2} . \tag{4.33}$$

Moreover, by a series of straightforward differentiations and algebraic manipulations utilizing the derivatives in equation (2.33), it is then discovered that

$$\sum_{p=1}^{3} \sum_{q=1}^{3} \frac{\partial^2}{\partial k_p \partial k_q} \left(\omega \frac{\partial^2 \Phi}{\partial x_p \partial x_q}\Big|_{\mathbf{x}=\mathbf{y}} \right) = -\frac{6k_3}{y_3 |\mathbf{k}|^2} . \tag{4.34}$$

We are now in a position to evaluate the complicated integral (4.31) that represents the third term in the expansion (4.17) for $\mathcal{Z}_0$. The following equation is obtained by substituting the above result (4.34) into the integral expression (4.31).

$$-\frac{i}{16\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{p=1}^{3} \sum_{q=1}^{3} \frac{\partial^2}{\partial k_p \partial k_q} \left(\omega \frac{\partial^2 \Phi}{\partial x_p \partial x_q}\Big|_{\mathbf{x}=\mathbf{y}} \right)$$
$$\cdot \psi(\mathbf{x})\, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\, d\mathbf{x}\, d\mathbf{k}$$

$$= -\frac{3}{8\pi^3 i} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\, \psi(\mathbf{x}) \frac{k_3}{y_3 |\mathbf{k}|^2}\, d\mathbf{x}\, d\mathbf{k} . \tag{4.35}$$

Thus, we have established the desired alternate form for the third term of the expansion (4.17) for $\mathcal{Z}_0$ by combining equations (4.31) and (4.35) above. Specifically,

$$\frac{i}{16\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \omega \sum_{p=1}^{3} \sum_{q=1}^{3} (x_p - y_p)(x_q - y_q) \frac{\partial^2 \Phi}{\partial x_p \partial x_q}\Big|_{\mathbf{x}=\mathbf{y}}$$

$$\cdot\ \psi(\mathbf{x})\ e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\ d\mathbf{x}\ d\mathbf{k}$$

$$= -\frac{3}{8\pi^3 i} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\ \psi(\mathbf{x})\ \frac{k_3}{y_3|\mathbf{k}|^2}\ d\mathbf{x}\ d\mathbf{k}\ . \tag{4.36}$$

The present section is concluded with the development of an alternate representation for the additional operator term appearing in the expansion (4.18) for the operator $\mathcal{Z}_1$. It turns out that this operator is the simplest of the terms to convert. Combining the relationship (4.32) for frequency ω with the equation (4.25) for the squared distance $|\mathbf{y} - \xi|^2$ results in the following.

$$\left(\frac{c_0}{2|\mathbf{y} - \xi|} \right) \frac{1}{i\omega} = \frac{y_3}{ik_3|\mathbf{y} - \xi|^2} = \frac{k_3}{iy_3|\mathbf{k}|^2}\ . \tag{4.37}$$

Thus, the desired expression for the additional operator term in equation (4.18) is found to be

$$-\frac{1}{8\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left(\frac{c_0}{2|\mathbf{y} - \xi|} \right) \frac{1}{i\omega}\ e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\ d\mathbf{x}\ d\mathbf{k}$$

$$= -\frac{1}{8\pi^3 i} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\ \psi(\mathbf{x})\ \frac{k_3}{y_3|\mathbf{k}|^2}\ d\mathbf{x}\ d\mathbf{k}\ . \tag{4.38}$$

With the alternate forms given in the equations (4.19), (4.28), (4.36), and (4.38) above, an understanding of the difference between the zero order and first order inversion algorithms can be produced. We now explore this in detail.

4.3 Inversion Algorithm Correction Term

Thus far, the inversion algorithms developed in the preceding chapter have been cast in terms of Fourier integral operators. These representations have been expanded into individual operator terms up to orders less than $\left(\frac{1}{\omega}\right)^2$. The individual operators were written purely in terms of the Beylkin transformation variable $\mathbf{k} = \omega \, \nabla_{\mathbf{x}} \phi |_{\mathbf{x}=\mathbf{y}}$ and are provided in equations (4.19), (4.28), (4.36), and (4.38).

In the present section, the results of the preceding sections are summarized. Combining the alternate representations from the previous section immediately shows the nature of the improvement of the first order inversion algorithm over the zero order algorithm.

