CWP-044

MATHEMATICS OF SEISMOLOGY

by

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Lecture notes from a course taught
at the Colorado School of Mines

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1. GREEN'S FUNCTIONS

In this chapter we discuss those properties of scalar Green's functions which make them useful as sources by themselves and as adjunct elemental sources in finding field distributions for more complicated problems involving surfaces and volumes. We begin with the Green's functions for the wave equation in n=1,2,3 spatial dimensions and 1 temporal dimension. Using different boundary conditions we show that in (n,1) dimensions there are five possible Green's functions and, using their interrelationships, only three independent ones. Each one has various uses depending on the problem at hand. Integration over time yields the corresponding Green's functions for the Helmholtz equation again in n=1,2,3 spatial dimensions. We also treat separately, and briefly, the causal Green's function for the parabolic wave equation.

Fourier transform representations of these Helmholtz Green's functions are often useful, as are additional integral representations formed by integration over one or two of the Fourier transform variables. This leads to integral representations associated with the names Weyl, Sommerfeld, and Weyrich. These are all spectral representations of some kind, the Weyl representation being a two-dimensional integral over the transverse wavenumber components, the Sommerfeld representation a one-dimensional representation over the (radial) transverse wavenumber, and the Weyrich representation a one-dimensional representation over the vertical wavenumber component. Both the latter use cylindrical symmetry properties. Plane wave spectral decompositions are also treated as are their interrelations with the above representations. An example is discussed of the use of the representations in the half-plane.

For complicated geometries the most straightforward approach to solving boundary value problems is to use integral equations. To develop surface integral equations using Green's functions as elemental sources, or their derivatives as dipole sources, it is necessary to know their analytic properties. In particular spatial singularities must be treated in such a way that the resulting integral equations may be solved using classical techniques. This is called regularization, and we demonstrate the regularization of the first and second vector derivatives of the Helmholtz Green's function.

Finally we discuss the Green's function in one-dimension using conventional methods here generalized to inhomogeneous media. We treat a general method of finding profiles for which the one-dimensional Helmholtz equation is solvable in terms of known classical functions.

1.1 GREEN'S FUNCTION FOR THE WAVE EQUATION

1.1.1 (3,1)-DIMENSIONS

The Green's function is defined as

$$G^{(3,1)}(\underset{\sim}{x},\underset{\sim}{x}';t,t') \tag{1.1}$$

in three spatial and one temporal dimensions. It satisfies the wave equation given by (c is the wave speed)

$$\left[\nabla_x^2 - c^{-2}\partial_t^2\right]G^{(3,1)}(\underset{\sim}{x},\underset{\sim}{x}';t,t') = -\delta(\underset{\sim}{x}-\underset{\sim}{x}')\delta(t-t') \quad , \tag{1.2}$$

where the Laplacian is defined by

$$\nabla_x^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 \quad . \tag{1.3}$$

It is convenient to do many of the manipulations in four-vector notation

$$x = (\underset{\sim}{x}, x_0) \qquad x_0 = ct \quad , \tag{1.4}$$

with the scalar product defined by

$$x \cdot x = \underset{\sim}{x} \cdot \underset{\sim}{x} - x_0^2 \begin{cases} > 0 & \text{space-like} \\ = 0 & \text{light-like} \\ < 0 & \text{time-like} \quad . \end{cases} \tag{1.5}$$

Using this notation the Green's function satisfies

$$\Box \, G^{(3,1)}(x,x') = -\delta(x-x') = -\delta(\underset{\sim}{x}-\underset{\sim}{x}')\delta(x_0-x_0') \quad , \tag{1.6}$$

where we have defined the d'Alembertian operator

$$\Box = \nabla_x^2 - c^{-2}\partial_t^2 = \nabla_x^2 - \partial_0^2 \quad . \tag{1.7}$$

Since we have $\delta(x_0-x_0') = c^{-1}\delta(t-t')$ we see from (1.2) and (1.6) that our Green's functions are related by

$$G^{(3,1)}(\underset{\sim}{x},\underset{\sim}{x}',t,t') = c\; G^{(3,1)}(x,x') \quad . \tag{1.8}$$

Since the delta function source term is a function of the difference between the space-time source point (x') and receiver point (x), and the coefficients of the differential operators in the d'Alembertian are constants (homogeneous medium), the Green's function only depends functionally on the difference x-x'. For convenience we write it as a function of this difference, and introduce the Fourier transform in the difference argument

$$G^{(3,1)}(x) = (2\pi)^{-4}\iiiint \exp(ik\cdot x)\ \tilde{G}^{(3,1)}(k)d^4k \quad , \tag{1.9}$$

with notation (ω is circular frequency)

$$k\cdot x = \underset{\sim}{k}\cdot\underset{\sim}{x} - k_0 x_0 \quad ; \quad k_0 = \omega/c$$

$$d^4k = d\underset{\sim}{k}\ dk_0 \quad . \tag{1.10}$$

Applying the d'Alembertian operator to (1.9) we see that (1.6) is satisfied provided

$$\tilde{G}^{(3,1)}(k) = (\underset{\sim}{k}^2 - k_0^2)^{-1} = k^{-2} \quad . \tag{1.11}$$

We note that the four-dimensional delta function is written as

$$\delta(x) = (2\pi)^{-4}\iiiint \exp(ik\cdot x)d^4k \quad . \tag{1.12}$$

Using (1.11) and defining $\omega_k = |\underset{\sim}{k}|$ we can write (1.9) as

$$G^{(3,1)}(x) = -\frac{1}{(2\pi)^4}\iiiint \frac{\exp(ik\cdot x)}{(k_0 - \omega_k)(k_0 + \omega_k)}\, d^4k \quad . \tag{1.13}$$

To evaluate (1.13) we must first evaluate the k_0 integral

$$I = \int_{-\infty}^{\infty} \frac{\exp(-ik_0 x_0)}{(k_0 - \omega_k)(k_0 + \omega_k)}\, dk_0 \quad . \tag{1.14}$$

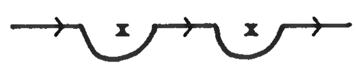
To do this we must define how to treat the pole terms at $k_0 = \pm\omega_k$ in the integrand. There are two equivalent ways to do this. The first is

(a) Fix the poles - offset the contour

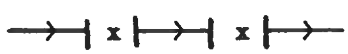
There are five ways to do this illustrated below:

1.  R: retarded

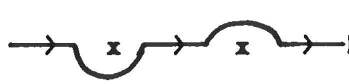
Here the integration contour is written as semicircles above the two poles at $\pm\omega_k$. It is called the retarded contour for reasons which will be clear later.

2.  A: advanced

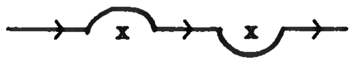
Here the integration contour is written as semicirlces below the two poles at $\pm\omega_k$.

3.  P: principal value

Here the poles are evaluated using the Cauchy principal value definition of the integral.

4.  D: Dyson

Here the integration contour is written using semicircles, one below the pole at $-\omega_k$ and one above the pole at $+\omega_k$. The name arises from F. Dyson in his work on quantum field theory.

5.  C: causal

Here the semicircles are reversed from the Dyson contour.

The second method is to:

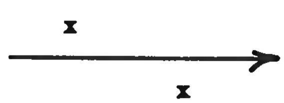
(b) <u>Fix the contour - offset the poles</u>

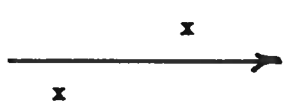
For this method we fix the contour along the real k_0 axis from $-\infty$ to ∞, and shift the poles. The above five become

1.  R: retarded

2.  A: advanced

3.  P: principal value

4.  D: Dyson

5.  C: causal

Each method makes clear that we will treat (1.14) as an integral in the complex plane. We use method (b), i.e. we shift the poles by an amount $i\varepsilon$ and consider the results in the limit as $\varepsilon \to 0$. That is we shift them into the imaginary part of the complex k_0 plane. We thus have

$$(k_o - \omega_k)(k_o + \omega_k) \rightarrow \left[k_o - (\omega_k + i\alpha\varepsilon)\right]\left[k_o + \omega_k - i\beta\varepsilon\right] \quad , \tag{1.15}$$

where $\alpha,\beta=\pm 1$, 0 depending on the shift. For the five cases we have that

$$\begin{aligned} &R: \quad \alpha = \beta = -1 \\ &A: \quad \alpha = \beta = +1 \\ &P: \quad \alpha = \beta = 0 \\ &D: \quad \alpha = -1,\ \beta = 1 \\ &C: \quad \alpha = 1,\ \beta = -1 \quad . \end{aligned} \tag{1.16}$$

To evaluate the poles, use the Dirac-Plemelj relations for distributions (we assume the limit $\varepsilon \rightarrow 0$)

$$\frac{1}{y \pm i\varepsilon} = P\frac{1}{y} \mp \pi i \delta(y) \quad . \tag{1.17}$$

That is, we express the poles at $y \pm i\varepsilon$ in terms of principal value (P) distributions and half-residue terms from the semicircles. (Ref. 1.4, p. 476.) We thus have for one pole

$$\frac{1}{k_o - \left[\omega_k + i\alpha\varepsilon\right]} = P\frac{1}{k_o - \omega_k} + \alpha\pi i\ \delta(k_o - \omega_k) \quad , \tag{1.18}$$

and for the product of two poles

$$\frac{1}{(k_o - \omega_k - i\alpha\varepsilon)(k_o + \omega_k - i\beta_\varepsilon)}$$

$$= P\frac{1}{k_o^2 - \omega_k^2} + \frac{\pi i}{2\omega_k}\left[\alpha\ \delta(k_o - \omega_k) - \beta\ \delta(k_o + \omega_k)\right] \quad . \tag{1.19}$$

Substituting (1.19) into (1.13) we can thus write all of our examples in terms of two integrals as

$$G^{(3,1)}(x) = -(2\pi)^{-4}\Big[I_1(x) + (\pi i/2)\big[\alpha\, I_2(\underset{\sim}{x},-x_0) - \beta\, I_2(\underset{\sim}{x},x_0)\big]\Big] \quad , \tag{1.20}$$

where the integrals are defined by

$$I_1(x) = P\iiiint \frac{\exp(ik\cdot x)}{k_0^2 - \omega_k^2}\, d^4k \tag{1.21}$$

and

$$I_2(\underset{\sim}{x},x_0) = \iiint \frac{\exp\left[i(\underset{\sim}{k}\cdot\underset{\sim}{x} + \omega_k x_0)\right]}{\omega_k}\, d\underset{\sim}{k} \quad . \tag{1.22}$$

To evaluate I_1 use the following distributional relation

$$P\,\frac{1}{\tau} = -\,\frac{i}{2}\int_{-\infty}^{\infty} \exp(ia\tau)\,\frac{a}{|a|}\,da \quad , \tag{1.23}$$

which can be easily proved as follows

$$\int_{-\infty}^{\infty} \exp(ia\tau)\,\frac{a}{|a|}\,da = \int_0^{\infty} \exp(ia\tau)da - \int_{-\infty}^{0} \exp(ia\tau)da$$

$$= \lim_{\varepsilon\to 0}\left[\int_0^{\infty} \exp\left[ia[\tau + i\varepsilon]\right]da - \int_{-\infty}^{0} \exp\left[ia[\tau - i\varepsilon]\right]da\right]$$

$$= \lim_{\varepsilon\to 0}\left[\frac{i}{\tau + i\varepsilon} + \frac{i}{\tau - i\varepsilon}\right]$$

$$= 2i\,P\,\frac{1}{\tau} \quad ,$$

where the latter step follows from the relations (1.17). I_1 can thus be written as

$$I_1(x) = -\frac{i}{2}\int da\,\frac{a}{|a|}\iiiint \exp(-iak\cdot k + ik\cdot x)d^4k \quad . \tag{1.24}$$

Completing the square in the four-dimensional integral and introducing the change of variables

$$k' = k - (2a)^{-1}x \tag{1.25}$$

and the integrals

$$\int_{-\infty}^{\infty} dp\, \exp(\mp iap^2) = \left[\pi/(\pm ia)\right]^{1/2} \quad , \tag{1.26}$$

we can evaluate the four-dimensional integral to get

$$I_1(x) = (\pi^2/2)\int da\, a^{-2}\, \exp\left[ix\cdot x(4a)^{-1}\right] \quad . \tag{1.27}$$

The further change of variables $\alpha=(4a)^{-1}$ then yields

$$I_1(x) = -4\pi^3\, \delta(x^2) \quad , \tag{1.28}$$

which can also be written as

$$I_1(x) = -4\pi^2(2r)^{-1}\left[\delta(r + x_0) + \delta(r - x_0)\right] \quad . \tag{1.29}$$

The latter follows from the general distributional result

$$\delta(f(x)) = \sum_i \delta(x - x_i)/|f'(x_i)| \tag{1.30}$$

where $f(x_i)=0$.

We evaluate I_2 using spherical coordinates with

$$d\underset{\sim}{k} = \omega_k^2 \, d\omega_k \, d(\cos\theta) d\phi$$

$$\underset{\sim}{k} \cdot \underset{\sim}{x} = \omega_k \, r \cos\theta \quad ; \quad r = |\underset{\sim}{x}| \quad . \tag{1.31}$$

The angular integrals are straightforward, and we get

$$I_2(\underset{\sim}{x}, x_0) = (2\pi/ir)\int_0^\infty \Big[\exp\big[i\omega_k(x_0 + r)\big] - \exp\big[i\omega_k(x_0 - r)\big]\Big] d\omega_k \quad . \tag{1.32}$$

Introduce a small convergence factor

$$I_2(\underset{\sim}{x}, x_0) = (2\pi/ir) \lim_{\varepsilon\to 0} \int_0^\infty \Big[\exp\big[i\omega_k(x_0 + r + i\varepsilon)\big]$$

$$- \exp\big[i\omega_k(x_0 - r + i\varepsilon)\big]\Big] d\omega_k$$

$$= (2\pi/r) \lim_{\varepsilon\to 0} \left[\frac{1}{x_0 + r + i\varepsilon} - \frac{1}{x_0 - r + i\varepsilon} \right] \quad . \tag{1.33}$$

If we use (1.17) and combine the terms we get

$$I_2(\underset{\sim}{x}, x_0) = 4\pi \, P \frac{1}{x^2} + \frac{2\pi^2 i}{r} \Big[\delta(x_0 - r) - \delta(x_0 + r) \Big] \quad . \tag{1.34}$$

By substituting $-x_0$ for x_0 and rewriting we get that

$$I_2(\underset{\sim}{x}, -x_0) = \big[I_2(\underset{\sim}{x}, x_0)\big]^*$$

$$= 4\pi \, P \frac{1}{x^2} - \frac{2\pi^2 i}{r} \big[\delta(x_0 - r) - \delta(x_0 + r) \big] \quad . \tag{1.35}$$

We thus have all the necessary integrals from (1.29), (1.34) and (1.35) and can evaluate our Green's functions from (1.20). The results are:

Eg. 1. **RETARDED GREEN'S FUNCTION** ($\alpha=\beta=-1$)

$$G_R^{(3,1)}(x) = (4\pi r)^{-1}\ \delta(x_0 - r) \quad . \tag{1.36}$$

This result means that after a time t, a pulse in three-dimensions is concentrated on the surface of a sphere of radius $r=x_0=ct$ (i.e. it is an outgoing spherical wave). Also we note that since

$$\delta(x_0 - r) = \delta(ct - r) = c^{-1}\ \delta(t - r/c) \quad ,$$

we have that following (1.8)

$$G_R^{(3,1)}(x) = c^{-1}\ G_R^{(3,1)}(\underset{\sim}{x},t) \ .$$

In addition this is called a retarded Green's function since any field u can be expressed as an integral over a source function f as

$$u(x) = u(\underset{\sim}{x},x_0) = \iiint d\underset{\sim}{x}' \int dx_0'\ G_R^{(3,1)}(x-x')f(\underset{\sim}{x}',x_0) \quad .$$

Substituting the Green's function from (1.36) and evaluating the x_0' integration yields

$$u(x) = \iiint d\underset{\sim}{x}'(4\pi r)^{-1}\ f(\underset{\sim}{x}',x_0 - r) \quad ,$$

where $r=|\underset{\sim}{x}-\underset{\sim}{x}'|$. The latter is an integral over a source function f evaluated at a retarded time.

Eg. 2. ADVANCED GREEN'S FUNCTION (α=β=1)

We have that

$$G_A^{(3,1)}(x) = (4\pi r)^{-1}\ \delta(x_0 + r) \tag{1.37}$$

This is an incoming spherical pulse concentrated on a sphere of radius r=-ct. Thus for a real pulse it must exist for negative times. It is called an advanced Green's function since any field can be written as the spatial integral over a source function f evaluated at an advanced time x_0+r in analogy to the previous discussion.

Eg. 3. PRINCIPAL VALUE GREEN'S FUNCTION (α=β=0)

From (1.20) this is directly related to I_1 so that

$$G_P^{(3,1)}(x) = (4\pi)^{-1}\ \delta(x^2) \quad . \tag{1.38}$$

We also note that it can be written as a linear combination of (1.36) and (1.37)

$$G_P^{(3,1)}(x) = \frac{1}{2}\left[G_R^{(3,1)}(x) + G_A^{(3,1)}(x)\right] \quad , \tag{1.39}$$

so that another representation is

$$G_P^{(3,1)}(x) = (8\pi r)^{-1}\left[\delta(x_0 - r) + \delta(x_0 + r)\right] \quad , \tag{1.40}$$

which also follows from (1.29) and (1.30). In a sense it is a standing wave Green's function since it balances both incoming and outgoing wave Green's functions.

Eg. 4. DYSON GREEN'S FUNCTION (α=-1, β=1)

By relating the I_1 integral to the principal value term (1.38) we get that

$$G_D^{(3,1)}(x) = G_P^{(3,1)}(x) + \frac{i}{4\pi^2} P \frac{1}{x^2} \quad . \tag{1.41}$$

Eg. 5. CAUSAL GREEN'S FUNCTION (α=1, β=-1)

This is just the complex conjugate of the Dyson function

$$G_C^{(3,1)}(x) = G_P^{(3,1)}(x) - \frac{i}{4\pi^2} P \frac{1}{x^2} \quad . \tag{1.42}$$

Finally we can easily conclude either from the explicit forms of the five functions (1.36), (1.37), (1.38), (1.41) and (1.42) or from the definitions (1.16) and (1.20) that we have the relations

$$G_P^{(3,1)}(x) = \frac{1}{2}\left[G_R^{(3,1)}(x) + G_A^{(3,1)}(x)\right] \quad ,$$

and

$$G_P^{(3,1)}(x) = \frac{1}{2}\left[G_R^{(3,1)}(x) + G_C^{(3,1)}(x)\right] \quad , \tag{1.43}$$

so that only three of the five functions are linearly independent. In addition we also note that the difference of any two of these Green's functions is a solution of the homogeneous equation, i.e. for example

$$g = G_R^{(3,1)} - G_A^{(3,1)} \tag{1.44}$$

satisfies

$$\Box\, g = 0 \quad . \tag{1.45}$$

1.1.2 (2,1)-DIMENSIONS

Here we compute the Green's functions in two spatial and one temporal dimension. We do this by identifying one spatial coordinate and integrating the (3,1) Green's functions over this coordinate. We choose the z-direction as our direction of integration and write the radius r as

$$r = \left[P^2 + (z - z')^2\right]^{1/2} \quad , \tag{1.46}$$

where

$$\underset{\sim}{P} = \underset{\sim}{\rho} - \underset{\sim}{\rho}' \quad , \quad \underset{\sim}{\rho} = (x, y) \quad , \tag{1.47}$$

will be the remaining two-dimensional vector. The Green's functions we define all satisfy the equation

$$(\partial_x^2 + \partial_y^2 - c^{-2}\partial_t^2)\, G^{(2,1)}(\underset{\sim}{P},\tau) = -\delta(\tau)\delta(\underset{\sim}{P}) \quad . \tag{1.48}$$

Eg. 1. RETARDED GREEN'S FUNCTION

We define $G_R^{(2,1)}$ as the spatial integral over $G_R^{(3,1)}$ given by (1.36) where $r=|\underset{\sim}{x}-\underset{\sim}{x}'|$ and $\tau=x_0-x_0'$. It is

$$G_R^{(2,1)}(\underset{\sim}{P},\tau) = \int_{-\infty}^{\infty} G_R^{(3,1)}(x,x')dz' \quad , \tag{1.49}$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\delta(r - \tau)}{r}\, dz' \quad .$$

We define

$$\zeta = z - z' \qquad d\zeta = -dz'$$

$$r^2 = P^2 + \zeta^2$$

$$d\zeta/r = dr/\zeta = dr(r^2 - P^2)^{-1/2} \quad , \tag{1.50}$$

so that

$$G_R^{(2,1)}(\underset{\sim}{P},\tau) = \frac{1}{2\pi}\int_0^\infty \frac{\delta(r-\tau)dr}{(r^2 - P^2)^{1/2}} \quad ,$$

and, evaluating the δ-function, we get

$$G_R^{(2,1)}(\underset{\sim}{P},\tau) = \frac{1}{2\pi}\frac{\theta(\tau - P)}{(\tau^2 - P^2)^{1/2}} \quad . \tag{1.51}$$

This illustrates the fact that, in two dimensions, the effect of an impulse after a (scaled) time τ has elapsed has spread over a region of spatial extent $P<\tau$. A line source in three dimensions produces a field which at any given point has a tail.

We can analogously define the other Green's functions as integrals over the corresponding (3,1) dimensional Green's functions. From (1.37), (1.40), (1.41) and (1.42) we get

Eg. 2. ADVANCED GREEN'S FUNCTION

$$G_A^{(2,1)}(\underset{\sim}{P},\tau) = \frac{1}{2\pi}\frac{\theta(-\tau-P)}{(\tau^2-P^2)^{1/2}} \quad . \tag{1.52}$$

Eg. 3. PRINCIPAL VALUE GREEN'S FUNCTION

$$G_P^{(2,1)}(\underset{\sim}{P},\tau) = \frac{1}{4\pi}\frac{1}{(\tau^2-P^2)^{1/2}}\left[\theta(\tau-P) + \theta(-\tau-P)\right] \quad . \tag{1.53}$$

Eg. 4. DYSON GREEN'S FUNCTION

$$G_D^{(2,1)}(\underset{\sim}{P},\tau) = G_P^{(2,1)}(\underset{\sim}{P},\tau) + \frac{i}{4\pi}\frac{\theta(P-|\tau|)}{[P^2-\tau^2]^{1/2}} \quad . \tag{1.54}$$

and finally

Eg. 5. CAUSAL GREEN'S FUNCTION

$$G_C^{(2,1)}(\underset{\sim}{P},\tau) = G_P^{(2,1)}(\underset{\sim}{P},\tau) - \frac{i}{4\pi}\frac{\theta(P-|\tau|)}{[P^2-\tau^2]^{1/2}} \quad . \tag{1.55}$$

1.1.3 (1,1)-DIMENSIONS

The Green's function in one spatial and one temporal dimension is defined as

$$G^{(1,1)}(\xi,\tau) \qquad \xi = x - x' \quad ; \quad \tau = x_0 - x_0' = c(t-t') \quad , \tag{1.56}$$

and satisfies the equation

$$(\partial_x^2 - \partial_0^2)\, G^{(1,1)}(\xi,\tau) = -\delta(\xi)\,\delta(\tau) \quad . \tag{1.57}$$

We can compute the five Green's functions either by integrating over the y-coordinate results $G^{(2,1)}(\underset{\sim}{P},\tau)$ from Sec. 1.1.2, or by using pole shifting and complex integration techniques as we used in Sec. 1.1.1. We choose the latter. Introduce the Fourier transform

$$G^{(1,1)}(\xi,\tau) = (2\pi)^{-2}\iint \exp\left[i(k_x\xi - k_0\tau)\right]\tilde{G}^{(1,1)}(k_x,k_0)dk_x dk_0 \quad , \tag{1.58}$$

and apply the differential operator in (1.57) to it. We can thus solve for the Fourier transform and write (1.58) as

$$G^{(1,1)}(\xi,\tau) = -(2\pi)^{-2}\int \exp(ik_x\xi)dk_x \int \frac{\exp(-ik_0\tau)}{(k_0^2 - k_x^2)} d\tau \quad . \tag{1.59}$$

Shift the poles of the integrand as in Sec. 1.1.1. That is we have that

$$\frac{1}{(k_0 - k_x)(k_0 + k_x)} \rightarrow \frac{1}{(k_0 - k_x - i\alpha\varepsilon)(k_0 + k_x - i\beta\varepsilon)}$$

$$= P\left[\frac{1}{k_0^2 - k_x^2}\right]$$

$$+ \frac{\pi i}{2k_x}\left[\alpha\delta(k_0 - k_x) - \beta\delta(k_0 + k_x)\right] \quad , \tag{1.60}$$

where we have used (1.17). The result for (1.59) is

$$G^{(1,1)}(\xi,\tau) = -(2\pi)^{-2}\left[I_1(\xi,\tau) + (\pi i/2)\left[\alpha\ I_2(\xi,-\tau) - \beta\ I_2(\xi,\tau)\right]\right] \quad , \tag{1.61}$$

in analogy to (1.20) where here

$$I_1(\xi,\tau) = P\iint_{-\infty}^{\infty} \frac{\exp\left[i(k_x\xi - k_0\tau)\right]}{k_0^2 - k_x^2} dk_x\ dk_0 \quad , \tag{1.62}$$

and

$$I_2(\xi,\tau) = \int_{-\infty}^{\infty} \frac{\exp\left[ik_x(\xi + \tau)\right]}{k_x} dk_x \quad . \tag{1.63}$$

The values α and β are given by (1.16).

The integrals can be easily evaluated using complex variable techniques. The results are

$$I_1(\xi,\tau) = -(\pi^2/2)\ \mathrm{sgn}\ \tau[\mathrm{sgn}(\xi + \tau) - \mathrm{sgn}(\xi - \tau)] \quad , \tag{1.64}$$

or

$$I_1(\xi,\tau) = -\pi^2[\theta(\tau - |\xi| + \theta(-\tau - |\xi|)] \quad , \tag{1.65}$$

and

$$I_2(\xi,\tau) = \pi i\ \mathrm{sgn}(\xi + \tau) \quad . \tag{1.66}$$

We combine these results using (1.64)-(1.66) and (1.16) in (1.61) to get the results:

Eg. 1. RETARDED GREEN'S FUNCTION

$$G_R^{(1,1)}(\xi,\tau) = \frac{1}{2}\,\theta(\tau - |\xi|) \quad . \tag{1.67}$$

Eg. 2. ADVANCED GREEN'S FUNCTION

$$G_A^{(1,1)}(\xi,\tau) = \frac{1}{2}\,\theta(-\tau - |\xi|) \quad . \tag{1.68}$$

Eg. 3. PRINCIPAL VALUE GREEN'S FUNCTION

$$G_P^{(1,1)}(\xi,\tau) = \frac{1}{4}\,[\theta(\tau - |\xi|) + \theta(-\tau - |\xi|)] \quad . \tag{1.69}$$

Eg. 4. DYSON GREEN'S FUNCTION

$$G_D^{(1,1)}(\xi,\tau) = -\frac{1}{4}\left[\theta(\tau)\,\mathrm{sgn}(\xi-\tau) + \theta(-\tau)\,\mathrm{sgn}(\xi+\tau)\right] \quad . \tag{1.70}$$

and

Eg. 5. CAUSAL GREENS FUNCTION

$$G_C^{(1,1)}(\xi,\tau) = \frac{1}{4}\left[\theta(\tau)\,\mathrm{sgn}(\xi+\tau) + \theta(-\tau)\,\mathrm{sgn}(\xi-\tau)\right] \quad . \tag{1.71}$$

Other compact values can also be derived, for example

$$G_D^{(1,1)}(\xi,\tau) = -\frac{1}{4}\,\mathrm{sgn}\left[\xi - |\tau|\right] \quad , \tag{1.72}$$

and

$$G_C^{(1,1)}(\xi,\tau) = \frac{1}{4}\,\mathrm{sgn}\left[\xi + |\tau|\right] \quad . \tag{1.73}$$

1.2 GREEN'S FUNCTIONS FOR THE HELMHOLTZ EQUATION

1.2.1 (3)-DIMENSIONS

We can define the Green's functions for the Helmholtz equation as the temporal Fourier transform of the Green's functions for the wave equation derived in Sec. 1. Using the four-dimensional formulation we have

$$G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \int_{-\infty}^{\infty} G^{(3,1)}(x,x') \exp(ik_0\tau)d\tau \quad , \tag{2.1}$$

where $\tau=x_0-x_0'$. They satisfy the Helmholtz equation given by the same Fourier transform operating on (1.6). It is

$$(\nabla_x^2 + k_0^2)G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = -\delta(\underset{\sim}{x} - \underset{\sim}{x}') \quad , \tag{2.2}$$

where $k_0=\omega/c$ and ω is circular frequency. We compute each of the Green's functions corresponding to the pole shifts in Sec. 1.

Eg. 1. RETARDED GREEN'S FUNCTION

From (1.36) we have that for a difference of arguments ($r=|\underset{\sim}{x}-\underset{\sim}{x}'|$)

$$G_R^{(3,1)}(x,x') = G_R^{(3,1)}(x - x') = (4\pi r)^{-1}\delta(\tau - r) \quad . \tag{2.3}$$

Substitute this in (2.1) to get

$$G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = (4\pi r)^{-1} \exp(ik_0 r) \quad , \tag{2.4}$$

which is an outgoing spherical wave, i.e. for harmonic time dependence

$$\exp(-i\omega t) = \exp(-ik_0 x_0) \quad , \tag{2.5}$$

the wave travels in a positive radial direction and satisfies an outgoing radiation condition of the form

$$\lim_{r\to\infty} \left[\frac{\partial}{\partial r} - ik_0 \right] G_R^{(3)} = 0(r^{-2}) \quad . \tag{2.6}$$

It expresses the field at the receiver point $\underset{\sim}{x}$ due to a point source located at $\underset{\sim}{x}'$ in a homogeneous medium.

Eg. 2. ADVANCED GREEN'S FUNCTION

From (1.37) we have that

$$G_A^{(3,1)}(x,x') = (4\pi r)^{-1}\delta(\tau + r) \quad , \tag{2.7}$$

which when substituted into (2.1) yields

$$G_A^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = (4\pi r)^{-1} \exp(-ik_0 r) \quad . \tag{2.8}$$

For harmonic time dependence this is an incoming radial wave satisfying the radiation condition

$$\lim_{r\to\infty} \left[\frac{\partial}{\partial r} + ik_0 \right] G_A^{(3)} = 0(r^{-2}) \quad . \tag{2.9}$$

Eg. 3. PRINCIPAL VALUE GREEN'S FUNCTION

This can be computed either directly from (1.40) using a difference of arguments or from (1.39) represented here as half the sum of (2.3) and (2.8). The result is

$$G_P^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = (4\pi r)^{-1}\cos(k_0 r) \quad , \tag{2.10}$$

which for harmonic time dependence represents a standing wave. Note also that in contrast to the retarded and advanced functions, the principal value Green's function is real.

Eg. 4. DYSON AND CAUSAL GREEN'S FUNCTIONS

From (1.41) and (1.42) we have that

$$G_{D,C}^{(3,1)}(x,x') = G_P^{(3,1)}(x,x') \pm \frac{i}{4\pi^2} P \frac{1}{(x-x')^2} \quad . \tag{2.11}$$

We substitute this into (2.1) and use (2.10) for the evaluation of the principal value term. The remaining integral can be evaluated using residue calculus methods. The result is

$$G_{D,C}^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = (4\pi r)^{-1}\left[\cos(k_0 r) \pm i\ \mathrm{sgn}(k_0)\ \sin(k_0 r)\right] \quad , \tag{2.12}$$

which can be rewritten as

$$G_D^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \begin{cases} G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') & k_0>0 \\ G_A^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') & k_0<0 \quad , \end{cases} \tag{2.13}$$

and

$$G_C^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \begin{cases} G_A^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') & k_0>0 \\ G_R^{(3)}(x,x') & k_0<0 \end{cases} \quad . \tag{2.14}$$

Because the Dyson and Causal Green's functions mix representations and do not satisfy either a well-defined radiation condition or a standing wave interpretation except for positive or negative frequency separately, they are not useful for our purposes. In addition negative frequency results are usually folded into positive frequency ones in applications, and neither the Dyson or Causal functions yield new results over and above those found from the retarded, advanced, and principal value functions. We do not compute them for (2) and (1) dimensions.

