CWP-173
May 1995

Thomsen operators and Thomsen matrices

Jack K. Cohen

Center for Wave Phenomena
Colorado School of Mines
Golden, Colorado 80401
303/273-3557

Thomsen Operators and Thomsen Matrices

Jack K. Cohen

ABSTRACT

Leon Thomsen introduced a set of parameters that allow specialization to weakly transverse isotropic (TI) media without losing the capability of treating the general TI medium. The Thomsen parameters have proven useful in a variety of transverse isotropic media studies—it turns out that they also lead to an elegant formulation of the TI wave equations. Indeed, the TI wave equation operator takes the form,

$$\mathcal{L} = \mathcal{L}^{(0)} + \gamma\mathcal{L}^{(\gamma)} + \tilde{\delta}\mathcal{L}^{(\delta)} + \epsilon\mathcal{L}^{(\epsilon)}.$$

The operator $\mathcal{L}^{(0)}$ is the isotropic wave equation operator, while the "Thomsen operators," $\mathcal{L}^{(\gamma)}$, $\mathcal{L}^{(\delta)}$, and $\mathcal{L}^{(\epsilon)}$, characterize the anisotropic contributions. The isotropic operator and the three Thomsen operators are *independent* of the Thomsen parameters γ, δ, and ϵ, so that the TI operator is *linear* in γ, ϵ, and the modified Thomsen parameter $\tilde{\delta}$. The parameter $\tilde{\delta}$ reduces to Thomsen's δ in the limit of weak transverse anisotropy. The three Thomsen operators are spatial differential operators and the "Thomsen matrices," $\mathsf{M}^{(\gamma)}$, $\mathsf{M}^{(\delta)}$, and $\mathsf{M}^{(\epsilon)}$ are their respective spatial Fourier transforms. The matrices $\mathsf{M}^{(\gamma)}$ and $\mathsf{M}^{(\epsilon)}$ have rank one, while the matrix $\mathsf{M}^{(\delta)}$ is of rank two.

Two simple applications are presented to illustrate the utility of the formulating the TI wave equations in terms of the Thomsen operators/matrices. The first is a direct derivation of the phase speeds in the limit of weak TI by application of the standard matrix perturbation theory for the eigenvalue-eigenvector problem. The second application is a derivation of the exact TI Green's tensor in the special case δ=ϵ=0.

THE TI WAVE EQUATIONS IN THOMSEN NOTATION

Leon Thomsen (1986) introduced a set of parameters that allow specialization to weakly transverse isotropic (TI) media without losing the capability of treating the general TI medium. The Thomsen parameters have proven useful in a variety of transversely isotropic media studies—the following considerations show that their use also leads to an elegant formulation of the TI wave equations.

The Love parameters are expressed in terms of the Thomsen parameters, γ, δ, ϵ, and the material parameters by the equations:

$$
\begin{aligned}
A &= \rho c_P^2(1+2\epsilon) \\
C &= \rho c_P^2 \\
F &= -\rho c_S^2 + \rho c_P^2\sqrt{f(f+2\delta)} = -\rho c_S^2 + \rho c_P^2 f\sqrt{1+2\delta/f} \\
L &= \rho c_S^2 \\
N &= \rho c_S^2(1+2\gamma)
\end{aligned}
\tag{1}
$$

Here, the quantity f is

$$
f = \frac{c_P^2 - c_S^2}{c_P^2} \tag{2}
$$

as introduced by Ilya Tsvankin (1994). In TI calculations, the following consequences of this definition are used repeatedly:

$$
c_P^2 - c_S^2 = f c_P^2, \qquad \text{and} \quad 1 - f = \frac{c_S^2}{c_P^2}. \tag{3}
$$

The material parameters are denoted by ρ for the density, and c_P and c_S for the speeds. These speeds represent the phase velocities along the distinguished axis (here the vertical or "3" axis)—alternately, they can be construed as the speeds that would prevail if the medium were isotropic. With this convention, the TI wave operators in Love notation are:

$$
\begin{aligned}
\mathcal{L}_1\boldsymbol{u} &= (A\partial_1^2 + N\partial_2^2 + L\partial_3^2)u_1 + (A-N)\partial_1\partial_2 u_2 + (F+L)\partial_1\partial_3 u_3 - \rho\partial_t^2 u_1 \\
\mathcal{L}_2\boldsymbol{u} &= (N\partial_1^2 + A\partial_2^2 + L\partial_3^2)u_2 + (A-N)\partial_1\partial_2 u_1 + (F+L)\partial_2\partial_3 u_3 - \rho\partial_t^2 u_2 \\
\mathcal{L}_3\boldsymbol{u} &= (L\partial_1^2 + L\partial_2^2 + C\partial_3^2)u_3 + (F+L)\partial_3(\partial_1 u_1 + \partial_2 u_2) - \rho\partial_t^2 u_3
\end{aligned}
\tag{4}
$$

Observe that δ enters the wave equations only through F, and, in turn, F enters the wave equations only in the combination $F+L$. Write this combination as

$$
F + L = \rho c_P^2 f\sqrt{1+2\delta/f} = \rho c_P^2(f+\tilde{\delta}), \tag{5}
$$

where we have introduced the modified Thomsen parameter,

$$
\tilde{\delta} = f(\sqrt{1+2\delta/f} - 1). \tag{6}
$$

Notice that for small δ, $\tilde{\delta} = \delta$ to first order. Thus, δ and $\tilde{\delta}$ are equally valid parameters for passing to the weak TI limit. The advantage of $\tilde{\delta}$ in the present study is that the TI wave equations are *linear* in this variable. Indeed, since the TI wave equations are already *linear* in γ and ϵ, we may write the TI operators in the form,

$$
\mathcal{L} = \mathcal{L}^{(0)} + \gamma\mathcal{L}^{(\gamma)} + \tilde{\delta}\mathcal{L}^{(\delta)} + \epsilon\mathcal{L}^{(\epsilon)}, \tag{7}
$$

where the matrix operators, $\mathcal{L}^{(0)}$, $\mathcal{L}^{(\gamma)}$, $\mathcal{L}^{(\delta)}$, and $\mathcal{L}^{(\epsilon)}$, are independent of the Thomsen parameters. Notice that this decomposition of the TI wave operator is valid for strong as well as for weak transverse anisotropy. Observe that a decomposition for weak TI using Thomsen's δ only requires expanding the scalar quantity $\tilde{\delta}$ to the desired order in δ—in the usual first order case, this amounts to merely replacing $\tilde{\delta}$ by δ.
Remark: For other purposes, $\tilde{\delta}$ is less useful that Thomsen's δ or other combinations of the Thomsen parameters, so I am *not* proposing a replacement of δ by $\tilde{\delta}$ in general TI studies.