The zero order inversion algorithm produces the following approximation to the index of refraction perturbation.

$$\tilde{\psi}_0(\mathbf{y}) = \int_{\mathbb{R}^2} \int_{\mathbb{R}} \frac{16 y_3}{\pi Q^2 c_0 |\mathbf{y} - \xi|} D(\omega, \xi) \, e^{-i\omega\left(\frac{2|\mathbf{y} - \xi|}{c_0}\right)} \, d\omega \, d\xi \, . \qquad (4.39)$$

This has been shown to have the following equivalent representation.

$$\tilde{\psi}_0(\mathbf{y}) = \left(\mathcal{Z}_0(\psi)\right)(\mathbf{y}) = \frac{1}{8\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi(\mathbf{x}) \, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}} \, d\mathbf{x} \, d\mathbf{k}$$

$$- \frac{1}{4\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(\mathbf{x} - \mathbf{y})\cdot(\mathbf{y} - \xi)}{|\mathbf{y} - \xi|^2} \psi(\mathbf{x}) \, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}} \, d\mathbf{x} \, d\mathbf{k}$$

$$+ \frac{i}{16\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \omega \sum_{p=1}^{3} \sum_{q=1}^{3} (x_p - y_p)(x_q - y_q) \left.\frac{\partial^2 \phi}{\partial x_p \partial x_q}\right|_{\mathbf{x}=\mathbf{y}}$$

$$\cdot \, \psi(\mathbf{x}) \, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}} \, d\mathbf{x} \, d\mathbf{k} + \cdots \, . \qquad (4.40)$$

In the above equation, terms involving $\left(\frac{1}{\omega}\right)^2$ and higher powers have not been explicitly written.

In the preceding section, the individual terms of equation (4.40) have been written in the alternate forms found in equations (4.19), (4.28), and (4.36). Thus, the zero order inversion algorithm produces the approximation to the index of refraction perturbation given by

$$\tilde{\psi}_0(\mathbf{y}) = \big(\mathcal{Z}_0(\psi)\big)(\mathbf{y})$$

$$= \psi(\mathbf{y}) + \frac{1}{2\pi^3 i}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{k_3}{y_3|\mathbf{k}|^2}\,\psi(\mathbf{x})\,e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\,d\mathbf{x}\,d\mathbf{k}$$

$$- \frac{3}{8\pi^3 i}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{k_3}{y_3|\mathbf{k}|^2}\,\psi(\mathbf{x})\,e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\,d\mathbf{x}\,d\mathbf{k} + \cdots . \tag{4.41}$$

It is immediately noticed that the two integral operators can be summed to yield

$$\tilde{\psi}_0(\mathbf{y}) = \big(\mathcal{Z}_0(\psi)\big)(\mathbf{y})$$

$$= \psi(\mathbf{y}) + \frac{1}{8\pi^3 i}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{k_3}{y_3|\mathbf{k}|^2}\,\psi(\mathbf{x})\,e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\,d\mathbf{x}\,d\mathbf{k} + \cdots . \tag{4.42}$$

Hence, the integral operator term above along with the higher order expressions that are not explicitly written represent the error in the approximation $\tilde{\psi}_0(\mathbf{x})$ to the exact index of refraction perturbation $\psi(\mathbf{x})$ in the zero order inversion algorithm.

Similarly, we examine the first order inversion algorithm given by

$$\tilde{\psi}_1(\mathbf{y}) = \int_{\mathbb{R}^2}\int_{\mathbb{R}} \frac{16 y_3}{\pi Q_0^2 c_0|\mathbf{y}-\xi|}\left(1 - \frac{c_0}{2i\omega|\mathbf{y}-\xi|}\right) \cdot D(\omega,\xi)\, e^{-i\omega\left(\frac{2|\mathbf{y}-\xi|}{c_0}\right)}\,d\omega\,d\xi . \tag{4.43}$$

The expression (4.43) has been shown to possess an equivalent representation of the form

$$\tilde{\psi}_1(\mathbf{y}) = \big(\mathcal{Z}_1(\psi)\big)(\mathbf{y}) = \big(\mathcal{Z}_0(\psi)\big)(\mathbf{y})$$

$$- \frac{1}{8\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left(\frac{c_0}{2|\mathbf{y} - \xi|} \right) \frac{1}{i\omega} \psi(\mathbf{x})\, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\, d\mathbf{x}\, d\mathbf{k} + \cdots \,. \tag{4.44}$$