1.2.2 (2)-DIMENSIONS

The two-dimensional Helmholtz Green's functions are the temporal Fourier transforms of the Green's functions for the two-dimensional wave equation in Sec. 1.1.2. They are defined as

$$G^{(2)}(\underset{\sim}{\rho},\underset{\sim}{\rho}') = \int_{-\infty}^{\infty} G^{(2,1)}(\underset{\sim}{P},\tau)\ \exp(ik_0\tau)d\tau \quad , \tag{2.15}$$

and satisfy the Helmholtz equation given by the corresponding transform of (1.48) which becomes

$$(\partial_x^2 + \partial_y^2 + k_0^2)G^{(2)}(\underset{\sim}{\rho},\underset{\sim}{\rho}') = -\delta(x - x')\delta(y - y') \quad . \tag{2.16}$$

Eg. 1. RETARDED GREEN'S FUNCTION

For this case we substitute (1.51) in (2.15) to get

$$G_R^{(2)}(\underset{\sim}{\rho},\underset{\sim}{\rho}') = \frac{1}{2\pi}\int_P^\infty \frac{\exp(ik_0\tau)}{(\tau^2 - P^2)^{1/2}}\, d\tau \quad . \tag{2.17}$$

If we make the substitution

$$\tau = P\cosh\phi \quad ; \quad d\tau = P\sinh\phi\, d\phi \quad ,$$

the integral becomes

$$G_R^{(3)}(\underset{\sim}{\rho},\underset{\sim}{\rho}') = \frac{1}{4\pi}\int_{-\infty}^{\infty} \exp(ik_0 P\cosh\phi)\, d\phi \quad , \tag{2.18}$$

written from $-\infty$ to ∞ since the integrand is an even function of ϕ. The integral (2.18) is a representation of the Hankel function

$$G_R^{(3)}(\underset{\sim}{\rho},\underset{\sim}{\rho}') = \frac{i}{4} H_0^{(1)}(k_0 P) \quad , \tag{2.19}$$

which is an outgoing cylindrical wave, i.e. asymptotically

$$H_0^{(1)}(k_0 P) \sim (2/\pi i k_0 P)^{1/2} \exp(ik_0 P) \quad , \tag{2.20}$$

which spreads like a cylindrical wave with amplitude factor $P^{-1/2}$.

Eg. 2. ADVANCED GREEN'S FUNCTION

Substitute (1.52) into (2.15). The integration can be performed as above with an additional sign change in τ. The result is the incoming Hankel function

$$G_A^{(2)}(\rho,\rho') = -\frac{i}{4} H_0^{(2)}(k_0 P) \quad , \tag{2.21}$$

which represents a cylindrical wave propagating in the direction of decreasing r.

Eg. 3. PRINCIPAL VALUE GREEN'S FUNCTION

This can be computed either directly by substituting (1.53) in (2.15) or as half the sum of (2.19) and (2.21). The result is

$$G_P^{(2)}(\underset{\sim}{\rho},\underset{\sim}{\rho}') = -\frac{1}{4} N_0(k_0 P) \quad , \tag{2.22}$$

where N_0 is the Neumann function.

1.2.3 (1)-DIMENSION

We define the one-dimensional Helmholtz Green's function as the temporal Fourier transform of the one-dimensional Green's functions for the wave equation in Sec. 1.1.3. It is

$$G^{(1)}(x,x') = \int_{-\infty}^{\infty} G^{(1,1)}(\xi,\tau)\exp(ik_0\tau)d\tau \quad , \tag{2.23}$$

where ξ=x−x'. Fourier transformation of the differential equation (1.57) yields the ordinary differential equation

$$\left[\frac{d^2}{dx^2} + k_0^2\right] G^{(1)}(x,x') = -\delta(x-x') \quad , \tag{2.24}$$

which is the one-dimensional version of the Helmholtz equation satisfied by all the one-dimensional Green's functions below.

Eg. 1. RETARDED GREEN'S FUNCTION

Substitute (1.67) in (2.23). Evaluation of the step function yields

$$G_R^{(1)}(x,x') = \frac{1}{2}\int_{|\xi|}^{\infty} \exp(ik_0\tau)d\tau \quad . \tag{2.25}$$

Recall that this Green's function for the wave equation was computed with k_0 shifted to k_{0+}. We can then directly evaluate the integral in (2.25) since the contribution at ∞ vanishes. The result is

$$G_R^{(1)}(x,x') = -(2ik_0)^{-1} \exp(ik_0|x-x'|) \quad , \tag{2.26}$$

which is a one-dimensional wave which travels to the right.

Eg. 2. ADVANCED GREEN'S FUNCTION

Substitute (1.69) in (2.23). The integral evaluation proceeds in the same manner as the previous example except that here we note that the advanced Green's function is computed with k_0 shifted to k_{0-}. The result is

$$G_A^{(1)}(x,x') = (2ik_0)^{-1}\exp(-ik_0|x-x'|) \quad , \tag{2.27}$$

which for harmonic time dependence is a one-dimensional wave travelling to the left.

Eg. 3. PRINCIPAL VALUE GREEN'S FUNCTION

This can be computed either directly from (1.69) or by combining half the sum of (2.26) and (2.27) to give

$$G_P^{(1)}(x,x') = -(4k_0)^{-1} \sin(k_0|x-x'|) \quad , \tag{2.28}$$

which represents a standing wave.

1.3 CAUSAL PARABOLIC GREEN'S FUNCTION

The causal Green's function which satisfies the parabolic wave equation

$$\nabla_x^2\, g(\underset{\sim}{x},\underset{\sim}{x}';t,t') - 2\gamma \frac{\partial g}{\partial t} = -\delta(\underset{\sim}{x}-\underset{\sim}{x}')\delta(t-t') \quad , \tag{3.1}$$

which contains only the first derivative in time and where γ is a constant can be computed as follows. Causality means that there is no measurable effect until the source turns on, i.e. g must vanish for times t less than the source turn-on time t'. This is

$$g(\underset{\sim}{x},\underset{\sim}{x}';t,t') = 0 \qquad t < t' \quad . \tag{3.2}$$

Introduce the Fourier transform in $\underset{\sim}{x}$

$$\tilde{g}(\underset{\sim}{k},\underset{\sim}{x}';t,t') = \iiint \exp(-i\underset{\sim}{k}\cdot\underset{\sim}{x})g(\underset{\sim}{x},\underset{\sim}{x}';t,t')d\underset{\sim}{x} \quad , \tag{3.3}$$

and correspondingly Fourier transform (3.1) to get

$$2\,\gamma \frac{\partial \tilde{g}}{\partial t} + k^2\tilde{g} = \exp(-i\underset{\sim}{k}\cdot\underset{\sim}{x}')\delta(t-t') \quad , \tag{3.4}$$

where $\underset{\sim}{k}$ is the Fourier transform variable and $k^2=\underset{\sim}{k}\cdot\underset{\sim}{k}$. We still require the condition (3.2) and the one-dimensional equation (3.4) has solution given by

$$\tilde{g}(\underset{\sim}{k},\underset{\sim}{x}';t,t') = (2\gamma)^{-1}\, \exp\left[-i\underset{\sim}{k}\cdot\underset{\sim}{x}' - k^2(t-t')/2\gamma\right]\theta(t-t') \quad , \tag{3.5}$$

where the step function defines causality. The inverse transform is

$$g(\underset{\sim}{x},\underset{\sim}{x}';t,t') = (2\pi)^{-3}\iiint \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x})\tilde{g}(\underset{\sim}{k},\underset{\sim}{x}';t,t')d\underset{\sim}{k} \quad , \tag{3.6}$$

and if we substitute (3.5) into (3.6) we note that the result is the Fourier transform of a Gaussian. The latter may be treated by cartesian coordinates to yield

$$g(\underset{\sim}{x},\underset{\sim}{x}';t,t') = \frac{1}{(2\pi)^3} \frac{\pi^{3/2}(2\gamma)^{1/2}}{(t-t')^{3/2}} \exp\left[-\frac{\gamma}{2}\frac{|\underset{\sim}{x}-\underset{\sim}{x}'|^2}{t-t'}\right]\theta(t-t') \quad . \qquad (3.7)$$

It can be shown that this result can be generalized to n-spatial dimensions to yield

$$g^{(n)}(\underset{\sim}{x},\underset{\sim}{x}';t,t') = \frac{1}{(2\pi)^n}\frac{1}{2\gamma}\left[\frac{2\gamma\pi}{t-t'}\right]^{n/2} \exp\left[-\frac{\gamma}{2}\frac{|\underset{\sim}{x}-\underset{\sim}{x}'|^2}{t-t'}\right]\theta(t-t') \ , \qquad (3.8)$$

where for

$n = 1$ $\quad \underset{\sim}{x} = (x)$

$n = 2$ $\quad \underset{\sim}{x} = (x,y)$

$n = 3$ $\quad \underset{\sim}{x} = (x,y,z)$

$\vdots$

n $\quad \underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$.

1.4 REPRESENTATIONS

There are several useful integral representations of the Green's function for the Helmholtz equation. The latter is a spherical wave and the representations amount to expanding a spherical wave into either plane waves (spectral integrals) or cylindrical waves (Sommerfeld and Weyrich representations).

1.4.1 WEYL REPRESENTATION

The first representation is an expansion of a spherical wave into plane waves. The Helmholtz Green's function in three dimensions satisfies the differential equation

$$(\nabla^2 + k_0^2)\, G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = -\delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad . \tag{4.1}$$

Fourier transform this equation with respect to $\underset{\sim}{x}$, i.e. multiply the equation by

$$\iiint \exp\left[-i(k_x x + k_y y + k_z z)\right] dxdydz \quad , \tag{4.2}$$

to get the mixed representation

$$\tilde{G}^{(3)}(\underset{\sim}{k},\underset{\sim}{x}') = \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x}')(k^2-k_0^2)^{-1} \quad , \tag{4.3}$$

where

$$k^2 = |\underset{\sim}{k}|^2 = k_x^2 + k_y^2 + k_z^2 = k_t^2 + k_z^2 \quad , \tag{4.4}$$

and $\underset{\sim}{k}$ is the Fourier transform variable. Note that in (4.4), by defining the transverse part of the wavenumber

$$k_t = (k_x^2 + k_y^2)^{1/2} \quad , \tag{4.5}$$

we have essentially picked out the z-direction as special. Of course we could do this with any of the three directions. Using (4.3) the inverse Fourier transform is thus

$$G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = (2\pi)^{-3}\iiint \exp\left[i(k_x x + k_y y + k_z z)\right]\tilde{G}(\underset{\sim}{k},\underset{\sim}{x}')d\underset{\sim}{k} \tag{4.6}$$

$$= \frac{1}{(2\pi)^3}\iiint \frac{\exp\left[i\left[k_x(x-x') + k_y(y-y') + k_z(z-z')\right]\right]}{(k_z - K)(k_z + K)}\, dk_x dk_y dk_z \quad , \tag{4.7}$$

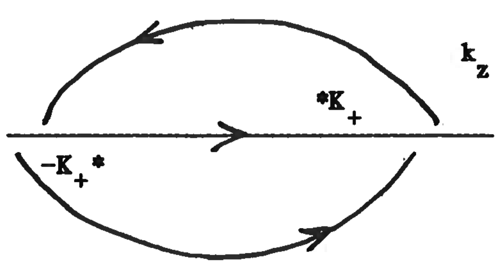
where

$$K = (k_0^2 - k_t^2)^{1/2} \quad , \tag{4.8}$$

and where we have distinguished the poles in the integrand of (4.7) as poles in k_z. We evaluate (4.7) cylindrically, i.e. we do the k_z integral first using complex variables. We shift the poles by adding a small positive imaginary part to k_0 and hence to K. That is

$$k_0 \rightarrow k_0 + i\varepsilon \doteq k_{0+} \quad ; \quad K \rightarrow K + i\varepsilon \doteq K_+ \quad .$$

Then in the complex k_z-plane we have



z − z' > 0

z − z' < 0

Fig. 1.1

We evaluate the k_z-integral in (4.7) by closing the contour in the upper half plane (z-z'>0) or in the lower half plane (z-z'<0). The result is

$$\int \frac{\exp\left[ik_z(z-z')\right]}{(k_z - K_+)(k_z + K_+)} dk_z = \frac{\pi i}{K} \exp[iK|z-z'|] \quad . \tag{4.9}$$

Using this in (4.7) and writing the remaining two integrals using the two-dimensional vectors

$$\underset{\sim}{k}_t = (k_x, k_y) \quad , \quad \underset{\sim}{x}_t = (x, y) \quad , \tag{4.10}$$

we get

$$G^{(3)}(\underset{\sim}{x}, \underset{\sim}{x}') = \frac{\pi i}{(2\pi)^3} \iint \frac{\exp\left[i\underset{\sim}{k}_t \cdot (\underset{\sim}{x}_t - \underset{\sim}{x}_t') + iK_+ |z-z'|\right]}{K_+} d\underset{\sim}{k}_t \quad , \tag{4.11}$$

where ImK>0.

Equation (4.11) is the two-dimensional plane-wave spectral representation or <u>Weyl representation</u>. In deriving it we have singled out the z-direction as special. This is appropriate if the z-direction in the application is special, for example if there is a discontinuity in z or if the variability in the medium is in the z-direction. We treat this further in the next section. Also notice that here we shifted <u>both</u> poles by $+i\varepsilon$, so that effectivey one shifted above the axis and one below. For the retarded Green's function is Sec. 1 we shifted both ω_k poles down. This is equivalent to what we have done here since we had

$$k_0 - \omega_k \rightarrow k_0 - (\omega_k - i\varepsilon) = k_0 + i\varepsilon - \omega_k \quad ,$$

$$k_0 + \omega_k \rightarrow k_0 + (\omega_k + i\varepsilon) = k_0 + i\varepsilon + \omega_k \quad .$$

In both terms we give a positive shift to the k_0 term. By shifting the poles in the manner above we have derived the Weyl representation for the retarded Green's function. Other pole shifts can be done to form a Weyl representation for the advanded or principal value Green's functions for example.

1.4.2 SOMMERFELD REPRESENTATION

For this case we expand the spherical wave in cylindrical waves. The expansion is essentially over the horizontal wave number. We begin by representing the Weyl representation (4.11) in cylindrical polar coordinates defined about x-x' and y-y'. We have that

$$\exp\left[i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t - \underset{\sim}{x}_t')\right] = \exp(ik_t\rho\cos\theta) \quad , \tag{4.12}$$

where

$$\rho = \left[(x-x')^2 + (y-y')^2\right]^{1/2} = |\underset{\sim}{x}_t - \underset{\sim}{x}_t'| \quad , \tag{4.13}$$

and θ is the angle between $\underset{\sim}{k}_t$ and $\underset{\sim}{x}_t - \underset{\sim}{x}_t'$. Using the cylindrical differential area element

$$d\underset{\sim}{k}_t = k_t\, dk_t\, d\theta \quad , \tag{4.14}$$

and the definition of the Bessel function (cylindrical wave)

$$J_0(k_t\rho) = (2\pi)^{-1}\int_0^{2\pi} \exp(ik_t\rho\cos\theta)d\theta \quad , \tag{4.15}$$

we evaluate the θ-integral as above to get from (4.11)

$$G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{i}{4\pi}\int_0^\infty \frac{J_0(k_t\rho)\exp\left[i(k_{0+}^2 - k_t^2)^{1/2}|z-z'|\right]}{(k_{0+}^2 - k_t^2)^{1/2}} k_t dk_t \quad , \tag{4.16}$$

where we have explicitly written out the square root K. The result (4.16) is a one-dimensional integral representation of a spherical wave in terms of cylindrical waves called the <u>Sommerfeld representation</u>. It is an integral written over the <u>horizontal wavenumber</u> k_t, and is only useful for problems which contain an analogous horizontal symmetry (i.e. a parametric independence of θ).

Alternatively, we can write (4.16) in terms of the Hankel function. The Bessel function J_0 can be written in terms of Hankel functions $H_0^{(1)}$ and $H_0^{(2)}$ as

$$J_0(k_t\rho) = 1/2\left[H_0^{(1)}(k_t\rho) + H_0^{(2)}(k_t r)\right] \quad . \tag{4.17}$$

Substitute this into (4.16), and use, in the $H_0^{(2)}$ integral, the result

$$H_0^{(2)}(k_t r) = -H_0^{(1)}(e^{\pi i}k_t r) \quad , \tag{4.18}$$

and, in this integral rotate the contour by defining a new variable

$$k_t' = e^{\pi i}k_t \quad , \tag{4.19}$$

so that the limits of integration go from $(0,\infty)$ to $(0,\infty e^{\pi i})=(0,-\infty)$. The result is an integral over only $H_0^{(1)}$ given by

$$G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{i}{8\pi}\int \frac{H_0^{(1)}(k_t\rho)\exp\left[i\left[k_{0+}^2-k_t^2\right]^{1/2}|z-z'|\right]}{\left[k_{0+}^2-k_t^2\right]^{1/2}} k_t dk_t \quad . \quad (4.20)$$

Since $H_0^{(1)}$ behaves like an outgoing wave asymptotically, the representation (4.20) is useful in problems which contain this type of geometry, e.g. the exterior problem of scattering from a bounded object. Equation (4.16) on the other hand is useful for an interior representation, i.e. one which contains standing waves rather than outgoing waves.

1.4.3 EXPLICIT EVALUATION OF $G^{(3)}$

We mentioned that our representations were for the retarded Green's function. We can explicitly exhibit this by evaluating all the integrals in $G^{(3)}$. From (4.7) we have that

$$G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{1}{(2\pi)^3}\iiint \frac{\exp\left[i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')\right]}{k^2-k_0^2}\, d\underset{\sim}{k} \quad (4.21)$$

Using spherical polar coordinates defined as in Fig. 1.2

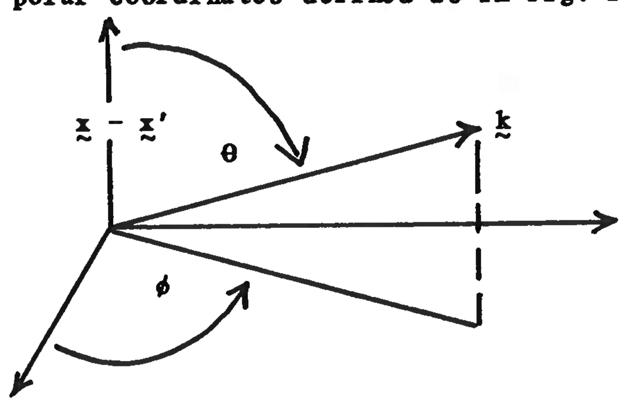


Fig. 1.2

we have that

$$\underset{\sim}{k}\cdot(\underset{\sim}{x} - \underset{\sim}{x}') = k|\underset{\sim}{x} - \underset{\sim}{x}'|\cos\theta = kr\ \cos\theta \quad ,$$
$$d\underset{\sim}{k} = k^2 dk\ \sin\theta\ d\theta d\phi \quad . \tag{4.22}$$

The ϕ-integral just yields 2π. The θ-integral is

$$\int_0^\pi \exp(ikr\ \cos\theta)\sin\theta d\theta = \left[\exp(ikr) - \exp(-ikr)\right](ikr)^{-1} \quad . \tag{4.23}$$

We evaluate the positive exponential integral using complex variables and pole shifts to $k_0+i\varepsilon$ to get (close in uhp)

$$\int_{-\infty}^{\infty} \frac{k\ \exp(ikr)}{(k-k_{0+})(k+k_{0+})} dk = \pi i\ \exp(ik_0 r) \quad . \tag{4.24}$$

The negative exponential is evaluated by closing in the lower half plane with the <u>same</u> pole shifts ($k_{0+}+i\varepsilon$) to yield

$$\int_{-\infty}^{\infty} \frac{k\ \exp(-ikr)}{(k-k_{0+})(k+k_{0+})} = -\pi i\ \exp(ik_0 r) \quad . \tag{4.25}$$

Combining all these results we get

$$G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \exp(ik_0 r)/4\pi r \quad , \tag{4.26}$$

which was the same result as we found using the retarded contour in Sec. 2.

1.4.4 WEYRICH REPRESENTATION

An alternative representation of spherical waves expanded in cylindrical waves can be found by expanding in the vertical (k_z) wavenumber rather than the horizontal wavenumber as in the Sommerfeld representation. Starting with (4.21) we don't do the k_z-integral. Instead do the k_x and k_y integrals using cylindrical symmetry. Using the definition

$$K_z = (k_0^2 - k_z^2)^{1/2} \quad , \tag{4.27}$$

(4.21) is written as

$$G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \exp\left[ik_z(z-z')\right]dk_z \iint \frac{\exp\left[i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t-\underset{\sim}{x}_t')\right]}{k_t^2 - K_z^2}\, dk_x dk_y . \tag{4.28}$$

The latter two integrals can be evaluated using the cylindrical coordinate and Bessel function definitions in (4.12)-(4.15). The θ-integral again yields a Bessel function so that we have

$$G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \exp\left[ik_z(z-z')\right]dk_z \int_0^{\infty} \frac{k_t J_0(k_t r)}{k_t^2 - K_z^2} dk_t \quad . \tag{4.29}$$

We use (4.17) for the Bessel function and in the integral for $H_0^{(2)}$ we again rotate the contour ($k_t \to - k_t$) so that using (4.18) we again have an integral only over $H_0^{(1)}$. We get

$$G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{1}{2(2\pi)^2}\int_{-\infty}^{\infty} \exp\left[ik_z(z-z')\right]dk_z \int_{-\infty}^{\infty} \frac{k_t H_0^{(1)}(k_t\rho)}{k_t^2 - K_z^2} dk_t \quad . \tag{4.30}$$

We have shifted our poles so that $\mathrm{Im}K_z > 0$. We evaluate the k_t-integral using complex variables and closing the contour in the uhp. Explicitly writing out K_z we get

$$G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{i}{8\pi}\int_{-\infty}^{\infty} \exp\left[ik_z(z-z')\right]H_0^{(1)}\left[\left[k_{0+}^2 - k_z^2\right]^{1/2}\rho\right]dk_z \quad , \qquad (4.31)$$

which is Weyrich's formula, a one-dimensional integral representation in terms of the vertical wavenumber k_z. The result (4.31) is often quoted using (4.26) as

$$\frac{\exp\left[ik_0(\rho^2 + z^2)^{1/2}\right]}{(\rho^2 + z^2)^{1/2}} = \frac{i}{2}\int_{-\infty}^{\infty} e^{i\alpha z}\, H_0^{(1)}\left[\rho(k_0^2 - \alpha^2)^{1/2}\right]d\alpha \quad , \qquad (4.32)$$

where ρ, z are real, $r=(\rho^2+z^2)^{1/2}$ and $0 \lesssim \arg\,(k_0^2-\alpha^2)^{1/2} < \pi$.

1.4.5 PLANE-WAVE DECOMPOSITION OF $G^{(2)}$

It is possible to derive two Weyl-type representations for the Helmholtz Green's function in two dimensions. In two dimensions the latter satisfies the Helmholtz equation

$$(\nabla_2^2 + k_0^2)G^{(2)}(\underset{\sim}{x}_t,\underset{\sim}{x}_t') = -\delta(\underset{\sim}{x}_t-\underset{\sim}{x}_t') \quad . \qquad (4.33)$$

We Fourier transform the equation by multiplying by

$$\iint \exp(-i\underset{\sim}{k}_t\cdot\underset{\sim}{x}_t)d\underset{\sim}{x}_t \quad ,$$

where $\underset{\sim}{k}_t=(k_x,k_y)$ is the Fourier transform variable. The result is the mixed representation

$$\tilde{G}^{(2)}(\underset{\sim}{k}_t,\underset{\sim}{x}_t') = \exp(-i\underset{\sim}{k}_t\cdot\underset{\sim}{x}_t')(k_t^2-k_0^2)^{-1} \quad . \qquad (4.34)$$

The inverse transform thus becomes

$$G^{(2)}(\underset{\sim}{x}_t,\underset{\sim}{x}_t') = \frac{1}{(2\pi)^2}\iint \exp(i\underset{\sim}{k}_t\cdot\underset{\sim}{x}_t)\tilde{G}^{(2)}(\underset{\sim}{k}_t,\underset{\sim}{x}_t')d\underset{\sim}{k}_t \quad , \tag{4.35}$$

$$= \frac{1}{(2\pi)^2}\iint \frac{\exp\left[i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t-\underset{\sim}{x}_t')\right]}{(k_x-K_y)(k_x+K_y)}d\underset{\sim}{k}_t \quad , \tag{4.36}$$

where

$$K_y = (k_0^2 - k_y^2)^{1/2} \quad . \tag{4.37}$$

We expressed the integrand of (4.36) in such a way as to do the k_x-integration using complex variable techniques just as we did the k_z-integration in (4.9). The result is

$$G^{(2)}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{\pi i}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\exp\left[ik_y(y-y') + iK_y|x-x'|\right]}{K_y}dk_y \quad . \tag{4.38}$$

Alternatively we could define

$$K_x = (k_0^2 - k_x^2)^{1/2} \quad , \tag{4.39}$$

so that the denominator of (4.36) becomes

$$(k_y - K_x)(k_y + K_x) \quad ,$$

and the obvious choice is to carry out the k_y-integration. The result is

$$G^{(2)}(\underset{\sim}{x}_t,\underset{\sim}{x}_t') = \frac{\pi i}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\exp\left[ik_x(x-x') + iK_x|y-y'|\right]}{K_x}dk_x \quad . \tag{4.40}$$

The representations (4.38) and (4.40) are both Weyl-type representations for the two-dimensional Green's function. Their use depends on exploiting the geometry of the particular application.

1.4.6 EXPLICIT EVALUATION OF $G^{(2)}$

Using (4.34) and (4.35) we can explicitly evaluate $G^{(2)}$ using cylindrical coordinates and complex integration. We have

$$G^{(2)}(\underset{\sim}{x}_t,\underset{\sim}{x}_t') = \frac{1}{(2\pi)^2}\iint \frac{\exp[i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t-\underset{\sim}{x}_t')]}{k_t^2-k_0^2}dk_x dk_y \quad . \tag{4.41}$$

In cylindrical coordinates we have that

$$\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t-\underset{\sim}{x}_t') = k_t|\underset{\sim}{x}_t-\underset{\sim}{x}_t'|\cos\theta = k_t r\cos\theta \quad , \tag{4.42}$$

$$dk_x dk_y = k_t dk_t d\theta \quad .$$

The θ-integration yields $2\pi\, J_0(k_t r)$, and we replace J_0 using (4.17). The integral involving $H_0^{(2)}$ is rewritten using (4.18) so that we only have an integral over $H_0^{(1)}$. It is

$$G^{(2)}(\underset{\sim}{x}_t,\underset{\sim}{x}_t') = \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{H_0^{(1)}(k_t r)}{k_t^2-k_0^2}k_t dk_t \quad . \tag{4.43}$$

Since $H_0^{(1)}$ behaves like an outgoing wave, we evaluate (4.43) using complex integration by closing the contour in the upper half plane. The poles are shifted by $k_0\rightarrow k_0+i\varepsilon$. The result is the cylindrical Green's function

$$G^{(2)}(\underset{\sim}{x}_t,\underset{\sim}{x}_t') = (i/4)\, H_0^{(1)}(k_0\rho) \quad . \tag{4.44}$$

It is most useful for problems in cylindrical coordinates which have no angular dependence.

1.4.7 EXPLICIT RELATION BETWEEN $G^{(3)}$ and $G^{(2)}$

In three dimensions we have that

$$G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{1}{(2\pi)^3}\iiint \frac{\exp\left[i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')\right]}{k^2-k_0^2}dk_x dk_y dk_z \quad . \tag{4.45}$$

Break up these three integrals into a k_z-integral and a two-dimensional transverse integral as

$$G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{1}{2\pi}\int \exp\left[ik_z(z-z')\right]dk_z \cdot$$

$$\frac{1}{(2\pi)^2}\iint \frac{\exp\left[i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t-\underset{\sim}{x}_t')\right]}{k_t^2-\left[k_0^2-k_z^2\right]}dk_x dk_y \quad . \tag{4.46}$$

The latter two integrals are just (4.41) but with k_0^2 replaced by $K_z^2=k_0^2-k_z^2$. Using (4.44) we thus have that

$$G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{i}{8\pi}\int_{-\infty}^{\infty} \exp\left[ik_t(z-z')\right]H_0^{(1)}\left[\left[k_0^2-k_t^2\right]^{1/2}\rho\right]dk_z \quad , \tag{4.47}$$

which is just Weyrich's formula (4.31).

1.4.8 THREE-DIMENSIONAL REPRESENTATIONS IN THE HALF-PLANE

In the previous sections we presented several integral representations for the Green's functions. We placed no restrictions on the regions of validity of these representations, and consequently they are valid in all space. Here we treat representations valid in one or the other half space.

We have singled out the z-direction as special, and we define the half-planes using this. The Weyl representation for the retarded Green's function is from (4.11)

$$G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{\pi i}{(2\pi)^3} \iint \frac{\exp\left[i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t - \underset{\sim}{x}_t') + iK_+|z-z'|\right]}{K_+} d\underset{\sim}{k}_t \quad . \tag{4.48}$$

If we restrict the region to $z-z'\geq 0$ so that the absolute value can be dropped we write the result as a three-dimensional integral as (the + sign indicates the region $z-z'\geq 0$).

$$\left[G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}')\right]_+ = (2\pi)^{-3} \iiint A_R^+(\underset{\sim}{k}) \exp\left[i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')\right] d\underset{\sim}{k} \quad , \tag{4.49}$$

where the amplitude function is defined by

$$A_R^+(\underset{\sim}{k}) = (\pi i/K)\delta(k_z - K) \quad . \tag{4.50}$$

The advantage of this representation is in a three-dimensional problem where however the boundary is planar.

Similarly, a representation for $z-z'\leq 0$ can be written as

$$\left[G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}')\right]_- = (2\pi)^{-3} \iiint A_R^-(\underset{\sim}{k}) \exp\left[i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')\right] d\underset{\sim}{k} \quad , \tag{4.51}$$

where the amplitude is defined by

$$A_R^-(\underset{\sim}{k}) = (\pi i/K)\delta(k_z + K) \quad . \tag{4.52}$$

We can combine the two representations (4.49) and (4.51) to yield a quasi-three-dimensional Fourier representation where however the amplitude is spatially dependent viz.

$$G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = (2\pi)^{-3}\iiint A_R(z-z',\underset{\sim}{k})\exp\left[i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')\right]d\underset{\sim}{k} \quad , \tag{4.53}$$

and is given by

$$\begin{aligned} A_R(z-z',\underset{\sim}{k}) &= \theta(z-z')A_R^+(\underset{\sim}{k}) + \theta(z'-z)A_R^-(\underset{\sim}{k}) \\ &= (\pi i/K)\left[\theta(z-z')\delta(k_z-K) + \theta(z'-z)\delta(k_t+K)\right] \quad . \end{aligned} \tag{4.54}$$

We could also derive a Weyl-representation for the advanced Green's function. It is

$$G_A^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{-\pi i}{(2\pi)^3}\iint \frac{\exp\left[i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t - \underset{\sim}{x}_t') - iK_-|z-z'|\right]}{K_-}d\underset{\sim}{k}_t \quad . \tag{4.55}$$

with analogous representations for $z-z'\gtrless 0$ and $z-z'\lesssim 0$ given by

$$\left[G_A^{(3)}(\underset{\sim}{x},\underset{\sim}{x}')\right]_\pm = (2\pi)^{-3}\iiint A_a^\pm(\underset{\sim}{k})\exp\left[i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')\right]d\underset{\sim}{k} \quad , \tag{4.56}$$

where

$$A_a^\pm(\underset{\sim}{k}) = -(\pi i/K)\delta(k_z \pm K) \quad , \tag{4.57}$$

with the full three-dimensional representation given by

$$G_A^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = (2\pi)^{-3}\iiint A_a(z-z',\underset{\sim}{k})\exp\left[i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')\right]d\underset{\sim}{k} \quad , \tag{4.58}$$

where the amplitude is

$$A_a(z-z',\underset{\sim}{k}) = -(\pi i/K)\left[\theta(z-z')\delta(k_z+K) + \theta(z+z')\delta(k_z-K)\right] \quad . \tag{4.59}$$

From these representations we can also compute the representation for the principal value using

$$G_P^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = 1/2\left[G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') + G_A^{(3)}(\underset{\sim}{x},\underset{\sim}{x}')\right] \quad , \tag{4.60}$$

so that

$$G_P^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = (2\pi)^{-3}\iiint A_P(z-z',\underset{\sim}{k})\exp\left[i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')\right]d\underset{\sim}{k} \quad , \tag{4.61}$$

where

$$A_P(z-z',\underset{\sim}{k}) = 1/2\left[A_R(z-z',\underset{\sim}{k}) + A_a(z-z',\underset{\sim}{k})\right] \tag{4.62}$$

$$= (\pi i/2K)\,\mathrm{sgn}(z-z')\left[\delta(k_z-K) - \delta(k_z+K)\right] \quad . \tag{4.63}$$

1.5 ANALYTIC PROPERTIES OF THE GREEN'S FUNCTIONS

1.5.1 ANALYTIC PROPERTIES OF $G_R^{(3)}$

We discuss the properties of $G_R^{(3)}$ using distributions. We have that

$$G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{\exp\left[ik_0\left|\underset{\sim}{x} - \underset{\sim}{x}'\right|\right]}{4\pi\left|\underset{\sim}{x} - \underset{\sim}{x}'\right|} \quad . \tag{5.1}$$

We begin with the Weyl representation presented in Sec. 4. Here we have

$K = \left[k_0^2 - k_t^2\right]$ and $k_t^2 = k_x^2 + k_y^2$. It is

$$G_R^{(3)}(\underset{\sim}{x} - \underset{\sim}{x}') = \frac{\pi i}{(2\pi)^3} \iint \frac{\exp\left[i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t - \underset{\sim}{x}_t') + iK_+\left|z-z'\right|\right]}{K_+} d\underset{\sim}{k}_t \ , \tag{5.2}$$

where $\underset{\sim}{x}'$ is the source point and $\underset{\sim}{x}$ the receiver point. We keep the representation as a difference in these coordinates. We use the term $K_+ = K + i\varepsilon$ to distinguish the square root having a positive imaginary part. The properties are as follows:

PROPERTY 1. $G_R^{(3)}$ $(\underset{\sim}{x} - \underset{\sim}{x}')$ is continuous as $\underline{z} - \underline{z}' \rightarrow \underline{0}$.