Explicitly, we have for the components of $\mathcal{L}^{(0)}$:

$$\begin{aligned}
\mathcal{L}_1^{(0)}\boldsymbol{u} &= (\rho c_P^2\partial_1^2 + \rho c_S^2\partial_2^2 + \rho c_S^2\partial_3^2)u_1 + \rho c_P^2 f\partial_1\partial_2 u_2 + \rho c_P^2 f\partial_1\partial_3 u_3 - \rho\partial_t^2 u_1 \\
\mathcal{L}_2^{(0)}\boldsymbol{u} &= (\rho c_S^2\partial_1^2 + \rho c_P^2\partial_2^2 + \rho c_S^2\partial_3^2)u_2 + \rho c_P^2 f\partial_1\partial_2 u_1 + \rho c_P^2 f\partial_2\partial_3 u_3 - \rho\partial_t^2 u_2 \\
\mathcal{L}_3^{(0)}\boldsymbol{u} &= (\rho c_S^2\partial_1^2 + \rho c_S^2\partial_2^2 + \rho c_P^2\partial_3^2)u_3 + \rho c_P^2 f\partial_3(\partial_1 u_1 + \partial_2 u_2) - \rho\partial_t^2 u_3
\end{aligned} \quad (8)$$

After writing the speeds in terms of the Lamé parameters as

$$c_S^2 = \frac{\mu}{\rho}, \quad (9)$$

$$c_P^2 = \frac{\lambda + 2\mu}{\rho}, \quad (10)$$

it is straightforward to show that these equations are just the ordinary *isotropic* elastic wave equations for a homogeneous medium.

The new operators, which characterize the anisotropic contributions, have the components,

$$\begin{aligned}
\mathcal{L}_1^{(\gamma)}\boldsymbol{u} &= 2\rho c_S^2\partial_2(\partial_2 u_1 - \partial_1 u_2) \\
\mathcal{L}_2^{(\gamma)}\boldsymbol{u} &= 2\rho c_S^2\partial_1(\partial_1 u_2 - \partial_2 u_1) \\
\mathcal{L}_3^{(\gamma)}\boldsymbol{u} &= 0,
\end{aligned} \quad (11)$$

$$\begin{aligned}
\mathcal{L}_1^{(\delta)}\boldsymbol{u} &= \rho c_P^2\partial_1\partial_3 u_3 \\
\mathcal{L}_2^{(\delta)}\boldsymbol{u} &= \rho c_P^2\partial_2\partial_3 u_3 \\
\mathcal{L}_3^{(\delta)}\boldsymbol{u} &= \rho c_P^2\partial_3(\partial_1 u_1 + \partial_2 u_2),
\end{aligned} \quad (12)$$

and,

$$\begin{aligned}
\mathcal{L}_1^{(\epsilon)}\boldsymbol{u} &= 2\rho c_P^2\partial_1(\partial_1 u_1 + \partial_2 u_2) \\
\mathcal{L}_2^{(\epsilon)}\boldsymbol{u} &= 2\rho c_P^2\partial_2(\partial_1 u_1 + \partial_2 u_2) \\
\mathcal{L}_3^{(\epsilon)}\boldsymbol{u} &= 0.
\end{aligned} \quad (13)$$

The explicit matrix form of the "Thomsen operators" introduced in equation (7) are:

$$\mathcal{L}^{(\gamma)} = 2\rho c_S^2 \begin{pmatrix} \partial_2^2 & -\partial_1\partial_2 & 0 \\ -\partial_1\partial_2 & \partial_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{14}$$

$$\mathcal{L}^{(\delta)} = \rho c_P^2 \begin{pmatrix} 0 & 0 & \partial_1\partial_3 \\ 0 & 0 & \partial_2\partial_3 \\ \partial_1\partial_3 & \partial_2\partial_3 & 0 \end{pmatrix}, \tag{15}$$

and,

$$\mathcal{L}^{(\epsilon)} = 2\rho c_P^2 \begin{pmatrix} \partial_1^2 & \partial_1\partial_2 & 0 \\ \partial_1\partial_2 & \partial_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{16}$$

The Thomsen operators can also be written in terms of dyadic differential operators. Indeed, $\mathcal{L}^{(\gamma)}$ is the rank one operator:

$$\mathcal{L}^{(\gamma)} = 2\rho c_S^2 \begin{pmatrix} -\partial_2 \\ \partial_1 \\ 0 \end{pmatrix} (-\partial_2\ \partial_1\ 0). \tag{17}$$

Thus

$$\mathcal{L}^{(\gamma)} = 2\rho c_S^2 \boldsymbol{D}^{\perp}\boldsymbol{D}^{\perp}, \qquad \text{where} \quad \boldsymbol{D}^{\perp} = \begin{pmatrix} -\partial_2 \\ \partial_1 \\ 0 \end{pmatrix}. \tag{18}$$

Similarly, the rank one representation of $\mathcal{L}^{(\epsilon)}$ is

$$\mathcal{L}^{(\epsilon)} = 2\rho c_P^2 \boldsymbol{D}\boldsymbol{D}, \qquad \text{where} \quad \boldsymbol{D} = \begin{pmatrix} \partial_1 \\ \partial_2 \\ 0 \end{pmatrix}, \tag{19}$$

and a rank two representation of $\mathcal{L}^{(\delta)}$ is

$$\mathcal{L}^{(\delta)} = \rho c_P^2 (\boldsymbol{D}\boldsymbol{D}_3 + \boldsymbol{D}_3\boldsymbol{D}), \qquad \text{where} \quad \boldsymbol{D}_3 = \begin{pmatrix} 0 \\ 0 \\ \partial_3 \end{pmatrix}. \tag{20}$$

