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ABSTRACT

If we assume the velocity profile is an image that can be compressed greatly using wavelets, then a relatively smaller *wavelet represented model* can be equivalent to a larger *physical model*; then we obtain the modified Cohen/Bleistein inversion formalism by restricting the perturbation to the *smaller wavelet represented model.* The method demonstrates three important features: first, all the Cohen/Bleistein technique can be modified easily; second, much larger inverse problems can be solved since they are expected to be greatly reduced; third, the ill-posedness can be alleviated by *ladder inversion*, i.e. inversion from low to high wave number versions.

INTRODUCTION

By a linearization process, most seismic inverse problems can be written as Fredholm integral equations of the first kind, as

$$\int k(\mathbf{x};\mathbf{x}_r;\mathbf{x}_s)m(\mathbf{x})d\mathbf{x} = b(\mathbf{x}_r;\mathbf{x}_s),$$

where $m(\mathbf{x})$ is the model to be determined, and the kernel $k(\mathbf{x};\mathbf{x}_r,\mathbf{x}_s)$ and the right-hand side $b(\mathbf{x}_r;\mathbf{x}_s)$ are known (at least in an interative manner). The use of such integral equations as a numerical tool in large-scale computations is rather limited, because such integral equations normally lead to dense systems of linear algebraic equations, and the latter have to be solved either directly or iteratively. Beylkin et al. (1991) introduced a very important and interesting idea, namely that *if* regularity permits, using compactly supported wavelets, such as Daubechies (1992) wavelets, dense matrices can often be converted to a sparse form. This technique is called the compression of operators.

In our paper, instead of compressing the *kernel,* we compress the *model,* $m(\mathbf{x})$. This demonstrates several important features: first, the assumption of the model

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$m(\mathbf{x})$ being an image is natural; second, the compression of $m(\mathbf{x})$ exactly also leads to the compression of the kernel, or in other words, the compression of the operator; third, the compression of the model naturally truncates the very high wavenumber components which cause ill-posedness; finally, since our aim is to obtain the model $m(\mathbf{x})$, we can simply set the cut-off error for compression of the model, while to set the cut-off error for the compression of the kernel to control the error of the model $m(\mathbf{x})$ is an open question (See Beylkin et al, 1991.).

MULTIDIMENSIONAL WAVELETS GENERATED BY SINGLE MOTHER WAVELET

By Multiresolution Analysis, $\mathcal{L}^2(\mathbf{R}^3)$–square integrable functions of three independent variables–can be decomposed into a ladder of subspaces with orthonormal wavelet bases by *fast decomposition algorithms.* Thus any function in $\mathcal{L}^2(\mathbf{R}^3)$ can be represented on these wavelet bases. The coefficients bear very striking physical features of both space and aperture, due to the space-aperture localization properties of wavelets. The signal can be reconstructed from these wavelet coefficients by *fast reconstruction algorithms.* Storing the nonzero wavelet coefficients can substantially compress the signals.

The Daubechies' (1992) orthogonal wavelets are compactly supported. The compactness of support translates to a saving of computation and storage. For geophysical problems, including migration and inversion, symmetry is also desirable. However simultaneous symmetry and compactness is impossible, thus we use the least asymmetric compactly supported wavelets with maximum number of vanishing moments (Daubechies, 1992). More vanishing moments mean better approximation for smooth functions (Wickerhauser, 1992). We choose wavelets with the above features to apply the velocity inversion by wavelet represented perturbation.

Suppose $\psi \in \mathcal{L}^2(\mathbf{R})$ and $\phi \in \mathcal{L}^2(\mathbf{R})$ are a pair of Daubechies' compactly supported wavelets and scaling functions. In the case of a real orthonormal mother wavelet and real signals to be processed, we can define the midpoint and radius of ψ in the space domains,

$$x^* = \int x \mid \psi(x) \mid^2 dx, \tag{1}$$

$$\Delta_\psi = \left\{ \int (x - x^*)^2 \mid \psi(x) \mid^2 dx \right\}^{1/2}, \tag{2}$$

and in the aperture domains as

$$k^* = (2\pi)^{-1} \int_0^\infty k \mid \hat{\psi}(k) \mid^2 dk, \tag{3}$$

$$\Delta_{\hat{\psi}} = (2\pi)^{-1/2} \left\{ \int_0^\infty (k - k^*)^2 \mid \hat{\psi}(k) \mid^2 dk \right\}^{1/2}, \tag{4}$$

The above definitions are slightly different from those of Chui (1992), and here the Fourier image of $f \in \mathcal{L}^2(\mathbf{R})$ is defined as

$$\widehat{f}(k) = \int e^{-ikx} f(x)dx.$$

Then, for example, the wavelet coefficients of $f \in \mathcal{L}^2(\mathbf{R})$, defined as

$$d_i^j = < f, \psi_{j,i} > = (2\pi)^{-1} < \widehat{f}, \widehat{\psi}_{j,i} > \tag{5}$$

represent the local information in a space-aperture window centered at $(2^j(i+x^*), \pm 2^{-j}\omega^*)$:

$$[2^j(i+x^*-\Delta_\psi), 2^j(i+x^*+\Delta_\psi)] \times \{[2^{-j}(-k^*-\Delta_{\widehat{\psi}}), 2^{-j}(-k^*+\Delta_{\widehat{\psi}})] \cup [2^{-j}(k^*-\Delta_{\widehat{\psi}}), 2^{-j}(k^*+\Delta_{\widehat{\psi}})]\}.$$