The proof is obvious. There are two cases, when $z - z' > 0$ and $z - z' < 0$. Both limits are the same. They are

$$\lim_{z-z'\rightarrow 0^+} G_R^{(3)}(\underset{\sim}{x} - \underset{\sim}{x}') = \lim_{z-z'\rightarrow 0^-} G_R^{(3)}(\underset{\sim}{x} - \underset{\sim}{x}') \quad ,$$

and equating the limits of (5.1) and (5.2) we get

$$\frac{e^{ik_0\rho}}{4\pi\rho} = \frac{\pi i}{(2\pi)^3} \iint \exp\left[i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t - \underset{\sim}{x}_t')\right] \frac{d\underset{\sim}{k}_t}{K_+} \quad , \tag{5.3}$$

where $\rho = |\underset{\sim}{x}_t - \underset{\sim}{x}'_t|$. Simply put the argument of the exponential in (5.2) vanishes independent of direction because of the absolute value.

PROPERTY 2. The first derivative in depth is discontinuous.

To prove this differentiate (5.2) with respect to z

$$\partial_z G_R^{(3)}(\underset{\sim}{x} - \underset{\sim}{x}') = \frac{-\pi \operatorname{sgn}(z-z')}{(2\pi)^3} \iint \exp\left[i\underset{\sim}{k}_t \cdot (\underset{\sim}{x}_t - \underset{\sim}{x}'_t) + iK_+ |z-z'|\right] d\underset{\sim}{k}_t \quad . \tag{5.4}$$

Note that the factor K cancels in the integrand. In the limit as $z - z' \to 0$ the exponent vanishes independent of direction but the antisymmetric signum function remains. Also, the integral for $z - z' = 0$ is just the two dimensional delta function multiplied by $(2\pi)^2$. The result is

$$\lim_{z-z'\to 0} \partial_z G_R^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') = \frac{-\pi}{(2\pi)^3} \lim_{z-z'\to 0} \operatorname{sgn}(z-z') \cdot \iint \exp\left[i\underset{\sim}{k}_t \cdot (\underset{\sim}{x}_t - \underset{\sim}{x}'_t)\right] d\underset{\sim}{k}_t \quad ,$$

$$\lim_{z-z'\to 0} \partial_z G_R^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') = -\frac{1}{2}\,\delta(\underset{\sim}{x}_t - \underset{\sim}{x}'_t) \begin{bmatrix} 1 & z - z' \to 0^+ \\ -1 & z - z' \to 0^- \end{bmatrix} \quad . \tag{5.5}$$

Note that the limits are independent of k_0, so the same discontinuous derivative behavior holds for static potential theory. Define:

$$\lim_{z-z'\to 0\pm} \partial_z G_R^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') = \left[\partial_z G_R^{(3)}(\underset{\sim}{x}_t, \underset{\sim}{x}'_t)\right]_\pm \quad , \tag{5.6}$$

so that the discontinuity is a distribution

$$\text{disc}\left[\partial_z G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}')\right]_{z\,=\,z'} = \left[\partial_z G_R^{(3)}\right]_+ - \left[\partial_z G_R^{(3)}\right]_-$$

$$= -\delta(\underset{\sim}{x}_t - \underset{\sim}{x}_t') \quad . \tag{5.7}$$

We show later that this is analogous to the discontinuous behavior for one-dimensional Green's functions. Here we also have an additional two-dimensional delta function.

PROPERTY 3. The transverse derivatives are continuous.

Define the transverse differential operator as

$$\partial_{jt} = \begin{cases} \partial/\partial x & j = 1 \\ \partial/\partial y & j = 2 \end{cases} \quad .$$

Differentiating (5.2) we get

$$\partial_{jt} G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{-\pi}{(2\pi)^3} \iint \exp\left[i\underset{\sim}{k}_t \cdot (\underset{\sim}{x}_t - \underset{\sim}{x}_t') + iK_+ |z-z'|\right] \frac{k_{jt}}{K_+}\, d\underset{\sim}{k}_t \quad . \tag{5.8}$$

This again approaches a finite value in the limit $z - z' \rightarrow 0$ independent of direction and is

$$\lim_{z-z'\rightarrow 0} \partial_{jt} G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{-\pi}{(2\pi)^3} \iint \exp\left[i\underset{\sim}{k}_t \cdot (\underset{\sim}{x}_t - \underset{\sim}{x}_t')\right] \frac{k_{jt}}{K_+}\, d\underset{\sim}{k}_t \quad , \tag{5.9}$$

which can be evaluated directly by doing the integral or simply by noting that we can interchange the derivative and the limiting process to yield

$$\lim_{z-z'\rightarrow 0} \partial_j G_R^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') = \partial_{jt} \frac{e^{ik_0\rho}}{4\pi\rho} \quad , \quad \rho = |x - x_t'| \; . \tag{5.10}$$

Next we want a representation for the full vector derivative. We will need this later to find the normal derivative of the function. For reasons which will be clear later it is convenient to have this in three-dimensional integral form which is regularized, i.e. which has no derivative singularity, plus a singular term. Differentiate (5.2) (where $j=1,2,3$ and $\partial_3 = \partial/\partial_z$)

$$\partial_j G_R^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') = \frac{\pi i}{(2\pi)^3} \iint \left[ik_{jt} + i\delta_{j3} K_t \, \mathrm{sgn}(z-z')\right] \cdot$$

$$\cdot \exp\left[i\underset{\sim}{k}_t \cdot (\underset{\sim}{x}_t - \underset{\sim}{x}_t') + iK_+ |z-z'|\right] \frac{d\underset{\sim}{k}_t}{K_+} \quad . \tag{5.11}$$

Now regularize the singularity in the z-derivative as follows:

$$\partial_j G_R (\underset{\sim}{x}-\underset{\sim}{x}') = \frac{\pi i}{(2\pi)^3} \iint ik_{jt} \exp\left[i\underset{\sim}{k}_t \cdot (\underset{\sim}{x}_t - \underset{\sim}{x}_t') + iK_+ |z-z'|\right] \frac{d\underset{\sim}{k}_t}{K_+}$$

$$+ i\delta_{j3} \, \mathrm{sgn}(z-z') \cdot$$

$$\cdot \left[\frac{\pi i}{(2\pi)^3} \iint d\underset{\sim}{k}_t \exp\left[i\underset{\sim}{k}_t \cdot (\underset{\sim}{x}_t - \underset{\sim}{x}_t')\right] \left[\exp(iK_+ |z-z'| - 1\right] \right.$$

$$\left. + \frac{\pi i}{(2\pi)^3} (2\pi)^2 \delta(\underset{\sim}{x}_t - \underset{\sim}{x}_t') \right] \quad . \tag{5.12}$$

In the singular term we subtracted and added the term 1. This brought out the δ function explicitly. We could have subtracted any function of $z - z'$ which has the limit 1 as $z - z' \to 0$, as for example $\cos[k_0(z - z')]$. Regularization is not unique, and this subtraction, or any appropriate subtraction, is a regularization in the sense that the resulting integral is not singular.

We next want to reintroduce the k_z-integration in (5.12). Recall that we eliminated this integration by evaluating it to derive the Weyl representation. In the first integral it is simple to recover the k_z integral. We use the result that ($k_0 \to k_0 + i\varepsilon$, $K \to K + i\varepsilon$)

$$e^{iK|z - z'|} = \frac{K}{\pi i} \int_{-\infty}^{\infty} \frac{\exp\left[ik_z(z - z')\right]}{k_z^2 - K_+^2} dk_z \quad . \tag{5.13}$$

In the second integral we use

$$\mathrm{sgn}(z - z')\left[\exp(iK|z-z'|)-1\right] = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\exp\left[ik_z(z-z')\right]}{k_z^2 - K_+^2} \, P\frac{K^2}{k_z} dk_z \quad , \tag{5.14}$$

which can also be derived using residue calculus methods. The integrand in (5.14) has three poles, the one at $k_z = 0$ evaluated using the principal value (P), and the other two using the shifts $k_0 + i\varepsilon$ or $K + i\varepsilon$. The result in (5.12) is, noting that $k_z^2 - K^2 = k^2 - k_0^2$

$$\begin{aligned}
\partial_j G_R^{(3)}(\underset{\sim}{x}, \underset{\sim}{x}') = {} & \frac{1}{(2\pi)^3} \iiint ik_{jt} \frac{\exp\left[i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')\right]}{k^2 - k_0^2} d\underset{\sim}{k} \\
& + i\delta_{j3} \iiint \frac{\exp\left[i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')\right]}{k^2 - k_0^2} \, P\frac{K^2}{k_z} d\underset{\sim}{k} \\
& - \frac{1}{2}\,\mathrm{sgn}(z-z')\,\delta(\underset{\sim}{x}_t - \underset{\sim}{x}_t')\,\delta_{j3} \quad .
\end{aligned} \tag{5.15}$$

Noting that the Fourier transform of $G_R^{(3)}$ is

$$\tilde{G}_R^{(3)}(k) = \left[k^2 - k_{0+}^2\right]^{-1} \quad , \tag{5.16}$$

with $k_0 \to k_0 + i\varepsilon$, we can write (5.15) as

$$\partial_j \; G_R^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') = \frac{1}{2}\;\frac{1}{(2\pi)^3}\iiint \exp\left[i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')\right]\;\tilde{G}_R^{(3)}(k)\;P_j(\underset{\sim}{k})d\underset{\sim}{k}$$

$$-\;\frac{1}{2}\;\delta_{j3}\mathrm{sgn}(z-z')\;\delta(\underset{\sim}{x}_t-\underset{\sim}{x}_t')\quad, \tag{5.17}$$

where

$$P_j(\underset{\sim}{k}) = 2i\left[k_{jt} + \delta_{j3}\;P\left[\frac{K^2}{k_z}\right]\right]\quad. \tag{5.18}$$

The integral term is not singular. (In fact note that if we set z-z' = 0 in the integral we get that the j = 3 term vanishes since the resulting integrand is an odd function of k_z.) It is a Cauchy principal value integral. The subtraction of the term 1 has led to this. Subtraction of another term will lead to an alternate principal value integral. The full discontinuity is proportional to the j = 3 term, i.e.

$$\left[\partial_j G_R^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}')\right]_+ - \left[\partial_j G_R^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}')\right]_- = -\delta_{j3}\delta(\underset{\sim}{x}_t-\underset{\sim}{x}_t')\quad, \tag{5.19}$$

where the + and - signs refer to the limits as z-z' approaches zero from positive or negative values respectively. An alternative way of writing (5.17) is

$$\partial_j G_R^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') = \frac{1}{2}\;R_j(\underset{\sim}{x}-\underset{\sim}{x}') - \frac{1}{2}\delta_{j3}\;\mathrm{sgn}(z-z')\delta(\underset{\sim}{x}_t-\underset{\sim}{x}_t')\quad, \tag{5.20}$$

where

$$R_j(\underset{\sim}{x}-\underset{\sim}{x}') = \frac{1}{(2\pi)^3}\iiint \exp\left[i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')\right]\tilde{G}_R^{(3)}(k)\;P_j(\underset{\sim}{k})d\underset{\sim}{k}\quad, \tag{5.21}$$

is the regular part of the derivative term. We can thus set both vectors $\underset{\sim}{x}$

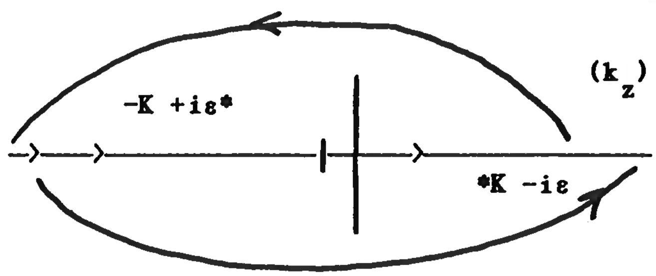
and $\underset{\sim}{x}'$ onto the surface in R_j and have a well defined limit.

1.5.2 ANALYTIC PROPERTIES OF $G_A^{(3)}$

The full three-dimensional Fourier representation for $G^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}')$ is given by

$$G^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') = \frac{1}{(2\pi)^3}\iint \exp\left[i\underset{\sim}{k}_t(\underset{\sim}{x}_t-\underset{\sim}{x}_t')\right]d\underset{\sim}{k}_t \int \frac{\exp\left[ik_z(z-z')\right]}{(k_z-K_+)(k_z+K_+)}\,dk_z \, , \qquad (5.22)$$

where $K = \left[k_0^2 - k_t^2\right]^{1/2}$ and $k_t^2 = k_x^2 + k_y^2$. The retarded Green's function was computed by shifting these poles using $k_0 \to k_{0+}$ or $K \to K_+$. The advanced Green's function is computed using the shift $k_0 \to k_0 - i\epsilon = k_{0-}$ or $K \to K - i\epsilon = K_-$. The singularity structure in the complex k_z-plane is thus



z-z'>0

z-z'<0

Fig. 1.3

We close the contour in the upper half plane for z-z'>0 and in the lower half plane for z-z'<0. The result is

$$\int \frac{\exp\left[ik_z(z-z')\right]}{(k_z-K)(k_z+K)}\,dk = \frac{-\pi i}{K}\exp\left[-iK|z-z'|\right] \, , \qquad (5.23)$$

so that the Weyl representation for the advanced Green's function

$$G_A^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') = \frac{\exp(-ik_0 r)}{4\pi r} \; ; \; r = |\underset{\sim}{x}-\underset{\sim}{x}'| \quad , \tag{5.24}$$

is given by

$$G_A^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') = \frac{-\pi i}{(2\pi)^3} \iint \exp\left[i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t - \underset{\sim}{x}_t') - iK_-|z-z'|\right] \frac{d\underset{\sim}{k}_t}{K_-} \quad , \tag{5.25}$$

where we distinguish the square root term using $K_- = K - i\varepsilon$. Its properties are

1. $G_A^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}')$ is continuous as $z-z' \rightarrow 0$. The limits from both directions are the same. Note that from the functional form the limit is

$$\frac{e^{-ik_0\rho}}{4\pi\rho} = \frac{-\pi i}{(2\pi)^3} \iint \exp\left[i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t - \underset{\sim}{x}_t')\right] \frac{d\underset{\sim}{k}_t}{K_-} \quad , \tag{5.26}$$

   with $\rho = |\underset{\sim}{x}_t - \underset{\sim}{x}_t'|$. The square root distinguishes the contribution.

2. The first derivative in depth is discontinuous. Differentiating we get from (5.25)

$$\partial_z G_A^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') = -\pi \frac{\mathrm{sgn}(z-z')}{(2\pi)^3} \iint \exp\left[i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t-\underset{\sim}{x}_t') + iK_-|z-z'|\right] d\underset{\sim}{k}_t \quad , \tag{5.27}$$

   which in the limit as $z-z' \rightarrow 0$ from the two directions is

$$\lim_{z-z\rightarrow 0} \partial_z G_A^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = -\frac{1}{2}\,\delta(\underset{\sim}{x}_t - \underset{\sim}{x}_t') \begin{bmatrix} 1 & z-z' \rightarrow 0^+ \\ -1 & z-z' \rightarrow 0^- \end{bmatrix} \quad , \tag{5.28}$$

   which is the same discontinuity as the derivative of the retarded Green's function in (5.5).

3. The transverse derivatives are continuous. From (5.23) we have that

$$\lim_{z-z' \to 0} \partial_{jt} G_A^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') = \frac{\pi}{(2\pi)^3} \iint \exp\left[i\underset{\sim}{k}_t \cdot (\underset{\sim}{x}_t - \underset{\sim}{x}_t')\right] \frac{k_{jt}}{K_-} \, d\underset{\sim}{k}_t \quad , \tag{5.29}$$

which from the functional form (5.24) equals

$$\partial_{jt}\left[\exp(-ik_0\rho)/4\pi\rho\right] \quad . \tag{5.30}$$

Hence we can write the full vector derivative as

$$\partial_j G_A^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') = \frac{-\pi i}{(2\pi)^3} \iint \left[ik_{jt} - i\delta_{j3} K_- \, \mathrm{sgn}(z-z')\right] \exp\left[i\underset{\sim}{k}_t \cdot [\underset{\sim}{x}_t - \underset{\sim}{x}_t'] - iK_- |z-z'|\right] \frac{d\underset{\sim}{k}_t}{K_-} \quad . \tag{5.31}$$

Now regularize in the z- derivative term. Rewrite (5.31) as

$$\begin{aligned}\partial_j G_A^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') = {} & \frac{-\pi i}{(2\pi)^3} \iint ik_{jt} \exp\left[i\underset{\sim}{k}_t \cdot (\underset{\sim}{x}_t - \underset{\sim}{x}_t') - iK_- |z-z'|\right] \frac{d\underset{\sim}{k}_t}{K_-} \\ & -i\delta_{j3} \, \mathrm{sgn}(z-z') \cdot \\ & \cdot \left[\frac{-\pi i}{(2\pi)^3} \iint \exp\left[i\underset{\sim}{k}_t \cdot (\underset{\sim}{x}_t - \underset{\sim}{x}_t'\right] \left[\exp(-iK_- |z-z'|) - 1\right] d\underset{\sim}{k}_t \right. \\ & \left. - \frac{\pi i}{(2\pi)^3} (2\pi)^2 \, \delta(\underset{\sim}{x}_t - \underset{\sim}{x}_t') \right] \quad . \end{aligned} \tag{5.32}$$

Reintroduce the k_z- integration (here $k_0 \to k_0 - i\varepsilon$ and $K \to K - i\varepsilon = K_-$) using

$$\exp\left[-iK|z-z'|\right] = \frac{-K}{\pi i}\int_{-\infty}^{\infty} dk_z \, \frac{\exp\left[ik_z(z-z')\right]}{k_z^2 - K_-^2} , \tag{5.33}$$

and

$$\mathrm{sgn}(z-z')\left[\exp\left[-iK|z-z'|\right]-1\right] = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\exp\left[ik_z(z-z')\right]}{k_z^2 - K_-^2} \, P\left[\frac{K^2}{k_z}\right] dk_z \quad . \tag{5.34}$$

Note the minus sign in front of (5.33) and the plus sign in front of (5.34). The principal value term in (5.34) is an odd function. Using the Fourier transform of $G_A^{(3)}$ which is

$$\tilde{G}_A^{(3)}(k) = \left[k^2 - k_{0-}^2\right]^{-1} , \tag{5.35}$$

where $k_0 \rightarrow k_0 - i\varepsilon$ the result using (5.33) and (5.34) in (5.32) is

$$\partial_j G_A^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') = \frac{1}{2}\frac{1}{(2\pi)^3}\iiint \exp\left[i\underset{\sim}{k}\cdot(x-x')\right]\tilde{G}_A^{(3)}(k) \; P_j(\underset{\sim}{k})d\underset{\sim}{k}$$

$$-\frac{1}{2}\,\mathrm{sgn}(z-z')\delta_{j3} \;\; \delta(\underset{\sim}{x}_t - \underset{\sim}{x}_t') \quad . \tag{5.36}$$

Note that the result is similar to that for the retarded Green's function, (5.17), except that here the Fourier transform of the advanced Green's function is under the integral. $P_j(\underset{\sim}{k})$ is defined by (5.18).

1.5.3 REGULARIZATION OF $\partial_j G_P^{(3)}$

We have that

$$G_P^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') = \frac{1}{2}\left[G_R^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') + G_A^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}')\right] , \tag{5.37}$$

so that combining (5.17) and (5.36) we get

$$\partial_j G_P^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') = \frac{1}{2}\frac{1}{[2\pi]^3}\iiint \exp\left[i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')\right] \tilde{G}_P^{(3)}(k)P_j(\underset{\sim}{k})d\underset{\sim}{k}$$
$$- \frac{1}{2}\,\mathrm{sgn}(z-z')\ \delta_{j3}\delta(\underset{\sim}{x}_t-\underset{\sim}{x}_t') , \tag{5.38}$$

where

$$\tilde{G}_P^{(3)}(k) = P\left[k^2 - k_o^2\right]^{-1} . \tag{5.39}$$

Hence the <u>principal value Green's function has the same discontinuity as the retarded and advanced Green's functions</u>.

1.5.4 REGULARIZATION OF $\partial_m\partial_j\ G_R^{(3)}$

We begin with the Weyl representation for the retarded Green's function in three dimensions

$$G_R^{(3)}(\underset{\sim}{x}) = \frac{\pi i}{(2\pi)^3}\iint \exp\left[i\underset{\sim}{k}_t\cdot\ \underset{\sim}{x}_t + iK_+|z|\right] \frac{d\underset{\sim}{k}_t}{K_+} . \tag{5.40}$$

Differentiate this representation to get

$$\partial_j G_R^{(3)}(\underset{\sim}{x}) = \frac{\pi i^2}{(2\pi)^3} \iint \left[k_{jt} + \delta_{j3}\ \mathrm{sgn}(z)K\right] \exp\left[i\underset{\sim}{k}_t \cdot \underset{\sim}{x}_t + iK_+|z|\right] \frac{d\underset{\sim}{k}_t}{K_+} \ , \tag{5.41}$$

and differentiate a second time to yield

$$\begin{aligned}\partial_m \partial_j G_R^{(3)}(\underset{\sim}{x}) = \frac{\pi i}{(2\pi)^3} \iint & \Big[2iK_+\ \delta(z)\ \delta_{j3}\ \delta_{m3} + \\ & +i^2\left[k_{jt} + \delta_{j3}K_+\ \mathrm{sgn}(z)\right]\left[k_{mt} + \delta_{m3}K_+ \mathrm{sgn}(z)\right]\Big] \ . \\ & \cdot \exp\left[i\underset{\sim}{k}_t \cdot \underset{\sim}{x}_t + iK_+|z|\right] \frac{d\underset{\sim}{k}_t}{K_+} \ ,\end{aligned} \tag{5.42}$$

which can be written as four terms

$$\begin{aligned}\partial_m \partial_j G_R^{(3)}(\underset{\sim}{x}) = \frac{\pi i}{(2\pi)^3} \int\int & \exp\left[i\underset{\sim}{k}_t \cdot \underset{\sim}{x}_t + iK_+|z|\right] \cdot \\ & \left[\frac{i^2 k_{jt} k_{mt}}{K_+} + i^2\ \mathrm{sgn}(z)\left[k_{mt}\delta_{j3} + k_{jt}\delta_{m3}\right]\right. \\ & \left. + i^2 K_+ \delta_{j3}\delta_{m3} + 2i\ \delta_{j3}\delta_{m3}\delta(z)\right] d\underset{\sim}{k}_t \ .\end{aligned} \tag{5.43}$$

The first term is not singular, and using the relation

$$\begin{aligned}\frac{\exp(iK_+|z|)}{K_+} &= \frac{1}{\pi i} \int \frac{\exp(izk_z)}{k^2 - k_{o+}^2}\ dk_z \\ &= \frac{1}{\pi i} \int \exp(izk_z)\ \tilde{G}_R^{(3)}(k)\ dk_z \ ,\end{aligned} \tag{5.44}$$

we can write it as a three dimensional integral

$$I_1 = \frac{1}{(2\pi)^3} \iiint i^2 k_{jt} k_{mt} \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x})\ \tilde{G}_R^{(3)}(k)\ d\underset{\sim}{k} \quad . \tag{5.45}$$

The second term containing the sgn(z) function has a possible delta function singularity in the limit as $z \to 0$. We regularize it as follows. First rewrite it as

$$\begin{aligned} I_2 = \frac{\pi i}{(2\pi)^3} \iint i^2 \operatorname{sgn}(z) \left[k_{mt}\delta_{j3} + k_{jt}\delta_{m3}\right] \exp(i\underset{\sim}{k}_t\cdot\underset{\sim}{x}_t) \quad \cdot \\ \cdot \left[\exp(iK_+|z|) - 1 + 1\right] d\underset{\sim}{k}_t \quad . \end{aligned} \tag{5.46}$$

The term involving the +1 in the bracket can be written as the derivatives of a two dimensional delta function. The remaining term can be written as a three dimensional integral using the relation (5.14) as

$$\operatorname{sgn}(z) \left[\exp(iK_+|z|)_{-1}\right] = \frac{1}{\pi i} \int \exp(ik_z z) \tilde{G}_R^{(3)}(k)\ P\left[\frac{K_+^2}{k_z}\right] dk_z \quad . \tag{5.47}$$

The result is

$$\begin{aligned} I_2 = -\frac{1}{2} \operatorname{sgn}(z) \left[\delta_{j3}\partial_{mt} + \delta_{m3}\partial_{jt}\right]\delta(\underset{\sim}{x}_t) \\ + \frac{1}{(2\pi)^3} \iiint i^2 \left[k_{mt}\delta_{j3} + k_{j3}\delta_{m3}\right] P\left[\frac{K_+^2}{k_z}\right] \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x}) G_R^{(3)}(k) d\underset{\sim}{k} \quad . \end{aligned} \tag{5.48}$$

The third term in (5.43) is not singular. Using the relation

$$K_+ \exp(iK_+|z|) = \frac{K_+^2}{\pi i} \int \exp(ik_z z)\ \tilde{G}_R^{(3)}(k) dk_z \quad , \tag{5.49}$$

we can write it as a three dimensional integral

$$I_3 = \frac{1}{(2\pi)^3} \iiint i^{-2} K_+^2 \tilde{G}_R^{(3)}(k) \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x}) d\underset{\sim}{k} \quad \delta_{j3}\,\delta_{m3} \quad , \tag{5.50}$$

which can be written using the identity

$$\begin{aligned} K_+^2 \tilde{G}_R^{(3)}(k) &= (k_{0+}^2 - k_t^2) \Big/ (k^2 - k_{0+}^2) \\ &= \left[k_{0+}^2 - (k_t^2 + k_z^2) + k_z^2\right] \Big/ (k^2 - k_{0+}^2) \\ &= -1 + k_z^2 \tilde{G}_R^{(3)}(k) \quad , \end{aligned} \tag{5.51}$$

to yield

$$I_3 = \frac{1}{(2\pi)^3} \iiint d\underset{\sim}{k}\; i^2 k_z^2 \tilde{G}_R^{(3)}(k) \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x}) \delta_{j3}\delta_{m3} + \delta_{j3}\delta_{m3}\delta(\underset{\sim}{x}) \quad . \tag{5.52}$$

The fourth term in (5.43) is a delta function. It is

$$\begin{aligned} I_4 &= \frac{\pi i}{(2\pi)^3} \iiint 2i\delta_{j3}\delta_{m3}\delta(z) \exp\left[i\underset{\sim}{k}_t \cdot \underset{\sim}{x}_t + iK_+|z|\right] d\underset{\sim}{k}_t \\ &= -\delta_{j3}\delta_{m3}\delta(\underset{\sim}{x}) \quad . \end{aligned} \tag{5.53}$$

The result (5.43) is given by the sum of I_1 thru I_4 from (5.45), (5.48), (5.52) and (5.53). The result can be written

$$\partial_m \partial_j G_R^{(3)}(\underset{\sim}{x}) = -\frac{1}{2} R_{mj}(\underset{\sim}{x}) \; - \frac{1}{2}\,\mathrm{sgn}(z)\left[\delta_{j3}\partial_{mt} + \delta_{m3}\delta_{jt}\right]\delta(\underset{\sim}{x}_t) \quad , \tag{5.54}$$

where $R_{mj}(\underset{\sim}{x})$ is the regular part of this mixed second derivative given by

$$R_{mj}(\underset{\sim}{x}) = \frac{1}{(2\pi)^3} \iiint \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x}) \; G_R^{(3)}(k) P_{mj}(\underset{\sim}{k}) \; d\underset{\sim}{k} \quad , \tag{5.55}$$

and where

$$\frac{1}{2} P_{mj}(\underset{\sim}{k}) = k_{mt}k_{jt} + \left[k_{mt}\delta_{j3} + k_{jt}\delta_{m3}\right] P\left[\frac{K^2}{k_z}\right] + k_z^2\delta_{m3}\delta_{j3} \quad . \tag{5.56}$$

Note that from this representation it is easy to show that

$$(\partial_1\partial_1 + \partial_2\partial_2 + \partial_3\partial_3) G_R^{(3)}(\underset{\sim}{x}) = -\frac{1}{(2\pi)^3} \iiint \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x})\tilde{G}_R^{(3)}(k)k^2 d\underset{\sim}{k} \ , \tag{5.57}$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$. From the identity

$$k^2/(k^2-k_0^2) = 1 + k_0^2\tilde{G}_R^{(3)}(k) \quad ,$$

we see that

$$\nabla^2 G_R^{(3)}(\underset{\sim}{x}) = -\delta(\underset{\sim}{x}) - k_0^2 \ \tilde{G}_R^{(3)}(k) \quad , \tag{5.58}$$

which serves as a check on our results.

Some properties of this representation are obvious. The first is the symmetry of the derivative operation

$$\partial_m\partial_j \ G_R^{(3)}(\underset{\sim}{x}) = \partial_j\partial_m G_R^{(3)}(\underset{\sim}{x}) \quad . \tag{5.59}$$

Since $G_R^{(3)}$ is a homogeneous function we can exchange derivatives with respect to field and source coordinates up to a minus sign so that

$$\partial_m\partial_j \ G_R^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') = \partial'_m\partial'_j \ G_R^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}') \quad . \tag{5.60}$$

From (5.54) it is obvious that P is symmetric

$$P_{mj}(\underset{\sim}{k}) = P_{jm}(\underset{\sim}{k}) \quad . \tag{5.61}$$

Since m and j run from 1 to 3 we thus have at most six independent components. Evaluating terms we get

$$P_{11} = 2k_x^2 \;,\quad P_{22} = 2k_y^2 \;,\quad P_{33} = 2k_z^2$$

$$P_{12} = 2k_x k_y$$

$$P_{13} = 2k_x P\left[\frac{K^2}{k_z}\right] \;,\quad P_{23} = 2k_y\, P\left[\frac{K^2}{k_z}\right] \;. \tag{5.62}$$

Other constraints are possible. Note that

$$P_{11}P_{22} = P_{12}^2 \tag{5.63}$$

and

$$[P_{13}/P_{23}]^2 = P_{11}/P_{22} \;. \tag{5.64}$$

Finally, note that the jump discontinuity in the representation (5.54) occurs only in the off-diagonal components. The discontinuity across the surface is

$$\left[\partial_m\partial_j\, G_R^{(3)}\right]_+ - \left[\partial_m\partial_j\, G_R^{(3)}\right]_- = -\left[\delta_{j3}\partial_{mt} + \delta_{m3}\partial_{jt}\right]\delta(\underset{\sim}{x}_t) \;. \tag{5.65}$$

Note also that if we spatially integrate these dipole terms by themselves the result is zero. For example

$$\int \partial_{jt}\, \delta(\underset{\sim}{x}_t) d\underset{\sim}{x}_t = 0_{jt} \;, \tag{5.66}$$

but with an additional term in the integral we get for example

$$\iint f(\underset{\sim}{x}_t)\, \partial_{jt}\delta(\underset{\sim}{x}_t)\, d\underset{\sim}{x}_t = -\partial_{jt}\, f(\underset{\sim}{x}_t)\Big|_{\underset{\sim}{x}_t = \underset{\sim}{0}_t} \;. \tag{5.67}$$

Note that if we have the difference of arguments we get

$$\partial'_m \partial'_j G_R^{(3)}(\underset{\sim}{x}' - \underset{\sim}{x}) = -\frac{1}{2} R_{mj}(\underset{\sim}{x}' - \underset{\sim}{x})$$
$$- \frac{1}{2} \, \mathrm{sgn}(z'-z)(\delta_{j3}\partial'_{mt} + \delta_{m3}\partial'_{jt})\delta(\underset{\sim}{x}'_t - \underset{\sim}{x}_t) \quad . \tag{5.68}$$

To differentiate on the second argument note that

$$\partial'_m \partial'_j G_R^{(3)}(\underset{\sim}{x}' - \underset{\sim}{x}) = \partial_m \partial_j G_R^{(3)}(\underset{\sim}{x}' - \underset{\sim}{x}) \quad , \tag{5.69}$$

by the homogeneity of the Green's function. Substituting this in (5.68) and writing the partial derivatives in terms of the unprimed coordinate on the rhs we get a sign change in the latter term. The result is

$$\partial_m \partial_j G_R^{(3)}(\underset{\sim}{x}' - \underset{\sim}{x}) = -\frac{1}{2} R_{mj}(\underset{\sim}{x} - \underset{\sim}{x})$$
$$+ \frac{1}{2} \, \mathrm{sgn}(z' - z)(\delta_{j3}\partial_{mt} + \delta_{m3}\partial_{jt})\delta(\underset{\sim}{x}'_t - \underset{\sim}{x}_t) \quad . \tag{5.70}$$

1.6 ONE-DIMENSIONAL PROBLEMS

1.6.1 GREEN'S FUNCTION IN 1-DIMENSION

We begin with a general second order linear differential equation with a delta function source term

$$\frac{d^2\phi}{dz^2} + p(z)\,\frac{d\phi}{dz} + q(z)\phi = -\delta(z-z') \quad . \tag{6.1}$$

The function ϕ is thus the Green's function for this one-dimensional problem. The source point z' is singled out and we have different solutions in the regions z>z' and z<z'. Thus the source point can be interpreted as introducing another boundary layer into the problem. We thus have two solutions and must say how they match at the layer interface. We assume that

(a) ϕ is continuous across the layer

(b) $d\phi/dz$ has a discontinuity across the layer.