Using the equivalent form of the above integral operator found in equation (4.38), the above equation can be rewritten as

$$\tilde{\psi}_1(\mathbf{y}) = \big(\mathcal{Z}_1(\psi)\big)(\mathbf{y}) = \big(\mathcal{Z}_0(\psi)\big)(\mathbf{y})$$

$$- \frac{1}{8\pi^3 i} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{k_3}{y_3|\mathbf{k}|^2} \psi(\mathbf{x})\, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\, d\mathbf{x}\, d\mathbf{k} + \cdots \,. \tag{4.45}$$

Comparing this result with equation (4.42) immediately shows that the $\frac{1}{\omega}$ error term in the first order approximation is cancelled. Specifically,

$$\tilde{\psi}_1(\mathbf{y}) = \psi(\mathbf{y}) + \frac{1}{8\pi^3 i} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{k_3}{y_3|\mathbf{k}|^2} \psi(\mathbf{x})\, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\, d\mathbf{x}\, d\mathbf{k}$$

$$- \frac{1}{8\pi^3 i} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{k_3}{y_3|\mathbf{k}|^2} \psi(\mathbf{x})\, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}\, d\mathbf{x}\, d\mathbf{k} + \cdots \,. \tag{4.46}$$

Thus, the error difference $\tilde{\psi}_1(\mathbf{y}) - \psi(\mathbf{y})$ between the first order inversion algorithm approximation $\tilde{\psi}_1(\mathbf{y})$ and the exact index of refraction perturbation $\psi(\mathbf{y})$ is given by higher order integral operators involving $\left(\frac{1}{\omega}\right)^m$ for $m \geq 2$. The $\frac{1}{\omega}$ error term contained in the zero order approximation $\tilde{\psi}_0(\mathbf{y})$ has been annihilated in the first order approximation $\tilde{\psi}_1(\mathbf{y})$. It is precisely in this general Fourier integral operator sense that the first order inversion algorithm (4.43) is seen to be more accurate than the zero order inversion algorithm (4.39).

The results of this section can be generalized to higher order inversion algorithms. The K^{th} order approximation $\tilde{\psi}_K(\mathbf{y})$ given by the methods of the preceding chapters is of the form

$$\tilde{\psi}_K(\mathbf{y}) = \left(\mathcal{Z}_K(\psi)\right)(\mathbf{y}) = \psi(\mathbf{y}) + \left(\mathcal{E}_{K+1}(\psi) + \mathcal{E}_{K+2}(\psi) + \cdots \right)(\mathbf{y}) \ , \tag{4.47}$$

where the operators $\mathcal{E}_{K+1}$, $\mathcal{E}_{K+2}$,... are defined in equation (2.56). Thus, the error term in the K^{th} order inversion algorithm involves integral operators of order $\left(\frac{1}{\omega}\right)^{K+1}$ and higher. This becomes apparent since by the construction of the K^{th} order algorithm, the Fourier integral operator $\mathcal{Z}_K$ that equivalently represents the K^{th} order algorithm is the identity operator up to K^{th} order. This follows as a result of the correction terms that the K^{th} order algorithm provides to cancel the corresponding error terms of the order zero algorithm.

4.4 Correction Term for Stratified Media

We now consider the backscattered data problem with constant reference velocity as before but with the additional restriction to wave propagation in a stratified medium. By a stratified medium, it is meant that the wave propagation velocity $c(\mathbf{x})$ is only a function of depth x_3. If the reference velocity is taken to be the constant c_0, then the index of refraction perturbation $\psi(\mathbf{x})$ is also only a function of depth x_3, that is,

$$\psi(\mathbf{x}) = \psi(x_3) \ . \tag{4.48}$$

It has been demonstrated in the previous section that the $\frac{1}{\omega}$ error term produced by the zero order inversion algorithm is given by the $\mathcal{Z}_0$ operator and takes the form

$$\tilde{\psi}_0(\mathbf{y}) - \psi(\mathbf{y}) = \big(\mathcal{Z}_0(\psi)\big)(\mathbf{y}) - \psi(\mathbf{y})$$

$$= \frac{1}{8\pi^3 i} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{k_3}{y_3 |\mathbf{k}|^2} e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}} \psi(\mathbf{x}) \, d\mathbf{x} \, d\mathbf{k} + \cdots . \tag{4.49}$$