In the 1D case, define the subspaces of $\mathcal{L}^2(\mathbf{R})$, as

$$\mathbf{V}^j = \text{span}\{\phi_{j,i}(x) = 2^{-j/2}\phi(2^j x - i), i \in \mathbf{Z}\} \tag{6}$$

and

$$\mathbf{W}^j = \text{span}\{\psi_{j,i}(x) = 2^{-j/2}\psi(2^j x - i), i \in \mathbf{Z}\}. \tag{7}$$

In the 2D case, define the subspaces of $\mathcal{L}^2(\mathbf{R}^2)$, as

$$\mathbf{V}^j = \text{span}\{\phi_{j,\mathbf{i}}(\mathbf{x}) = \phi_{j,i_1}(x_1)\phi_{j,i_2}(x_2), \mathbf{i} \in \mathbf{Z}^2, \mathbf{x} \in \mathbf{R}^2\}, \tag{8}$$

and

$$\mathbf{W}_1^j = \text{span}\{\psi_{j,\mathbf{i}}^1(\mathbf{x}) = \psi_{j,i_1}(x_1)\phi_{j,i_2}(x_2), \mathbf{i} \in \mathbf{Z}^2, \mathbf{x} \in \mathbf{R}^2\}, \tag{9}$$

$$\mathbf{W}_2^j = \text{span}\{\psi_{j,\mathbf{i}}^2(\mathbf{x}) = \phi_{j,i_1}(x_1)\psi_{j,i_2}(x_2), \mathbf{i} \in \mathbf{Z}^2, \mathbf{x} \in \mathbf{R}^2\}, \tag{10}$$

$$\mathbf{W}_3^j = \text{span}\{\psi_{j,\mathbf{i}}^3(\mathbf{x}) = \psi_{j,i_1}(x_1)\psi_{j,i_2}(x_2), \mathbf{i} \in \mathbf{Z}^2, \mathbf{x} \in \mathbf{R}^2\}, \tag{11}$$

and

$$\mathbf{W}^j = \oplus_{l=1}^3 \mathbf{W}_l^j.$$

Here, $\oplus$ is the direct sum. In the 3D case, similarly, we define the linear subspaces of $\mathcal{L}^2(\mathbf{R}^3)$, as

$$\mathbf{V}^j = \text{span}\{\phi_{j,\mathbf{i}}(\mathbf{x}) = \phi_{j,i_1}(x_1)\phi_{j,i_2}(x_2)\phi_{j,i_3}(x_3), \mathbf{i} \in \mathbf{Z}^3, \mathbf{x} \in \mathbf{R}^3\}, \tag{12}$$

and

$$\mathbf{W}_1^j = \text{span}\{\psi_{j,\mathbf{i}}^1(\mathbf{x}) = \psi_{j,i_1}(x_1)\phi_{j,i_2}(x_2)\phi_{j,i_3}(x_3), \mathbf{i} \in \mathbf{Z}^3, \mathbf{x} \in \mathbf{R}^3\}, \tag{13}$$

$$\mathbf{W}_2^j = \text{span}\{\psi_{j,\mathbf{i}}^2(\mathbf{x}) = \phi_{j,i_1}(x_1)\psi_{j,i_2}(x_2)\phi_{j,i_3}(x_3), \mathbf{i} \in \mathbf{Z}^3, \mathbf{x} \in \mathbf{R}^3\}, \tag{14}$$

$$\mathbf{W}_3^j = \text{span}\{\psi_{j,\mathbf{i}}^3(\mathbf{x}) = \phi_{j,i_1}(x_1)\phi_{j,i_2}(x_2)\psi_{j,i_3}(x_3), \mathbf{i} \in \mathbf{Z}^3, \mathbf{x} \in \mathbf{R}^3\}, \tag{15}$$

$$\mathbf{W}_4^j = \text{span}\{\psi_{j,\mathbf{i}}^4(\mathbf{x}) = \phi_{j,i_1}(x_1)\psi_{j,i_2}(x_2)\psi_{j,i_3}(x_3), \mathbf{i} \in \mathbf{Z}^3, \mathbf{x} \in \mathbf{R}^3\}, \tag{16}$$

$$\mathbf{W}_5^j = \text{span}\{\psi_{j,\mathbf{i}}^5(\mathbf{x}) = \psi_{j,i_1}(x_1)\phi_{j,i_2}(x_2)\psi_{j,i_3}(x_3), \mathbf{i} \in \mathbf{Z}^3, \mathbf{x} \in \mathbf{R}^3\}, \tag{17}$$