We use the continuity property to find the discontinuity as follows. First rewrite (6.1) in the form

$$\frac{d^2\phi}{dz^2} + \frac{d}{dz}\,[p(z)\phi] + [q(z) - p'(z)]\phi(z) = 0 \quad . \tag{6.2}$$

Next integrate (6.2) across the layer from $z'-\varepsilon$ to $z'+\varepsilon$ where ε is small and shrinks to zero. For the second derivative term in (6.2) we get

$$\lim_{\varepsilon \to 0} \int_{z'-\varepsilon}^{z'+\varepsilon} \frac{d^2\phi}{d^2z}\,dz = \frac{d\phi}{dz}\Bigg|_{z'-\varepsilon}^{z'+\varepsilon} = \frac{d\phi^+}{dz} - \frac{d\phi^-}{dz} \quad , \tag{6.3}$$

where we have defined

$$\lim_{\varepsilon \to 0} \frac{d\phi}{dz}(z'\pm\varepsilon) = \frac{d\phi^{\pm}}{dz} \quad . \tag{6.4}$$

The second term on the lhs of (6.2) becomes

$$\int_{z'-\varepsilon}^{z'+\varepsilon} \frac{d}{dz}\left[p(z)\phi(z)\right]dz = p(z)\phi(z)\Big|_{z'-\varepsilon}^{z'+\varepsilon} \to 0 \quad ,$$

which vanishes as $\varepsilon \to 0$ since ϕ is continuous and we assume p is also continuous. It is not necessary to assume the latter in which case our result is

$$(p^{+} - p^{-})\ \phi(z')$$

where

$$p^{\pm} = \lim_{\varepsilon \to 0} \quad p(z' \pm \varepsilon) \quad .$$

The third term on the lhs of (6.2) is

$$\int_{z'-\varepsilon}^{z'+\varepsilon} [q(z) - p'(z)]\ \phi(z)dz \quad ,$$

which vanishes in the limit as $\varepsilon \to 0$ unless q(z) or p'(z) are discontinuous. If q(z) is discontinuous it becomes

$$(q^{+} - q^{-})\ \phi(z')$$

where

$$q^{\pm} = \lim_{\varepsilon \to 0} \quad q(z' \pm \varepsilon) \quad ,$$

and if p is discontinuous so that

$$p(z) = p_0(z)\ \theta(z-z') + p_1(z)\ \theta(z'-z) \quad ,$$

where θ is the step function

$$\theta(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases} ,$$

we have that

$$p'(z) = p_0'(z)\ \theta(z-z') + p_1'(z)\ \theta(z-z')$$

$$+ p_0(z)\ \delta(z-z') - p_1(z)\ \delta(z-z') \quad ,$$

so that we get for the result

$$\int_{z'-\varepsilon}^{z'+\varepsilon} p'(z)\phi(z)dz$$

$$= p(z)\phi(z)\Big|_{z'-\varepsilon}^{z'+\varepsilon} - \int_{z'-\varepsilon}^{z'+\varepsilon} p(z)\ \frac{d\phi}{dz}\ dz \quad .$$

$$\rightarrow \left[p_0(z') - p_1(z')\right]\phi(z') - \left[p_0(z')\ \frac{d\phi^+}{dz} - p_1(z')\ \frac{d\phi^-}{dz}\right]\ \varepsilon \quad .$$

The second term vanishes as $\varepsilon \rightarrow 0$ provided neither of $d\phi^{\pm}/dz$ is singular.

Essentially all of these discontinuous properties merely complicate our algebra and we drop them. That is, we assume p, p', and q are continuous at the interface. The result is, from (6.3)

$$\frac{d\phi^+}{dz} - \frac{d\phi^-}{dz} = -1 \qquad (z = z') \quad , \tag{6.5}$$

where the -1 results from integrating the delta function. The continuity of ϕ is expressed as

$$\phi^+ - \phi^- = 0 \qquad (z = z') \quad , \tag{6.6}$$

and it is these latter two equations we use in the analysis. Now we must define the field boundary value problem. We do two examples.

Eg. 1. INFINITE SPACE EXAMPLE

Here the boundary layer is the only finite boundary. Above that boundary layer we have solutions of the homogeneous version of (6.1) (no delta function term). For wave like solutions asymptotically we choose the solution which satisfies an outgoing radiation condition. To insure these wave-like solutions p(z) and q(z) are required to have certain asymptotic properties. The simplest are that as $z \to \infty$

$$p(z) \to 0 \quad \text{and} \quad q(z) \to \text{constant} > 0 \quad .$$

Strictly speaking we also have to require that p(z) and q(z) are monotonic functions. If there are any kinks in these profiles, waves can be trapped, and we must effectively introduce further layers into the problem. For simplicity we assume these properties are satisfied.

Thus, in the upper layer (U) where $z > z'$ the solution of the homogeneous version of (6.1) satisfying the outgoing radiation condition can be written as

$$\phi_U(z) = A\, u_+(z) \qquad z > z' \quad , \tag{6.7}$$

where A is an unknown constant. Similarly in the lower (L) region ($z < z'$) the solution satisfying the outgoing radiation condition is

$$\phi_L(z) = B\, u_-(z) \qquad z < z' \quad , \tag{6.8}$$

where B is an unknown constant. Our conditions (6.5) and (6.6) are then satisfied by (6.7) and (6.8) provided that at $z = z'$

$$A \frac{du_+}{dz} - B \frac{du_-}{dz} = -1 \quad , \tag{6.9}$$

and

$$Au_+ - Bu_- = 0 \quad . \tag{6.10}$$

These are coupled equations for A and B whose solution is

$$A = -u_-(z')/W \quad , \quad B = -u_+(z')/W \quad , \tag{6.11}$$

where W is the Wronskian

$$W = u_+'u_- - u_+u_-' \quad . \tag{6.12}$$

The full solution can thus be written as

$$\phi(z) = \begin{cases} - u_+(z)u_-(z')/W & z>z' \\ - u_+(z')u_-(z)/W & z<z' \end{cases} \quad . \tag{6.13}$$

To specify the solution further we must know the eigenfunctions. As a simple example choose $p = 0$ and $q = k_0^2$ in (6.1). Then the eigenfunctions of

$$\frac{d^2\phi}{dz^2} + k_0^2 \phi = 0 \quad , \tag{6.14}$$

are either $\exp(\pm ik_0 z)$ or $[\cos k_0 z, \sin k_0 z]$. We choose the former set since they correspond to outgoing waves. We thus have that

$$u_+(z) = e^{ik_0 z} \; ; \; u_-(z) = e^{-ik_0 z} \quad ; \quad W = 2ik_0 \quad , \tag{6.15}$$

so that

$$\phi(z) = \begin{cases} -\dfrac{e^{ik_0(z-z')}}{2ik_0} & z>z' \\[2ex] -\dfrac{e^{-ik_0(z-z')}}{2ik_0} & z<z' \quad , \end{cases} \tag{6.16}$$

or rewriting

$$\phi(z) = -\frac{1}{2ik_0}\, e^{ik_0|z-z'|} \quad , \tag{6.17}$$

which is our standard retarded one-dimensional Green's function. Note that we got the retarded Green's function because we chose outgoing radiation solutions.

Eg. 2. BOUNDARY EXAMPLE

In this example we introduce upper and lower boundaries at a finite distance from the source plane. Geometrically we have that

———————— z_U Upper Region 1: $z' < z < z_U$

———————— z'

———————— z_L Lower Region 2: $z_L < z < z'$

Now we must choose our solutions as

$$\phi_U(z) = A_+\, u_+^{(1)}(z) + A_-\, u_-^{(1)}(z) \quad , \tag{6.18}$$

including both linearly independent solutions $u_\pm^{(1)}$ in region 1, and in the lower region 2

$$\phi_L(z) = B_+\, u_+^{(2)}(z) + B_-\, u_-^{(2)}(z) \quad , \tag{6.19}$$

also including both linearly independent solutions in this region $u_\pm^{(2)}$. We have the same continuity and jump conditions at the interface as before.

Equations (6.5) and (6.6) become at $z=z'$

$$\frac{d\phi_U}{dz} - \frac{d\phi_L}{dz} = -1 \quad , \tag{6.20}$$

and

$$\phi_U - \phi_L = 0 \quad . \tag{6.21}$$

These are two conditions on the four constants $A_\pm$ and $B_\pm$. In addition there are boundary conditions at z_U and z_L. Assume for simplicity that

$$\phi_U(z_U) = 0 \quad , \tag{6.22}$$

$$\frac{d\phi_L}{dz}(z_L) = 0 \quad . \tag{6.23}$$

Again our formalism can accommodate much more complicated impedance type boundary conditions at these surfaces. We again take our previous example where $p = 0$ and $q = k_0^2$ in (6.1). Because of the form of the boundary conditions it is convenient (but not necessary) to choose our eigenfunctions as

$$u_\pm^{(1)}(z) = \begin{bmatrix} \sin k_0(z-z_U) \\ \cos k_0(z-z_U) \end{bmatrix} \quad , \tag{6.24}$$

and

$$u_\pm^{(2)}(z) = \begin{bmatrix} \sin k_0(z-z_L) \\ \cos k_0(z-z_L) \end{bmatrix} \quad . \tag{6.25}$$

Using (6.18) and (6.22) yields $A_-=0$. Using (6.19) and (6.23) yields $B_+=0$. We thus have satisfied the boundary conditions with the fields

$$\phi_U(z) = A_+ \sin\left[k_o(z-z_U)\right] \quad , \tag{6.26}$$

and

$$\phi_L(z) = B_- \cos\left[k_o(z-z_L)\right] \quad . \tag{6.27}$$

We now must satisfy conditions (6.20) and (6.21) which become

$$A_+k_o \cos\left[k_o(z'-z_U)\right] + B_-k_o \sin\left[k_o(z'-z_L)\right] = -1 \tag{6.28}$$

$$A_+ \sin\left[k_o(z'-z_U)\right] - B_- \cos\left[k_o(z'-z_L)\right] = 0 \quad , \tag{6.29}$$

whose solution is

$$A_+ = \frac{u_-^{(z)}(z')}{W} = \cos\left[k_o(z'-z_L)\right] \Big/ W \quad , \tag{6.30}$$

$$B_- = \frac{u_+^{(1)}(z')}{W} = \sin\left[k_o(z'-z_u)\right] \Big/ W \quad , \tag{6.31}$$

where

$$W = -k_o \cos\left[k_o(z_L-z_u)\right] \quad , \tag{6.32}$$

so that

$$\phi(z) = \begin{cases} -\dfrac{\sin\left[k_o(z-z_U)\right] \cos\left[k_o(z'-z_L)\right]}{k_o \cos\left[k_o(z_L-z_U)\right]} & z>z' \\[2ex] -\dfrac{\sin\left[k_o(z'-z_U)\right] \cos\left[k_o(z-z_L)\right]}{k_o \cos\left[k_o(z_L-z_U)\right]} & z<z' \end{cases} \quad ,$$

or

$$\phi(z) = \begin{cases} \dfrac{\sin\left[k_o(z_U-z)\right] \cos\left[k_o(z'-z_L)\right]}{k_o \cos\left[k_o(z_L-z_U)\right]} & z>z' \\[2ex] \dfrac{\sin\left[k_o(z_U-z')\right] \cos\left[k_o(z-z_L)\right]}{k_o \cos\left[k_o(z_L-z_U)\right]} & z<z' \end{cases} \quad . \tag{6.33}$$

1.6.2 SOLVABLE PROFILES - INHOMOGENEOUS MEDIA

The examples in the previous development in this section were for homogeneous media. Here we develop a general method to find the eigenfunctions for various one-dimensionally inhomogeneous media. In a sense we do an inverse problem, first choosing the eigenfunctions and then finding the medium index of refraction.

We begin with a general linear second order differential equation

$$\frac{d^2h}{dx^2} + p(x)\,\frac{dh}{dx} + q(x)h(x) = 0 \quad , \tag{6.34}$$

whose solutions are assumed to be known in terms of special functions for example. Transform both independent and dependent variables as

$$x = u(z) \qquad\qquad u'(z) \neq 0 \tag{6.35}$$

$$h(x) = w(z)\phi(z) \qquad\qquad w(z) \neq 0 \quad , \tag{6.36}$$

so that the new equation on ϕ is given by

$$\frac{d^2\varphi}{dz^2} + A(z)\,\frac{d\phi}{dz} + B(z)\phi(z) = 0 \quad , \tag{6.37}$$

where A and B are given by

$$A(z) = 2\,\frac{w'}{w} - \frac{u''}{u'} + u'P(z) \quad , \tag{6.38}$$

and

$$B(z) = \frac{w''}{w} + \frac{w'}{w}\left[u'\;P(z) - \frac{u''}{u'}\right] + (u')^2Q(z) \quad , \tag{6.39}$$

where we have defined

$$P(z) = p(u(z)) \;\; ; \;\; Q(z) = q(u(z)) \quad . \tag{6.40}$$

To prove this note that

$$\frac{d}{dx} h(x) = \frac{dz}{dx}\frac{d}{dz}[w(z)\phi(z)] = \frac{1}{u'}[w'\phi + w\phi']$$

and

$$\frac{d^2}{dx^2} h(x) = \frac{dz}{dx}\frac{d}{dx}\left[\frac{df}{dx}\right]$$

$$= \frac{1}{u'}\left[-\frac{u''}{(u')^2}[w'\phi + w\phi'] + \frac{1}{u'}[w''\phi + 2w'\phi' + w\phi'']\right] .$$

Combining these in (6.34) yields

$$\frac{[w\phi'' + 2w'\phi' + w''\phi]}{(u')^2} - \frac{u''}{(u')^3}(w\phi' + w'\phi)$$

$$+ \frac{P(z)}{u'}[w\phi' + w'\phi] + Q(z)w\phi = 0 .$$

Multiplying by $(u')^2$ yields

$$w\phi'' + 2w'\phi + w''\phi - \frac{u''}{u'}[w\phi' + w'\phi]$$

$$+ P(z)u'[w\phi' + w'\phi] + Q(z)(u')^2 w\phi = 0$$

Dividing by w yields

$$\phi'' + \phi'\left[2\frac{w'}{w} - \frac{u''}{u'} + u' P(z)\right]$$

$$+ \phi\left[\frac{w''}{w} - \frac{u''}{u'}\frac{w'}{w} + P(z)u'\frac{w'}{w} + Q(z)(u')^2\right] = 0 .$$

The <u>basic idea of this development is as follows</u>:

1. Choose p(x) and q(x) such that the differential equation (6.34) has known solutions in terms of special functions.

2. Choose A(z) = 0 to find one of the functions u or w in terms of the other.

3. Then B(z) is known in terms of one of the transformation functions from (6.38) and (6.39) as

$$B(z) = \frac{w''}{w} - 2\left[\frac{w'}{w}\right]^2 + [u']^2\,Q(z)$$

$$= \frac{d}{dz}\left[\frac{w'}{w}\right] - \left[\frac{w'}{w}\right]^2 + [u']^2\,Q(z) \quad , \tag{6.41}$$

with say u known as a function of w from the condition A(z) = 0.

4. Then choose the second transformation function, w say, so that B(z) is known.

5. Write $B(z) = k_0^2\,n^2(z)$ where n(z) is the index of refraction in the inhomogeneous media described by the differential equation (A = 0 and B $= k_0^2 n^2(z)$ from (6.37)) as

$$\frac{d^2\phi}{dz^2} + k_0^2\,n^2(z)\phi = 0 \quad . \tag{6.42}$$

We can integrate the equation A(z) = 0 which is

$$u'\,P(z) + 2\,\frac{w'}{w} - \frac{u''}{u'} = 0 \quad ,$$

$$\frac{d}{dz}\ln\,[u'] - 2\,\frac{d}{dz}\ln\,[w] = u'\,P(z) \quad ,$$

and

$$\frac{d}{dz}\ln\left[\frac{u'}{w^2}\right] = u'\,P(z) \quad . \tag{6.43}$$

However, rather than integrate the equation for a general P(z) we will find that the differential form is most useful as we choose particular values of p(x) and hence P(z).

Eg. 1. **HYPERGEOMETRIC EQUATION**

We begin with our differential equation (6.34) where we choose

$$p(x) = \frac{c-[a+b+1]x}{x(1-x)} \quad , \quad q(x) = \frac{-ab}{x(1-x)} \quad . \tag{6.44}$$

Here a,b and c are constants. For convenience we define α = a+b+1. The result is a second order linear differential equation with three regular singular points which is called the hypergeometric equation. Solutions can be found in the form of power series, convergent for $|x|<1$. The two linearly independent solutions can be written as

$$h_1 = F(a,b;c;x) \quad , \tag{6.45}$$

and

$$h_2 = x^{1-c}\, F(a+1-c,\ b+1-c;\ 2-c;\ x) \quad , \tag{6.46}$$

where the notation for the hypergeometric function is (Ref. 1.1)

$$F(a,b,c,x) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n \quad , \tag{6.47}$$

with

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1) \qquad n \geq 1 \tag{6.48}$$

$$(\alpha)_0 = 1 \quad .$$

Note that the series are finite for negative integers or zero.

From (6.40) we have that

$$P(z) = \frac{c-\alpha u(z)}{u(z)[1-u(z)]} \quad , \quad Q(z) = \frac{-ab}{u(z)[1-u(z)]} \quad , \tag{6.49}$$

so that (6.43) can be written as

$$\frac{d}{dz} \ln \left[\frac{u'}{w^2}\right] = u' \frac{c-\alpha u}{u(1-u)} \; .$$

$$= c\,\frac{u'}{u} - (\alpha-c)\,\frac{u'}{1-u}$$

$$= \frac{d}{dz} \ln \left[u^c (1-u)^{\alpha-c}\right] \quad ,$$

which can be integrated to yield a relation between w and u

$$w^2(z) = D_0 \frac{u'}{u^c(1-u)^{a-c}} , \qquad (6.50)$$

where D_0 is an integration constant. Alternatively we could write from (6.38) with A(z) = 0

$$\frac{w'}{w} = \frac{1}{2}\left[\frac{u''}{u'} - \frac{u'(c-au(z))}{u(1-u)}\right] , \qquad (6.51)$$

so that our representation for B(z) from (6.41) is

$$B(z) = \frac{d}{dz}\left[\frac{w'}{w}\right] - \left[\frac{w'}{w}\right]^2 - \frac{ab[u']^2}{u(1-u)} , \qquad (6.52)$$

which, using (6.51), expresses B entirely in terms of u(z), which we now choose.

To motivate a choice of u(z), note that from (6.52) we would like a constant background term in the index of refraction. The differential equation

$$\frac{(u')^2}{u(1-u)} = 4f^2 , \qquad (6.53)$$

where f is a constant, has the solution

$$u(z) = \sin^2(fz + g) , \qquad (6.54)$$

where g is an integration constant. From (6.50) we get

$$w^2(z) = 2 D_0 f[\sin(fz + g)]^{1-2c} [\cos(fz + g)]^{1+2c-2a} , \qquad (6.55)$$

so that

$$\frac{w'}{w} = \frac{f}{2}\left[(1-2c)\ \mathrm{ctn}(fz+g) - (1+2c-2\alpha)\ \tan(fz+g)\right] ,$$

and hence

$$\frac{d}{dz}\left[\frac{w'}{w}\right] = -\frac{f^2}{2}\left[(1-2c)[\csc(fz+g)]^2 + (1+2c-2\alpha)[\sec(fz+g)]^2\right] ,$$

and

$$\left[\frac{w'}{w}\right]^2 = \frac{f^2}{4}\left[(1-2c)^2[\mathrm{ctn}(fz+g)]^2 + (1-2c-2\alpha)^2[\tan(fz+g)]^2\right.$$

$$\left. - 2(1-2c)(1+2c-2\alpha)\right] ,$$

so that from (6.52) we get

$$\begin{aligned} B(z) = {} & f^2\left[\frac{1}{2}(1-2c)(1+2c-2\alpha) - 4ab\right] \\ & - (f^2/2)\left[(1-2c)[\csc(fz+g)]^2 + (1+2c-2\alpha)[\sec(fz+g)]^2\right] \\ & - (f^2/4)\left[(1-2c)^2[\mathrm{ctn}(fz+g)]^2 + (1+2c-2\alpha)^2[\tan(fz+g)]^2\right] . \end{aligned} \tag{6.56}$$

An alternate version of this expression is

$$B(z) = f^2\left[a_1 + \frac{a_2}{[\sin(fz+g)]^2} + \frac{a_3}{[\cos(fz+g)]^2}\right] , \tag{6.57}$$

where

$$a_1 = (a-b)^2 , \tag{6.58}$$

$$a_2 = -(c-1/2)(c-3/2) , \tag{6.59}$$

and

$$a_3 = 1/4 - (c-a-b)^2 . \tag{6.60}$$

<u>Some simple properties of the profile (6.57) are</u>:

(a) All the coefficients, hence the profile, are positive if we require $a+b$, $1/2 < c < 3/2$, and $-1/2 < c-a-b < 1/2$.

(b) The derivative of B vanishes at $z = z_0$ if

$$\left[\tan(fz_0+g)\right]^4 = a_2/a_3 \quad . \tag{6.61}$$

An a priori choice of z_0 can be used to fix the other parameters. The values of the function and its second derivative at $z = z_0$ are

$$B(z_0) = f^2\left[a_1+a_2+a_3+2\sqrt{a_2a_3}\,\right] \quad , \tag{6.62}$$

and

$$B''(z_0) = 8f^4\left[a_2+a_3+2\sqrt{a_2a_3}\,\right] = 8f^2\left[B(z_0) - a_1f^2\right] \quad . \tag{6.63}$$

A profile minimum occurs at z_0 if $B''(z_0) > 0$ or if $B(z_0) > a_1f^2$ which implies $a_2+a_3+2\sqrt{a_2a_3} > 0$. A profile maximum occurs at z_0 if $B''(z_0) < 0$ or if $a_2+a_3+2\sqrt{a_2a_3} < 0$, in which case both a_2 and a_3 must be negative in order for the rhs of (6.61) to be positive.

(c) From (6.45) and (6.46) the solutions corresponding to the profile (6.57) are

$$h_1 = F(a,b;c;\ \sin^2(fz+g)) \quad , \tag{6.64}$$

and

$$h_2 = \left[\sin(fz+g)\right]^{2(1-c)} F(a+1-c,b+1-c;2-c;\ \sin^2(fz+g)) \quad . \tag{6.65}$$

(d) Note that from (6.57) if we choose $f = k_0$ to cancel the k_0 terms in $B(z) = k_0^2\, n^2(z)$, the resulting profile $n(z)$ is frequency dependent due to the remaining f terms in the denominator sine and cosine functions.

However, we can scale this out. See the remarks in Appendix 6A.

Eg. 1a.

As a simple example and check on the method, we should recover the solutions for a constant profile. Let $c=1/2$ and $a=-b=1/2$ in (6.57) then

$$B(z) = k_0^2 \, n^2(z) = f^2 \quad . \tag{6.66}$$

The solutions are from (6.45)

$$h_1 = F(1/2,-1/2;1/2;\ \sin^2(fz+g)) = \cos(fz+g) \quad , \tag{6.67}$$

and

$$h_2 = \sin(fz+g)\ F(1,0;3/2;\ \sin^{-2}(fz+g)) = \sin(fz+g) \quad , \tag{6.68}$$

both evaluations of which can be found in Ref. 1.2, pg. 1040.

Eg. 1b.

Suppose in the representation (6.56) we choose $c=1/2$, then we get

$$B(z) = f^2\left[-4ab - (1-\alpha)[\sec(fz+g)]^2 - (1-\alpha)^2[\tan(fz+g)]^2\right] \quad .$$

For convenience let $g=0$ and choose f to be pure imaginary, $f=if_1$, f_1 real. Then using $\cos(if_1 z)= \cosh(f_1 z)$ and $\tan(if_1 z)= i\ \tanh(f_1 z)$ we get

$$B(z) = f_1^2 \left[\ 4ab + (1-\alpha)[\mathrm{sech}(f_1 z)]^2 - (1-\alpha)^2[\tanh(f_1 z)]^2\right] \quad . \tag{6.69}$$

which bears a resemblance to the Epstein profile illustrated later. Here however, the eigenfunctions are

$$h_1 = F(a,b;c; -\sinh^2(f_1 z)) \quad , \quad c = 1/2 \quad ,$$

and

$$h_2 = [i \sinh(f_1 z)] \; F(a+1-c, b+1-c; 2-c; -\sinh^2(f_1 z)) \ .$$

For example, for $a=b=c=1/2$ we get (Ref. 1.2, pg. 1042)

$$h_1 = \operatorname{sech}(f_1 z) \quad , \tag{6.70}$$

and

$$h_2 = \tanh(f_1 z) \quad , \tag{6.71}$$

each of which is a solution of $h''-f_1^2 h = 0$. This makes sense because $\alpha = 2$ and the only profile remaining is $B(z) = f_1^2$. The minus sign in the equation results from rotating z to iz, or equivalently $f \rightarrow if_1$.

<u>**Eg. 1c.**</u>

For this example we directly pick the transformation function w(z) as

$$w(z) = e^{\beta z} \quad . \tag{6.72}$$

Then from (6.50) we have that

$$u'(z) = \frac{1}{D_o} e^{2\beta z} \; u^c (1-u)^{\alpha - c} \quad .$$

Let $\alpha = c$ and choose

$$u(z) = e^{\alpha z} \quad .$$

We have a solution provided

$$\gamma = \frac{1}{D_o} = 2\beta + c\gamma \quad \text{or} \quad \beta = (1-c)\ \gamma/2 \quad .$$

Then from (6.52) we get

$$B(z) = -\left[\frac{(1-c)\gamma}{2}\right]^2 - ab\,\frac{\gamma^2 e^{\gamma z}}{1-e^{\gamma z}} \quad . \tag{6.73}$$

If we want to shift the origin from z=0 define

$$u(z) = e^{\gamma(z-z_0)} \quad , \tag{6.74}$$

then the constraints are

$$\gamma = 2\beta + c\gamma \quad \text{and} \quad \gamma D_0 = e^{\gamma(1-c)z_0} \quad ,$$

with

$$B(z) = -\beta^2 - \frac{ab\,\gamma^2\, e^{\gamma(z-z_0)}}{1 - e^{\gamma(z-z_0)}} \quad . \tag{6.75}$$

Eg. 1d. Epstein Profile

Here we choose our transformation function as

$$u(z) = 1/2(1 + \tanh(z/2) \quad , \tag{6.76}$$

so that

$$1 - u = 1/2(1 - \tanh(z/2)) \quad ,$$

and

$$u(1-u) = 1/4\,[\mathrm{sech}(z/2)]^2 \quad ,$$

with the results

$$u'(z) = 1/4\,\mathrm{sech}^2(z/2) \quad ,$$

$$u''(z) = -\,1/4\,\mathrm{sech}^2(z/2)\,\tanh(z/2) \quad ,$$

$$\frac{u''(z)}{u'(z)} = -\tanh(z/2) \quad ,$$

$$\frac{u'(z)}{u(z)} = \frac{1}{2}\,\frac{\mathrm{sech}^2(z/2)}{1+\tanh(z/2)} = \frac{1}{2}\,\frac{1-\tanh^2(z/2)}{1+\tanh(z/2)}$$

$$= \frac{1}{2}\,[1-\tanh(z/2)] \quad ,$$

and

$$\frac{u'}{1-u} = \frac{1/4\ \mathrm{sech}^2(z/2)}{1/2[1-\tanh(z/2)]} = \frac{1}{2}[1+\tanh(z/2)] \quad .$$

We thus have from (6.51) that

$$\frac{w'}{w} = \frac{1}{2}\left[\frac{u''}{u'} - c\,\frac{u'}{u} - (c-\alpha)\,\frac{u'}{1-u}\right]$$

$$= -\frac{1}{2}\left[c - \frac{\alpha}{2} + (1-\frac{\alpha}{2})\ \tanh(z/2)\right] \quad ,$$

$$\frac{d}{dz}\left[\frac{w'}{w}\right] = -\frac{1}{4}\,(1-\frac{\alpha}{2})\ \mathrm{sech}^2(z/2) \quad ,$$

and

$$\left[\frac{w'}{w}\right]^2 = \frac{1}{4}\left[(c-\alpha/2)^2 + 2(c-\alpha/2)(1-\alpha/2)\tanh(z/2)\right.$$

$$\left. + (1-\alpha/2)^2[\tanh(z/2)]^2\right] \quad ,$$

and if we replace

$$[\tanh(z/2)]^2 = 1 - [\mathrm{sech}(z/2)]^2$$

we have

$$\left[\frac{w'}{w}\right]^2 = \frac{1}{4}\left[(c-\alpha/2)^2 + (1-\alpha/2)^2\right.$$

$$+ 2(c-\alpha/2)(1-\alpha/2)\ \tanh(z/2)$$

$$\left. - (1-\alpha/2)^2[\mathrm{sech}(z/2)]^2\right] \quad .$$

We can thus write from (6.52) that

$$B(z) = B_0 + B_1[\mathrm{sech}(z/2)]^2 + B_2 \tanh(z/2) \quad , \tag{6.77}$$

where

$$B_0 = \frac{1}{4}\left[(c-\alpha/2)^2 + (1-\alpha/2)^2\right] \quad , \tag{6.78}$$

$$B_1 = -\frac{1}{4}(1-\frac{\alpha}{2}) + \frac{1}{4}(1-\frac{\alpha}{2})^2 - \frac{ab}{4} \quad , \tag{6.79}$$

and

$$B_2 = -\frac{1}{2}(c-\alpha/2)(1-a/2) \quad . \tag{6.80}$$

Equation (6.77) has the form of the Epstein profile (Ref. 1.3). Some properties of the profile are:

(a)
$$B(o) = \frac{1}{4}(1-\alpha/2)^2 - \frac{1}{4}(1-\alpha/2) - \frac{ab}{4} - \frac{1}{4}(c-\alpha/2)^2 - \frac{1}{4}(1-\alpha/2)^2$$
$$= -\frac{1}{4}\left[1-\frac{\alpha}{2} + ab + [c-\alpha/2]^2\right]$$
$$= B_0 + B_1 \quad , \tag{6.81}$$

(b)
$$B'(z) = -B_1 \mathrm{sech}^2(z/2)\tanh(z/2) + 1/2\, B_2\, \mathrm{sech}^2(z/2) \quad ,$$

$$B'(z) = [\mathrm{sech}(z/2)]^2[1/2\, B_2 - B_1 \tanh(z/2)] \quad , \tag{6.82}$$

so that $B'(z_0) = 0$ if

$$T \equiv \tanh(z_0/2) = B_2/2B_1 \quad . \tag{6.83}$$

Since the tanh is positive for $z_0 > 0$, B_2 and B_1 must have the same sign.