The $\frac{1}{\omega}$ correction term provided by the first order inversion operator is precisely the opposite of the $\frac{1}{\omega}$ error term of the zero order algorithm.

$$\tilde{\psi}_1(\mathbf{y}) - \tilde{\psi}_0(\mathbf{y}) = \big(\mathcal{Z}_1(\psi)\big)(\mathbf{y}) - \big(\mathcal{Z}_0(\psi)\big)(\mathbf{y})$$

$$= -\frac{1}{8\pi^3 i} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{k_3}{y_3 |\mathbf{k}|^2} e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}} \psi(\mathbf{x}) \, d\mathbf{x} \, d\mathbf{k} + \cdots . \tag{4.50}$$

For a stratified medium, the index of refraction perturbation is of the form (4.48). Consequently, the correction term (4.50) can be integrated with respect to x_1 and x_2 to yield Dirac delta functions.

$$\tilde{\psi}_1(\mathbf{y}) - \tilde{\psi}_0(\mathbf{y}) = -\frac{1}{8\pi^3 i y_3} \int_{\mathbb{R}^3} \frac{k_3}{|\mathbf{k}|^2} e^{-i\mathbf{y}\cdot\mathbf{k}} \, 4\pi^2 \, \delta(k_1) \, \delta(k_2)$$

$$\cdot \int_{\mathbb{R}} e^{ix_3k_3} \psi(x_3) \, dx_1 \, d\mathbf{k} + \cdots . \tag{4.51}$$

Denoting the 1-dimensional Fourier transform of $\psi(x_3)$ by $\hat{\psi}(k_3)$, we have the following equivalent expression.

$$\tilde{\psi}_1(\mathbf{y}) - \tilde{\psi}_0(\mathbf{y}) = -\frac{1}{2\pi i y_3} \int_{\mathbb{R}^3} \frac{k_3}{|\mathbf{k}|^2} e^{-i\mathbf{y}\cdot\mathbf{k}} \, \delta(k_1) \, \delta(k_2) \, \hat{\psi}(k_3) \, d\mathbf{k}$$

$$+ \cdots . \tag{4.52}$$

Carrying out the k_1 and k_2 integrations and noting the Dirac delta function properties results in

$$\tilde{\psi}_1(\mathbf{y}) - \tilde{\psi}_0(\mathbf{y}) = -\frac{1}{2\pi i y_3} \int_{\mathbb{R}} \frac{1}{k_3} \hat{\psi}(k_3)\, e^{-iy_3k_3}\, dk_3 + \cdots . \qquad (4.53)$$

However, $1/(ik_3)$ can immediately be recognized as corresponding to the Fourier transform of an integral. Hence,

$$\tilde{\psi}_1(\mathbf{y}) - \tilde{\psi}_0(\mathbf{y}) = \frac{1}{y_3} \int_{\rho}^{y_3} \psi(y_3')\, dy_3' + \cdots . \qquad (4.54)$$

where ρ is a constant. Thus, the correction term in the first order inversion algorithm in a stratified medium with constant reference velocity is seen to be essentially an integration of the index of refraction perturbation.

In particular, for a simple discontinuous step perturbation arising from a single horizontal reflector at a depth ρ in the medium,

$$\psi(y_3) = H(y_3 - \rho) , \qquad (4.55)$$

where $H(x)$ denotes the Heaviside unit step function, the corresponding correction term is essentially a linear ramp in a neighborhood of the reflector.

$$\tilde{\psi}_1(\mathbf{y}) - \tilde{\psi}_0(\mathbf{y}) = \frac{1}{y_3} (y_3 - \rho)\Big|_{y_3 \geqq \rho} + \cdots . \qquad (4.56)$$

We now make these concepts more precise by comparing the correction term to the exact solutions generated by the Cagniard-de Hoop method.

4.5 Cagniard-de Hoop Data

In the present section, the Cagniard-de Hoop method is used to further analyze the action of the first order inversion algorithm (4.43) in a stratified medium with constant reference velocity. Gray, Cohen, and Bleistein [16] showed that a closed form solution can be obtained for the exact backscattered data in this particular situation. This result is established from the Cagniard-de Hoop method described by Achenbach [1].