THE THOMSEN MATRICES

Now study the Thomsen form of the TI wave equations in Fourier domain. Apply the four-fold Fourier transform,

$$\boldsymbol{U}(\boldsymbol{k}, t) = \int d\boldsymbol{r} \int dt\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}\, \boldsymbol{u}(\boldsymbol{r}, t), \tag{21}$$

and define M, $\mathsf{M}^{(0)}$, $\mathsf{M}^{(\gamma)}$, $\mathsf{M}^{(\delta)}$, and $\mathsf{M}^{(\epsilon)}$ as the *negatives* of the transforms of the corresponding differential operators in equation (7). From equations (14), (15), and (16), the negatives of the transformed Thomsen operators can be written down at once in matrix form as

$$\mathsf{M}^{(\gamma)} = 2\rho c_S^2 \begin{pmatrix} k_2^2 & -k_1k_2 & 0 \\ -k_1k_2 & k_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{22}$$

$$\mathsf{M}^{(\delta)} = \rho c_P^2 \begin{pmatrix} 0 & 0 & k_1k_3 \\ 0 & 0 & k_2k_3 \\ k_1k_3 & k_2k_3 & 0 \end{pmatrix}, \tag{23}$$

and,

$$\mathsf{M}^{(\epsilon)} = 2\rho c_P^2 \begin{pmatrix} k_1^2 & k_1k_2 & 0 \\ k_1k_2 & k_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{24}$$

Similarly, the transforms of equations (18), (19), and, (20) yield the dyadic representations of the Thomsen matrices as

$$\mathsf{M}^{(\gamma)} = 2\rho c_S^2 \boldsymbol{\kappa}^\perp \boldsymbol{\kappa}^\perp, \qquad \text{where} \quad \boldsymbol{\kappa}^\perp = \begin{pmatrix} -k_2 \\ k_1 \\ 0 \end{pmatrix}, \tag{25}$$

$$\mathsf{M}^{(\epsilon)} = 2\rho c_P^2 \boldsymbol{\kappa}\boldsymbol{\kappa}, \qquad \text{where} \quad \boldsymbol{\kappa} = \begin{pmatrix} k_1 \\ k_2 \\ 0 \end{pmatrix}, \tag{26}$$

and,

$$\mathsf{M}^{(\delta)} = \rho c_P^2 (\boldsymbol{\kappa}\boldsymbol{k}_3 + \boldsymbol{k}_3\boldsymbol{\kappa}), \qquad \text{where} \quad \boldsymbol{k}_3 = \begin{pmatrix} 0 \\ 0 \\ k_3 \end{pmatrix}. \tag{27}$$

The vector system $\boldsymbol{\kappa}$, $\boldsymbol{\kappa}^\perp$, and $\boldsymbol{k}_3$ is closely related to the ordinary cylindrical unit vector basis, here denoted by $\hat{\boldsymbol{\kappa}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{k}}_3$. Indeed,

$$\boldsymbol{\kappa} = \kappa \hat{\boldsymbol{\kappa}}, \qquad \hat{\boldsymbol{\kappa}} = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}, \tag{28}$$

$$\boldsymbol{\kappa}^\perp = \kappa \hat{\boldsymbol{\phi}}, \qquad \hat{\boldsymbol{\phi}} = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \tag{29}$$

and,

$$\boldsymbol{k}_3 = k_3 \hat{\boldsymbol{k}}_3, \qquad \hat{\boldsymbol{k}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{30}$$

In cylindrical notation, the Thomsen matrices are

$$\mathsf{M}^{(\gamma)} = 2\rho c_S^2 \kappa^2 \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}}, \tag{31}$$

$$\mathsf{M}^{(\epsilon)} = 2\rho c_P^2 \kappa^2 \hat{\boldsymbol{\kappa}}\hat{\boldsymbol{\kappa}}, \tag{32}$$

and,

$$\mathsf{M}^{(\delta)} = \rho c_P^2 \kappa k_3 (\hat{\boldsymbol{\kappa}}\hat{\boldsymbol{k}}_3 + \hat{\boldsymbol{k}}_3\hat{\boldsymbol{\kappa}}). \tag{33}$$

To get the Thomsen matrices in terms of the spherical coordinate unit vectors, here denoted by $\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{\theta}}$, and the aforementioned $\hat{\boldsymbol{\phi}}$, observe the relations,

$$\hat{\boldsymbol{k}} = \begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix} = \sin\theta\,\hat{\boldsymbol{\kappa}} + \cos\theta\,\hat{\boldsymbol{k}}_3$$

$$\hat{\boldsymbol{\theta}} = \begin{pmatrix} \cos\phi\cos\theta \\ \sin\phi\cos\theta \\ -\sin\theta \end{pmatrix} = \cos\theta\,\hat{\boldsymbol{\kappa}} - \sin\theta\,\hat{\boldsymbol{k}}_3, \tag{34}$$

or,

$$\hat{\boldsymbol{\kappa}} = \sin\theta\,\hat{\boldsymbol{k}} + \cos\theta\,\hat{\boldsymbol{\theta}}$$

$$\hat{\boldsymbol{k}}_3 = \cos\theta\,\hat{\boldsymbol{k}} - \sin\theta\,\hat{\boldsymbol{\theta}}. \tag{35}$$

In terms of spherical coordinates, the Thomsen matrices are

$$\mathsf{M}^{(\gamma)} = 2\rho c_S^2 k^2 \sin^2\theta\,\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}}, \tag{36}$$

$$\mathsf{M}^{(\epsilon)} = 2\rho c_P^2 k^2 \sin^2\theta \left(\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\sin^2\theta + (\hat{\boldsymbol{k}}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\theta}}\hat{\boldsymbol{k}})\sin\theta\cos\theta + \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}\cos^2\theta\right), \tag{37}$$

and,

$$\mathsf{M}^{(\delta)} = \rho c_P^2 k^2 \sin\theta\cos\theta \left(\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\sin 2\theta + (\hat{\boldsymbol{k}}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\theta}}\hat{\boldsymbol{k}})\cos 2\theta - \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}\sin 2\theta\right). \tag{38}$$

For the application given in the next section, it is convenient to also have explicit representations for the isotropic matrix, $\mathsf{M}^{(0)}$:

$$\begin{aligned} \mathsf{M}^{(0)} &= \rho(c_S^2 k^2 - \omega^2)\mathbb{I} + \rho(c_P^2 - c_S^2)\boldsymbol{k}\boldsymbol{k} && (39) \\ &= \lambda_S \mathbb{I} + (\lambda_P - \lambda_S)\hat{\boldsymbol{k}}\hat{\boldsymbol{k}} \\ &= \lambda_S(\mathbb{I} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}) + \lambda_P \hat{\boldsymbol{k}}\hat{\boldsymbol{k}} && (40) \end{aligned}$$

Here $\mathbb{I}$ denotes the identity matrix.