$$\mathbf{W}_6^j = \text{span}\{\psi_{j,\mathbf{i}}^6(\mathbf{x}) = \psi_{j,i_1}(x_1)\psi_{j,i_2}(x_2)\phi_{j,i_3}(x_3), \mathbf{i} \in \mathbf{Z}^3, \mathbf{x} \in \mathbf{R}^3\}, \tag{18}$$

$$\mathbf{W}_7^j = \text{span}\{\psi_{j,\mathbf{i}}^7(\mathbf{x}) = \psi_{j,i_1}(x_1)\psi_{j,i_2}(x_2)\psi_{j,i_3}(x_3), \mathbf{i} \in \mathbf{Z}^3, \mathbf{x} \in \mathbf{R}^3\}, \tag{19}$$

and

$$\mathbf{W}^j = \oplus_{l=1}^7 \mathbf{W}_l^j.$$

We extract some results of Meyer's Multiresolution Analysis, as follows:

$$\overline{\mathbf{V}}^j \to \mathcal{L}^2(\mathbf{R}^3), \quad \text{as } j \to -\infty;$$

$\{\phi_{j,\mathbf{i}}, \mathbf{i} \in \mathbf{Z}^3\}$ constitute an orthonomal base for $\mathbf{V}^j, j \in \mathbf{Z}$; and

$$\mathbf{V}^{j-1} = \mathbf{V}^j \oplus \mathbf{W}^j. \tag{20}$$

The set of functions, $\{\psi_{j,\mathbf{i}}^l, \mathbf{i} \in \mathbf{Z}^3\}$, constitute an orthonomal base for $\mathbf{W}_l^j, j \in \mathbf{Z}, 1 \leq l \leq 7$, and $\{\psi_{j,\mathbf{i}}^l, \mathbf{i} \in \mathbf{Z}^3, 1 \leq l \leq 7, j \in \mathbf{Z},\}$ constitute an orthonomal base for $\mathcal{L}^2(\mathbf{R}^3)$; etc.

ON THE COMPRESSIBILITY OF THE MODEL

The success of data compression with wavelets suggests that only a few wavelet coefficients may contain sufficient information to reconstruct the signals (Meng and Yang, 1995). Thus, it is a good idea to calculate the few wavelet coefficients of a velocity model v instead of the fully spatially discretized model through a large, and often seriously ill-posed computation (Yang and Meng, 1995).

For example, Yang and Meng (1995) compressed a bulk modulus κ with 1024 samplings by only storing the wavelet coefficients with absolute values greater than the *cutoff*$= \varepsilon\|\kappa\|^\infty$ and determined compression ratios with respect to different cutoff levels. For example, the compression ratios 92% and 90% were obtained for cutoff rates 2.56% and 1.024% respectively. Yang and Meng (1995) concluded that in the 1D case, the "size" of an inverse problem can be greatly reduced if the inversion of parameters is carried out in wavelet represented form.

3D WAVELET REPRESENTED CB METHOD

As in Cohen and Bleistein (1979), we introduce a right-handed coordinate system $\mathbf{x}=(x_1, x_2, x_3)$, with x_3 being positive in the downward direction into the earth. The observed field is the backscattered response from acoustic point sources set off at every point $\mathbf{x}_s = (x_1^s, x_2^s, 0)$ on the surface of the earth. We assume the total field $u(\mathbf{x}; \mathbf{x}_s; \omega)$ is a solution of the Helmholtz equation,

$$\Delta u(\mathbf{x}; \mathbf{x}_s; \omega) + \frac{\omega^2}{v^2(\mathbf{x})} u(\mathbf{x}; \mathbf{x}_s; \omega) = -\delta(x_1 - x_1^s)\delta(x_2 - x_2^s)\delta(x_3). \tag{21}$$

In this equation, $v(\mathbf{x})$ is the variable reference speed we seek. For simplicity, we will omit the overhat for the Fourier transforms from this section.

As in Cohen/Bleistein (1979), we model the problem as if the medium is extended to negative infinity in x_3. Cohen and Bleistein introduced a reference velocity $c_0(\mathbf{x})$ and an arbitrary perturbation. However, here we restrict the perturbation to be a linear combination of certain dilated and translated versions of a mother wavelet, inspired by the ladder inversion method developed in Yang and Meng (1995), based on the fundamental ideas of Daubechies (1990) and others, thus we set

$$\frac{1}{v^2(\mathbf{x})} = \frac{1}{c_0^2(\mathbf{x})}\left[1 + \sum_{j,l,\mathbf{i}} d_{j,\mathbf{i}}^l \psi_{j,\mathbf{i}}^l(\mathbf{x})\right]. \tag{22}$$