We consider a piecewise constant velocity profile in a stratified medium given by

$$c(\mathbf{x}) = c(x_3) = \begin{cases} c_0 & 0 < x_3 < h \\ c_1 & x_3 > h \end{cases} , \tag{4.57}$$

where $c_1 - c_0$ is assumed to be a small velocity perturbation. Taking the reference velocity $c_0(\mathbf{x})$ to be the constant c_0, then the index of refraction perturbation defined in equation (1.10) takes the form

$$\psi(\mathbf{x}) = \psi(x_3) = \begin{cases} 0 & 0 < x_3 < h \\ c_0^2 \left(\frac{1}{c_1^2} - \frac{1}{c_0^2} \right) & x_3 > h \end{cases} . \tag{4.58}$$

If the impulsive wave source is normalized ($Q_0 = 1$), then the scattered field at the data surface $x_3 = 0$ is given by the Cagniard-de Hoop formula:

$$U_S(t,\xi) = \frac{1}{8\pi h} \frac{d}{dt} \left(R(t)\, H\left(t - \frac{2h}{c_0}\right) \right) , \tag{4.59}$$

where ξ is the coincident source and receiver point. The function $R(t)$ is given by

$$R(t) = \frac{t - \left(t^2 + 4h^2 \left(\frac{1}{c_1^2} - \frac{1}{c_0^2} \right) \right)^{\frac{1}{2}}}{t + \left(t^2 + 4h^2 \left(\frac{1}{c_1^2} - \frac{1}{c_0^2} \right) \right)^{\frac{1}{2}}} . \tag{4.60}$$

Hence, by taking the 1-dimensional Fourier transform $\mathfrak{F}_1$ given in equation (1.2) of the scattered field $U_S(t,\boldsymbol{\xi})$ in the time-domain, we obtain the scattered field in the frequency-domain,

$$u_S(\omega,\boldsymbol{\xi}) = \mathfrak{F}_1 [U_S(t,\boldsymbol{\xi})] . \tag{4.61}$$

It turns out in the subsequent analysis that it is not necessary to explicitly calculate $u_S(\omega,\boldsymbol{\xi})$.

We now consider the zero order inversion algorithm acting on the data $u_S(\omega,\boldsymbol{\xi})$. Specifically,

$$\tilde{\psi}_0(y_3) = \frac{16y_3}{\pi c_0} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \frac{1}{r} e^{-\left(\frac{2i\omega r}{c_0}\right)} u_S(\omega,\boldsymbol{\xi})\, d\omega\, d\boldsymbol{\xi} , \tag{4.62}$$

where

$$r = |\mathbf{y} - \boldsymbol{\xi}| . \tag{4.63}$$

Recognizing that the integration with respect to ω is essentially an inverse Fourier transform, then it follows that

$$\tilde{\psi}_0(y_3) = \frac{32y_3}{c_0} \int_{\mathbb{R}^2} \frac{1}{r} U_S\left(\frac{2r}{c_0}, \boldsymbol{\xi}\right) d\boldsymbol{\xi} . \tag{4.64}$$

Making a change of variables to cylindrical coordinates (ρ,θ,y_3), then the zero order inversion approximation becomes

$$\tilde{\psi}_0(y_3) = \frac{32y_3}{c_0} \int_{y_3}^{\infty} \int_0^{2\pi} \frac{1}{r} U_S\left(\frac{2r}{c_0}, \boldsymbol{\xi}\right) d\theta\, \rho\, d\rho . \tag{4.65}$$

Moreover, in cylindrical coordinates we have the relation

$$r^2 = \rho^2 + y_3^2 \ . \tag{4.66}$$

Using the closed form expression (4.59) for the Cagniard-de Hoop data $U_S(t,\xi)$, the zero order inversion approximation (4.65) can be explicitly integrated. Hence,

$$\tilde{\psi}_0(y_3) = \frac{32y_3}{c_0} \int_{y_3}^{\infty} \int_0^{2\pi} \frac{1}{8\pi hr} \frac{d}{dt} \left(R(t)\ H\left(t - \frac{2h}{c_0} \right) \right) \Big|_{t=\frac{2r}{c_0}} dr$$

$$= \frac{4y_3}{h} \int_{y_3}^{\infty} \frac{d}{dr} \left(R(t)\ H\left(\frac{2r}{c_0} - \frac{2h}{c_0} \right) \right) dr \ . \tag{4.67}$$