Remark: The representation of $\mathsf{M}^{(0)}$ in terms of the Lamé parameters is

$$\mathsf{M}^{(0)} = (\mu k^2 - \rho\omega^2)\mathbb{I} + (\lambda + \mu)\boldsymbol{k}\boldsymbol{k}. \tag{41}$$

Before proceeding, review the theory of the "spectral representation" for a real symmetric matrix A (say 3 by 3, for simplicity). The eigenvectors of such a matrix can be taken as an orthonormal basis of $\mathcal{R}^3$, say e_1, e_2, and e_3. Denoting the corresponding eigenvalues by λ_1, λ_2, and λ_3, the spectral representation of A is

$$\mathsf{A} = \lambda_1 e_1 e_1 + \lambda_2 e_2 e_2 + \lambda_3 e_3 e_3. \tag{42}$$

Then, for any function $f(z)$ defined on the λ_j, we have

$$f(\mathsf{A}) = f(\lambda_1) e_1 e_1 + f(\lambda_2) e_2 e_2 + f(\lambda_3) e_3 e_3. \tag{43}$$

The form of the matrix $\mathsf{M}^{(0)}$ in equation (40) is its spectral representation with the eigenvalues being λ_S (double eigenvalue) and λ_P. The rank two tensor $\mathbb{I} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}$ may be replaced by a sum of dyadics based on any pair of orthonormal vectors that are orthogonal to $\hat{\boldsymbol{k}}$, but we have no immediate need to introduce a specific pair, so we allow this mild generalization of the spectral representation and write the matrix functions of $\mathsf{M}^{(0)}$ as

$$f(\mathsf{M}^{(0)}) = f(\lambda_S)(\mathbb{I} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}) + f(\lambda_P)\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}. \tag{44}$$

A principal application of the spectral form of $\mathsf{M}^{(0)}$ is in finding the isotropic Green's tensor $\mathsf{G}^{(0)}$ satisfying the differential system

$$\mathcal{L}^{(0)} \mathbf{g}^{(0)} = -\mathbb{I}\, \delta(\boldsymbol{r})\delta(t). \tag{45}$$

In transform domain, this is

$$\mathsf{M}^{(0)} \cdot \mathsf{G}^{(0)} = \mathbb{I}. \tag{46}$$

Thus, $\mathsf{G}^{(0)}$ is just the matrix inverse of $\mathsf{M}^{(0)}$ and we can apply the spectral theory with $f(z) = 1/z$ to obtain

$$\mathsf{G}^{(0)} = \frac{1}{\lambda_S}(\mathbb{I} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}) + \frac{1}{\lambda_P}\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}. \tag{47}$$

APPLICATION: PHASE SPEEDS IN THE WEAK TI LIMIT

As an application of the Thomsen matrices, seek the plane wave solutions in the weak TI limit. These are the solutions $\boldsymbol{u}$ of the homogeneous equation,

$$\mathcal{L}\boldsymbol{u} = \mathbf{o}, \tag{48}$$

that have the form

$$\boldsymbol{u} = \boldsymbol{v} e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}, \tag{49}$$

where $\boldsymbol{v}$ is a *constant* amplitude vector. Insert this plane wave ansätz into the TI wave equation to obtain

$$\mathsf{M} \cdot \boldsymbol{v} = 0. \tag{50}$$

Here, to first order in the Thomsen parameters, M is given by

$$\mathsf{M} \approx \mathsf{M}^{(0)} + \gamma \mathsf{M}^{(\gamma)} + \delta \mathsf{M}^{(\delta)} + \epsilon \mathsf{M}^{(\epsilon)}. \tag{51}$$

Again, $\mathsf{M}^{(0)}$ is the transform of the isotropic elastic wave operator and the remaining matrices on the right are the Thomsen matrices defined earlier. From equation (39), write

$$\mathsf{M}^{(0)} = \rho(c_P^2 - c_S^2)\boldsymbol{k}\boldsymbol{k} + \rho c_S^2 k^2 \mathbb{I} - \rho\omega^2 \mathbb{I}. \tag{52}$$

Thus, the plane wave problem is related to the eigenvalue-eigenvector problem by equating $\rho\omega^2$ to the eigenvalue λ in the zero order matrix. Indeed, we consider both the unperturbed eigen-problem,

$$\mathsf{N}^{(0)}\boldsymbol{v}^{(0)} = \lambda^{(0)}\boldsymbol{v}^{(0)}, \qquad \mathsf{N}^{(0)} = \rho(c_P^2 - c_S^2)\boldsymbol{k}\boldsymbol{k} + \rho c_S^2 k^2 \mathbb{I}, \tag{53}$$

and the perturbed problem with perturbation specified by the Thomsen matrices,

$$\mathsf{N}\boldsymbol{v} = \lambda\boldsymbol{v}, \qquad \mathsf{N} = \mathsf{N}^{(0)} + \gamma\mathsf{M}^{(\gamma)} + \delta\mathsf{M}^{(\delta)} + \epsilon\mathsf{M}^{(\epsilon)}. \tag{54}$$

Then, we obtain the plane wave solutions by using the eigenvectors as the amplitudes of the plane waves with dispersion relations obtained by setting the eigenvalues equal to $\rho\omega^2$. Notice that $\mathsf{N}^{(0)}$ and N are respectively the same as $\mathsf{M}^{(0)}$ and M except for the omitted $\rho\omega^2\mathbb{I}$ term.

The results cited below for the perturbed eigen-problem rely on the theory expounded in the classic Courant-Hilbert text (Courant & Hilbert, 1953). Computing the perturbation corrections in the present application has two complications over the simplest case:

1. The eigenvalues in the unperturbed (isotropic) case are degenerate.
2. There are *three* small parameters instead of just one.