We decompose the total field into an incident and scattered field

$$u(\mathbf{x};\mathbf{x}_s;\omega) = u_I(\mathbf{x};\mathbf{x}_s;\omega) + u_S(\mathbf{x};\mathbf{x}_s;\omega), \tag{23}$$

in which $u_I(\mathbf{x};\mathbf{x}_s;\omega)$ is the response to the source in the unperturbed medium,

$$\Delta u_I(\mathbf{x};\mathbf{x}_s;\omega) + \frac{\omega^2}{c_0^2(\mathbf{x})} u_I(\mathbf{x};\mathbf{x}_s;\omega) = -\delta(x_1 - x_1^s)\delta(x_2 - x_2^s)\delta(x_3) \tag{24}$$

and u_S for small perturbation must then satisfy

$$\Delta u_S(\mathbf{x};\mathbf{x}_s;\omega) + \frac{\omega^2}{c_0^2(\mathbf{x})} u_S(\mathbf{x};\mathbf{x}_s;\omega) = -\frac{\omega^2}{c_0^2(\mathbf{x})} \sum_{j,l,\mathbf{i}} d_{j,\mathbf{i}}^l \psi_{j,\mathbf{i}}^l(\mathbf{x}) u_I(\mathbf{x};\mathbf{x}_s;\omega). \tag{25}$$

Cohen and Bleistein used the Green's function to write down a representation of the backscattered field $u_S(\mathbf{x}_r;\mathbf{x}_s;\omega)$ and obtained

$$u_S(\mathbf{x}_r;\mathbf{x}_s;\omega) = \omega^2 \sum_{j,l,\mathbf{i}} d_{j,\mathbf{i}}^l \int d\mathbf{x} \frac{\psi_{j,\mathbf{i}}^l(\mathbf{x})}{c_0^2(\mathbf{x})} u_I(\mathbf{x};\mathbf{x}_s;\omega) u_I(\mathbf{x};\mathbf{x}_r;\omega). \tag{26}$$

Again, this is an equation with unknown $\mathbf{d} = \{d_{j,\mathbf{i}}^l, 1 \le l \le 7; j \in \mathbf{Z}\}$. The Green's function u_I is a solution of

$$\left[\Delta + \frac{\omega^2}{c^2(\mathbf{x})}\right] u_I(\mathbf{x},\mathbf{x}_s,\omega) = -\delta(\mathbf{x} - \mathbf{x}_s). \tag{27}$$

The WKB approximation to u_I is

$$u_I(\mathbf{x},\mathbf{x}_s,\omega) \sim A(\mathbf{x},\mathbf{x}_s)\exp[i\omega\tau(\mathbf{x},\mathbf{x}_s)], \tag{28}$$

Here, τ is a solution of the Eikonal equation

$$\nabla\tau \cdot \nabla\tau = \frac{1}{c^2(\mathbf{x})}, \tag{29}$$

and A is a solution of the transport equation,

$$2\nabla\tau \cdot \nabla A + A\nabla^2\tau = 0. \tag{30}$$

In general we can obtain the asymptotic Green's function by 3D ray theory. Then (26) becomes

$$u_S(\mathbf{x}_r;\mathbf{x}_s;\omega) = \sum_{j,l,\mathbf{i}} d^l_{j,\mathbf{i}} g^l_{j,\mathbf{i}}(\mathbf{x}_r;\mathbf{x}_s;\omega). \tag{31}$$

where

$$g^l_{j,\mathbf{i}}(\mathbf{x}_r;\mathbf{x}_s;\omega) = \omega^2 \int d\mathbf{x} \frac{\psi^l_{j,\mathbf{i}}(\mathbf{x})}{c_0^2(\mathbf{x})} A(\mathbf{x},\mathbf{x}_s)A(\mathbf{x},\mathbf{x}_r)\exp\{i\omega[\tau(\mathbf{x},\mathbf{x}_s)+\tau(\mathbf{x},\mathbf{x}_r)]\}. \tag{32}$$

One can derive a linear system of equations with constant coefficients for $d^l_{j,\mathbf{i}}$, by applying wavelet transforms with respect to $\mathbf{x}_r$ and $\mathbf{x}_s$. This yields

$$u^{l_r;l_s}_{j_r,i_1^r,i_2^r;j_s,i_1^s,i_2^s}(\omega) = \sum_{j,l,\mathbf{i}} d^l_{j,\mathbf{i}} g^{l;l_r;l_s}_{j,\mathbf{i};j_r,i_1^r,i_2^r;j_s,i_1^s,i_2^s}(\omega), \tag{33}$$

where

$$u^{l_r;l_s}_{j_r,i_1^r,i_2^r;j_s,i_1^s,i_2^s}(\omega) = \int d\mathbf{x}_r \psi^{l_r}_{j_r,i_1^r,i_2^r}(x_1^r,x_2^r) \int dx_1^s dx_2^s \psi^{l_s}_{j_s,i_1^s,i_2^s}(x_1^s,x_2^s) u_s(\mathbf{x}_r;\mathbf{x}_s;\omega), \tag{34}$$

and

$$\begin{aligned} g^{l;l_r;l_s}_{j,\mathbf{i};j_r,i_1^r,i_2^r;j_s,i_1^s,i_2^s}(\omega) &= \omega^2 \int dx_1^r dx_2^r \psi^{l_r}_{j_r,i_1^r,i_2^r}(x_1^r,x_2^r) \\ & \int dx_1^s dx_2^s \psi^{l_s}_{j_s,i_1^s,i_2^s}(x_1^s,x_2^s) g^l_{j,\mathbf{i}}(\mathbf{x}_r;\mathbf{x}_s;\omega), \end{aligned} \tag{35}$$

Equation (33) is the linear system we seek for the set of coefficients, $d^l_{j,\mathbf{i}}$. We do not know anything about the coefficient matrix for this system in general. However, below, we show that this system is readily invertible for the zero offset constant background case. Furthermore, each coefficient will be given in terms of a "localization" of the observed data in the Fourier domain. These observations support continued research on this approach to inversion.