Thus,

$$\tilde{\psi}_0(y_3) = \frac{4y_3}{h} R\left(\frac{2r}{c_0} \right)\ H(r - h) \Big|_{y_3}^{\infty}$$

$$= - \frac{4y_3}{h} R\left(\frac{2y_3}{c_0} \right)\ H(y_3 - h) \ . \tag{4.68}$$

In a neighborhood of the reflector (that is, for $y_3 \to h$), the zero order inversion approximation can be written in terms of a linear Taylor polynomial.

$$\tilde{\psi}_0(y_3) \to \tilde{\psi}_0(h) + \frac{d\tilde{\psi}_0}{dy_3}(h)\ (y_3 - h) \ . \tag{4.69}$$

Hence, by evaluating the expressions in the above Taylor polynomial, it is discovered that

$$\tilde{\psi}_0(y_3) \to - 4R_0\ H(y_3 - h) - \frac{4R_0}{h} \left(1 - \frac{2c_1}{c_0} \right) (y_3 - h) \Big|_{y_3 \geq h} , \tag{4.70}$$

where R_0 is recognized as the reflection coefficient,

$$R_0 = R\left(\frac{2h}{c_0} \right) = \frac{c_1 - c_0}{c_1 + c_0} , \tag{4.71}$$

and where

$$\frac{dR_0}{dt}\left(\frac{2h}{c_0}\right) = -\frac{c_1}{h} R_0 \ . \tag{4.72}$$

The factor of four multiplying the reflection coefficient in the result (4.70) is in exact agreement with results obtained by Bleistein, Cohen, and Hagin [5].

Now consider the first order inversion approximation given by

$$\tilde{\psi}_1(y_3) = \frac{16y_3}{\pi c_0} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \frac{1}{r} \left(1 - \frac{c_0}{2i\omega r} \right) e^{-\left(\frac{2i\omega r}{c_0}\right)} u_S(\omega,\xi)\ d\omega\ d\xi \ . \tag{4.73}$$

The first order inversion approximation above involves two individual terms. The first term has been calculated in equation (4.70) since it is simply the zero order inversion approximation. The second term in equation (4.73) is the correction term. Specifically,

$$\begin{aligned}\Delta\tilde{\psi} &= \tilde{\psi}_1(y_3) - \tilde{\psi}_0(y_3) \\ &= -\frac{8y_3}{\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \frac{1}{i\omega r^2} e^{-\left(\frac{2i\omega r}{c_0}\right)} u_S(\omega,\xi)\ d\omega\ d\xi \ .\end{aligned} \tag{4.74}$$

It is important to notice that the factor $-\frac{1}{i\omega}$ in the frequency-domain corresponds to integration in the time-domain. In particular,

$$\mathfrak{F}_1 \left(\int_{-\infty}^{t} U_S(t',\xi)\ dt' + C \right) = -\frac{1}{i\omega} u_S(\omega,\xi) \ , \tag{4.75}$$

where C is an arbitrary constant. Thus, the correction term $\Delta\tilde{\psi}$ in equation (4.74) can be explicitly integrated to yield

$$\Delta\tilde{\psi} = 16y_3 \int_{\mathbb{R}^2} \frac{1}{r^2} \left(\int_{-\infty}^{\frac{2r}{c_0}} U_S(t',\xi)\ dt' + C \right) d\xi \ . \tag{4.76}$$

Using the Cagniard-de Hoop formula (4.59) in equation (4.76) results in the following.

$$\Delta\tilde{\psi} = 16y_3 \int_{\mathbb{R}^2} \frac{1}{r^2} \left(\int_{-\infty}^{\frac{2r}{c_0}} \frac{1}{8\pi h} \frac{d}{dt} \left\{ R(t')\, H\!\left(t' - \frac{2h}{c_0}\right) \right\} dt' + C \right) d\xi \; . \tag{4.77}$$

Carrying out the integration inside the parentheses produces

$$\Delta\tilde{\psi} = 16y_3 \int_{\mathbb{R}^2} \frac{1}{r^2} \left(\frac{1}{8\pi h} R\!\left(\frac{2r}{c_0} \right) H\!\left(\frac{2r}{c_0} - \frac{2h}{c_0} \right) + C \right) d\xi \; . \tag{4.78}$$