The second problem isn't serious: the corrections corresponding to each Thomsen parameter can be computed separately and the total correction is just the sum of the individual ones. We overcome the associated notational problem by first stating the results for a generic small parameter β and then applying the generic result for each of the perturbations γ, δ, and ϵ. In particular, denote the expansions to first order of the generic eigenvalue and eigenvector by

$$\lambda = \lambda^{(0)} + \beta\lambda^{(\beta)}, \tag{55}$$

and

$$\boldsymbol{v} = \boldsymbol{v}^{(0)} + \beta\boldsymbol{v}^{(\beta)}. \tag{56}$$

For the unperturbed eigen-problem in equation (53), the eigenvectors and associated eigenvalues are:

$$\boldsymbol{v}^{(0)} = \begin{cases} \hat{\boldsymbol{k}} \\ c_{11}\hat{\boldsymbol{\theta}} + c_{12}\hat{\boldsymbol{\phi}} \equiv \hat{\boldsymbol{l}} \\ c_{21}\hat{\boldsymbol{\theta}} + c_{22}\hat{\boldsymbol{\phi}} \equiv \hat{\boldsymbol{m}} \end{cases}, \qquad \lambda^{(0)} = \begin{cases} \rho c_P^2 k^2 \\ \rho c_S^2 k^2 \\ \rho c_S^2 k^2 \end{cases}. \tag{57}$$

As expected, the second eigenvalue is repeated. Putting the eigenvalues equal to $\rho\omega^2$, obtain the plane wave dispersion relations $\omega^2 = c_P^2 k^2$ and $\omega^2 = c_S^2 k^2$ with the associated phase speeds ω/k being c_P and c_S, also as expected.

As far as the unperturbed problem is concerned, the matrix $\mathsf{C} = (c_{ij})$ can be *any* orthogonal matrix. It is perhaps surprising that a consistent perturbation theory for the degenerate case puts the following constraint on C:

Theorem 1 C *must be chosen such that*

$$\hat{\boldsymbol{l}} \cdot \mathsf{M}^{(\beta)} \cdot \hat{\boldsymbol{m}} = \mathsf{D}, \tag{58}$$

where D *is a diagonal matrix,*

$$\mathsf{D} = \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}. \tag{59}$$

Observe that equations (36-38) imply that this condition is satisfied simultaneously for all three Thomsen matrices with the choices $\hat{\boldsymbol{l}} = \hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{m}} = \hat{\boldsymbol{\phi}}$. Thus, we can dispense with the matrix C and simplify the result for the unperturbed eigen-problem to:

$$\boldsymbol{v}^{(0)} = \begin{Bmatrix} \hat{\boldsymbol{k}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{Bmatrix}, \qquad \lambda^{(0)} = \begin{Bmatrix} \lambda_k^{(0)} \\ \lambda_\theta^{(0)} \\ \lambda_\phi^{(0)} \end{Bmatrix} = \begin{Bmatrix} \rho c_P^2 k^2 \\ \rho c_S^2 k^2 \\ \rho c_S^2 k^2 \end{Bmatrix}. \tag{60}$$

The theory dictates expanding the first order perturbation in the eigenvectors in terms of the zeroth order eigenvectors (chosen consistent with Theorem 1). In generic notation, specialized to the case of perturbations from our unperturbed results in equation (60):

Theorem 2 *The perturbations in the eigenvalues are*

$$\begin{aligned} \lambda_k^{(\beta)} &= \hat{\boldsymbol{k}} \cdot \mathsf{M}^{(\beta)} \cdot \hat{\boldsymbol{k}} \\ \lambda_\theta^{(\beta)} &= \hat{\boldsymbol{\theta}} \cdot \mathsf{M}^{(\beta)} \cdot \hat{\boldsymbol{\theta}} \\ \lambda_\phi^{(\beta)} &= \hat{\boldsymbol{\phi}} \cdot \mathsf{M}^{(\beta)} \cdot \hat{\boldsymbol{\phi}} \end{aligned} \tag{61}$$

Moreover, the perturbations in the eigenvectors have the anti-symmetric form

$$\begin{aligned} \hat{\boldsymbol{k}}^{(\beta)} &= & b_{k\theta}^{(\beta)}\,\hat{\boldsymbol{\theta}} &+ b_{k\phi}^{(\beta)}\,\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{\theta}}^{(\beta)} &= -b_{k\theta}^{(\beta)}\,\hat{\boldsymbol{k}} & &+ b_{\theta\phi}^{(\beta)}\,\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{\phi}}^{(\beta)} &= -b_{k\phi}^{(\beta)}\,\hat{\boldsymbol{k}} & - b_{\theta\phi}^{(\beta)}\,\hat{\boldsymbol{\theta}} & \end{aligned} \tag{62}$$

with

$$
\begin{aligned}
b_{k\theta}^{(\beta)} &= \frac{\hat{\boldsymbol{k}} \cdot \mathsf{M}^{(\beta)} \cdot \hat{\boldsymbol{\theta}}}{\lambda_k - \lambda_\theta} \\
b_{k\phi}^{(\beta)} &= \frac{\hat{\boldsymbol{k}} \cdot \mathsf{M}^{(\beta)} \cdot \hat{\boldsymbol{\phi}}}{\lambda_k - \lambda_\phi} \\
b_{\theta\phi}^{(\beta)} &= \frac{(\hat{\boldsymbol{k}} \cdot \mathsf{M}^{(\beta)} \cdot \hat{\boldsymbol{\theta}})\,(\hat{\boldsymbol{\phi}} \cdot \mathsf{M}^{(\beta)} \cdot \hat{\boldsymbol{\theta}})}{(\lambda_\theta^{(\beta)} - \lambda_\phi^{(\beta)})\,(\lambda_\theta - \lambda_k)}.
\end{aligned} \tag{63}
$$