In practice data covering all $\mathbf{x}_s$ and $\mathbf{x}_r$ are often not available. Instead, we have different gathers of data of limited extent. As we shall see, we can modify (33) for particular data acquisition geometrics.

3D CONSTANT REFERENCE WAVELET REPRESENTED CB METHOD

In this section, we specialize to the case in which c_0 is a constant, and consider the zero-offset case. We set $\boldsymbol{\xi} = \mathbf{x}_s = \mathbf{x}_r$. In this case, $u_I(\mathbf{x};\boldsymbol{\xi};\omega)$ can be expressed explicitly, since it is proportional to the spherical Hankel function of the first kind. Then equation (26) becomes

$$u_S(\boldsymbol{\xi};\boldsymbol{\xi};\omega) = \frac{\omega^2}{(4\pi c_0)^2}\sum_{j,l,\mathbf{i}} d^l_{j,\mathbf{i}} \int d\mathbf{x}\, \psi^l_{j,\mathbf{i}}(\mathbf{x}) \frac{\exp[2i\omega|\mathbf{x}-\boldsymbol{\xi}|/c_0]}{|\mathbf{x}-\boldsymbol{\xi}|^2}. \tag{36}$$

To solve equation (36) for $\mathbf{d}$ by Fourier methods, following Cohen and Bleistein (1979), we introduce the function,

$$\Theta(\boldsymbol{\xi};\omega) = -i\frac{\partial}{\partial\omega}\left[\frac{u_S(\boldsymbol{\xi};\boldsymbol{\xi};\omega)}{\omega^2}\right]. \tag{37}$$

Then we have

$$\Theta(\boldsymbol{\xi};\omega) = \frac{1}{2\pi c_0^3}\sum_{j,l,\mathbf{i}} d^l_{j,\mathbf{i}} \int d\mathbf{x}\, \psi^l_{j,\mathbf{i}}(\mathbf{x}) \frac{\exp[2i\omega|\mathbf{x}-\boldsymbol{\xi}|/c_0]}{4\pi|\mathbf{x}-\boldsymbol{\xi}|}. \tag{38}$$

This is the analog of equation (26), although we have already modified that equation by differentiating with respect to ω. In this case, the simple convolution form of the result makes Fourier transformation in $\boldsymbol{\xi}$—the specialization of $\mathbf{x}_s$ and $\mathbf{x}_r$ to zero offset–more attractive. Accordingly, we introduce the transverse Fourier transform in the variables ξ_1 and ξ_2:

$$\Theta(k_1,k_2;\omega) = \int d\xi_1 d\xi_2 \Theta(\boldsymbol{\xi};\omega)\exp[-2i\{k_1\xi_1 + k_2\xi_2\}]. \tag{39}$$

The factor of two in the phase here is a convenience, since without it a factor of two woumd have to be introduced into the familiar dispersion relation (41), below.

We now apply this Fourier transform to equation (38) to obtain an integral equation for $\sum_{j,l,\mathbf{i}} d^l_{j,\mathbf{i}}\psi^l_{i,\mathbf{i}}(\mathbf{k})$. The result of transforming this equation is

$$-8\pi i c_0^3 k_3 \Theta(k_1,k_2;\omega) = \sum_{j,l,\mathbf{i}} d^l_{j,\mathbf{i}} \int_{-\infty}^{\infty} dx_3 \psi^l_{j,\mathbf{i}}(k_1,k_2,x_3)\exp(2ik_3|x_3|). \tag{40}$$

In this equation

$$k_3 = \begin{cases} (\mathrm{sgn}\ \omega)\sqrt{\omega^2/c_0^2 - k_1^2 - k_2^2}, & \omega^2/c_0^2 \geq k_1^2 + k_2^2, \\ i\sqrt{k_1^2 + k_2^2 - \omega^2/c_0^2}, & \omega^2/c_0^2 < k_1^2 + k_2^2. \end{cases} \tag{41}$$