(c) $$B''(z) = [\mathrm{sech}(z/2)]^2 \left[2B_1[\tanh(z/2)]^2 - \frac{B_2}{2}\tanh(z/2) - B_1\right], \quad (6.84)$$

so that using (6.83) we get

$$B''(z_0) = B_1[\mathrm{sech}(z_0/2)]^2 \left[T^2 - 1\right]$$
$$= -B_1 \ [\mathrm{sech}(z_0/2)]^4 \quad , \qquad (6.85)$$

so that z_0 is a minimum if $B_1 < 0$ and a maximum if $B_1 > 0$.

Eg. 2. CONFLUENT HYPERGEOMETRIC EQUATION

As our second example we again start with (6.34) where we now choose

$$p(x) = (c-x)/x \quad , \quad q(x) = a/x \quad , \qquad (6.86)$$

The resulting equation is the confluent hypergeometric equation with a and c constants.

Eg. 2a.

Choose both transformation functions as

$$u(z) = \beta z \quad , \quad w(z) = z^{\gamma} e^{\delta z} \quad , \qquad (6.87)$$

where from (6.43) we have the constraint

$$2\,\frac{w'}{w} = \frac{u''}{u'} - u'\ P(z)$$
$$= \frac{u''}{u'} - \frac{u'}{u}(c-u) = \frac{u''}{u'} - c\,\frac{u'}{u} + u'$$
$$= -\frac{c}{z} + \beta \quad , \qquad (6.88)$$

with

$$w' = \gamma z^{\gamma-1} e^{\delta z} + \delta z^{\gamma} e^{\delta z} = (\frac{\gamma}{z} + \delta) W \quad ,$$

so that

$$2(\frac{\gamma}{z} + \delta) = \beta - \frac{c}{z} \quad ,$$

and

$$\delta = \beta/2 \qquad \gamma = -c/2 \quad . \tag{6.89}$$

We have that

$$\left[\frac{w'}{w}\right]^2 = \frac{1}{4} (\beta^2 - 2\beta \frac{c}{z} + \frac{c}{z^2})$$

and

$$\frac{d}{dz}\left[\frac{w'}{w}\right] = \frac{c}{2z^2} \quad ,$$

so that from (6.41)

$$\begin{aligned} B(z) &= \frac{c}{2z^2} - \frac{1}{4}\left[\beta^2 - \frac{2\beta c}{z} + \frac{c}{z^2}\right] - \frac{\beta^2}{z} \\ &= -\frac{1}{4}\beta^2 + \frac{1}{2}\beta\frac{c}{z} - \frac{\beta^2}{z} + \frac{c}{4z^2} \\ &= B_0 + \frac{B_1}{z} + \frac{B_2}{z^2} \quad , \end{aligned} \tag{6.90}$$

where

$$B_0 = -\frac{1}{4}\beta^2 \quad , \tag{6.91}$$

$$B_1 = \beta(c/2 - \beta) \tag{6.92}$$

and

$$B_2 = c/4 \quad , \tag{6.93}$$

and thus for an algebraic u(z) we get an algebraic profile. If we want a profile in a region including z=0 we must shift this origin away.

Eg. 2b.

As a second example we choose an exponential transformation function

$$u(z) = e^{-\beta z} \quad , \tag{6.94}$$

and from the constraint (6.43)

$$2\,\frac{w'}{w} = \frac{u''}{u'} - u'\,P(z) \quad ,$$

we have that

$$\frac{w'}{w} = \frac{\beta}{2}\,(c-1-e^{-\beta z}) \quad , \tag{6.95}$$

so that B(z) from (6.41) can be written as

$$B(z) = a_0 + a_1\;e^{-1\beta z} + a_2\;e^{-2bz} \quad , \tag{6.96}$$

where

$$a_0 = -\frac{1}{4}\,\beta^2\;(c-1)^2 \quad ,$$

$$a_1 = \beta^2(\frac{c}{2} - a) \quad , \tag{6.97}$$

and

$$a_2 = -\beta^2/4 \quad . \tag{6.99}$$

Thus for an exponential choice of transformation function we get an exponential profile.

Eg. 3. BESSEL EQUATION

As a third example, start with (6.34) with

$$p(x) = 1/x \quad , \quad q(x) = 1-\alpha^2/x^2 \quad , \tag{6.100}$$

which is the Bessel differential equation with solutions $Z_\alpha(x)$ where Z_α represents the appropriate Bessel and Hankel function for a given boundary value problem.

Eq. 3a.

Choose the transformation function as

$$u(z) = \beta z \quad , \tag{6.101}$$

so that the constraint (6.43) is

$$2\,\frac{w'}{w} = \frac{u''}{u'} - u'\,P(z) = \frac{u''}{u'} - \frac{u'}{u} = -\beta \quad , \tag{6.102}$$

and

$$w(z) = e^{-\beta z/2} \quad . \tag{6.103}$$

The profile B(z) from (6.41) is

$$\begin{aligned} B(z) &= \frac{d}{dz}\frac{w'}{w} - \left[\frac{w'}{w}\right]^2 + (u')^2\,Q(z) \ . \\ &= -\frac{\beta^2}{4} + \beta^2 + \beta^2(1-\alpha^2/\beta^2 z^2) \ . \\ &= \frac{3}{4}\beta^2 - \frac{\alpha^2}{z^2} \quad , \end{aligned} \tag{6.104}$$

with solutions

$$\phi = Z_\alpha(\beta z) \ . \tag{6.105}$$

Eq. 3b.

Choose an exponential transformation function

$$x = u(z) = e^{-\beta z} \quad , \tag{6.106}$$

and the constraint (6.43) is

$$2\,\frac{w'}{w} = \frac{u''}{u'} - \frac{u'}{u} = -\beta + \beta = 0 \quad ,$$

so that

$$w(z) = w_0 = \text{constant} \quad . \tag{6.107}$$

Thus from (6.41)

$$B(z) = (u')^2\, Q(z) = \beta^2\, e^{-2\beta z}\left[1 - \frac{\alpha^2}{e^{-2\beta z}}\right] \tag{6.108}$$

$$= \beta^2 \left[\, e^{-2\beta z} - \alpha^2 \,\right] \quad ,$$

so that the solutions of

$$\frac{d^2\phi}{dz^2} + B(z)\phi = 0 \quad , \tag{6.109}$$

are

$$\phi = Z_\alpha\,(e^{-\beta z}) \; . \tag{6.110}$$

APPENDIX 1A. SCALING AND FREQUENCY INDEPENDENT n(z)

In general we have that

$$B(z) = k_0^2\, n^2(z) = \text{rhs} \quad ,$$

where the rhs contains various constants. We want n(z) to be frequency independent (i.e. independent of k_0), as we previously remarked. We can't do this for a direct choice of constants since the k_0 is buried in the functional dependence of the profile. We scale out this frequency dependence as follows: Our differential equation on ϕ is

$$\frac{d^2\phi}{dz^2} + k_0^2 n^2(z)\phi = 0 \quad .$$

Scale the z-coordinate to give $z^*=k_0 z$ so that

$$\frac{d^2\phi}{dz^{*2}} + n^2(z^*)\phi = 0 \quad ,$$

so we are really finding $B(z^*)=n^2(z^*)$. But this has the same functional dependence as $n^2(z)$. So once we find $n^2(z^*)$ simply replace z^* by z (not by $k_0 z$) to get n(z). The correct scaling occurs in the solution since there we automatically replace z^* by $k_0 z$.

APPENDIX 1B. Darboux Theorem

Once we know one profile and its solution it is possible to find other profiles in a systematic way. It goes like this:

If the general solution of the second order linear differential equation

$$\frac{d^2 f}{dz^2} = [h + a(z)]f \quad ,$$

is known for all values of h, and v(z) is a particular solution of this equation for $h=h_1$, i.e.

$$\frac{d^2 v}{dz^2} = [h_1 + a(z)]v \quad ,$$

then the general solution of

$$\frac{d^2 g}{dz^2} = \left[h - h_1 + v(z) \frac{d^2}{dz^2} \left[\frac{1}{v(z)}\right] \right] g \quad ,$$

for $h \neq h_1$ is given by

$$g(z) = v(z) \frac{d}{dz} \left[\frac{f(z)}{v(z)}\right] \quad .$$

2. SOLUTION OF INITIAL AND BOUNDARY VALUE PROBLEMS

In this chapter we study the initial and boundary value problems for the wave equation, Helmholtz equation, and the parabolic equation. We discuss the integral representations of the first two in detail. In addition we briefly mention the Rayleigh-Sommerfield integral representations, the extended boundary condition or extinction coefficient method, the T-matrix approach, and the Kirchhoff approximation.

2.1 WAVE EQUATION

We write the wave equation for the Green's function as

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] G(\underset{\sim}{x},t;\underset{\sim}{x}',t') = -\delta(\underset{\sim}{x}-\underset{\sim}{x}')\delta(t-t') \quad , \tag{1.1}$$

which is related to our four-dimensional formulation of the Green's function by a factor c^{-1}. We assume the Green's function satisfies causality given by

$$G(\underset{\sim}{x},t;\underset{\sim}{x}',t') = 0 \qquad t' > t \quad . \tag{1.2}$$

That is, no signal is present for measurement times in the field, t, less than the initial time of the source, t'. We first prove reciprocity given by

$$G(\underset{\sim}{x},t;\underset{\sim}{x}',t') = G(\underset{\sim}{x}',-t';\underset{\sim}{x},-t) \quad , \tag{1.3}$$

which yields a relation between exchanging source and receiver positions and times. The proof goes as follows. Since we have a second derivative in time we can write an equation analogous to (1.1) in the form

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] G(\underset{\sim}{x},-t;\underset{\sim}{x}'',-t'') = -\delta(\underset{\sim}{x}-\underset{\sim}{x}'')\delta(t-t'') \quad . \tag{1.4}$$

Next, multiply (1.1) by $G(\underset{\sim}{x},-t;\underset{\sim}{x}'',-t'')$ and (1.4) by $G(\underset{\sim}{x},t;\underset{\sim}{x}',t')$ and subtract the resulting equations. Integrate the result over all space and time to yield

$$I_1(\underset{\sim}{x}',\underset{\sim}{x}'';t',t'') - \frac{1}{c^2} I_2(\underset{\sim}{x}',\underset{\sim}{x}'';t',t'')$$

$$= G(\underset{\sim}{x}'',t'';\underset{\sim}{x}',t') - G(\underset{\sim}{x}',-t';\underset{\sim}{x}'',-t'') \quad , \tag{1.5}$$

where I_1 and I_2 are defined as

$$I_1(\underset{\sim}{x}',\underset{\sim}{x}'';t',t'') = \int dt \iiint d\underset{\sim}{x}\Big[G(\underset{\sim}{x},-t;\underset{\sim}{x}'',-t'')\, \nabla^2 G(\underset{\sim}{x},t;\underset{\sim}{x}',t')$$
$$-G(\underset{\sim}{x},t;\underset{\sim}{x}',t')\, \nabla^2 G(\underset{\sim}{x},-t;\underset{\sim}{x}'',-t'')\Big] \quad , \tag{1.6}$$

and

$$I_2(\underset{\sim}{x}',\underset{\sim}{x}'';t',t'') = \iiint d\underset{\sim}{x}\int dt \Big[G(\underset{\sim}{x},-t;\underset{\sim}{x}'',-t'')\, \frac{\partial^2}{\partial t^2}\, G(\underset{\sim}{x},t;\underset{\sim}{x}',t')$$
$$-G(\underset{\sim}{x},t;\underset{\sim}{x}',t')\, \frac{\partial^2}{\partial t^2}\, G(\underset{\sim}{x},-t;\underset{\sim}{x}'',-t'')\Big] \quad . \tag{1.7}$$

We prove both integrals vanish, and reciprocity follows from (1.5). In I_1 use Green's theorem on the volume integral part to yield a surface integral of the form

$$\iint dS \Big[G(\underset{\sim}{x}_s,-t;\underset{\sim}{x}'',-t'')\, \frac{\partial}{\partial n}\, G(\underset{\sim}{x}_s,t;\underset{\sim}{x}',t')$$
$$-G(\underset{\sim}{x}_s,t;\underset{\sim}{x}',t')\, \frac{\partial}{\partial n}\, G(\underset{\sim}{x}_s,-t;\underset{\sim}{x}'',-t'')\Big] \quad , \tag{1.8}$$

where n is the outward normal over any bounded closed surfaces in the problem. The contribution from the surface at infinity vanishes because the functions satisfy the radiation condition. The surface is specified by evaluating $\underset{\sim}{x}$ on S, i.e. x_S. If we assume that either G, its normal derivative, or a homogeneous linear combination of the two vanish on the surface, then (1.8) is identically zero. Next, write the temporal integral in I_2, using the fact that the integrand is an exact differential, as

$$\int_{-\infty}^{\infty} dt \frac{\partial}{\partial t} \Big[G(\underset{\sim}{x},-t;\underset{\sim}{x}'',-t'')\frac{\partial}{\partial t} G(\underset{\sim}{x},t;\underset{\sim}{x}',t') \\ -G(\underset{\sim}{x},t;\underset{\sim}{x}',t')\frac{\partial}{\partial t} G(\underset{\sim}{x},-t;\underset{\sim}{x}'',-t'') \Big] \quad . \tag{1.9}$$

When integrated and evaluated at $\pm\infty$ the terms vanish by causality. Thus both I_1 and I_2 vanish and reciprocity follows from (1.5).

In order to derive the integral relation for the wave equation we begin with equations on the field function ϕ (arising from a source S) and the Green's function as follows

$$\left[\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right] \phi(\underset{\sim}{x}',t') = -S(\underset{\sim}{x}',t') \quad , \tag{1.10}$$

$$\left[\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right] G(\underset{\sim}{x},t;\underset{\sim}{x}',t') = -\delta(\underset{\sim}{x}-\underset{\sim}{x}')\delta(t-t') \quad , \tag{1.11}$$

where (1.11) follows from (1.1) by interchanging variables, viz.

$$\left[\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right] G(\underset{\sim}{x}',t';\underset{\sim}{x},t) = -\delta(\underset{\sim}{x}-\underset{\sim}{x}')\delta(t-t') \quad . \tag{1.12}$$

If we now let $t\rightarrow -t$ and $t'\rightarrow -t'$ we get

$$\left[\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right] G(\underset{\sim}{x}',-t';\underset{\sim}{x},-t) = -\delta(\underset{\sim}{x}-\underset{\sim}{x}')\delta(t-t') \quad , \tag{1.13}$$

and, by reciprocity, (1.11) follows from this. Next, multiply (1.10) by $G(\underset{\sim}{x},t;\underset{\sim}{x}',t')$ and (1.11) by $\phi(\underset{\sim}{x}',t')$ and subtract the resulting equations.

Integrate the results over all space and from t=0 to t=∞ in time. The result is written as (t=0 is the initial time).

$$\phi(\underset{\sim}{x},t) = \int_0^\infty dt' \iiint d\underset{\sim}{x}' G(\underset{\sim}{x},t;\underset{\sim}{x}',t') S(\underset{\sim}{x}',t')$$

$$+ \int_0^\infty dt' \iiint d\underset{\sim}{x}' \Big[G(\underset{\sim}{x},t;\underset{\sim}{x}',t') \nabla'^2 \phi(\underset{\sim}{x}',t')$$

$$- \phi(\underset{\sim}{x}',t') \nabla'^2 G(\underset{\sim}{x},t;\underset{\sim}{x}',t') \Big]$$

$$- \frac{1}{c^2} \iiint d\underset{\sim}{x}' \int_0^\infty dt' \frac{\partial}{\partial t'} \Big[G(\underset{\sim}{x},t;\underset{\sim}{x}',t') \frac{\partial}{\partial t'} \phi(\underset{\sim}{x}',t')$$

$$- \phi(\underset{\sim}{x}',t') \frac{\partial}{\partial t'} G(\underset{\sim}{x},t;\underset{\sim}{x}',t') \Big] \quad . \qquad (1.14)$$

In the second integral term in (1.14) we use the spatial Green's theorem as in (1.8), and we integrate the temporal part of the third integral whose integrand is an exact differential. The infinite surface contributions vanish by the radiation condition, and the infinite time contributions vanish using causality. The resulting integrals can be restricted using causality to give the final result

$$\phi(\underset{\sim}{x},t) = \int_0^{t+} dt' \iiint d\underset{\sim}{x}' G(\underset{\sim}{x},t;\underset{\sim}{x}',t') S(\underset{\sim}{x}',t')$$

$$+ \int_0^{t+} dt' \iint dS' \Big[G(\underset{\sim}{x},t;\underset{\sim}{x}'_s,t') \frac{\partial}{\partial n'} \phi(\underset{\sim}{x}'_s,t')$$

$$- \phi(\underset{\sim}{x}'_s,t') \frac{\partial}{\partial n'} G(\underset{\sim}{x},t;\underset{\sim}{x}'_s,t') \Big]$$

$$+ \frac{1}{c^2} \iiint d\underset{\sim}{x}' \Big[G(\underset{\sim}{x},t;\underset{\sim}{x}',0) \frac{\partial}{\partial t'} \phi(\underset{\sim}{x}',0)$$

$$- \phi(\underset{\sim}{x}',0) \frac{\partial}{\partial t'} G(\underset{\sim}{x},t;\underset{\sim}{x}',0) \Big] \quad . \qquad (1.15)$$

in terms of a surface integral over finite surfaces having an outward normal $\underset{\sim}{n}'$. The result for the field function is that it is expressed as a superposition of wavelets from the source S, the boundary surface (or surfaces), and the initial conditions $\phi(\underset{\sim}{x}',0)$ and $\partial\phi(\underset{\sim}{x}',0)/\partial t'$. Both initial conditions and one boundary condition specify the problem uniquely.

We now want to examine the separate terms in (1.15). We choose for our Green's function the retarded Green's function in Ch. 1. It is written as

$$G(\underset{\sim}{x},t;\underset{\sim}{x}',t') = c\, G_R^{(3,1)}(x,x')$$

$$= \frac{c\ \delta(\tau-r)}{4\pi r}\ ;\ \tau = c(t-t'),\quad r = |\underset{\sim}{x}-\underset{\sim}{x}'|$$

$$= \frac{1}{4\pi r}\,\delta(t-t'-\frac{1}{c}\,|\underset{\sim}{x}-\underset{\sim}{x}'|)\quad . \tag{1.16}$$

The first term in (1.15) is

$$\phi_1(\underset{\sim}{x},t) = \int_0^{t^+}dt'\iiint d\underset{\sim}{x}'G(\underset{\sim}{x},t;\underset{\sim}{x}',t')S(\underset{\sim}{x}',t')\quad , \tag{1.17}$$

which is an integral over the domain of the source function S. Recall that G was a retarded Green's function and yielded the representation

$$\phi_1(\underset{\sim}{x},t) = \frac{1}{4\pi r}\iiint d\underset{\sim}{x}'\ S(\underset{\sim}{x}',t-r/c)\quad , \tag{1.18}$$

where we have used (1.16) in (1.17) and integrated over time. The result is a field ϕ_1 due to sources integrated over those spatial values such that its temporal values occur before the measurement time t, i.e. at retarded time values.

We next discuss several examples.

Eg. 1. MOVING POINT SOURCE

Although many applications relate to fixed sources, we can treat the signal received by a moving point source as follows. Choose

$$\delta(\underset{\sim}{x}',t') = \delta(\underset{\sim}{x}'-\underset{\sim}{R}(t')) \quad , \tag{1.19}$$

which is a point source moving on a path $\underset{\sim}{R}(t')$. See Fig. 2.1 below.

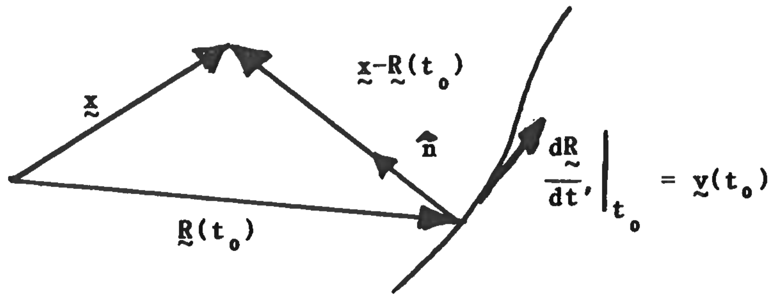


Fig. 2.1

Using (1.16) and (1.19) the result of (1.17) is

$$\phi_1(\underset{\sim}{x},t) = \int_0^{t+} dt' \iiint d\underset{\sim}{x}' \, \frac{\delta(t-t'-\frac{1}{c}|\underset{\sim}{x}-\underset{\sim}{x}'|)}{4\pi|\underset{\sim}{x}-\underset{\sim}{x}'|} \, \delta(\underset{\sim}{x}'-\underset{\sim}{R}(t')) \quad . \tag{1.20}$$

The volume integral can be easily evaluated to yield

$$\varphi_1(\underset{\sim}{x},t) = \int_0^{t+} dt' \, \frac{\delta(t-t'-\frac{1}{c}|\underset{\sim}{x}-\underset{\sim}{R}(t')|)}{4\pi|\underset{\sim}{x}-\underset{\sim}{R}(t')|} \quad . \tag{1.21}$$

To evaluate this final integral use the relation

$$\int g(t')\delta(f(t'))dt' = \int g(t') \frac{\delta(t'-t_0)}{|df/dt'|} dt' = \frac{g(t_0)}{|df/dt'|_{t'=t_0}} \quad , \tag{1.22}$$

where

$$g(t') = \left[4\pi \, |\underset{\sim}{x}-\underset{\sim}{R}(t')|\right]^{-1} \quad , \tag{1.23}$$

$$f(t') = t - t' - \frac{1}{c}|\underset{\sim}{x}-\underset{\sim}{R}(t')| \quad , \tag{1.24}$$

and t_0 is given by the solution $f(t_0) = 0$ so that it satisfies

$$t_0 = t - \frac{1}{c} |\underset{\sim}{x} - \underset{\sim}{R}(t_0)| \quad . \tag{1.25}$$

We also have the result

$$\frac{df}{dt'} = -1 + \frac{1}{c} \frac{d\underset{\sim}{R}}{dt'} \cdot \frac{[\underset{\sim}{x} - \underset{\sim}{R}(t')]}{|\underset{\sim}{x} - \underset{\sim}{R}(t')|} \quad . \tag{1.26}$$

To carry out the integral we require that $0<t_0<t$. The result is the contribution from a moving point source

$$\phi_1(\underset{\sim}{x},t) = \frac{1}{4\pi} \frac{\theta(t-t_0)\theta(t_0)}{\left| |\underset{\sim}{x} - \underset{\sim}{R}(t_0)| - \frac{1}{c} \left[\frac{d\underset{\sim}{R}}{dt'}\right]_{t_0} \cdot [\underset{\sim}{x} - \underset{\sim}{R}(t_0)] \right|} \quad , \tag{1.27}$$

which is called the Lienard-Wiechert potential. It can also be written as

$$\phi_1(\underset{\sim}{x},t) = \frac{1}{4\pi} \frac{\theta(t-t_0)}{|\underset{\sim}{x} - \underset{\sim}{R}(t_0)|} \frac{\theta(t_0)}{\left|1 - \frac{1}{c} \underset{\sim}{v}(t_0) \cdot \hat{n}(t_0)\right|} \quad . \tag{1.28}$$

For the special case $\underset{\sim}{v} = \underset{\sim}{0}$, the result of the spatial and temporal superposition of spherical waves is a pulse-like solution. Note that we have evaluated the last delta function in (1.21) assuming only one solution. For a homogeneous medium this is always true. The reason is that there is one arrival time and thus a minimum path for the signal. A signal along any other path would arrive at a later time. However, in an inhomogeneous medium this might not be true. Even though the distance may be longer for a second path, the point source speed v in an inhomogeneous medium might be faster than the wave speed c in the homogeneous region. The result could be the same arrival time for two waves, one which radiates solely into the homogeneous region to the receiver, the other where the source travels a distance in the inhomogeneous medium, then radiates into the homogeneous medium to the receiver. An example in electromagnetic theory is the Cerenkov effect, where the signal travels faster than the speed of light in the inhomogeneous medium. Note this is phase velocity.

Eg. 2. MOVING DIPOLE SOURCE

As a second example, consider a moving dipole source term. For simplicity we write the dipole in only one direction, the x-direction, as

$$S(\underset{\sim}{x}',t') = \delta'(x'-R(t'))\delta(y')\delta(z') \quad . \tag{1.29}$$

in the x-direction. A sum of dipoles in the x, y, and z directions would correspond to an approximation of an explosive source. The moving dipole in a single direction could correspond to a very simple model of a fault or earthquake. The result for (1.21) is

$$\phi_1(\underset{\sim}{x},t) = \int_0^{t+} dt' \iiint dx' \frac{\delta(t-t'-\frac{1}{c}|\underset{\sim}{x}-\underset{\sim}{x}'|)}{4\pi|\underset{\sim}{x}-\underset{\sim}{x}'|} \delta'(x'-R(t'))\delta(y')\delta(z') \quad . \tag{1.30}$$

If we integrate by parts we get

$$\phi_1(\underset{\sim}{x},t) = -\int_0^{t+} dt' \iiint d\underset{\sim}{x}' \frac{\partial}{\partial x'} \left[\frac{\delta(t-t'-\frac{1}{c}|\underset{\sim}{x}-\underset{\sim}{x}'|)}{4\pi|\underset{\sim}{x}-\underset{\sim}{x}'|}\right] \delta(\underset{\sim}{x}'-\underset{\sim}{R}(t')\delta(y')\delta(z') \quad . \tag{1.31}$$

The integrand has two terms

$$\frac{\partial}{\partial x'}\left[\frac{\delta(t-t')-\frac{1}{c}|\underset{\sim}{x}-\underset{\sim}{x}'|)}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\right] = \frac{\delta'(t-t'-\frac{1}{c}|\underset{\sim}{x}-\underset{\sim}{x}'|)}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\frac{1}{c}\frac{(x-x')}{|\underset{\sim}{x}-\underset{\sim}{x}'|}$$

$$+ \delta(t-t'-\frac{1}{c}|\underset{\sim}{x}-\underset{\sim}{x}'|)\frac{(x-x')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} \quad . \tag{1.32}$$

The first term can be written as

$$\phi_a(\underset{\sim}{x},t) = -\int dt' \iiint d\underset{\sim}{x}' \frac{\delta'(t-t'-\frac{1}{c}|\underset{\sim}{x}-\underset{\sim}{x}'|)}{4\pi|x-x'|^2} \frac{(x-x')}{c} \delta(x'-R(t'))\delta(y')\delta(z')$$

$$= -\frac{1}{4\pi}\int dt' \, \delta'(f(t')) \, g(t') \quad , \tag{1.33}$$

where

$$f(t') = t-t'-\frac{1}{c}\,|\underset{\sim}{x}-\underset{\sim}{x}'|$$

$$= t-t'-\frac{1}{c}\left[[x-R(t')]^2 + y^2 + z^2\right]^{1/2} \quad , \tag{1.34}$$

and

$$g(t') = \frac{x-R(t')}{c}\;\frac{1}{[x-R(t')]^2 + y^2 + z^2} \quad . \tag{1.35}$$

We can evaluate this integral using integration by parts

$$\int dt'\;\delta'(f(t'))g(t') = \int dt'\;\frac{d}{dt'}\;\delta(f(t'))\frac{g(t')}{df/ft'}$$

$$= -\int dt'\;\delta(f(t'))\frac{d}{dt'}\left[\frac{g(t')}{f'(t)}\right] \tag{1.36}$$

$$= -\frac{1}{|df/dt'|_{t_0}}\;\frac{d}{dt'}\left[\frac{g(t')}{f'(t')}\right]\Bigg|_{t\,=\,t_0} \quad ,$$

where t_0 arises from $f(t_0)=0$ and hence solves (1.25). We thus have that

$$\phi_a(\underset{\sim}{x},t) = \frac{1}{4\pi}\,\frac{1}{|f'(t_0)|}\left[\frac{g'(t_0)}{f'(t_0)} - \frac{g(t_0)}{[f'(t_0)]^2}\;f''(t_0)\right] \quad . \tag{1.37}$$

To evaluate (1.37) define

$$D(t') = \left[[x-R(t')]^2 + y^2 + z^2\right]^{1/2} \quad . \tag{1.38}$$

Then we can write

$$f(t') = t-t'-\frac{1}{c}\,D(t') \quad ,$$

and

$$g(t') = \frac{1}{c}\,[x-R(t')]\;D^{-2}(t') \quad ,$$

so that

$$f' = -1 + \frac{1}{c}\frac{R'(x-R)}{D} = \frac{R'(x-R)-c\,D}{c\,D} ,$$

$$f'' = [R'(x-R) - c\,D]\,\frac{(x-R)R'}{c\,D^3} + \frac{1}{c\,D}\left[R''(x-R) - (R')^2 + c\,\frac{R'(x-R)}{D}\right]$$

$$= \frac{R''(x-R)}{c\,D} - \frac{(R')^2}{c\,D} + \frac{[R'(x-R)]^2}{c\,D^3} ,$$

and

$$g' = \frac{1}{c}(-R')\,D^{-2} + \frac{x-R}{c}(-2)D^{-3}\left[\frac{-R'(x-R)}{D}\right] .$$

$$= \frac{-R'}{c\,D^2} + \frac{2\,R'(x-R)^2}{c\,D^4} .$$

Combining these we get

$$\phi_a = \frac{c^2\,R'\,D}{4\pi\,|R'(x-R) - c\,D|^3} \cdot \left[1 - \left[\frac{R''}{R'}\right]\frac{(x-R)^2}{c\,D} - 2\,\frac{(x-R)^2}{D^2} + \frac{R'(x-R)^3}{c\,D^3}\right] , \quad (1.39)$$

where each term is evaluated at $t'=t_0$ with t_0 a solution to $f(t_0)=0$ or

$$t_0 = t - \frac{1}{c}\,|\underset{\sim}{x}-R(t_0)\hat{1}| , \quad (1.40)$$

with D defined in (1.38). ϕ_a has the dimensions $(\text{length})^{-2}$.

The second term in (1.31) is defined as

$$\phi_b = \frac{-1}{4\pi}\int_0^{t+}dt'\iiint d\underset{\sim}{x}'\;\delta(t-t'-\frac{1}{c}\,|\underset{\sim}{x}-\underset{\sim}{x}'|) \cdot \frac{(x-x')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3}\,\delta(x'-R(t')\delta(y')\delta(z') . \quad (1.41)$$

Spatial integration yields

$$\phi_b = -\frac{1}{4\pi}\int_0^{t+} dt'\ \delta(t-t'-\frac{1}{c}D(t'))\ \frac{[x-R(t')]}{[D(t')]^3}\ , \qquad (1.42)$$

where D(t') is defined in (1.38). To evaulate this use (1.22) to get

$$\phi_b = \frac{-1}{4\pi}\frac{[x-R(t_0)]}{[D(t_0)]^3}\frac{1}{|df/dt'|_{t_0}}$$

$$= -\frac{1}{4\pi}\frac{c[x-R(t_0)]\quad D(t_0)}{|R'(x-R)\ -\ c\ D|}\ \frac{1}{[D(t_0)]^3}$$

$$= -\frac{c}{4\pi}\frac{x-R(t_0)}{D^2(t_0)\ |R'(x-R)\ -\ c\ D|}\ , \qquad (1.43)$$

which also has dimensions $(\text{length})^{-2}$. This can be rewritten as

$$\phi_b = -\frac{c^2 R'D}{4\pi\,|R'(x-R)\ -\ c\ D|^3}\left[\frac{[R'(x-R)\ -\ c\ D]^2\ (x-R)}{cR'D^3}\right]$$

$$= -\frac{c^2}{4\pi}\frac{R'D\quad (x-R)}{|R'(x-R)\ -\ c\ D|^3}\left[\frac{R'^2(x-R)^2\ -\ 2\ c\ D\ R'(x-R)\ +\ c^2\ D^2}{c\ R'\ D^3}\right]$$

$$= -\frac{c^2}{4\pi}\frac{R'D}{|R'(x-R)\ -\ c\ D|^3}\left[\frac{c(x-R)}{R'D} - \frac{2(x-R)^2}{D^2} + \frac{R'(x-R)^3}{c\ D^3}\right]\ . \qquad (1.44)$$

Thus ϕ_1 from (1.31) and (1.32) is

$$\phi_1 = \phi_a + \phi_b\ ,$$

and from (1.39) and (1.44) we get

$$\phi_1 = \frac{c^2}{4\pi} \frac{R'D}{|R'(x-R) - c\,D|^3} \left[1 - \frac{x-R}{cR'D}\,[R''(x-R) + c^2]\right] \quad . \tag{1.45}$$

The first term in the brackets in (1.45) is the far field term and the second term a near-field term.

As a simple check on this result, suppose the dipole is fixed at $x'=0$, i.e. let $R(t')=0$. Then

$$D = r = (x^2 + y^2 + z^2)^{1/2} \quad ,$$

and only the second term in (1.45) contributes. The result is

$$\varphi_1\Big|_{R=0} = - \frac{x}{4\pi r^3} = \frac{1}{4\pi}\frac{\partial}{\partial x}\left[\frac{1}{r}\right] \quad ,$$

which is the standard result for a fixed dipole source.