Apply this generic form of the result to each of the Thomsen perturbations given in equations (36-38) to obtain for the γ corrections:

$$
\lambda_k^{(\gamma)} = 0, \quad \lambda_\theta^{(\gamma)} = 0, \quad \lambda_\phi^{(\gamma)} = 2\rho c_S^2 k^2 \sin^2\theta, \tag{64}
$$

and

$$
b_{k\theta}^{(\gamma)} = 0, \quad b_{k\phi}^{(\gamma)} = 0, \quad b_{\theta\phi}^{(\gamma)} = 0, \tag{65}
$$

for the δ corrections:

$$
\lambda_k^{(\delta)} = 2\rho c_P^2 k^2 \sin^2\theta\cos^2\theta, \quad \lambda_\theta^{(\delta)} = -2\rho c_P^2 k^2 \sin^2\theta\cos^2\theta, \quad \lambda_\phi^{(\delta)} = 0, \tag{66}
$$

and

$$
b_{k\theta}^{(\delta)} = \frac{1}{f}\sin\theta\cos\theta(1 - 2\sin^2\theta), \quad b_{k\phi}^{(\delta)} = 0, \quad b_{\theta\phi}^{(\delta)} = 0, \tag{67}
$$

and for the ϵ corrections:

$$
\lambda_k^{(\epsilon)} = 2\rho c_P^2 k^2 \sin^4\theta, \quad \lambda_\theta^{(\epsilon)} = 2\rho c_P^2 k^2 \sin^2\theta\cos^2\theta, \quad \lambda_\phi^{(\epsilon)} = 0, \tag{68}
$$

and

$$
b_{k\theta}^{(\epsilon)} = \frac{2}{f}\sin^3\theta\cos\theta, \quad b_{k\phi}^{(\epsilon)} = 0, \quad b_{\theta\phi}^{(\epsilon)} = 0. \tag{69}
$$

In these equations, f is the quantity defined in equation (2).

Adding these results gives the following first order solution to the eigen-problem:

$$
\lambda_k \approx \rho c_P^2 k^2 \left(1 + 2\sin^2\theta\left[\delta + (\epsilon - \delta)\sin^2\theta\right]\right) \tag{70}
$$

$$
\boldsymbol{v}_k \approx \hat{\boldsymbol{k}} + q\hat{\boldsymbol{\theta}} \tag{71}
$$

$$
\lambda_\theta \approx \rho k^2 \left(c_S^2 + 2c_P^2 \sin^2\theta\cos^2\theta\,(\epsilon - \delta)\right) \tag{72}
$$

$$
\boldsymbol{v}_\theta \approx \hat{\boldsymbol{\theta}} - q\hat{\boldsymbol{k}} \tag{73}
$$

$$
\lambda_\phi \approx \rho c_S^2 k^2 (1 + 2\gamma\sin^2\theta) \tag{74}
$$

$$
\boldsymbol{v}_\phi \approx \hat{\boldsymbol{\phi}}. \tag{75}
$$

Here, the shorthand notation

$$q = \frac{\sin\theta\cos\theta}{f}\left[\delta + 2(\epsilon - \delta)\sin^2\theta\right] \tag{76}$$

has been introduced.

Equating the eigenvalues to $\rho\omega^2$ yields the phase velocities:

$$V_k = V_{QP} \approx c_P\left\{1 + \left[\delta + (\epsilon - \delta)\sin^2\theta\right]\sin^2\theta\right\} \tag{77}$$

$$V_\theta = V_{QS} \approx c_S\left\{1 + \frac{c_P^2}{c_S^2}(\epsilon - \delta)\sin^2\theta\cos^2\theta\right\} \tag{78}$$

$$V_\phi = V_{SP} \approx c_S\left\{1 + \gamma\sin^2\theta\right\}. \tag{79}$$

Remark: These results could also be obtained by power series expansion of the eigenvalues of the full TI wave equation (Thomsen, 1986).

APPLICATION: THE GREEN'S TENSOR FOR δ=ϵ=0

As another example of using the Thomsen matrices, derive the Green's tensor in the special case when δ=ϵ=0. Note that this case is distinctly easier than the general case; the calculations here should be regarded as only a "warm-up" to obtaining fuller results. There is a substantial literature on the TI Green's function, however, most of it assumes the far field (or high frequency or ray) approximation. Some excellent papers on this topic are (Ben-Menahem & Sena, 1990; Ben-Menahem *et al.*, 1991; Buchwald, 1959; Kazi-Aoual *et al.*, 1988; Tverdokhlebov & Rose, 1988). Here, although only a special case of the Thomsen parameters is treated, the near field or low frequency terms are included.

Begin with the defining equation for the Green's tensor **g**,

$$\mathcal{L}\mathbf{g} = -\mathbb{I}\,\delta(r)\delta(t), \tag{80}$$

where $\mathbb{I}$ denotes the three by three identity matrix. After the Fourier transform defined in equation (21), this becomes

$$\mathsf{M}\cdot\mathsf{G} = \mathbb{I}. \tag{81}$$

Here the matrix M is given by

$$\mathsf{M} = \mathsf{M}^{(0)} + \gamma\mathsf{M}^{(\gamma)} + \tilde{\delta}\mathsf{M}^{(\delta)} + \epsilon\mathsf{M}^{(\epsilon)}, \tag{82}$$

where $\mathsf{M}^{(0)}$ is the transform of the isotropic elastic wave operator and the remaining matrices on the right are the Thomsen matrices defined earlier.

Observe that

$$\mathsf{M}^{(0)}\cdot\mathsf{G}^{(0)} = \mathbb{I}, \tag{83}$$

where $\mathsf{G}^{(0)}$ represents the *isotropic* Green's tensor in transform domain—given explicitly in equation (47).

Equation (83) implies that $\mathsf{M}^{(0)}$ and $\mathsf{G}^{(0)}$ are inverses, thus equation (81) can be written as

$$(\mathbb{I} + \mathsf{G}^{(0)} \cdot (\gamma\mathsf{M}^{(\gamma)} + \tilde{\delta}\mathsf{M}^{(\delta)} + \epsilon\mathsf{M}^{(\epsilon)})) \cdot \mathsf{G} = \mathsf{G}^{(0)}. \tag{84}$$

Hence,

$$\mathsf{G} = \mathsf{C}^{-1} \cdot \mathsf{G}^{(0)}, \tag{85}$$

where

$$\mathsf{C} = \mathbb{I} + \mathsf{G}^{(0)} \cdot (\gamma\mathsf{M}^{(\gamma)} + \tilde{\delta}\mathsf{M}^{(\delta)} + \epsilon\mathsf{M}^{(\epsilon)}). \tag{86}$$