The function $\sum_{j,l,\mathbf{i}} d^l_{j,\mathbf{i}}\psi^l_{j,\mathbf{i}}(\mathbf{x})$ and also $\sum_{j,l,\mathbf{i}} d^l_{j,\mathbf{i}}\psi^l_{j,\mathbf{i}}(k_1, k_2, x_3)$ are assumed non-zero only for x_3 positive. Therefore, there is no need for the absolute value sign in equation (40) and the right side of this equation becomes the 3D spatial Fourier transform of $\sum_{j,l,\mathbf{i}} d^l_{j,\mathbf{i}}\psi^l_{j,\mathbf{i}}(\mathbf{x})$; or, Then

$$\sum_{j,l,\mathbf{i}} d^l_{j,\mathbf{i}}\psi^l_{j,\mathbf{i}}(\mathbf{k}) = -8\pi i c_0^3 k_3 \Theta(k_1, k_2; \omega), \quad \mathbf{k} = (k_1, k_2, k_3), \tag{42}$$

with k_3 defined by the dispersion relation, which is the upper choice in equation (41). The function $\Theta(k_1, k_2; \omega)$ is defined in terms of the observed surface data through equation (39). By the orthonormality of the wavelets we obtain

$$d^l_{j,\mathbf{i}} = -8\pi i c_0^3 \int d\mathbf{k} k_3 \Theta(k_1, k_2; \omega)\psi^l_{j,\mathbf{i}}(\mathbf{k}), \quad \text{for all } l, j \text{ and } \mathbf{i}. \tag{43}$$

Thus, we have not *avoided* the use of wavelet transform by introducing the Fourier transform in this case; we have merely postponed its application to the Fourier domain. Furthermore, because this is the zero offset case, we apply the wavelet transform only in the common source/receiver variables, not in two sets of variables as in the general discussion, above. This result provides an indication of the likelihood of inversion of equation (33); for the zero-offset case, the Fourier transform has effected the inversion of this system of equations.

One striking feature of the standard Cohen/Bleistein method (Bleistein el al, 1985) is the technique of obtaining the reflection coefficient $r(\mathbf{x})$ by simply multiplying by $i\omega/2c_0$ in (43), yielding

$$r^l_{j,\mathbf{i}} = 4\pi c_0^2 \int d\mathbf{k} k_3 \omega \Theta(k_1, k_2; \omega)\psi^l_{j,\mathbf{i}}(\mathbf{k}), \tag{44}$$

or equivalently,

$$r^l_{j,\mathbf{i}} = \frac{4c_0^2}{\pi^2} \int dk_1 dk_2 dk_3\, k_3 \omega \Theta(k_1, k_2; \omega)\psi^l_{j,\mathbf{i}}(\mathbf{k}) \exp[2i\{k_1 x_1 + k_2 x_2 - k_3 x_3\}], \tag{45}$$

where

$$r(\mathbf{x}) = \sum_{j,l,\mathbf{i}} r^l_{j,\mathbf{i}}\psi^l_{j,\mathbf{i}}(\mathbf{x}), \tag{46}$$

is the reflection coefficient, these results are obtained by considering only the leading-order asymptotic approximation and interpreting the output in terms of the reflectivity function of the subsurface. An immediate simplification can be achieved by extracting only the leading term of Θ as defined by equation (37), we have

$$\Theta(k_1, k_2; \omega) = -\frac{i}{\omega^2}\frac{\partial u_S(k_1, k_2; \omega)}{\partial \omega}. \tag{47}$$

Thus, we can rewrite (45) as

$$\begin{aligned} r^l_{j,\mathbf{i}} &= \frac{4c_0^2}{\pi^2} \int dk_1 dk_2 dk_3 d\xi_1 d\xi_2 dt \frac{k_3}{\omega} t u_S(\boldsymbol{\xi}; \boldsymbol{\xi}; t)\psi^l_{j,\mathbf{i}}(\mathbf{k}) \\ &\quad \exp[\{2i[k_1(x_1 - \xi_1) + k_2(x_2 - \xi_2) - k_3 x_3] + i\omega t\}], \end{aligned} \tag{48}$$

or

$$r^l_{j,\mathbf{i}} = \frac{4}{\pi^2}\int dk_1 dk_2 d\omega d\xi_1 d\xi_2 dt\, t u_S(\boldsymbol{\xi};\boldsymbol{\xi};t)\psi^l_{j,\mathbf{i}}(\mathbf{k}) \\ \exp[\{2i[k_1(x_1-\xi_1)+k_2(x_2-\xi_2)-k_3x_3]+i\omega t\}], \tag{49}$$

where we integrate over the bandwidth of the data and over those values of k_1 and k_2 for which $k_3 = k_3(k_1, k_2; \omega)$ as defined by (41) is real.

Formula (43) or (48) has a clearer physical meaning than the standard Cohen/Bleistein formula. The correction of the velocity confined to a distinct spatial-aperture window is obtained from field data truncated to the corresponding wave number window. In other words, if we want to get the information about the earth structure in a certain spatial-aperture window, we only need the data in the corresponding wave number-aperture window. An important conclusion is that the inversion of wavelet coefficients $d^l_{j,\mathbf{i}}$ can be carried out independently. That is, the equivalent *system matrix* of the first order perturbation system is essentially diagonal for a constant reference model, and, should therefore be "almost" diagonal for a low wave number reference model.