We can again make a change of variables to cylindrical coordinates and then relate the result to integration with respect to r. The result is given by

$$\Delta\tilde{\psi} = 32\pi y_3 \int_{y_3}^{\infty} \frac{1}{r} \left(\frac{1}{8\pi h} R\!\left(\frac{2r}{c_0} \right) H(r - h) + C \right) dr \; . \tag{4.79}$$

Thus far, we have not identified the constant C. We now choose the arbitrary constant C such that $\Delta\tilde{\psi} = 0$ at $y_3 = h$. Then for $y_3 \geqq h$, we discover that

$$\Delta\tilde{\psi} = 32\pi y_3 \left(\int_{y_3}^{\infty} \frac{1}{8\pi hr} R\!\left(\frac{2r}{c_0} \right) dr - \int_{h}^{\infty} \frac{1}{8\pi hr} R\!\left(\frac{2r}{c_0} \right) dr \right) . \tag{4.80}$$

Equivalently,

$$\Delta\tilde{\psi} = -\frac{4y}{h} \int_{h}^{y_3} \frac{1}{r} R\!\left(\frac{2r}{c_0} \right) dr \; . \tag{4.81}$$

In a neighborhood of the reflector $(y_3 \to h)$, we obtain a similar approximation to that of equation (4.70).

$$\Delta\tilde{\psi} \to -\frac{4}{h} R\!\left(\frac{2h}{c_0} \right) (y_3 - h) = -\frac{4R_0}{h} (y_3 - h) \; . \tag{4.82}$$

In the Born approximation used in the linearization of the backscattered data integral equation (1.24), the index of refraction perturbation is considered to be small. This is equivalent to the assumption that $c_1 - c_0$ is small. Hence, the ratio c_1/c_0 is approximately unity. In this case, we observe that equation (4.70) becomes

$$\tilde{\psi}_0(y_3) \to -\,4R_0\;H(y_3 - h) + \frac{4R_0}{h}\,(y_3 - h)\Big|_{y_3 \geq h} \,. \tag{4.83}$$

Moreover, we observe that the correction term of the first order inversion algorithm produces precisely the linear correction to annihilate the linear error in equation (4.83). Specifically, we see from equation (4.82) that the correction is given by

$$\tilde{\psi}_1(y_3) - \tilde{\psi}_0(y_3) = -\,\frac{4R_0}{h}\,(y_3 - h)\Big|_{y_3 \geq h} \tag{4.84}$$

Thus, the first order inversion algorithm eliminates local linear approximation errors that the zero order inversion algorithm generates. However, it does not correct for the linear error derived from the Born approximation in linearizing the backscattered data integral equation.

4.6 Inversion Algorithm for General Initial Conditions

In this concluding section, we generalize the first order inverse algorithm to account for more complex impulsive source waves. It was assumed in Chapter 3 that the source wave is asymptotically a spherical wave in the near-field. Specifically, the amplitude is given by

$$A(\mathbf{x},\omega,\boldsymbol{\xi}) \to \frac{Q_0}{4\pi\sigma}\,, \tag{4.85}$$

as $\sigma \to 0$. However, it is possible for the source wave in the near-field to have an asymptotic expansion of the form

$$A(\mathbf{x},\omega,\boldsymbol{\xi}) \to \frac{1}{4\pi\sigma}\left(Q_0 + \frac{Q_1}{i\omega} + \cdots \right) . \tag{4.86}$$

For a constant reference velocity, the rays propagate along straight lines as in equation (3.8). Solving the transport equations for these more general asymptotic initial conditions by the methods of Chapter 3 results in

$$A_j(\sigma) = \frac{Q_j}{4\pi\sigma} \qquad (j = 0,1,\ldots) , \tag{4.87}$$

where the notation $A_j(\sigma) = A_j(\mathbf{x}(\sigma),\boldsymbol{\xi})$ is used.

The backscattered data integral equation for constant reference velocity again takes the form

$$D(\omega,\boldsymbol{\xi}) = \frac{\omega^2}{c_0^2} \int_{\mathbb{R}^3} a(\mathbf{x},\omega,\boldsymbol{\xi})\; e^{i\omega\phi(\mathbf{x},\boldsymbol{\xi})}\; \psi(\mathbf{x})\; d\mathbf{x} , \tag{4.88}$$

where the phase is given by

$$\phi(\mathbf{x},\boldsymbol{\xi}) = 2\tau(\mathbf{x},\boldsymbol{\xi}) = \frac{2|\mathbf{x} - \boldsymbol{\xi}|}{c_0} . \tag{4.89}$$