At this point, make the simplifying assumption that δ and ϵ vanish. From equations (47) and (25), and the relation $\boldsymbol{k} \cdot \boldsymbol{\kappa}^{\perp} = 0$, conclude

$$\mathsf{G}^{(0)} \cdot \mathsf{M}^{(\gamma)} = \frac{2\rho c_S^2}{\lambda_S}\boldsymbol{\kappa}^{\perp}\boldsymbol{\kappa}^{\perp}. \tag{87}$$

Thus, in our special case,

$$\mathsf{C} = \mathbb{I} + \frac{2\gamma\rho c_S^2}{\lambda_S}\boldsymbol{\kappa}^{\perp}\boldsymbol{\kappa}^{\perp} = \begin{pmatrix} 1+\nu k_2^2 & -\nu k_1 k_2 & 0 \\ -\nu k_1 k_2 & 1+\nu k_2^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{88}$$

where

$$\nu = \frac{2\gamma\rho c_S^2}{\lambda_S}. \tag{89}$$

Using the block structure of C, obtain

$$\mathsf{C}^{-1} = \begin{pmatrix} \frac{1+\nu k_1^2}{\Delta} & \frac{\nu k_1 k_2}{\Delta} & 0 \\ \frac{\nu k_1 k_2}{\Delta} & \frac{1+\nu k_2^2}{\Delta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{90}$$

where

$$\Delta = (1+\nu k_2^2)(1+\nu k_1^2) - \nu^2 k_1^2 k_2^2 = 1 + \nu\kappa^2. \tag{91}$$

On using this result in equation (85), after some calculations, find that the formula for G can be written

$$\mathsf{G} = \mathsf{G}^{(0)} + \mathsf{G}^{(1)}, \tag{92}$$

where the exact "correction" for anisotropy in the special case $\delta = \epsilon = 0$ is given by

$$\mathsf{G}^{(1)} = -\frac{\nu}{\lambda_S\Delta}\boldsymbol{\kappa}^{\perp}\boldsymbol{\kappa}^{\perp}. \tag{93}$$

The factor $\frac{\nu}{\lambda_S\Delta}$ expands to

$$\frac{\nu}{\lambda_S\Delta} = \frac{2\gamma c_S^2}{(\omega^2 - \omega_0^2)(\omega^2 - \omega_1^2)}, \tag{94}$$

where

$$\omega_0^2 = c_S^2 k^2, \text{ and } \omega_1^2 = c_S^2(k_3^2 + (1+2\gamma)\kappa^2). \tag{95}$$

It is convenient to express these quantities as

$$\omega_n^2 = c_S^2(k_3^2 + a_n^2\kappa^2), \tag{96}$$

with

$$a_0^2 = 1, \text{ and } a_1^2 = 1 + 2\gamma. \tag{97}$$

The above results lead to the Fourier inversion of $\mathsf{G}^{(1)}$ as the integral

$$\mathbf{g}^{(1)} = \frac{-2\gamma c_S^2}{(2\pi)^4\rho} \int dk\, \boldsymbol{\kappa}^{\perp}\boldsymbol{\kappa}^{\perp} \int d\omega\, \frac{e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}}{(\omega^2-\omega_0^2)(\omega^2-\omega_1^2)}, \tag{98}$$

or by use of partial fractions as

$$\mathbf{g}^{(1)} = \frac{1}{(2\pi)^4\rho} \int dk\, \hat{\boldsymbol{\kappa}}^{\perp}\hat{\boldsymbol{\kappa}}^{\perp} \int d\omega\, e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} \left(\frac{1}{\omega^2-\omega_0^2} - \frac{1}{\omega^2-\omega_1^2}\right), \tag{99}$$

where $\hat{\boldsymbol{\kappa}}^{\perp}$ denotes the unit vector $\boldsymbol{\kappa}^{\perp}/\kappa$. The ω integrals are done by residue integration, yielding

$$\int d\omega\, \frac{e^{-i\omega t}}{\omega^2-\omega_n^2} = -2\pi H(t)\frac{\sin\omega_n t}{\omega_n}, \qquad n = 0, 1. \tag{100}$$

The integral over k_3 is done using a cosine transform result as

$$\int dk_3\, e^{i\,k_3 z}\frac{\sin\omega_n t}{\omega_n} = 2\int_0^\infty dk_3 \cos k_3|z|\, \frac{\sin\omega_n t}{\omega_n} = H(c_S t - z)\frac{\pi}{c_S}J_0(a_n\kappa\sqrt{c_S t - z}). \tag{101}$$

To accomplish the remaining integrations over k_1, k_2, introduce the plane polar coordinates κ and ϕ. Since $\hat{\boldsymbol{\kappa}}^{\perp}$ is just $\hat{\boldsymbol{\phi}}$ (see equation 29), the ϕ integration can be written

$$\begin{aligned}\int_{-\pi}^{\pi} d\phi & \begin{pmatrix} \sin^2\phi & -\sin\phi\cos\phi & 0 \\ -\sin\phi\cos\phi & \cos^2\phi & 0 \\ 0 & 0 & 0 \end{pmatrix} e^{i\,\kappa R\cos\phi} \\ &= 2\int_0^{\pi} d\phi \begin{pmatrix} 1-\cos^2\phi & 0 & 0 \\ 0 & \cos^2\phi & 0 \\ 0 & 0 & 0 \end{pmatrix} e^{i\,\kappa R\cos\phi},\end{aligned} \tag{102}$$

where the reduction on the right follows from elementary properties of the trigonometric functions. Introducing the unit vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ along the transverse axes, and recalling the integral representation of the Bessel function, this integral reduces to

$$2\pi J_0(\kappa R)\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + 2(\hat{\boldsymbol{y}}\hat{\boldsymbol{y}} - \hat{\boldsymbol{x}}\hat{\boldsymbol{x}})\int_0^{\pi} d\phi \cos^2\phi\, e^{i\,\kappa R\cos\phi}. \tag{103}$$

The remaining angular integral can also be reduced to an explicit Bessel function as follows:

$$\int_0^\pi d\phi \cos^2\phi\, e^{i\kappa R\cos\phi} = -\frac{1}{\kappa^2}\frac{d^2}{dR^2}\int_0^\pi d\phi\, e^{i\kappa R\cos\phi} = -\frac{1}{\kappa^2}\frac{d^2}{dR^2}J_0(\kappa R). \tag{104}$$