2.5D WAVELET REPRESENTED CB METHOD

In 2.5D, equation (31) is replaced by

$$u_S(\mathbf{x}_r;\mathbf{x}_s;\omega) = \sum_{j,l,\mathbf{i}} d^l_{j,\mathbf{i}} g^l_{j,i_1,i_3}(\mathbf{x}_r;\mathbf{x}_s;\omega), \tag{50}$$

where

$$g^l_{j,i_1,i_3}(\mathbf{x}_r;\mathbf{x}_s;\omega) = \omega^2\int dx_1 x_3 \frac{\psi^l_{j,i_1,i_3}(x_1,x_3)}{c_0^2(x_1,x_3)} \\ \int dx_2 A(\mathbf{x},\mathbf{x}_s)A(\mathbf{x},\mathbf{x}_r)\exp\{i\omega[\tau(\mathbf{x},\mathbf{x}_s)+\tau(\mathbf{x},\mathbf{x}_r)]\}. \tag{51}$$

Here τ is the solution of the 2D eikonal equation, and A is a solution of the in-plane 3D (2.5D) transport equation, see Bleistein et al. (1987).

To solve the equation for d^l_{j,i_1,i_3}, we apply the wavelet transforms with respect to $\mathbf{x}_r$ and $\mathbf{x}_s$. For a common source gather $\mathbf{x}_r = (x_r, 0, 0)$ and $\mathbf{x}_s = \mathbf{0}$, we solve for $\mathbf{d}$ by solving

$$u^{l_r}_{j_r,i_r}(\omega) = \sum_{j,l,i_1,i_3} d^l_{j,i_1,i_3} g^{l;l_r}_{j,i_1,i_3;j_r,i_r}(\omega), \tag{52}$$

where

$$u^{l_r}_{j_r,i_r}(\omega) = \int dx_r \psi^{l_r}_{j_r,i_r}(x_r) u_s(\mathbf{x}_r;\mathbf{0};\omega), \tag{53}$$

and

$$g^{l;l_r}_{j,i_1,i_3;j_r,i_r}(\omega) = \omega^2\int dx_r \psi^{l_r}_{j_r,i_r}(x_r) g^l_{j,i_1,i_3}(\mathbf{x}_r;\mathbf{0};\omega). \tag{54}$$

For a common receiver gather, $\mathbf{x}_r = \mathbf{0}$ and $\mathbf{x}_s = (x_s, 0, 0)$:

$$u^{l_s}_{j_s,i_s}(\omega) = \sum_{j,l,i_1,i_3} d^l_{j,i_1,i_3} g^{l;l_s}_{j,i_1,i_3;j_s,i_s}(\omega), \tag{55}$$

where

$$u^{l_s}_{j_s,i_s}(\omega) = \int dx_s \psi^{l_s}_{j_s,i_s}(x_s) u_s(\mathbf{0}; \mathbf{x}_s; \omega), \tag{56}$$

and

$$g^{l;l_s}_{j,i_1,i_3;j_s,i_s}(\omega) = \omega^2 \int dx_s \psi^{l_s}_{j_s,i_s}(x_s) g^l_{j,i_1,i_3}(\mathbf{0}; \mathbf{x}_s; \omega). \tag{57}$$

For a common midpoint gather, $\mathbf{x}_r = -\mathbf{x}_s = (x_r, 0, 0)$:

$$u^{l_r}_{j_r,i_r}(\omega) = \sum_{j,l,i_1,i_3} d^l_{j,i_1,i_3} g^{l;l_r}_{j,i_1,i_3;j_r,i_r}(\omega), \tag{58}$$

where

$$u^{l_r}_{j_r,i_r}(\omega) = \int d\mathbf{x}_r \psi^{l_r}_{j_r,i_r}(x_r) u_s(\mathbf{x}_r; -\mathbf{x}_r; \omega), \tag{59}$$

and

$$g^{l;l_r}_{j,i_1,i_3;j_r,i_r}(\omega) = \omega^2 \int dx_r \psi^{l_r}_{j_r,i_r}(x_r) g^l_{j,i_1,i_3}(\mathbf{x}_r; -\mathbf{x}_r; \omega). \tag{60}$$

For the zero offset case, $\mathbf{x}_r = \mathbf{x}_s = (x_r, 0, 0)$:

$$u^{l_r}_{j_r,i_r}(\omega) = \sum_{j,l,i_1,i_3} d^l_{j,i_1,i_3} g^{l;l_r}_{j,i_1,i_3;j_r,i_r}(\omega), \tag{61}$$

where

$$u^{l_r}_{j_r,i_r}(\omega) = \int dx_r \psi^{l_r}_{j_r,i_r}(x_r) u_s(\mathbf{x}_r; \mathbf{x}_r; \omega), \tag{62}$$

and

$$g^{l;l_r}_{j,i_1,i_3;j_r,i_r}(\omega) = \omega^2 \int dx_r \psi^{l_r}_{j_r,i_r}(x_r) g^l_{j,i_1,i_3}(\mathbf{x}_r; \mathbf{x}_r; \omega). \tag{63}$$

Equation (52) [or (55),(58),(61)] is a linear system for the wavelet coefficients. Solving the linear system (52) is equivalent to calculating the fundamental determinant $h(\mathbf{x}, \boldsymbol{\xi})$; thus solving this system is the crucial step in the practical implementation.