Dipoles moving in other directions can easily be computed using the form (1.45). We relable the source in (1.29) indicating that it is in the x-direction as

$$S_x(\underset{\sim}{x}',t') = \delta'(x'-R_1(t'))\delta(y')\delta(z') \quad .$$

If we also define from (1.38)

$$D_1(t') = \left[[x-R_1(t')]^2 + y^2 + z^2\right]^{1/2} \quad .$$

Then (1.45) is for a moving dipole in the x-direction

$$\phi_1^{Dx} = \frac{c^2}{4\pi} \frac{R_1'\,D_1}{|R_1'(x-R_1) - c\,D_1|^3} \left[1 - \frac{x-R_1}{CR_1'D_1}\,[R''_1(x-R_1) + c^2]\right] \quad .$$

A moving dipole in the y-direction is

$$S_y(\underset{\sim}{x}',t') = \delta(x')\delta(y'-R_2(t'))\delta(z')$$

and if we define

$$D_2(t') = \left[x^2 + [y-R_2(t')]^2 + z^2\right]^{1/2}$$

the result analogous to (1.45) is

$$\phi_1^{Dy} = \frac{c^2}{4\pi} \frac{R_2' D_2}{|R_2'(y-R_2) - c D_2|^3} \left[1 - \frac{y-R_2}{cR_2'D_2} [R_2''(y-R_2) + c^2]\right] .$$

Similarly in the z-direction we have

$$S_z(\underset{\sim}{x}',t') = \delta(x')\delta(y')\delta(z'-R_3(t')) \quad ,$$

and

$$D_3(t') = \left[x^2 + y^2 + [z-R_3(t')]^2\right]^{1/2} \quad ,$$

and the result we get is

$$\phi_1^{Dz} = \frac{c^2}{4\pi} \frac{R_3' D_3}{|R_3'(z-R_3) - c D_3|^3} \left[1 - \frac{z-R_3}{cR_3'D_3} [R_3''(z-R_3) + c^2]\right] .$$

For the second term in (1.15) we write

$$\phi_2(\underset{\sim}{x},t) = \int_0^{t+} dt' \iint dS' \left[G(\underset{\sim}{x},t;\underset{\sim}{x}_s',t') \frac{\partial}{\partial n'}\phi(\underset{\sim}{x}_s',t') - \phi(\underset{\sim}{x}_s',t') \frac{\partial}{\partial n'} G(\underset{\sim}{x},t;\underset{\sim}{x}_s',t')\right] . \tag{1.46}$$

For the boundary condition we have either the Dirichlet boundary value problem

$$\phi(\underset{\sim}{x}_s',t') = 0 \quad , \tag{1.47}$$

or the Neumann boundary value problem

$$\frac{\partial}{\partial n'} \phi(\underset{\sim}{x}_s', t') = 0 \quad . \tag{1.48}$$

If we choose the Green's function such that it has the corresponding boundary value $G(\underset{\sim}{x}, t; \underset{\sim}{x}_s', t') = 0$ or $\partial G(\underset{\sim}{x}, t; \underset{\sim}{x}_s', t')/\partial n' = 0$ then ϕ_2 vanishes. The proviso with this convenient method is that one can find a Green's function which say vanishes on the boundary. It is usually only possible to find this image Green's function for simple geometries involving flat planes, cylinders, and spheres for example. For an arbitrary boundary this in effect would amount to fully solving the problem. In any case we get one term to drop by our choice of boundary condition on ϕ or $\partial\phi/\partial n'$. To solve the problem we must write an integral equation on the remaining boundary value (either ϕ or $\partial\phi/\partial n'$). We do this in the next section for the Helmholtz equation. We also treat, in the next section, an example for a flat interface where we can find an image Green's function. The result will be the Rayleigh–Sommerfeld diffraction formulae.

The third term in (1.15) is

$$\phi_3(\underset{\sim}{x}, t) = \frac{1}{c^2} \iiint d\underset{\sim}{x}' \left[G(\underset{\sim}{x}, t; \underset{\sim}{x}', 0) \frac{\partial}{\partial t'} \phi(\underset{\sim}{x}', 0) - \phi(\underset{\sim}{x}', 0) \frac{\partial}{\partial t'} G(\underset{\sim}{x}, t; \underset{\sim}{x}', 0) \right] \quad , \tag{1.49}$$

in terms of initial conditions on the field, i.e. $\phi(\underset{\sim}{x}', 0)$ and $\partial\phi(\underset{\sim}{x}', 0)/\partial t'$. Again if we are able to choose the Green's function such that its initial values matched those of ϕ, the term would vanish. Regardless of the choice of G, both initial values on ϕ must be known, so once we specify the choice of G, this term is a known function. We specify the initial conditions on the retarded Green's function later in this section.

To summarize, we have that

$$\phi(\underset{\sim}{x},t) = \phi_1(\underset{\sim}{x},t) + \phi_2(\underset{\sim}{x},t) + \phi_3(\underset{\sim}{x},t) \quad , \tag{1.50}$$

with ϕ_1 given by (1.17), ϕ_2 by (1.46) and ϕ_3 by (1.49). ϕ_1 is known if G and S are known. ϕ_2 is only partially known since either $\phi(\underset{\sim}{x}_s',t')$ or $\partial\phi(\underset{\sim}{x}_s',t')/\partial n'$ is known but not both. ϕ_3 is known if G and both initial conditions $\phi(\underset{\sim}{x}',0)$ and $\partial\phi(\underset{\sim}{x}',0)/\partial t'$ are known.

2.1.1 FULL (BOUNDARY) GREEN'S FUNCTION

Rather than find the field function ϕ due to a general source term S, we instead find the Green's function Γ due to a point source, i.e. we want to solve for the function Γ, the full Green's function, which satisfies the equation

$$\left[\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right]\Gamma(\underset{\sim}{x}',t';\underset{\sim}{x}'',t'') = -\delta(\underset{\sim}{x}'-\underset{\sim}{x}'')\delta(t-t'') \quad . \tag{1.51}$$

This is the same as (1.10) if we replace ϕ by Γ and S by the point source on the rhs of (1.51). Equation (1.15) then becomes

$$\begin{aligned}\Gamma(\underset{\sim}{x},t;\underset{\sim}{x}'',t'') &= G(\underset{\sim}{x},t;\underset{\sim}{x}'',t'') \\ &+ \int_0^{t+}dt'\iint dS'\left[G(\underset{\sim}{x},t;\underset{\sim}{x}_s',t')\frac{\partial}{\partial n'}\Gamma(\underset{\sim}{x}_s',t';\underset{\sim}{x}'',t'')\right. \\ &\qquad\left. - \Gamma(\underset{\sim}{x}_s',t';\underset{\sim}{x}'',t'')\frac{\partial}{\partial n'}G(\underset{\sim}{x},t;\underset{\sim}{x}_s',t')\right] \\ &+ \frac{1}{c^2}\iiint d\underset{\sim}{x}'\left[G(\underset{\sim}{x},t;\underset{\sim}{x}',0)\frac{\partial}{\partial t'}\Gamma(\underset{\sim}{x}',0;\underset{\sim}{x}'',t'')\right. \\ &\qquad\left. - \Gamma(\underset{\sim}{x}',0;\underset{\sim}{x}'',t'')\frac{\partial}{\partial t'}G(\underset{\sim}{x},t;\underset{\sim}{x}',0)\right] \quad ,\end{aligned} \tag{1.52}$$

which is the integral representation for the full Green's function of the problem. Once we know its solution we can find the solution for any source $S(\underset{\sim}{x}',t'')$ by multiplying (1.52) by $S(\underset{\sim}{x}'',t'')$ and integrating over $\underset{\sim}{x}''$ and t''. For example, the result due to a source S is just

$$\phi(\underset{\sim}{x},t) = \int dt'' \iiint d\underset{\sim}{x}'' \; \Upsilon(\underset{\sim}{x},t;\underset{\sim}{x}'',t'') S(\underset{\sim}{x}'',t'') \quad . \tag{1.53}$$

Our specification of the initial-boundary value problem is analogous to before. We assume G is known, and specify one boundary condition on either Υ or its normal derivative, as well as both initial conditions on Υ. The full solution of the problem requires us to solve an integral equation on the remaining value evaluated on the surface. Note that if Υ and G satisfy the same initial and boundary conditions, both integral terms vanish, and $\Upsilon = G$ which is the complete solution of the problem.

2.1.2 INITIAL CONDITIONS ON $G^{(3,1)}$

From Ch. 1, Eq. (1.12), we have a representation valid for any Green's function solving the wave equation. It is written in terms of the pole shifts and is given by ($x = (\underset{\sim}{x},x_0)$, $\omega_k = |\underset{\sim}{k}|$)

$$G^{(3,1)}(x) = \frac{-1}{(2\pi)^4} \left[P \iiiint \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} \, e^{-ik_0 x_0}}{k_0^2 - \omega_k^2} \, d\underset{\sim}{k} \, dk_0 \right.$$
$$\left. + \frac{\pi i}{2} \iiint \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}}{\omega_k} \left[\alpha \, e^{-i\omega_k x_0} - \beta \, e^{i\omega_k x_0} \right] d\underset{\sim}{k} \right] \quad . \tag{1.54}$$

The initial condition on $G^{(3,1)}$ is specified at $x_0 = 0$ so that

$$G^{(3,1)}(\underset{\sim}{x},0) = -\frac{1}{(2\pi)^4}\left[P\iiiint \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}}{k_0^2-\omega_k^2}\,d\underset{\sim}{k}\,dk_0\right.$$

$$\left.+\frac{\pi i}{2}(\alpha-\beta)\iiint \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}}{\omega_k}\,d\underset{\sim}{k}\right] \quad . \tag{1.55}$$

We can evaluate the k_0 integral in the first term directly

$$P\int \frac{dk_0}{k_0^2-\omega_k^2} = \pi i\left[\frac{1}{2\omega_k}+\frac{1}{(-2\omega_k)}\right] = 0 \quad , \tag{1.56}$$

so that it always vanishes. The second term vanishes if $\alpha = \beta$, i.e. for the retarded (R), advanced (A), and principal value (P) Green's functions. So we have the initial conditions

$$G^{(3,1)}(\underset{\sim}{x},0) = 0 \qquad (R,A,P) \quad . \tag{1.57}$$

The time derivative of $G^{(3,1)}$ is from (1.54)

$$\partial_0 G^{(3,1)}(x) = -\frac{1}{(2\pi)^4}\left[P\iiiint \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}(-ik_0)\,e^{-ik_0x_0}}{k_0^2-\omega_k^2}\,d\underset{\sim}{k}\,dk_0\right.$$

$$\left.+\frac{\pi i}{2}\iiint \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}}{\omega_k}[-i\omega_k]\left[\alpha e^{i\omega_k x_0}+\beta e^{i\omega_k x_0}\right]d\underset{\sim}{k}\right] \quad . \tag{1.58}$$

Next set $x_0 = 0$. The first term vanishes identically because the integrand of the k_0-integral is an odd function. The second integral becomes

$$\frac{\pi}{2}(\alpha+\beta)\iiint e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\,d\underset{\sim}{k} = \frac{\pi}{2}(2\pi)^3(\alpha+\beta)\,\delta(\underset{\sim}{x}) \quad , \tag{1.59}$$

so that

$$\partial_0 G^{(3,1)}(\underset{\sim}{x},0) = -\frac{1}{4}(\alpha+\beta)\,\delta(\underset{\sim}{x}) \quad . \tag{1.60}$$

For the retarded Green's function $\alpha = \beta = -1$, for the advanced Green's function $\alpha = \beta = 1$, and for the principal value Green's function $\alpha = \beta = 0$, so we have

$$\partial_0 G^{(3,1)}(\underset{\sim}{x},0) = \begin{cases} 1/2\ \delta(\underset{\sim}{x}) & \text{(R)} \\ -1/2\ \delta(\underset{\sim}{x}) & \text{(A)} \\ 0 & \text{(P)} \end{cases} \quad . \tag{1.61}$$

By (1.16) we thus have that for $t = t' = 0$

$$G(\underset{\sim}{x},0;\underset{\sim}{x}',0) = c\ G_R^{(3,1)}(\underset{\sim}{x}-\underset{\sim}{x}',0) = 0 \tag{1.62}$$

and that

$$\left.\frac{\partial}{\partial t'} G(\underset{\sim}{x},t;\underset{\sim}{x}',t')\right|_{t=t'=0} = \partial_0\ G_R^{(3,1)}(\underset{\sim}{x}-\underset{\sim}{x}',0)$$

$$= 1/2\ \delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad . \tag{1.63}$$

Note that in our integral representation (1.15) the initial condition is set at $t' = 0$ for the source. This was a matter of choice and led us to integrate the representation from 0 to ∞. The integration was reduced by causality. The initial condition on the field was thus at $t = 0$, expressed under the integral by $\phi(\underset{\sim}{x}',0)$ but on the field function as $\phi(\underset{\sim}{x},0)$ for example.

We can derive these initial conditions another way. We have specific functional forms for the Green's functions which enable us to derive these results directly. For example from Ch. 1 we have

$$G_R^{(3,1)}(x,x') = \frac{\delta(r-t)}{4\pi r} \quad , \tag{1.64}$$

where

$$x = (\underset{\sim}{x}, x_0), \quad r = |\underset{\sim}{x}-\underset{\sim}{x}'|, \quad \text{and} \quad \tau = c(t-t') \quad .$$

Evaluating this function directly we have

$$G_R^{(3,1)}(x,x')\Big|_{\tau = 0} = \frac{\delta(r)}{4\pi r} \quad , \tag{1.65}$$

and

$$\partial_0 G_R^{(3,1)}(x,x')\Big|_{\tau = 0} = -\frac{\delta'(r)}{4\pi r} \quad . \tag{1.66}$$

If we evaluate these terms as distributions and integrate over all space we get for the rhs of (1.65)

$$\int_0^\infty r^2 \, dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \, \frac{\delta(r)}{4\pi r} = 0 \quad , \tag{1.67}$$

and for the rhs of (1.66)

$$\int_0^\infty r^2 \, dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \left[-\frac{\delta'(r)}{4\pi r}\right]$$

$$= -\int_0^\infty r\delta'(r) \, dr = \int_0^\infty \delta(r) dr = 1/2 \quad . \tag{1.68}$$

Note that if we wrote $G_R^{(3,1)}$ with a step function $\theta(\tau)$, the derivative contains $\delta(r)/4\pi r$ which vanishes as a distribution in three dimensions.

Hence as distributions

(a) $\delta(r)/4\pi r$ is equivalent to zero in three dimensions and

(b) $-\delta'(r)/4\pi r$ is equivalent to $1/2\ \delta(\underset{\sim}{x})$ in three dimensions

so that our result is for G

$$G(\underset{\sim}{x},0;\underset{\sim}{x}',0) = 0 \quad ,$$

$$\partial_{t'} G(\underset{\sim}{x},0;\underset{\sim}{x}'0) = 1/2\ \delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad .$$

as before.

2.2 HELMHOLTZ EQUATION

In this section we construct the integral representation for the solution of the Helmholtz equation as well as the integral equations on a surface field value necessary to solve the boundary value problem. We do it in a different way from the wave equation, by using index notation. We assume the field ϕ satisfies a Helmholtz equation with a source S

$$(\partial_j'\partial_j' + k_o^2)\ \phi(\underset{\sim}{x}') = -S(\underset{\sim}{x}') \quad , \tag{2.1}$$

and the equation for the Green's function satisfies the same equation but with a delta function source

$$(\partial_j'\partial_j' + k_o^2)\ G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = -\delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad . \tag{2.2}$$

Note that the differential operators in (2.2) operate on the source coordinate, which is explicitly permitted by reciprocity.

We assume that $G^{(3)}$ is a known function and that ϕ satisfies certain boundary conditions which we specify later. Also here we assume that the differential equation (2.1) and source S are valid and exist only in a half-space. We assume perfectly reflecting boundary conditions on this half-space illustrated below.

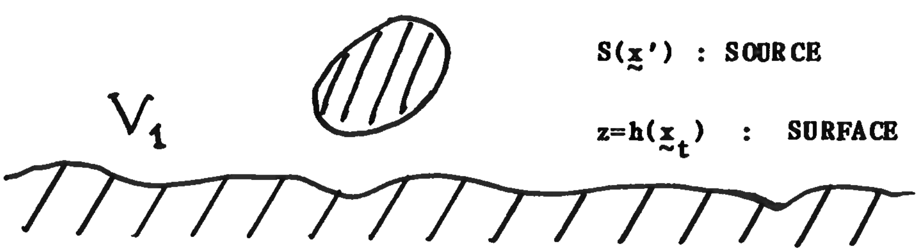


Fig. 2.2

Next cross multiply the equations to form

$$G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}')(\partial'_j\partial'_j + k_0^2)\phi(\underset{\sim}{x}')$$

$$-\left[(\partial'_j\partial'_j + k_0^2)G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}')\right]\phi(\underset{\sim}{x}') = G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}')S(\underset{\sim}{x}')$$

$$+ \phi(\underset{\sim}{x}')\delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad . \qquad (2.3)$$

The k_0^2 terms cancel and the left hand side can be written as a divergence. The result is

$$\phi(\underset{\sim}{x}')\delta(\underset{\sim}{x}-\underset{\sim}{x}') = G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}')S(\underset{\sim}{x}') + \partial'_j F_j(\underset{\sim}{x},\underset{\sim}{x}') \quad , \qquad (2.4)$$

where

$$F_j(\underset{\sim}{x},\underset{\sim}{x}') = G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}')\partial'_j\,\phi(\underset{\sim}{x}') - \left[\partial'_j\,G(\underset{\sim}{x},\underset{\sim}{x}')\right]\phi(\underset{\sim}{x}') \quad . \qquad (2.5)$$

If we multiply (2.4) by the step function

$$\Theta(z' - h(\underset{\sim}{x}'_t)) \quad ,$$

and integrate the result over $\underset{\sim}{x}'$, the result is restricted to V_1 in the figure by the step function, and we get

$$\phi(\underset{\sim}{x})\Theta(z - h(\underset{\sim}{x}_t)) = \iiint_{V_1} G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}')S(\underset{\sim}{x}')d\underset{\sim}{x}'$$

$$+ \iiint \left[\partial'_j\,F_j(\underset{\sim}{x},\underset{\sim}{x}')\right]\Theta(z' - h(\underset{\sim}{x}'_t))d\underset{\sim}{x}' \quad , \qquad (2.6)$$

where the source integral is restricted even further by its support (which must be in V_1). The latter integral in (2.6) can be integrated by parts to give

$$\iiint \left[\partial_j' \, F_j(\underset{\sim}{x},\underset{\sim}{x}')\right]\theta(z' - h(\underset{\sim}{x}_t'))d\underset{\sim}{x}'$$

$$= - \iiint F_j(\underset{\sim}{x},\underset{\sim}{x}') \, \partial_j' \, \theta(z' - h(\underset{\sim}{x}_t'))d\underset{\sim}{x}' \quad . \qquad (2.7)$$

It can easily be seen that since the integral is over all space, each integrated term vanishes. For example the j=1 integrated term is

$$F_1(\underset{\sim}{x},\underset{\sim}{x}')\theta(z' - h(\underset{\sim}{x}_t'))\Big|_{x'=-\infty}^{x=+\infty} = 0 \quad .$$

The step function derivative in (2.7) is

$$\partial_j' \, \theta(z' - h(\underset{\sim}{x}_t')) = \delta(z' - h(\underset{\sim}{x}_t'))n_j(\underset{\sim}{x}_t') \quad , \qquad (2.8)$$

where the delta function is the characteristic function of the surface and n_j is

$$n_j(\underset{\sim}{x}_t') = \delta_{j3} - \partial_{jt}' h(\underset{\sim}{x}_t') \quad , \qquad (2.9)$$

which is a vector in the direction of the surface normal (but not a unit vector) and the derivative terms for j=1 and j=2 are

$$h_{x'} = \frac{\partial}{\partial x'} h(\underset{\sim}{x}_t') \quad , \quad h_{y'} = \frac{\partial}{\partial y'} h(\underset{\sim}{x}_t') \quad , \qquad (2.10)$$

which are the surface slopes. We assume the surface is differentiable. The result yields for (2.6)

$$\phi(\underset{\sim}{x})\theta(z - h(\underset{\sim}{x}_t')) = \phi^{in}(\underset{\sim}{x}) + \phi^{sc}(\underset{\sim}{x}) \quad , \qquad (2.11)$$

with the total field (on the lhs) given in terms of the incident field

$$\phi^{in}(\underset{\sim}{x}) = \iiint\limits_{V_1} G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}')S(\underset{\sim}{x}')d\underset{\sim}{x}' \quad , \tag{2.12}$$

and the scattered field

$$\phi^{sc}(\underset{\sim}{x}) = - \iiint F_j(\underset{\sim}{x},\underset{\sim}{x}')n_j(\underset{\sim}{x}_t')\delta(z' - h(\underset{\sim}{x}_t'))d\underset{\sim}{x}' \quad . \tag{2.13}$$

We can evaluate the delta function in (2.13) by setting the vector $\underset{\sim}{x}'$ on the surface, $\underset{\sim}{x}_s'=(\underset{\sim}{x}_t,h(\underset{\sim}{x}_t))$, to yield

$$\phi^{sc}(\underset{\sim}{x}) = - \iint F_j(\underset{\sim}{x},\underset{\sim}{x}_s')n_j(\underset{\sim}{x}_t')d\underset{\sim}{x}_t' \quad . \tag{2.14}$$

As an example, if S is a point source

$$S(\underset{\sim}{x}') = \delta(\underset{\sim}{x}' - \underset{\sim}{x}'') \quad ,$$

the incident field is that field due to the point source evaluated at the field point $\underset{\sim}{x}$. Since $G^{(3)}$ satisfies our outgoing radiation condition it is

$$\phi^{in}(\underset{\sim}{x}) = G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}'') \quad .$$

In general we can write the integral representation for ϕ with $z\varepsilon V_1$ as (from (2.11), (2.13) and (2.5))

$$\phi(\underset{\sim}{x}) = \phi^{in}(\underset{\sim}{x}) + \iint\Big[N^{(3)}(\underset{\sim}{x},\underset{\sim}{x}')\phi(x_s') - G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}_s')N(\underset{\sim}{x}_s')\Big]d\underset{\sim}{x}_t' \quad , \tag{2.15}$$

in terms of the normal derivative of the field evaluated on the surface

$$N(\underset{\sim}{x}'_s) = n_j(\underset{\sim}{x}'_t)\partial'_j\, \phi(\underset{\sim}{x}'_s) \quad , \tag{2.16}$$

and the normal derivative of the Green's function

$$N^{(3)}(\underset{\sim}{x},\underset{\sim}{x}'_s) = n_j(\underset{\sim}{x}'_t)\partial'_j\, G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}'_s) \quad . \tag{2.17}$$

The representation (2.15) is called the Helmholtz-Kirchhoff Representation of the field. Our boundary value problem consists in specifying either N or ϕ on the surface and then constructing an integral equation on the remaining boundary value. There are several ways to do this which we now describe.

Eg. 1. FIRST KIND EQUATION FOR N

We assume that ϕ satisfies a Dirichlet boundary condition on the surface, i.e.

$$\phi(\underset{\sim}{x}_s) = 0 \quad . \tag{2.18}$$

From (2.15) we thus have that in the limit as $z \to h(\underset{\sim}{x}_t)$ (where $z \varepsilon V_1$) so that $\underset{\sim}{x} \to \underset{\sim}{x}_s$, the lhs vanishes and we get

$$\phi^{in}(\underset{\sim}{x}_s) = \iint G^{(3)}(\underset{\sim}{x}_s, x'_s) N(\underset{\sim}{x}'_s) d\underset{\sim}{x}'_t \quad , \tag{2.19}$$

which is an integral equation of first kind for N. Both ϕ^{in} and $G^{(3)}$ are known, and, as we noted in Ch. 1, $G^{(3)}$ is continuous at the boundary. The square root term in its denominator is an integrable singularity. Once we solve (2.19) for N, we substitute the result into (2.15) using (2.18) to yield the field representation

$$\phi(\underset{\sim}{x}) = \phi^{in}(\underset{\sim}{x}) - \iint G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}_s')N(\underset{\sim}{x}_s')d\underset{\sim}{x}_t' \quad . \tag{2.20}$$

Eg. 2. SECOND KIND EQUATION FOR ϕ

ϕ satisfies the Neumann boundary condition on the surface, i.e.

$$N(\underset{\sim}{x}_s) = 0 \quad . \tag{2.21}$$

In the limit as $z \to h(\underset{\sim}{x}_t)$, $\underset{\sim}{x} \to \underset{\sim}{x}_s$ the lhs of (2.15) goes to $\phi(\underset{\sim}{x}_s)$. To find the limit of the normal derivative of the Green's function in (2.15) we recall from Ch. 1 Sec. 5 that we can write

$$\partial_j' G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}_s') = -\frac{1}{2} R_j(\underset{\sim}{x}-\underset{\sim}{x}_s') + \frac{1}{2}\delta_{j3}\,\mathrm{sgn}(z'-h(\underset{\sim}{x}_t'))\delta(\underset{\sim}{x}_t-\underset{\sim}{x}_t') \quad , \tag{2.22}$$

which is analogous to Eq. (5.20) in Ch. 1 except that here we are differentiating on the source coordinate in $G^{(3)}$ and we thus have an overall minus sign. The limit of the integral resulting from (2.15) thus yields

$$\frac{1}{2}\phi(\underset{\sim}{x}_s) = \phi^{in}(\underset{\sim}{x}_s) - \frac{1}{2}\iint P(\underset{\sim}{x}_s,\underset{\sim}{x}')\phi(\underset{\sim}{x}_s)d\underset{\sim}{x}_t' \quad ,$$

or

$$\phi(\underset{\sim}{x}_s) = 2\phi^{in}(\underset{\sim}{x}_s) - \iint P(\underset{\sim}{x}_s,\underset{\sim}{x}_s')\phi(\underset{\sim}{x}_s')d\underset{\sim}{x}_t' \quad , \tag{2.23}$$

where the function P is defined as

$$P(\underset{\sim}{x}_s,\underset{\sim}{x}_s') = n_j(\underset{\sim}{x}_t')R_j(\underset{\sim}{x}_s-\underset{\sim}{x}_s') \quad , \tag{2.24}$$

in terms of the regular part R_j. The latter is defined in Eq. (5.21) in Ch. 1. The result, (2.23), is an <u>integral equation of second kind for ϕ</u>. The result when solved is substituted back into (2.15) to give with (2.21) the resulting field expression for ϕ

$$\phi(\underset{\sim}{x}) = \phi^{in}(\underset{\sim}{x}) + \iint N^{(3)}(\underset{\sim}{x},\underset{\sim}{x}_s')\phi(\underset{\sim}{x}_s')d\underset{\sim}{x}_t' \quad . \tag{2.25}$$

A Born approximation to (2.23) (i.e. neglecting the integral term) illustrates that for a vanishing normal derivative on the surface, the field on the surface is twice the incident field. This is also equivalent to a Kirchhoff approximation for a reflection coefficient equal to one. There is a well developed theory for solving integral equations of second kind. First kind equations are more difficult to solve in general. (Refs. 2.2, 2.3 and 2.4.)

We have found an integral equation of first kind for N for the Dirichlet problem, (2.19), and a second kind equation for ϕ for the Neumann problem, (2.23). We can also find a second kind equation for N and a first kind equation for ϕ. To do this, differentiate (2.15) and multiply by the normal to get

$$\begin{aligned} n_m(\underset{\sim}{x}_t)\partial_m\phi(\underset{\sim}{x}) = {} & n_m(\underset{\sim}{x}_t)\partial_m\phi^{in}(\underset{\sim}{x}) \\ & + \iint\Big[n_m(\underset{\sim}{x}_t)\partial_m N^{(3)}(\underset{\sim}{x},\underset{\sim}{x}_s')\phi(\underset{\sim}{x}_s') \\ & - n_m(\underset{\sim}{x}_t)\partial_m G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}_s')N(\underset{\sim}{x}_s')\Big]d\underset{\sim}{x}_t' \quad . \end{aligned} \tag{2.26}$$

<u>**Eg. 3. SECOND KIND EQUATION ON N**</u>

Let ϕ satisfy the Dirichlet boundary condition (2.18). Substitute the result in (2.26) and take the surface limit as $\underset{\sim}{x}\rightarrow\underset{\sim}{x}_s$. The normal derivative of $G^{(3)}$ produces a regular part plus a jump discontinuity. Here the differentiation is on the field variable of $G^{(3)}$ and the result (5.20) applies. The resuting limit of (2.26) is

$$N(\underset{\sim}{x}_s) = 2\, N^{in}(\underset{\sim}{x}_s) - \iint \bar{P}(\underset{\sim}{x}_s, \underset{\sim}{x}_s') N(\underset{\sim}{x}_s') d\underset{\sim}{x}_t' \quad , \tag{2.27}$$

where $N(\underset{\sim}{x}_s)$ is defined by (2.16), N^{in} is defined by

$$N^{in}(\underset{\sim}{x}_s) = n_m(\underset{\sim}{x}_t)\partial_m \phi^{in}(\underset{\sim}{x}_s) \quad , \tag{2.28}$$

and $\bar{P}$ differs from P in (2.24) in that the normal is a function of the exterior variable, i.e.

$$\bar{P}(\underset{\sim}{x}_s, \underset{\sim}{x}_s') = n_m(\underset{\sim}{x}_t) R_m(\underset{\sim}{x}_s - \underset{\sim}{x}_s') \quad , \tag{2.29}$$

defined in terms of the regular part R_m in (5.20) in Ch. 1. The result (2.27) is an <u>integral equation of second kind for N</u>. Its Born approximation is that N on the surface is just twice the normal derivative of the incident field, and this is the same as the Kirchhoff approximation for a reflection coefficient equal to −1.

<u>Eg. 4. FIRST KIND EQUATION FOR ϕ</u>

For the Neumann boundary condition (2.21) the limit as $\underset{\sim}{x} \to \underset{\sim}{x}_s$ of the lhs of (2.26) vanishes as does the second term in the integral. Recall from Ch. 1 that we can regularize the second derivative of $G^{(s)}$ as

$$\partial_m \partial_j' G^{(s)}(\underset{\sim}{x}, \underset{\sim}{x}') = R_{mj}(\underset{\sim}{x} - \underset{\sim}{x}')$$

$$+ \frac{1}{2}\, \mathrm{sgn}(z - z') \left[\delta_{m3} \partial_{jt} + \delta_{j3} \partial_{mt} \right] \delta(\underset{\sim}{x}_t - \underset{\sim}{x}_t') \quad , \tag{2.30}$$

where here we are differentiating once with respect to the source (primed) variable and once with respect to the field (unprimed) variable (see (2.16)). The regular part R_{mj} is defined in (5.55) of Ch. 1. The resulting limit of (2.26) becomes

$$N^{in}(\underset{\approx}{x}_s) = -\iint Q(\underset{\approx}{x}_s, \underset{\approx}{x}_s')\phi(\underset{\approx}{x}_s')d\underset{\approx}{x}_t'$$

$$-\frac{1}{2}\iint\left[(n_{jt}(\underset{\approx}{x}_t')\partial_{jt} + n_m(\underset{\approx}{x}_t)\partial_{mt})\delta(\underset{\approx}{x}_t-\underset{\approx}{x}_t')\right]\phi(\underset{\approx}{x}_s')d\underset{\approx}{x}_t' \quad , \tag{2.31}$$

where

$$Q(\underset{\approx}{x}_s, \underset{\approx}{x}_s') = n_m(\underset{\approx}{x}_t)R_{mj}(\underset{\approx}{x}_s-\underset{\approx}{x}_s')n_j(\underset{\approx}{x}_t') \quad . \tag{2.32}$$

The delta function terms in (2.31) all vanish because they integrate to the normal derivative of the field evaluated on the surface and this is assumed to vanish. The final result is an <u>integral equation of first kind on</u> ϕ,

$$-N^{in}(\underset{\approx}{x}_s) = \iint Q(\underset{\approx}{x}_s, \underset{\approx}{x}_s')\phi(\underset{\approx}{x}_s')d\underset{\approx}{x}_t' \quad . \tag{2.33}$$

2.3 RAYLEIGH-SOMMERFELD DIFFRACTION FORMULAE

From (2.15) if we assume that S=0 so $\phi^{in}=0$ (zero source condition), and that we have a problem with a flat geometry, it is possible to find a Green's function using an image source which either vanishes on the (flat) boundary or whose normal (i.e. z) derivative vanishes. Our free-space retarded Green's function is

$$G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{e^{ik_o|\underset{\sim}{x}-\underset{\sim}{x}'|}}{4\pi|\underset{\sim}{x}-\underset{\sim}{x}'|} \quad . \tag{3.1}$$

Boundary Green's functions can be written as

$$G_\pm(\underset{\sim}{x},\underset{\sim}{x}') = G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}') \pm G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}_i') \quad , \tag{3.2}$$

where $\underset{\sim}{x}=(x,y,z)$, $\underset{\sim}{x}'=(x',y',z')$, and $\underset{\sim}{x}_i'=(x',y',-z')$. It is easily seen that on the z'=0 plane

$$G_-(\underset{\sim}{x},\underset{\sim}{x}_t') = 0 \quad , \tag{3.3}$$

and

$$\frac{\partial}{\partial z'} G_+(\underset{\sim}{x},\underset{\sim}{x}_t') = 0 \quad . \tag{3.4}$$

Similarly it can be shown that for z'=0

$$G_+(\underset{\sim}{x},\underset{\sim}{x}_t') = 2G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}_t') \quad , \tag{3.5}$$

and

$$\partial_j' G_-(\underset{\sim}{x},\underset{\sim}{x}_t')n_j(\underset{\sim}{x}_t') = \frac{\partial}{\partial z'} G_-(\underset{\sim}{x},\underset{\sim}{x}_t') = -2\frac{\partial}{\partial z} G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}_t') \quad , \tag{3.6}$$

where the latter relation in (3.6) is written in terms of differentiation on the field coordinate z.