Using the defining differential equation for J_0 and the relation $J_0'(z) = J_1(z)$, derive

$$\frac{1}{\kappa^2}\frac{d^2}{dR^2}J_0(\kappa R) = \frac{1}{\kappa R}J_1(\kappa R) - J_0(\kappa R). \tag{105}$$

Assembling our results, we have

$$\begin{aligned} \mathbf{g}^{(1)} &= \frac{H(t)H(c_S t - z)}{4\pi\rho c_S}\int_0^\infty d\kappa\,\kappa\left(J_0(a\kappa\sqrt{c_S^2t^2 - z^2}) - J_0(\kappa\sqrt{c_S^2t^2 - z^2})\right)\cdot \\ &\qquad \left(J_0(\kappa R)\hat{\boldsymbol{y}}\hat{\boldsymbol{y}} + \frac{1}{\kappa R}J_1(\kappa R)(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{y}})\right), \end{aligned} \tag{106}$$

where $a = a_1 = \sqrt{1+2\gamma}$.

The final integration over κ is accomplished with the aid of the identities

$$\int_0^\infty d\kappa\,\kappa J_0(\kappa R)J_0(\kappa S) = \frac{\delta(R-S)}{\sqrt{RS}} = \frac{\delta(R-S)}{R}, \tag{107}$$

and

$$\int_0^\infty d\kappa\, J_0(\kappa R)J_1(\kappa S) = \frac{H(R-S)}{R}. \tag{108}$$

These lead to the closed form result

$$\begin{aligned} \mathbf{g}^{(1)} &= \frac{H(t)}{4\pi\rho c_S}\left(\left[\delta(R - a\sqrt{c_S^2t^2 - z^2}) - \delta(R - \sqrt{c_S^2t^2 - z^2})\right]\frac{\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}}{R}\right. \\ &\qquad \left. + \left[H(R - a\sqrt{c_S^2t^2 - z^2}) - H(R - \sqrt{c_S^2t^2 - z^2})\right]\frac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}}{R^2}\right), \end{aligned} \tag{109}$$

which, using standard properties of the Dirac and Heaviside functions, may be cast as

$$\begin{aligned} \mathbf{g}^{(1)} &= \frac{H(t)}{4\pi\rho}\left(\left[\frac{\delta(t - \frac{1}{c_S}\sqrt{\frac{R^2}{a^2} + z^2})}{c_S^2a^2\sqrt{\frac{R^2}{a^2} + z^2}} - \frac{\delta(t - \frac{r}{c_S})}{c_S^2\, r}\right]\frac{\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}}{R}\right. \\ &\qquad \left. + \left[H(t - \frac{1}{c_S}\sqrt{\frac{R^2}{a^2} + z^2}) - H(t - \frac{r}{c_S})\right]\frac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}}{c_S\, R^2}\right). \end{aligned} \tag{110}$$

To complete the determination of $\mathbf{g} = \mathbf{g}^{(0)} + \mathbf{g}^{(1)}$, note that the Fourier inversion of $\mathsf{G}^{(0)}$ is the isotropic Green's function which may be written as

$$\begin{aligned} \mathbf{g}^{(0)} &= \frac{H(t)}{4\pi r\rho}\left[\left(\frac{1}{c_P^2}\,\delta(t - t_P) + \frac{2t}{r^2}\,\mathcal{X}_{[t_P,t_S]}(t)\right)\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\right. \\ &\qquad \left. + \left(\frac{1}{c_S^2}\,\delta(t - t_S) - \frac{t}{r^2}\,\mathcal{X}_{[t_P,t_S]}(t)\right)(\mathbb{I} - \hat{\boldsymbol{r}}\hat{\boldsymbol{r}})\right]. \end{aligned} \tag{111}$$

Here the indicator function $\mathcal{X}_{[a,b]}$ is defined as

$$\mathcal{X}_{[a,b]}(x) = \begin{cases} 1 & a < x < b \\ 0 & \text{otherwise} \end{cases}, \tag{112}$$

and the P-wave and S-wave arrival times at r are defined as

$$t_P = \frac{r}{c_P}, \quad \text{and} \quad t_S = \frac{r}{c_S}. \tag{113}$$

Discussion of results

It should be emphasized again that the purpose of these Green's tensor calculations is only to suggest of the utility of using Thomsen notation as a starting point in TI wave equation studies. The full result would entail inverting the entire C matrix which would "scramble" the terms considered above. However, the result is sufficient to indicate that, in contrast to the phase velocity application, a perturbation expansion of the Green's tensor in the Thomsen parameters is only valid under restricted circumstances—for example, the difference of the Dirac functions can only be replaced by a derivative of the Dirac function for small γ under the restriction of small R (or in frequency domain under the restriction of low frequency).

REFERENCES

Ben-Menahem, A., & Sena, A. G. 1990. Seismic source theory in stratified anisotropic media. *J. Geophys. Res.*, **95**, 15.395–15.427.

Ben-Menahem, A., Gibson, R. L., & Sena, A. G. 1991. Green's tensor and radiation patterns of point sources in general anisotropic inhomogeneous elastic media. *Geophys. J. Int.*, **107**, 297–308.

Buchwald, V. T. 1959. Elastic waves in anisotropic media. *Proc. R. Soc. Lond. A*, **253**, 563–580.

Courant, R., & Hilbert, D. 1953. *Methods of Mathematical Physics, Volume I.* New York: Interscience.

Kazi-Aoual, M. N., Bonnet, G., & Jouanna, P. 1988. Response of an infinite elastic transversely isotropic medium to a point force. An analytic solution in Hankel space. *Geophys. J.*, **93**, 587–590.

Thomsen, Leon. 1986. Weak elastic anisotropy. *Geophysics*, **51**(10), 1954–1966.

Tsvankin, Ilya. 1994 (May). P-wave signatures and parametrization of transversely isotropic media: An overview. *Pages 349–384 of:* Bleistein, Norman, *et al.* (eds), *Project Review.* Center for Wave Phenomena, Golden, Colorado.

Tverdokhlebov, A., & Rose, J. 1988. On Green's functions for elastic waves in anisotropic media. *J. Acoust. Soc. Am.*, **83**, 118–121.