However, the amplitude is modified for the more general imposed initial conditions (4.86). Specifically,

$$\begin{aligned} a(\mathbf{x},\omega,\boldsymbol{\xi}) &= \frac{Q_0^2}{16\pi^2\sigma^2} + \frac{Q_0^2 Q_1^2}{8\pi^2\sigma^2 i\omega} + \cdots \\ &= \frac{Q_0^2}{16\pi^2|\mathbf{x} - \boldsymbol{\xi}|^2} + \frac{Q_0^2 Q_1^2}{8\pi^2|\mathbf{x} - \boldsymbol{\xi}|^2 i\omega} + \cdots . \end{aligned} \tag{4.90}$$

The first order inversion algorithm is obtained by following the procedure of Chapter 3. This involves the calculation of the first order kernel function $b_1(\mathbf{y},\boldsymbol{\xi})$ provided in equation (3.46). There is an additional term included since $a_1(\mathbf{y},\boldsymbol{\xi})$ does not vanish for the initial conditions (4.86). The kernel function is given by

$$\begin{aligned} b_1(\mathbf{y},\boldsymbol{\xi}) &= \frac{128\pi^2 y_3}{Q_0^2 c_0 r} \left(- \frac{a_1(\mathbf{y},\boldsymbol{\xi})}{a_0(\mathbf{y},\boldsymbol{\xi})} - \frac{c_0}{2r} \right) \\ &= \frac{128\pi^2 y_3}{Q_0^2 c_0 r} \left(- \frac{2Q_1}{Q_0} - \frac{c_0}{2r} \right) . \end{aligned} \tag{4.91}$$

Thus, the first order inversion algorithm for constant reference velocity with the extended initial conditions (4.86) has the following form

$$\psi(\mathbf{y}) \sim \int_{\mathbb{R}^2} \int_{\mathbb{R}} \frac{16y_3}{\pi Q_0^2 c_0 r} \left(1 - \frac{1}{i\omega} \left\{ \frac{2Q_1}{Q_0} + \frac{c_0}{2r} \right\} \right) \cdot D(\omega,\xi)\, e^{-i\left(\frac{2r\omega}{c_0}\right)} \, d\omega \, d\xi \ . \tag{4.92}$$

The modified inversion algorithm (4.92) provides an additional correction term for the first order term of the asymptotic expansion of the impulsive source wave.

CONCLUSION

The present work has addressed the velocity inverse problem of geometric acoustics for an inhomogeneous non-dispersive medium. The velocity inverse problem involves the imaging of surfaces of discontinuous index of refraction through the analysis of the scattered waves at an accessible surface of the medium. The velocity inverse problem is mathematically formulated in terms of a linear integral equation based on the perturbation approach of the Born approximation of theoretical physics.

The elegant and powerful theory of Fourier integral operators is utilized to generate an explicit asymptotic expansion of the generalized asymptotic integral equation inversion operator. These results are specialized to the velocity inverse problem in 3-dimensions for constant reference velocity. Explicit zero order and first order inversion algorithms are developed. The first order algorithm is extensively analyzed using Fourier integral operator and Cagniard-de Hoop methods. It is shown that the first order inversion algorithm annihilates the linear error terms that are generated by the zero order inversion algorithm. However, it does not eliminate the error due to the Born approximation itself. The elimination of the Born approximation error requires the use of higher order Born approximations. However, if the index of refraction perturbation is sufficiently small, then the Born approximation error is negligible. Consequently, the first order inversion algorithm developed in this work represents a potentially valuable improvement over existing inversion algorithms. Furthermore, the results and methods of the present work are informative in demonstrating that the general pseudodifferential operator and Fourier integral operator theory can be applied to the problems of geometric acoustics.

It is to be noted that the Fourier integral operator methods and results extend to the much more difficult problem of non-constant reference velocity. The inversion algorithms developed in Chapter 2 are still valid for the non-constant reference velocity case, but the explicit calculation of the required kernel functions is extremely complex and remains unsolved. Stickler, Tavantzis, and Ammicht [30] showed that for certain reference velocity profiles, the second term of the geometric acoustics expansion can be implicitly obtained. It remains unknown whether the detailed calculations can be performed or whether the surprising simplifications and cancellations seen in the constant reference velocity case will occur for non-constant reference velocity examples.

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