If we choose $G^{(3)} = G_+$ in (2.15) we get the result that, for the flat boundary, $(\partial G_+/\partial z' = N^{(3)} = 0)$ the field representation is

$$\phi(\underset{\sim}{x}) = -2 \iint G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}_t')N(\underset{\sim}{x}_t')d\underset{\sim}{x}_t' \quad , \tag{3.7}$$

where

$$N(\underset{\sim}{x}_t') = \frac{\partial}{\partial z'}\phi(\underset{\sim}{x}_t') \quad . \tag{3.8}$$

If we choose $G^{(3)}=G_-$ in (2.15) so that on the flat surface $G^{(3)}=G_-=0$ and $N^{(3)}=-2\ \partial G_R^{(3)}/\partial z$ we get the result

$$\phi(\underset{\sim}{x}) = -2 \frac{\partial}{\partial z} \iint G_R^{(3)}(\underset{\sim}{x},\underset{\sim}{x}_t')\phi(\underset{\sim}{x}_t')d\underset{\sim}{x}_t' \quad . \tag{3.9}$$

Equations (3.7) and (3.9) are the <u>Rayleigh-Sommerfeld diffraction formulae</u> which yields the field value in terms of either boundary conditions on N or ϕ. Note that the formulae are <u>not</u> useful for Dirichlet or Neumann type problems. (Refs. 2.5 and 2.6.)

One advantage these formulas have is that they are self-consistent, i.e. as $\underset{\sim}{x}\rightarrow\underset{\sim}{x}_t$ $(z\rightarrow 0)$ the limit of the function $\phi(\underset{\sim}{x})$ in the field is equal to whatever is assumed for ϕ on the surface, and similarly for N. For example, the limit of the lhs of (3.9) is $\phi(\underset{\sim}{x}_t)$. The limit of the rhs can be found from our regularization of the derivative of $G_R^{(3)}$ from (5.20). For j=3 it is

$$\frac{\partial}{\partial z} G_R^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}_t') = \frac{1}{2} R_3(\underset{\sim}{x}-\underset{\sim}{x}_t) - \frac{1}{2} \operatorname{sgn}(z)\delta(\underset{\sim}{x}_t-\underset{\sim}{x}_t') \quad ,$$

where from (5.20) and (5.18)

$$R_3(\underset{\sim}{x}-\underset{\sim}{x}_t') = \frac{2i}{(2\pi)^3}\iiint e^{i\underset{\sim}{k}\cdot[\underset{\sim}{x}-\underset{\sim}{x}_t']}\,\tilde{G}_R^{(3)}(k)\;P\left[\frac{K_t^2}{k_z}\right]d\underset{\sim}{k} \quad .$$

In the limit of z=0, R_3=0 since it is the integral of an odd function of k_z $\tilde{G}_R^{(3)}$ is an even function of k_z, and the exponential is not a function of k_z. The result is

$$\lim_{z\to 0}\ \frac{\partial}{\partial z}\ G_R^{(3)}(\underset{\sim}{x}-\underset{\sim}{x}_t') = -\frac{1}{2}\,\delta(\underset{\sim}{x}_t-\underset{\sim}{x}_t') \quad ,$$

which when substituted into (3.9) yields the self-consistent result that $\phi(\underset{\sim}{x}_t) = \phi(\underset{\sim}{x}_t)$. Similarly, if we differentiate (3.7) with respect to z and take the limit as z→0 we get the self-consistent result $N(\underset{\sim}{x}_t) = N(\underset{\sim}{x}_t)$.

The Rayleigh-Sommerfeld formulae can be used for other geometries where it is possible to find image-type Green's functions, i.e. geometries containing canonical shapes such as cylinders, spheres, etc. They are also a useful starting point for geometries where the shape is nearly canonical, i.e. where the shape can be defined in a perturbation sense as canonical plus a small correction.

It can also be shown that the Rayleigh-Sommerfeld diffraction formulae are consistent with the Kirchhoff boundary conditions on ϕ or N (see Sec. 6) except at the edge of an aperture. (Ref. 2.7.)

2.4 EXTENDED BOUNDARY CONDITION

In (2.11) we established the integral relation for the field

$$\phi(\underset{\sim}{x})\Theta(z - h(\underset{\sim}{x}_t)) = \phi^{in}(\underset{\sim}{x}) + \phi^{sc}(\underset{\sim}{x}) \quad , \tag{4.1}$$

where

$$\phi^{in}(\underset{\sim}{x}) = \iiint_{V_1} G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}'_s)S(\underset{\sim}{x}')d\underset{\sim}{x}' \quad , \tag{4.2}$$

and

$$\phi^{sc}(\underset{\sim}{x}) = \iint\left[N^{(3)}(\underset{\sim}{x},\underset{\sim}{x}'_s)\phi(\underset{\sim}{x}'_s) - G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}'_s)N(\underset{\sim}{x}'_s)\right]d\underset{\sim}{x}'_t \quad . \tag{4.3}$$

The total field in V_1 is thus due to volume sources S, and to a layer of point and dipole sources on the surface h with source densities $N(x'_s)$ and $\phi(\underset{\sim}{x}'_s)$ respectively.

If we assume that $z\varepsilon V_2$, the lhs of (4.1) is zero and the result is a volume boundary condition

$$\phi^{in}(\underset{\sim}{x}) + \phi^{sc}(\underset{\sim}{x}) = 0 \qquad \underset{\sim}{x}\varepsilon V_2 \quad , \tag{4.4}$$

called either the extended boundary condition (Ref. 2.8), the extinction coefficient (Ref. 2.9) or the null field equation (Ref. 2.10). It is just that boundary condition on the scattered field, hence on the point and dipole sources, necessary to extinguish the incident field <u>everywhere</u> in V_2. Hence it is a volume and not a surface boundary condition. The induced surface fields extinguish the incident field everywhere below the surface, and directly incorporates into the solution of the problem the fact that the field in region V_2 must vanish identically.

2.5 T-MATRIX

Our field representation from (2.11) is

$$\phi(\underset{\sim}{x}) = \phi^{in}(\underset{\sim}{x}) + \phi^{sc}(\underset{\sim}{x}) \qquad \underset{\sim}{x} \varepsilon V_1 \quad . \tag{5.1}$$

We choose the Dirichlet boundary condition (2.18) so that

$$\phi^{sc}(\underset{\sim}{x}) = - \iint G^{(3)}(\underset{\sim}{x}, \underset{\sim}{x}'_s) N(\underset{\sim}{x}'_s) d\underset{\sim}{x}'_t \quad , \tag{5.2}$$

which follows from (2.15). Use the Weyl representation for $G^{(3)}=G_R^{(3)}$ from (4.11) in Ch. 1 where z' is evaluated on the surface

$$G_R^{(3)}(\underset{\sim}{x}, \underset{\sim}{x}'_s) = \frac{\pi i}{(2\pi)^3} \iint \frac{\exp\left[i\underset{\sim}{k}_t \cdot (\underset{\sim}{x}_t - \underset{\sim}{x}'_t) + iK_+ \left| z - h(\underset{\sim}{x}'_t)\right|\right]}{K_+} d\underset{\sim}{k}_t \quad , \tag{5.3}$$

where $K_+=(k_{o+}^2-k_t^2)^{1/2}$. Assume z is greater than the highest surface excursion so that the absolute value in (5.3) can be dropped. Substitute the result in (5.2) so that we can write

$$\phi^{sc}(\underset{\sim}{x}) = \iint \exp\left[i(\underset{\sim}{k}_t \cdot \underset{\sim}{x}_t + K_+ z)\right] T(\underset{\sim}{k}_t)\, d\underset{\sim}{k}_t \quad , \tag{5.4}$$

where

$$T(\underset{\sim}{k}_t) = \frac{-\pi i}{(2\pi)^3 K} \iint \exp\left[-i\left[\underset{\sim}{k}_t \cdot \underset{\sim}{x}'_t - K_+ h(\underset{\sim}{x}'_t)\right]\right] N(\underset{\sim}{x}'_s) d\underset{\sim}{x}'_t \quad . \tag{5.5}$$

Thus ϕ^{sc} can be expressed as a sum (integral) of upgoing (propagating) or decaying (evanescent) waves in the positive z-direction. T is derived from the surface source density N and is called the T-matrix. It is directly related to the scattering cross section.

In order to solve for N and hence T (by (5.5)) and hence the scattered field in V_1 (by (5.4)) we use the extended boundary condition (4.4). Then for $\underset{\sim}{x} \varepsilon V_2$, (5.2) becomes

$$\phi^{in}(\underset{\sim}{x}) = \iint G^{(3)}(\underset{\sim}{x}, \underset{\sim}{x}_s') \; N(\underset{\sim}{x}_s') \; d\underset{\sim}{x}_t' \quad . \tag{5.6}$$

Assume z is less than the lowest surface excursion so that the absolute value in (5.3) can be dropped. The result inserted in (5.6) yields

$$(2\pi)^3(\pi i)^{-1}K \; \tilde{\phi}^{in}(\underset{\sim}{k}_t) = \iint \exp\left[-i\underset{\sim}{k}_t \cdot \underset{\sim}{x}_t' + iK_+ h(\underset{\sim}{x}_t')\right] N(\underset{\sim}{x}_s') d\underset{\sim}{x}_t' \tag{5.7}$$

which is used to determine the surface density N in terms of the Fourier transform of the incident field.

This method has been extended to scattering from a multi-layered single body and to multi-bodies. It is also possible to develop a way to find T directly in terms of quantities which do not directly involve the surface fields. (Ref. 2.11.)

2.6 KIRCHHOFF APPROXIMATION

The Helmholtz-Kirchhoff integral representation for the field is given by (2.15) as

$$\phi(\underset{\sim}{x}) = \phi^{in}(\underset{\sim}{x}) + \iint\left[N^{(3)}(\underset{\sim}{x},\underset{\sim}{x}_s')\phi(\underset{\sim}{x}_s') - G^{(3)}(\underset{\sim}{x},\underset{\sim}{x}_s')N(\underset{\sim}{x}_s')\right]d\underset{\sim}{x}_t' \quad . \tag{6.1}$$

The Kirchhoff approximation consists in assuming <u>both</u> the surface and normal surface derivative values of the fields, i.e. both $\phi(\underset{\sim}{x}_s')$ and $N(\underset{\sim}{x}_s')$. To motivate the choice of boundary conditions we consider plane wave scattering from a flat interface. The total field is

$$\phi(\underset{\sim}{x}) = \phi^{in}(\underset{\sim}{x}) + R\,\phi^{sc}(\underset{\sim}{x}) \quad , \tag{6.2}$$

where R is the reflection coefficient and ϕ^{in} and ϕ^{sc} are incident and scattered plane waves, the latter of which is specular so that

$$\phi^{in}(\underset{\sim}{x}) = \exp\left[ik_1\left[\alpha_0 x + \beta_0 y - \gamma_0 z\right]\right] \quad , \tag{6.3}$$

and

$$\phi^{sc}(\underset{\sim}{x}) = \exp\left[ik_1\left[\alpha_0 x + \beta_0 y + \gamma_0 z\right]\right] \quad , \tag{6.4}$$

where α_0, β_0, γ_0 are the direction cosines of the waves. On the surface z=0

$$\phi(\underset{\sim}{x}_t,0) = (1 + R)\phi^{in}(\underset{\sim}{x}_t,0) \quad , \tag{6.5}$$

and

$$\frac{\partial}{\partial z}\,\phi(\underset{\sim}{x}_t,0) = -ik_1\gamma_0(1 - R)\phi^{in}(\underset{\sim}{x}_t,0) \quad . \tag{6.6}$$

We assume the true interface is gently undulating so that we can replace the z-derivative by the normal derivative. Also we evaluate the terms on the true surface, not z=0. We thus have the approximate boundary conditions

$$\phi(\underset{\sim}{x}_s') = (1 + R)\,\phi^{in}(\underset{\sim}{x}_s') \equiv \phi^{o}(\underset{\sim}{x}_s') \quad , \tag{6.7}$$

and

$$N(\underset{\sim}{x}_s') = -\,ik_1\gamma_o(1 - R)\phi^{in}(\underset{\sim}{x}_s') \equiv N^{o}(\underset{\sim}{x}_s') \quad . \tag{6.8}$$

Then the field is known from (6.1).

Note that in the limit as $\underset{\sim}{x} \to \underset{\sim}{x}_s$ or $z \to h(\underset{\sim}{x}_t)$ the result of (6.1) is using (6.7) and (6.8)

$$\begin{aligned}\phi(\underset{\sim}{x}_s) = \phi^{in}(\underset{\sim}{x}_s) &- \iint G^{(3)}(\underset{\sim}{x}_s,\underset{\sim}{x}_s')N^{o}(\underset{\sim}{x}_s')d\underset{\sim}{x}_t' \\ &- \frac{1}{2}\iint P(\underset{\sim}{x}_s,\underset{\sim}{x}_s')\phi^{o}(\underset{\sim}{x}_s')d\underset{\sim}{x}_t' + \frac{1}{2}\,\phi^{o}(\underset{\sim}{x}_s) \quad ,\end{aligned} \tag{6.9}$$

so that we do not in general recover the assumed surface value i.e. $\phi(\underset{\sim}{x}_s)$ <u>is not</u> necessarily $\phi^{o}(\underset{\sim}{x}_s)$. Differentiating (6.1) using the normal derivative merely yields another equation (which is linearly dependent) and no recovery of surface field values which have been assumed.

If we do take the normal derivative of (6.1), i.e. multiply by $n_m(\underset{\sim}{x}_t)\partial_m$ and pass to the surface limit we get

$$N(\underset{\sim}{x}_s) = N^{in}(\underset{\sim}{x}_s) + \iint Q(\underset{\sim}{x}_s,\underset{\sim}{x}_s')\phi^0(\underset{\sim}{x}_s')d\underset{\sim}{x}_t'$$

$$+ \frac{1}{2}\iiint\Big[(n_{jt}(\underset{\sim}{x}_t')\partial_{jt} + n_{mt}(\underset{\sim}{x}_t)\partial_{mt})\delta(\underset{\sim}{x}_t - \underset{\sim}{x}_t')\Big]\phi^0(\underset{\sim}{x}_s')d\underset{\sim}{x}_t$$

$$- \frac{1}{2}\iint\bar{P}(\underset{\sim}{x}_s,\underset{\sim}{x}_s')N^0(\underset{\sim}{x}_s')d\underset{\sim}{x}_t'$$

$$+ \frac{1}{2}N^0(\underset{\sim}{x}_s).$$

Integrating the delta function terms gives an additional term N^0 so that

$$N(\underset{\sim}{x}_s) = N^{in}(\underset{\sim}{x}_s) + \frac{3}{2}N^0(\underset{\sim}{x}_s)$$

$$+ \iint\Big[Q(\underset{\sim}{x}_s,\underset{\sim}{x}_s')\phi^0(\underset{\sim}{x}_s') - \frac{1}{2}\bar{P}(\underset{\sim}{x}_s,\underset{\sim}{x}_s')N^0(\underset{\sim}{x}_s')\Big]d\underset{\sim}{x}_t' \quad ,$$

where Q and $\bar{P}$ are defined in (2.32) and (2.29) respectively. Again we do not in general recover the assumed boundary condition.

If however we do assume that we recover the boundary condition so that (6.9) becomes

$$\phi^0(\underset{\sim}{x}_s) = 2\,\phi^{in}(\underset{\sim}{x}_s) - 2\iint G^{(3)}(\underset{\sim}{x}_s,\underset{\sim}{x}_s')N^0(\underset{\sim}{x}_s')d\underset{\sim}{x}_t' \quad ,$$

$$- \iint P(\underset{\sim}{x}_s,\underset{\sim}{x}_s')\phi^0(\underset{\sim}{x}_s')d\underset{\sim}{x}_t' \quad ,$$

and we substitute in (6.7) and (6.8) we can solve for the reflection coefficient to get

$$R = \frac{\phi^{in}(\underset{\sim}{x}_s) - \iint P(\underset{\sim}{x}_s,\underset{\sim}{x}_s')\phi^{in}(\underset{\sim}{x}_s')d\underset{\sim}{x}_t' + 2\,ik_1\gamma_0\iint G^{(3)}(\underset{\sim}{x}_s,\underset{\sim}{x}_s')\phi^{in}(\underset{\sim}{x}_s')d\underset{\sim}{x}_t'}{\phi^{in}(\underset{\sim}{x}_s) + \iint P(\underset{\sim}{x}_s,\underset{\sim}{x}_s')\phi^{in}(\underset{\sim}{x}_s')d\underset{\sim}{x}_t' + 2\,ik_1\gamma_0\iint G^{(3)}(\underset{\sim}{x}_s,\underset{\sim}{x}_s')\phi^{in}(\underset{\sim}{x}_s')d\underset{\sim}{x}_t'} \quad ,$$

which in theory should be independent of x, y and h.

3. ELASTICITY

In this chapter we study the propagation and scattering of waves in elastic media. To do this we derive equations satisfied by the longitudinal and transverse displacement components, discuss the free-space elastic Green's function, and use it to construct integral representations for the full Green's function (or displacement) in terms of values of displacement and traction (stress) on the surrounding surfaces, and other sources which may be present. We illustrate how to find the integral equations for these surface values in terms of a regularized kernel. We further discuss the possible boundary conditions, the plane wave states convenient for layered media problems, and the representation of the displacement in terms of potentials. We also treat the scattering problem at a plane interface for various compressional and shear wave combinations.

3.1 PRELIMINARIES AND EQUATIONS

Define the orthogonal Cartesian coordinate system x_j, $j=1,2,3$, which is sometimes written in vector notation $\underset{\sim}{x} = (x_1, x_2, x_3)$. We use the symbol $u_j(\underset{\sim}{x},t)$ for the three components of elastic displacement. The (symmetric) strain tensor is given by

$$e_{jk} = \tfrac{1}{2}(\partial_j u_k + \partial_k u_j) \quad . \tag{1.1}$$

The stress tensor τ_{jk} is related to the strain by Hooke's law

$$\tau_{jk} = C_{jkpm}\, e_{pm} \quad , \tag{1.2}$$

where repeated indices are summed over (from 1 to 3) and where we have defined the elastic constants C_{jkpm}. Since each subscript runs from 1 to 3, there are in the most general case 81 independent elastic constants. They are really only constant in a homogeneous elastic medium, and we assume this here. For an inhomogeneous elastic medium, the elasticities are in general material functions of position. (See Appendix 3C.)

We reduce the number of independent elastic constants further by using the following symmetry restrictions derived from infinitesimal stress-strain theory:

(a) stress-strain symmetry given by

$$\tau_{jk} = \tau_{kj} \qquad \text{and} \qquad e_{pm} = e_{mp} \quad . \tag{1.3}$$

This implies

$$C_{jkpm} = C_{kjpm} = C_{jkmp} \quad . \tag{1.4}$$

If the indices are thus taken in pairs, eq. $C_{(jk)(pm)}$ then since each pair has 6 independent values (11,22,33,12,13,23) there exist a total of $6 \cdot 6 = 36$

independent elasticities for a body without material symmetry.

(b) There is a second definition of classical linear elasticity resting on the Postulate: The work done by the stress in a deformation depends only on the strain and is recoverable work. This implies an additional symmetry

$$C_{jkpm} = C_{pmjk} \quad . \tag{1.5}$$

Again thinking in terms of pairs of indices, the above imply a number of constraints given by the combinations of 6 things (independent pair values) taken 2 at a time or $\binom{6}{2}=15$ constraints. There are thus 36-15=21 independent elasticities remaining.

(c) A final large number of constraints is introduced by isotropy. For sufficient material symmetry of the body so that the body is an isotropic elastic material, the number of independent elasticities reduces to 2. We can write the remaining non-zero terms as

$$C_{jjjj} = \lambda + 2\mu \quad , \tag{1.6}$$

$$C_{jjkk} = \lambda \quad , \tag{1.7}$$

$$C_{jkjk} = \mu = C_{jkkj} \quad , \tag{1.8}$$

in terms of the Lamé modulus λ and the shear modulus μ. The latter are also often written in terms of Poisson's ratio σ and Young's modulus E as $\lambda=2\mu\sigma/(1-2\sigma)$ and $\mu=E/2(1+\sigma)$. The resulting elastic constants can be summarized as

$$C_{jkpm} = \lambda\delta_{jk}\delta_{pm} + \mu\left[\delta_{jp}\delta_{km} + \delta_{jm}\delta_{kp}\right] \quad , \tag{1.9}$$

which is the most general fourth-rank isotropic tensor with the above symmetries, and Hooke's law written as

$$\tau_{jk} = \lambda\theta\delta_{jk} + \mu\left[\partial_j u_k + \partial_k u_j\right] \quad , \tag{1.10}$$

in terms of the dilatation $\theta = \partial_j u_j$.

EQUATIONS OF MOTION

The basic equations of motion of the vector displacement are just Newton's law, F=ma. The force is the spatial divergence of the stress tensor. Using mass density ρ we can write the equations of motion as

$$\partial_k \tau_{jk} = \rho \frac{\partial^2}{\partial t^2} u_j \quad . \tag{1.11}$$

If we let $u_j(\underset{\sim}{x},t) = \exp(-i\omega t) u_j(\underset{\sim}{x})$, i.e. we factor out a harmonic time dependence (so we essentially work in frequency space) we get

$$\partial_k \tau_{jk} + K^2 u_j = 0 \quad , \quad K^2 = \omega^2 \rho \quad . \tag{1.12}$$

Substituting for the stress using Hooke's law

$$\frac{1}{2} \partial_k C_{jkpm} \left[\partial_p u_m + \partial_m u_p\right] + K^2 u_j = 0 \quad . \tag{1.13}$$

For an inhomogeneous medium, the C's would be differentiated. For a homogeneous and anisotropic medium (eq. a crystal) we use the fact that the C's are constant and that $C_{jkpm} = C_{jkmp}$ to write

$$C_{jkpm} \partial_k \partial_p u_m + K^2 u_j = 0 \quad , \tag{1.14}$$

which is the set of equations we work with. If we assume

$$u_j(\underset{\sim}{x}) = \overset{\circ}{u}_j \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x}) \quad ,$$

then the set of equations can be written as

$$(K^2\delta_{jm} - C_{jkpm}k_k k_p)\ \overset{\circ}{u}_m = 0 \quad .$$

This is a set of three homogeneous equations of first degree for $\overset{\circ}{u}_m$. Solutions exist if

$$|K^2\delta_{jm} - C_{jkpm}\ k_k k_p| = 0$$

i.e. if the determinant of coefficients vanishes. This is a cubic equation in ω^2(or K^2) and has three roots, $\omega_i(k)$. Now ω is linear in k so that the wave velocities (group velocities) $\partial\omega/\partial k_j$ are independent of k_j. Velocity of the wave is a function of its direction, not of its frequency. In general in anisotropic bodies we have <u>three</u> different velocities of propagation.

For an <u>isotropic</u> <u>body</u> we will find only <u>two</u> different velocities of propagation. Substitute (1.9) into (1.14) to get

$$\mu\ \partial_m\partial_m u_j + (\lambda + \mu)\ \partial_j\partial_m u_m + K^2 u_j = 0 \quad , \tag{1.15}$$

or in the notation of vector analysis

$$\mu\ \Delta\ \underset{\sim}{u} + (\lambda + \mu)\,\text{grad}\ \underset{\sim}{\nabla}\cdot\underset{\sim}{u} + K^2\underset{\sim}{u} = \underset{\sim}{0} \quad . \tag{1.16}$$

Equivalently we could define the operator

$$\Delta^* = \mu\,\Delta + (\lambda + \mu)\text{ grad div} = (\lambda + 2\mu)\text{ grad div} - \mu\text{ curl curl} \quad , \tag{1.17}$$

which plays the same role in elastic theory that the Laplacian Δ plays in harmonic function theory (e.g. for $\mu=1=-\lambda$, $\Delta^*=\Delta$). Our equation is thus

$$(\Delta^* u)_j + K^2 u_j = 0_j \quad . \tag{1.18}$$

If we decompose the displacement into longitudinal (L) and transverse (T) parts

$$u_j = u_j^L + u_j^T \tag{1.19}$$

where the transverse part is divergenceless (solenoidal) and the longitudinal part curlless (irrotational)

$$\partial_j u_j^T = 0 \quad ; \quad \varepsilon_{imj}\partial_m u_j^L = 0_i \quad , \tag{1.20}$$

then we can write the longitudinal displacement as the divergence of a scalar potential ϕ

$$u_j^L = \partial_j \phi \quad , \tag{1.21}$$

and the transverse part as the curl of a vector potential A_p

$$u_j^T = \varepsilon_{jmp}\,\partial_m A_p \quad . \tag{1.22}$$

We discuss these potentials later in this chapter.

Substituting these results in (1.18) using (1.17) we get

$$(\lambda + 2\mu)\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{u}^{L}) - \mu\underset{\sim}{\nabla}x\underset{\sim}{\nabla}x\underset{\sim}{u}^{T} + K^{2}(\underset{\sim}{u}^{L} + \underset{\sim}{u}^{T}) = \underset{\sim}{0} \quad . \tag{1.23}$$

If we take the curl of this equation we get an equation in only the transverse displacement

$$\underset{\sim}{\nabla}x\left[K^{2}\underset{\sim}{u}^{T} - \mu\underset{\sim}{\nabla}x\underset{\sim}{\nabla}x\underset{\sim}{u}^{T}\right] = \underset{\sim}{0} \quad . \tag{1.24}$$

The divergence of the bracket in (1.24) also vanishes. Up to an additional scalar potential (the Laplacian of which vanishes) we can set the bracket to zero. Using (1.20) this is

$$\Delta\, u_{j}^{T} + k_{T}^{2}\, u_{j}^{T} = 0_{j} \quad , \tag{1.25}$$

where $k_T^2 = K^2/\mu = \omega^2\rho/\mu = \omega^2/c_T^2$ is the square of the transverse wave number, and $c_T = (\mu/\rho)^{1/2}$ the transverse wave speed. Similarly, taking the divergence of (1.23) we wind up setting another solenoidal and irrotational bracket to zero

$$K^{2}\underset{\sim}{u}^{L} + (\lambda + 2\mu)\,\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{u}^{L}) = \underset{\sim}{0} \quad , \tag{1.26}$$

and if we use the relation Δ = grad div - curlcurl we get

$$\Delta u_{j}^{L} + k_{L}^{2}u_{j}^{2} = 0_{j} \quad , \tag{1.27}$$

where $k_L^2 = K^2/(\lambda + 2\mu) = \omega^2\rho/(\lambda + 2\mu) = \omega^2/c_L$ is the square of the longitudinal wave number, and $c_L = [(\lambda + 2\mu)/\rho]^{1/2}$ is the longitudinal wave speed.

3.2 FREE-SPACE ELASTIC GREEN'S FUNCTION

The free-space elastic Green's function is the tensor solution to the point source generalization of (1.18) which is

$$\left[\Delta^* G^o(\underset{\sim}{x},\underset{\sim}{x}')\right]_{ij} + K^2 G^o_{ij}(\underset{\sim}{x},\underset{\sim}{x}') = -\delta_{ij}\delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad , \tag{2.1}$$

and is explicity given by (see Appendix 3B)

$$G^o_{ij}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{1}{\mu}\,\delta_{ij}G^T(\underset{\sim}{x},\underset{\sim}{x}') + \frac{1}{K^2}\,\partial_i\partial_j\left[G^T(\underset{\sim}{x},\underset{\sim}{x}') - G^L(\underset{\sim}{x},\underset{\sim}{x}')\right] \quad . \tag{2.2}$$

Where G^T and G^L are the scalar free space Green's function with wave numbers k_T and k_L respectively. That is they are

$$G^{T,L}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{\exp(ik_{T,L}|\underset{\sim}{x}-\underset{\sim}{x}'|)}{4\pi|\underset{\sim}{x}-\underset{\sim}{x}'|} \quad . \tag{2.3}$$

We choose the solution satisfying the outgoing radiation condition, i.e. the retarded solution. Note that G^o_{ij} is regular in the sense of the singularities we discussed in Ch. 1. G^T is regular, and the second derivative of G^T contains the same singular terms as the second derivative of G^L, i.e. the singularities are independent of wavenumber and thus cancel. The derivative of (2.2) will occur in later integral equations we derive, and this is regularized in App. 3A.

To prove that G^o is a solution of (2.1) substitute it and use the definition of Δ^* in (1.17) to get

$$\left[\Delta^{*}G^{o}(\underset{\sim}{x},\underset{\sim}{x}')\right]_{ij} = \mu\,\partial_m\partial_m G^{o}_{ij} + (\lambda + \mu)\,\partial_i\partial_m G^{o}_{mj}$$

$$= \mu\,\partial_m\partial_m\left[\mu^{-1}\delta_{ij}G^{T} + K^{-2}\partial_i\partial_j(G^{T} - G^{L})\right]$$

$$+ (\lambda + \mu)\partial_i\left[\mu^{-1}\partial_j G^{T} + K^{-2}\partial_j\partial_m\partial_m(G^{T} - G^{L})\right] \quad .$$

Use the differential equations satisfied by G^{T} and G^{L}, viz.

$$\left[\partial_m\partial_m + k^2_{T,L}\right] G^{T,L}(\underset{\sim}{x},\underset{\sim}{x}') = -\delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad , \tag{2.4}$$

and the wave number values $k_T^2 = K^2/\mu$ and $k_L^2 = K^2/(\lambda + 2\mu)$ to get

$$\left[\Delta^{*}G^{o}(\underset{\sim}{x},\underset{\sim}{x}')\right]_{ij} = -\partial_i\partial_j(G^{T} - G^{L}) - \delta_{ij}k_T^2 G^{T} - \delta_{ij}\delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad ,$$

which can be written as (2.1), i.e.

$$\left[\Delta^{*}G^{o}(\underset{\sim}{x},\underset{\sim}{x}')\right]_{ij} = -K^2 G^{o}_{ij}(\underset{\sim}{x},\underset{\sim}{x}') - \delta_{ij}\delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad .$$

We can also derive the divergence and curl of (2.2) as

$$\partial_m G^{o}_{mp}(\underset{\sim}{x},\underset{\sim}{x}') = (\lambda + 2\mu)^{-1}\partial_p G^{L}(\underset{\sim}{x},\underset{\sim}{x}') \quad , \tag{2.5}$$

and

$$\varepsilon_{ijm}\partial_j G^{o}_{mp}(\underset{\sim}{x},\underset{\sim}{x}') = \mu^{-1}\varepsilon_{ijp}\partial_j G^{T}(\underset{\sim}{x},\underset{\sim}{x}') \quad . \tag{2.6}$$

Note that just as in the scalar case $G^{o}_{ij}(\underset{\sim}{x},\underset{\sim}{x}')$ is a homogeneous function of its arguments and can be written as $G^{o}_{ij}(\underset{\sim}{x}-\underset{\sim}{x}')$.