Notice that, since we assume the velocity model is an image, due to the high compressibility of the wavelet coefficients, only a small number of unknown wavelet coefficients are sufficient, to effect the recovery of the velocity model. Thus, we expect that the system can be reduced in size substantially. For example, in Yang and Meng (1995), by solving a linear system similar to (61) for 1/20 of the unknowns, the 1D density and bulk modulus profiles were reasonably reconstructed. If we assume the model is a multidimensional image, then the ratio of the number of unknowns that need to be calculated to the number of total unknowns should be relatively small.

One important step for us to "compress" the model to be determined successfully is to know where both in space and aperture we need to know about the model. This is not a problem in applications.

2.5D CONSTANT REFERENCE WAVELET REPRESENTED CB METHOD

Recall that 2.5D inversion is obtained from 3D inversion under the assumption that there is no out-of-plane variation in the background velocity or in the velocity perturbation. Thus, only the operator depends on the out-of-plane variable, not the data. Thus, the inversion formulas for the 2.5D case are readily obtained from the results for the 3D zero offset case, as

$$\begin{aligned} r^l_{j,i_1,i_2} &= \frac{4c_0^2}{\pi}\int dk_1 dk_3 d\xi_1 dt \frac{k_3}{\omega} t u_S(\xi_1;\xi_1;t)\psi^l_{j,i_1,i_2}(k_1,k_3) \\ &\quad \cdot \exp[\{2i[k_1(x_1-\xi_1)-k_3x_3]+i\omega t\}], \end{aligned} \tag{64}$$

or

$$\begin{aligned} r^l_{j,i_1,i_2} &= \frac{4}{\pi}\int dk_1 d\omega d\xi_1 dt\, t u_S(\xi_1;\xi_1;t)\psi^l_{j,i_1,i_2}(k_1,k_3) \\ &\quad \cdot \exp[\{2i[k_1(x_1-\xi_1)-k_3x_3]+i\omega t\}], \end{aligned} \tag{65}$$

where the dispersion relation

$$k_3 = \operatorname{sgn}\omega\left(\sqrt{\frac{\omega^2}{c_0^2}-k_1^2}\right), \quad \frac{\omega^2}{c_0^2} \geq k_1^2. \tag{66}$$

The range of ω and k_1 for which k_3 is imaginary is not used.

STRATIFIED REFERENCE WAVELET REPRESENTED CB METHOD

In the stratified reference case, we have

$$u_S(\mathbf{x}_r;\omega) = \sum_{j,l,\mathbf{i}} d^l_{j,\mathbf{i}} g^l_{j,\mathbf{i}}(\mathbf{x}_r,\omega), \tag{67}$$

where

$$g^l_{j,\mathbf{i}}(\mathbf{x}_r;\omega) = \frac{\omega^2}{16\pi^2}\int d\mathbf{x}\frac{\psi^l_{j,\mathbf{i}}(\mathbf{x})\exp[2i\omega\tau(K,x_3)]}{c^2(x_3)k_3(K,0)k_3(K,x_3)E(K,x_3)H(K,x_3)}, \tag{68}$$

here the quantities k_3, E, H and the traveltime τ are defined in Cohen and Hagin (1985). Again, we solve for $\mathbf{d}$ from the equations,

$$u^{l_r}_{j_r,i^r_1,i^r_2}(\omega) = \sum_{j,l,\mathbf{i}} d^l_{j,\mathbf{i}} g^{l;l_r}_{j,\mathbf{i};j_r,i^r_1,i^r_2}(\omega), \tag{69}$$

where

$$u^{l_r}_{j_r,i^r_1,i^r_2}(\omega) = \int dx^r_1 dx^r_2 \psi^{l_r}_{j,i^r_1,i^r_2}(x^r_1,x^r_2)u_s(\mathbf{x}_r,\omega), \tag{70}$$

and

$$g^{l;l_r}_{j,\mathbf{i};j_r,i^r_1,i^r_2}(\omega) = \omega^2\int dx^r_1 dx^r_2 \psi^{l_r}_{j_r,i^r_1,i^r_2}(x^r_1,x^r_2) g^l_{j,\mathbf{i}}(\mathbf{x}_r,\omega). \tag{71}$$

CONCLUSIONS

We have derived a system of equations for coefficients in a wavelet expansion of the perturbation in propagation speed. Our method is based on the Bleistein/Cohen inversion formalism. The system matrix is a multi-scale, multi-variable decomposition of the inversion operator operating on the observed data. We have no *a priori* estimates of the condition number of this matrix. However, when we specialize to the zero-offset, constant background case, we find that Fourier transform leads to a well-conditioned solution for the wavelet coefficients we seek, leading us to believe that something similar is likely in the case of a slowly varying background velocity and small offset, at the very least. The Fourier solution also indicates that localization of the solution in space is directly tied to localization of the observed data in the wave number domain.

When the perturbation consists primarily of reflectors (functions with nearly two dimensional support in a three dimensional domain, for example), we expect significant compression in the wavelet domain compared to a pointwise description of the velocity function. Thus, we expect that the number of wavelet decomposion coefficients that we need to determine will be relatively small compared to the total number of data points in a Cartesian description of the velocity variation.

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