3.3 SURFACE INTEGRAL EQUATIONS

We derive the surface integral equation satisfied by the full elastic tensor Green's function G_{ij} which satisfies the same differential equation (2.1) as G^o_{ij} and in addition certain boundary conditions which we specify later. The development is analogous to that in the scalar case. G_{ij} satisfies the equation

$$\left[\Delta^* G(\underset{\sim}{x},\underset{\sim}{x}'')\right]_{in} + K^2 G_{in}(\underset{\sim}{x},\underset{\sim}{x}'') = -\delta_{in}\delta(\underset{\sim}{x}-\underset{\sim}{x}'') \quad , \tag{3.1}$$

and we write the equation (2.1) on G^o_{ij} but here differentiate on the source variable

$$\left[\Delta^* G^o(\underset{\sim}{x}',\underset{\sim}{x})\right]_{ij} + K^2 G^o_{ij}(\underset{\sim}{x}',\underset{\sim}{x}) = -\delta_{ij}\delta(\underset{\sim}{x}'-\underset{\sim}{x}) \quad . \tag{3.2}$$

By cross multiplication we form the quantity

$$G^o_{ij}(\underset{\sim}{x}',\underset{\sim}{x})\left[\Delta^* G(\underset{\sim}{x},\underset{\sim}{x}'')\right]_{in} - \left[\Delta^* G^o(\underset{\sim}{x}',\underset{\sim}{x})\right]_{ij} G_{in}(\underset{\sim}{x},\underset{\sim}{x}'') \quad . \tag{3.3}$$

Next write (3.3) in two ways which we then equate. The first way is

$$\begin{aligned}
(a) \quad &= G^o_{ij}(\underset{\sim}{x}',\underset{\sim}{x})\left[-K^2 G_{in}(\underset{\sim}{x},\underset{\sim}{x}'') - \delta_{in}\delta(\underset{\sim}{x}-\underset{\sim}{x}'')\right] \\
&\quad - \left[-K^2 G^o_{ij}(\underset{\sim}{x}',\underset{\sim}{x}) - \delta_{ij}\delta(\underset{\sim}{x}'-\underset{\sim}{x})\right] G_{in}(\underset{\sim}{x},\underset{\sim}{x}'') \\
&= G_{jn}(\underset{\sim}{x},\underset{\sim}{x}'')\delta(\underset{\sim}{x}'-\underset{\sim}{x}) - G^o_{jn}(\underset{\sim}{x}',\underset{\sim}{x})\delta(\underset{\sim}{x}-\underset{\sim}{x}'') \quad ,
\end{aligned} \tag{3.4}$$

where we have used the symmetry of G^o as

$$G^{o}_{nj} = G^{o}_{jn} \quad , \tag{3.5}$$

to write the last term in (3.4). The second way is to use the specific definition of the differential operator given by (1.17) to write

$$\begin{aligned}(b) = {} & G^{o}_{ij}(\underset{\sim}{x}',\underset{\sim}{x})\left[\mu\partial_m\partial_m G_{in}(\underset{\sim}{x},\underset{\sim}{x}'') + (\lambda+\mu)\partial_i\partial_m G_{mn}(\underset{\sim}{x},\underset{\sim}{x}'')\right] \\ & - \left[\mu\partial_m\partial_m G^{o}_{ij}(\underset{\sim}{x}',\underset{\sim}{x}) + (\lambda+\mu)\partial_i\partial_m G^{o}_{mj}(\underset{\sim}{x}',\underset{\sim}{x})\right] G_{in}(\underset{\sim}{x},\underset{\sim}{x}'') \quad .\end{aligned} \tag{3.6}$$

Next factor a divergence term out of this as

$$\begin{aligned}(b) = {} & G^{o}_{ij}(\underset{\sim}{x}',\underset{\sim}{x})\partial_p\Big[\mu\delta_{pm}\partial_m G_{in}(\underset{\sim}{x},\underset{\sim}{x}'') + \mu\delta_{pm}\partial_i G_{mn}(\underset{\sim}{x},\underset{\sim}{x}'') \\ & \qquad + \lambda\delta_{ip}\partial_m G_{mn}(\underset{\sim}{x},\underset{\sim}{x}'')\Big] \\ & - \partial_p\Big[\mu\delta_{pm}\partial_m G^{o}_{ij}(\underset{\sim}{x}',\underset{\sim}{x}) + \mu\delta_{pm}\partial_i G^{o}_{mj}(\underset{\sim}{x}',\underset{\sim}{x}) \\ & \qquad + \lambda\delta_{ip}\partial_m G^{o}_{mj}(\underset{\sim}{x}',\underset{\sim}{x})\Big] G_{in}(\underset{\sim}{x},\underset{\sim}{x}'') \quad ,\end{aligned} \tag{3.7}$$

and finally note that we can factor the divergence out of the full term

$$(b) = \partial_p \Bigg[\Big[G^o_{ij}(\underset{\sim}{x}', \underset{\sim}{x}) \big[\mu\delta_{pm}\partial_m G_{in}(\underset{\sim}{x}, \underset{\sim}{x}'') + \mu\delta_{pm}\partial_i G_{mn}(\underset{\sim}{x}, \underset{\sim}{x}'')$$

$$+ \lambda\delta_{ip}\partial_m G_{mn}(\underset{\sim}{x}, \underset{\sim}{x}'') \big]$$

$$- \big[\mu\delta_{pm}\partial_m G^o_{ij}(\underset{\sim}{x}', \underset{\sim}{x}) + \mu\delta_{pm}\partial_i G^o_{mj}(\underset{\sim}{x}', \underset{\sim}{x})$$

$$+ \lambda\delta_{ip}\partial_m G^o_{mj}(\underset{\sim}{x}', \underset{\sim}{x}) \big] G_{in}(\underset{\sim}{x}, \underset{\sim}{x}'') \Big] \Bigg]$$

$$- \big[\partial_p G^o_{ij}(\underset{\sim}{x}', \underset{\sim}{x}) \big] \big[\mu\delta_{pm}\partial_m G_{in}(\underset{\sim}{x}, \underset{\sim}{x}'') + \mu\delta_{pm}\partial_i G_{mn}(\underset{\sim}{x}, \underset{\sim}{x}'')$$

$$+ \lambda\delta_{ip}\partial_m G_{mn}(\underset{\sim}{x}, \underset{\sim}{x}'') \big]$$

$$+ \big[\mu\delta_{pm}\partial_m G^o_{ij}(\underset{\sim}{x}', \underset{\sim}{x}) + \mu\delta_{pm}\partial_i G^o_{mj}(\underset{\sim}{x}', \underset{\sim}{x})$$

$$+ \lambda\delta_{ip}\partial_m G^o_{mj}(\underset{\sim}{x}', \underset{\sim}{x}) \big] \partial_p G_{in}(\underset{\sim}{x}, \underset{\sim}{x}'') \quad . \tag{3.8}$$

It can easily be seen that the last six terms all cancel so that (b) is the divergence of a triple index object. We write it as

$$(b) = \partial_p \big[\underset{\sim}{x}', \underset{\sim}{x}, \underset{\sim}{x}'' \big]_{pjn} \quad , \tag{3.9}$$

where we define the symbol

$$\big[\underset{\sim}{x}', \underset{\sim}{x}, \underset{\sim}{x}'' \big]_{pjn} = G^o_{ij}(\underset{\sim}{x}', \underset{\sim}{x}) \big[TG(\underset{\sim}{x}, \underset{\sim}{x}'') \big]_{pin}$$

$$- \big[TG^o(\underset{\sim}{x}', \underset{\sim}{x}) \big]_{pij} G_{in}(\underset{\sim}{x}, \underset{\sim}{x}'') \quad , \tag{3.10}$$

with the operator T defined as

$$\left[TG(\underset{\sim}{x},\underset{\sim}{x}'')\right]_{pin} = \mu\partial_p G_{in}(\underset{\sim}{x},\underset{\sim}{x}'') + \mu\partial_i G_{pn}(\underset{\sim}{x},\underset{\sim}{x}'') + \lambda\delta_{ip}\partial_m G_{mn}(\underset{\sim}{x},\underset{\sim}{x}'') \quad , \qquad (3.11)$$

which will be related to the traction of G across the surface. The definition (3.11) follows from the first set of terms in (3.8). Analogously, for the free space Green's function we have

$$\left[T\ G^o(\underset{\sim}{x}',\underset{\sim}{x})\right]_{pij} = \mu\partial_p G^o_{ij}(\underset{\sim}{x}',\underset{\sim}{x}) + \mu\partial_i G^o_{pj}(\underset{\sim}{x}',\underset{\sim}{x}) + \lambda\delta_{ip}\partial_m G^o_{mj}(\underset{\sim}{x}',\underset{\sim}{x}) \quad . \qquad (3.12)$$

Note that the differential operators in (3.11) act on the "field" coordinate $\underset{\sim}{x}$, and in (3.12) act on the "source" coordinate $\underset{\sim}{x}$ of the function. Also note that (3.11) and (3.12) are symmetric in p and i. We could also use (2.5) to simplify the last term in (3.12). Equating (3.4) and (3.9) we get finally

$$G_{jn}(\underset{\sim}{x},\underset{\sim}{x}'')\delta(\underset{\sim}{x}'-\underset{\sim}{x}) = G^o_{jn}(\underset{\sim}{x}',\underset{\sim}{x})\delta(\underset{\sim}{x}-\underset{\sim}{x}'') + \partial_p\left[\underset{\sim}{x}',\underset{\sim}{x},\underset{\sim}{x}''\right]_{pjn} \quad . \qquad (3.13)$$

From this basic identity we are able to start forming integral equations for many different problems.

Eg. 1. TRACTION FREE SURFACE

We specify the surface $z=h(\underset{\sim}{x}_t)$ and we want to find the field above this surface. Multiply (3.13) by the step function $\theta(z-h(\underset{\sim}{x}_t))$ and integrate over all space $\iiint d\underset{\sim}{x}$. The result is

$$G_{jn}(\underset{\sim}{x}',\underset{\sim}{x}'')\theta(z' - h(\underset{\sim}{x}_t'))$$

$$= G^o_{jn}(\underset{\sim}{x}',\underset{\sim}{x}'')\theta(z'' - h(\underset{\sim}{x}_t''))$$

$$+ \iiint \partial_p\left[\underset{\sim}{x}',\underset{\sim}{x},\underset{\sim}{x}''\right]_{pjn} \theta(z - h(\underset{\sim}{x}_t))d\underset{\sim}{x} \quad . \qquad (3.14)$$

The field term on the lhs of (3.14) thus exists provided $z'>h(x_t)$, i.e. assuming the vector field position $\underset{\sim}{x}'$ is above the surface. The source term exists provided $z''>h(\underset{\sim}{x}_t'')$, i.e. provided the source point $\underset{\sim}{x}''$ is above the surface. For the integral term we integrate by parts. Surface terms at infinity vanish either because of the step function or because of the radiation condition, and if we use the fact that

$$\partial_p \theta(z - h(\underset{\sim}{x}_t)) = \delta(z - h(\underset{\sim}{x}_t))n_p(\underset{\sim}{x}_t) \quad , \qquad (3.15)$$

where

$$n_p(\underset{\sim}{x}_t) = \delta_{p3} - \partial_{pt}h(\underset{\sim}{x}_t) \quad , \qquad (3.16)$$

is a vector in the direction of the surface normal (<u>not</u> a unit vector) we get that for $\underset{\sim}{x}'$ and $\underset{\sim}{x}''$ above the surface

$$G_{jn}(\underset{\sim}{x}',\underset{\sim}{x}'') = G^o_{jn}(\underset{\sim}{x}',\underset{\sim}{x}'') - \iint n_p(\underset{\sim}{x}_t)\left[\underset{\sim}{x}',\underset{\sim}{x}_s,\underset{\sim}{x}''\right]_{pjn}d\underset{\sim}{x}_t \quad , \qquad (3.17)$$

where we have evaluated the delta function and $\underset{\sim}{x}_s = (\underset{\sim}{x}_t, h(\underset{\sim}{x}_t))$ is a three-vector evaluated on the surface. Explicitly the symbol term in (3.17) is from (3.10)

$$n_p(\underset{\sim}{x}_t)\left[\underset{\sim}{x}',\underset{\sim}{x}_s,\underset{\sim}{x}''\right]_{pjn}$$

$$= G^o_{ij}(\underset{\sim}{x}',\underset{\sim}{x}_s)n_p(\underset{\sim}{x}_t)\left[T\ G(\underset{\sim}{x}_s,\underset{\sim}{x}'')\right]_{pin}$$

$$- n_p(\underset{\sim}{x}_t)\left[T\ G^o(\underset{\sim}{x}',\underset{\sim}{x}_s)\right]_{pij}G_{in}(\underset{\sim}{x}_s,\underset{\sim}{x}'') \quad . \tag{3.18}$$

If we assume a zero traction (or traction-free or just free) boundary condition given by

$$n_p(\underset{\sim}{x}_t)\left[T\ G(\underset{\sim}{x}_s,\underset{\sim}{x}'')\right]_{pin} = 0_{in} \quad , \tag{3.19}$$

then (3.17) becomes

$$G_{jn}(\underset{\sim}{x}',\underset{\sim}{x}'') = G^o_{jn}(\underset{\sim}{x}',\underset{\sim}{x}'')$$

$$+ \iint n_p(\underset{\sim}{x}_t)\left[T\ G^o(\underset{\sim}{x}',\underset{\sim}{x}_s)\right]_{pij}G_{in}(\underset{\sim}{x}_s,\underset{\sim}{x}'')d\underset{\sim}{x}_t \quad , \tag{3.20}$$

which is our first result, <u>the integral representation of the Green's function at the field point $\underset{\sim}{x}'$ above the surface in terms of its "value" on the surface</u> $G_{in}(\underset{\sim}{x}_s,\underset{\sim}{x}'')$ which however, is unknown. To find an integral equation for the surface value take the surface limit, i.e. let $\underset{\sim}{x}' \rightarrow \underset{\sim}{x}_s'$, a point on the surface. As in the scalar cases we treated, we must regularize the kernel of the transform in (3.20). We do this in Appendix 3A, where from Eq. (A.31) we find that we can write

$$n_p(\underset{\sim}{x}_t)\left[T\ G^{o}(\underset{\sim}{x}',\underset{\sim}{x}_s)\right]_{pij}$$

$$= K_{ji}(\underset{\sim}{x}',\underset{\sim}{x}_s)$$

$$+ 1/2\ \mathrm{sgn}(z'-h(\underset{\sim}{x}_t'))\delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)\ \cdot$$

$$\cdot \left[\delta_{ij} - \delta_{i3}\delta_{jt}h(\underset{\sim}{x}_t) - \Lambda\delta_{j3}\partial_{it}h(\underset{\sim}{x}_t)\right] , \qquad (3.21)$$

where K_{ji} is the regular part and is given explicitly by (A.32) and $\Lambda=\lambda/(\lambda+2\mu)$. Substituting this result in (3.20) and taking the surface limit we get

$$Q_{ji}(\underset{\sim}{x}_t')G_{in}(\underset{\sim}{x}_s',\underset{\sim}{x}'') = G^{o}{}_{jn}(\underset{\sim}{x}_s',\underset{\sim}{x}) + \iint K_{ji}(\underset{\sim}{x}_s',\underset{\sim}{x}_s)G_{in}(\underset{\sim}{x}_s,\underset{\sim}{x}'')d\underset{\sim}{x}_t , \qquad (3.22)$$

where the matrix Q_{ji} is defined as

$$Q_{ji}(\underset{\sim}{x}_t') = 1/2\left[\delta_{ij} + \delta_{i3}\partial_{jt}'h(\underset{\sim}{x}_t') + \Lambda\delta_{j3}\partial_{it}'h(\underset{\sim}{x}_t)\right] . \qquad (3.23)$$

Equation (3.22) is the <u>integral equation for the value of the Green's function on the surface</u>, $G_{in}(\underset{\sim}{x}_s,\underset{\sim}{x}'')$. The procedure is to solve these coupled integral equations (in general computationally, and in general difficult) for the surface values and then substitute them into the integral representation (3.20). An alternative integral equation can be formed if we define the surface Green's function as the lhs of (3.22)

$$G^{s}_{jn}(\underset{\sim}{x}_s',\underset{\sim}{x}'') = Q_{ji}(\underset{\sim}{x}_t)G_{in}(\underset{\sim}{x}_s',\underset{\sim}{x}'') . \qquad (3.24)$$

Using the inverse matrix to Q given by

$$Q_{jm}(\underset{\sim}{x}_t')U_{mp}(\underset{\sim}{x}_t') = \delta_{jp} \quad , \tag{3.25}$$

we can write

$$G_{mn}(\underset{\sim}{x}_s',\underset{\sim}{x}'') = U_{mp}(\underset{\sim}{x}_t')G^s_{pn}(\underset{\sim}{x}_s',\underset{\sim}{x}'') \quad , \tag{3.26}$$

so that (3.22) becomes

$$G^s_{jn}(\underset{\sim}{x}_s',\underset{\sim}{x}'') = G^o_{jn}(\underset{\sim}{x}_s',\underset{\sim}{x}'') + \iint K_{ji}(\underset{\sim}{x}_s',\underset{\sim}{x}_s)U_{ip}(\underset{\sim}{x}_t)G^s_{pn}(\underset{\sim}{x}_s,\underset{\sim}{x}'')d\underset{\sim}{x}_t \quad . \tag{3.27}$$

Explicitly we have, in matrix notion

$$Q(\underset{\sim}{x}_t) = \frac{1}{2}\begin{bmatrix} 1 & 0 & \partial_1 h(\underset{\sim}{x}_t) \\ 0 & 1 & \partial_2 h(\underset{\sim}{x}_t) \\ \Lambda\partial_1 h(\underset{\sim}{x}_t) & \Lambda\partial_2 h(\underset{\sim}{x}_t) & 1 \end{bmatrix} \quad , \tag{3.28}$$

and

$$U(\underset{\sim}{x}_t) = \frac{2}{1-\Lambda\left[\partial_t h(\underset{\sim}{x}_t)\right]^2}\begin{bmatrix} 1-\Lambda\left[\partial_2 h(\underset{\sim}{x}_t)\right]^2 & \Lambda\partial h_1(\underset{\sim}{x}_t)\partial_2 h(\underset{\sim}{x}_t) & -\partial_1 h(\underset{\sim}{x}_t) \\ \Lambda\partial_1 h(\underset{\sim}{x}_t)\partial_2 h(\underset{\sim}{x}_t) & 1-\Lambda\left[\partial_1 h(\underset{\sim}{x}_t)\right]^2 & -\partial_2 h(\underset{\sim}{x}_t) \\ -\Lambda\partial_1 h(\underset{\sim}{x}_t) & -\Lambda\partial_2 h(\underset{\sim}{x}_t) & 1 \end{bmatrix} \tag{3.29}$$

and the field representation (3.20) can be written as

$$G_{jn}(\underset{\sim}{x}',\underset{\sim}{x}'') = G^o_{jn}(\underset{\sim}{x}',\underset{\sim}{x}'') + \iint n_m(\underset{\sim}{x}_t)\left[T\ G^o(\underset{\sim}{x}',\underset{\sim}{x}_s)\right]_{mij}U_{ip}(\underset{\sim}{x}_t)G^s_{pn}(\underset{\sim}{x}_s,\underset{\sim}{x}'')d\underset{\sim}{x}_t \quad . \tag{3.30}$$

Eg. 2. VOLUME SOURCES - TRACTION FREE SURFACE

The source terms in (3.20), (3.22), (3.27) or (3.30) are all due to the free space Green's function G^0. If instead we want to solve for an arbitrary tensor field $u_{in}(\underset{\sim}{x})$ due to the tensor sources $S_{in}(\underset{\sim}{x})$, i.e. when we have the differential equation

$$\left[\Delta^* u(\underset{\sim}{x})\right]_{in} + K^2 u_{in}(\underset{\sim}{x}) = -S_{in}(\underset{\sim}{x}) \quad , \tag{3.31}$$

we multiply (3.20), (3.22), (3.27), or (3.30) from the right by $S_{nq}(\underset{\sim}{x}'')$ and integrate over $\underset{\sim}{x}''$. From (3.27) for example we have that

$$u^s_{jn}(\underset{\sim}{x}'_s) = u^{in}_{jn}(\underset{\sim}{x}'_s) + \iint K_{ji}(\underset{\sim}{x}'_s, x_s) U_{ip}(\underset{\sim}{x}_t) u^s_{pn}(\underset{\sim}{x}_s) d\underset{\sim}{x}_t \quad , \tag{3.32}$$

where

$$u^s_{jn}(\underset{\sim}{x}'_s) = \iiint G^s_{jm}(\underset{\sim}{x}'_s, \underset{\sim}{x}'') \; S_{mn}(\underset{\sim}{x}'') d\underset{\sim}{x}'' \quad , \tag{3.33}$$

and the incident field is

$$u^{in}_{jn}(\underset{\sim}{x}'_s) = \iiint G^0_{jm}(\underset{\sim}{x}'_s, \underset{\sim}{x}'') \; S_{mn}(\underset{\sim}{x}'') d\underset{\sim}{x}'' \quad . \tag{3.34}$$

The function evaluated at a point in the field, $\underset{\sim}{x}'$, i.e. off the surface, is defined via

$$u_{jn}(\underset{\sim}{x}') = \iiint G_{jm}(\underset{\sim}{x}', \underset{\sim}{x}'') \; S_{mn}(\underset{\sim}{x}'') d\underset{\sim}{x}'' \quad . \tag{3.35}$$

For a vector equation, for example, for the displacement u_i we have that including vector source terms

$$(\Delta^* \underset{\sim}{u}(\underset{\sim}{x}))_i + K^2 u_i(\underset{\sim}{x}) = -S_i(\underset{\sim}{x}) \quad , \tag{3.36}$$

the field representation can be found from (3.20) by multiplying from the right by $S_n(\underset{\sim}{x}'')$ and integrating over $\underset{\sim}{x}''$. Using the definition

$$u_j(\underset{\sim}{x}') = \iiint G_{jn}(\underset{\sim}{x}', \underset{\sim}{x}'') \; S_n(\underset{\sim}{x}'') d\underset{\sim}{x}'' \quad , \tag{3.37}$$

we can write this field displacement value in terms of its surface value as

$$u_j(\underset{\sim}{x}') = u_j^{in}(\underset{\sim}{x}') + \iint n_p(\underset{\sim}{x}_t) \left[T \; G^0(\underset{\sim}{x}', \underset{\sim}{x}_s)\right]_{pij} u_i(\underset{\sim}{x}_s) d\underset{\sim}{x}_t \quad , \tag{3.38}$$

where the incident field is analogous to (3.34) if we simply drop the n-index. The surface integral equation can be found from (3.22) for example by the same procedure. It is

$$Q_{ji}(\underset{\sim}{x}_t')u_i(\underset{\sim}{x}_s') = u_j^{in}(x_s') + \iint K_{ji}(\underset{\sim}{x}_s', \underset{\sim}{x}_s) u_i(\underset{\sim}{x}_s) d\underset{\sim}{x}_t \quad . \tag{3.39}$$

Eg. 3. TWO ELASTIC MEDIA

The traction free boundary in examples 1 and 2 was essentially a perfectly reflecting boundary condition. If instead we have two different elastic media joined at our rough interface $z = h(\underset{\sim}{x}_t)$ as in Fig. 3.1

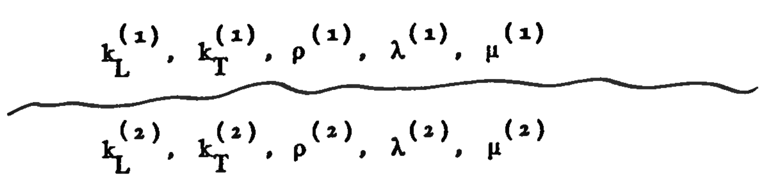


Fig. 3.1

we can still use our identity (3.13) but now must apply it twice. First, multiply by $\theta(z-h(\underset{\sim}{x}_t))$ and integrate over all space $\underset{\sim}{x}$. We get essentially the results of Eg. 1 except that the traction is not zero and all the fields and parameters have superscript (1). We assume, of course, that the source is in region (1). The field representation in region (1) is thus for the displacement vector in analogy to (3.38)

$$u_j^{(1)}(\underset{\sim}{x}') = u_j^{in}(\underset{\sim}{x}') + \iint n_p(\underset{\sim}{x}_t)\left[T\ G^o(\underset{\sim}{x}',\underset{\sim}{x}_s)\right]_{pij}^{(1)} u_i^{(1)}(\underset{\sim}{x}_s)d\underset{\sim}{x}_t$$

$$- \iint G_{ji}^{o(1)}(\underset{\sim}{x}',\underset{\sim}{x}_s)\ T_i^{(1)}(\underset{\sim}{x}_s)d\underset{\sim}{x}_t \quad , \tag{3.40}$$

where we have abbreviated the traction as

$$T_i^{(1)}(\underset{\sim}{x}_s) = n_p(\underset{\sim}{x}_t)\left[T\ u(\underset{\sim}{x}_s)\right]_{pi}^{(1)} \quad . \tag{3.41}$$

The field in region (1) is thus given once its surface values and the surface values of its traction are known. An integral equation for the surface values can be found by letting $\underset{\sim}{x}' \rightarrow \underset{\sim}{x}_s'$ from above the surface. This is exactly the result we had before with the addition of the surface traction term. No regularization of $G^{o(1)}$ is required since it is not singular. The result is

$$Q_{ji}^{(1)}(\underset{\sim}{x}_t')u_i^{(1)}(\underset{\sim}{x}_s') = u_j^{in}(\underset{\sim}{x}_s') + \iint K_{ji}^{(1)}(\underset{\sim}{x}_s',\underset{\sim}{x}_s)u_i^{(1)}(\underset{\sim}{x}_s)d\underset{\sim}{x}_t$$

$$- \iint G_{ji}^{o(1)}(\underset{\sim}{x}_s',\underset{\sim}{x}_s)T_i^{(1)}(\underset{\sim}{x}_s)d\underset{\sim}{x}_t \quad , \tag{3.42}$$

where the Q-matrix follows from (3.23) if we replace the elastic parameters by $\lambda^{(1)}$ and $\mu^{(1)}$.

In region (2) we multiply (3.13) by $\theta(h(\underset{\sim}{x}_t)-z)$ and integrate over all space. There is no source term now since we assumed it to be in region (1) and the field representation in region (2) is thus

$$u_j^{(2)}(\underset{\sim}{x}') = -\iint n_p(\underset{\sim}{x}_t)\left[T\ G^o(\underset{\sim}{x}',\underset{\sim}{x}_s)\right]_{pij}^{(2)} u_i^{(2)}(\underset{\sim}{x}_s)d\underset{\sim}{x}_t$$

$$+\iint G_{ji}^{o(2)}(\underset{\sim}{x}',\underset{\sim}{x}_s)\ T_i^{(2)}(\underset{\sim}{x}_s)d\underset{\sim}{x}_t \quad , \tag{3.43}$$

in terms of surface displacements and tractions which result as we take the limit from region (2). The additional minus sign on the rhs of (3.43) results from the fact that when we integrate the divergence term by parts and differentiate the step function, it has the negative of the argument it had in region (1). The surface integral equation follows analogously by taking the limit $\underset{\sim}{x}' \rightarrow \underset{\sim}{x}_s'$ as $z' \rightarrow h(\underset{\sim}{x}_t')$ from below. We thus get an additional minus sign from the singular terms since the signum function is negative. The resulting surface integral equation is

$$Q_{ji}^{(2)}(\underset{\sim}{x}_t')u_i^{(2)}(\underset{\sim}{x}_s') = -\iint K_{ji}^{(2)}(\underset{\sim}{x}_s',\underset{\sim}{x}_s)u_i^{(2)}(\underset{\sim}{x}_s)d\underset{\sim}{x}_t$$

$$+\iint G_{ji}^{o(2)}(\underset{\sim}{x}',\underset{\sim}{x}_s)T_i^{(2)}(\underset{\sim}{x}_s)d\underset{\sim}{x}_t \quad , \tag{3.44}$$

where the Q-matrix here follows from (3.23) by replacing the elastic paramters by $\lambda^{(2)}$ and $\mu^{(2)}$.

We thus have two vector integral equation (3.42) and (3.44) with four unknown vector quantities on the surface. We require two additional vector boundary conditions, actually continuity conditions, at the boundary interface. They are the continuity of vector displacement and traction (stress) given by

$$u_m^{(1)}(\underset{\sim}{x}_s) = u_m^{(2)}(\underset{\sim}{x}_s) \quad , \tag{3.45}$$

$$T_i^{(1)}(\underset{\sim}{x}_s) = T_i^{(2)}(\underset{\sim}{x}_s) \quad . \tag{3.46}$$

The resulting equations (3.42) and (3.44) are thus two coupled equations for surface displacements and tractions. These are solved and the results used to find the fields in the upper region using (3.40) and in the lower region using (3.43).

Eg. 4. ELASTIC LAYER

For an elastic layer sandwiched between two rough surfaces $h_1(x_t)$ and $h_2(x_t)$ as in Fig. 3.2

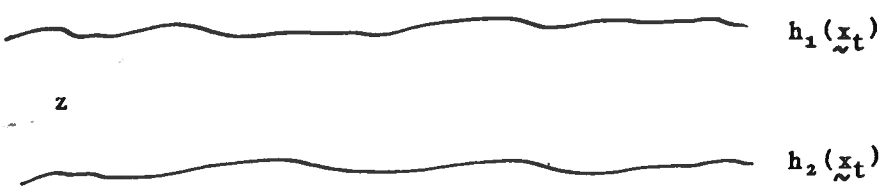


Fig. 3.2

we can again use the identity (3.13). Now however, we multiply it with the product of two step functions

$$\theta(h_1(\underset{\sim}{x}_t)-z)\ \theta(z-h_2(\underset{\sim}{x}_t)) \quad , \tag{3.47}$$

and integrate on $\underset{\sim}{x}$ over all space. The result is for $\underset{\sim}{x}'$ and $\underset{\sim}{x}''$ in the layer

$$G_{jn}(\underset{\sim}{x}',\underset{\sim}{x}'') = G^{o}_{jn}(\underset{\sim}{x}',\underset{\sim}{x}'')$$
$$+ \iiint \partial_p [\underset{\sim}{x}',\underset{\sim}{x},\underset{\sim}{x}'']_{pjn}\theta(h_1(\underset{\sim}{x}_t) - z)\theta(z - h_2(\underset{\sim}{x}_t))d\underset{\sim}{x}_t \tag{3.48}$$

which is written in terms of the symbol defined in (3.10). Integration by parts of this integral produces the derivative of the product of step functions which is given by

$$\partial_p \theta(h_1(\underset{\sim}{x}_t)-z)\theta(z-h_2(\underset{\sim}{x}_t))$$

$$= -n_p^{(1)}(\underset{\sim}{x}_t)\delta(z-h_1(\underset{\sim}{x}_t))\theta(z-h_2(\underset{\sim}{x}_t))$$

$$+ n_p^{(2)}(\underset{\sim}{x}_t)\delta(z-h_2(\underset{\sim}{x}_t))\theta(h_1(\underset{\sim}{x}_t)-z) \quad , \tag{3.49}$$

where the normal vectors are defined as in (3.16) with the superscript indicating either h_1 or h_2. Evaluating the step functions in (3.49) at the value of the delta function we see that each is equal to 1 since $h_1 > h_2$. The result is that the Green's tensor in (3.48) in the layer is given by contibutions from two surface integrals

$$G_{jn}(\underset{\sim}{x}',\underset{\sim}{x}'') = G^{o}_{jn}(\underset{\sim}{x}',\underset{\sim}{x}'') - \iint n_m^{(2)}(\underset{\sim}{x}_t)\left[\underset{\sim}{x}',\underset{\sim}{x}_s^{(2)},\underset{\sim}{x}''\right]_{mjn} d\underset{\sim}{x}_t$$

$$+ \iint n_m^{(1)}(\underset{\sim}{x}_t)\left[\underset{\sim}{x}',\underset{\sim}{x}_s^{(1)},\underset{\sim}{x}''\right]_{mjn} d\underset{\sim}{x}_t \quad , \tag{3.50}$$

where the symbol terms are

$$n_m^{(p)}(\underset{\sim}{x}_t)\left[\underset{\sim}{x}',\underset{\sim}{x}_s^{(p)},\underset{\sim}{x}''\right]_{mjn}$$

$$= n_m^{(p)}(\underset{\sim}{x}_t)\left[\left[G^{o}_{ij}(\underset{\sim}{x}',\underset{\sim}{x}_s^{(p)})\left[TG(\underset{\sim}{x}_s^{(p)},\underset{\sim}{x}'')\right]_{min}\right.\right.$$

$$\left.- \left[TG^{o}(\underset{\sim}{x}',\underset{\sim}{x}_s^{(p)})\right]_{mij} G_{in}(\underset{\sim}{x}_s^{(p)},\underset{\sim}{x}'')\right] \quad , \tag{3.51}$$

for the p^{th} surface.

If for simplicity we again choose <u>both</u> surfaces to be traction free then using (3.19) the symbol terms reduce to

$$n_m^{(p)}(\underset{\sim}{x}_t)\left[\underset{\sim}{x}',\underset{\sim}{x}_s^{(p)},\underset{\sim}{x}''\right]_{mjn}$$

$$= - n_m^{(p)}(\underset{\sim}{x}_t)\left[TG^o(\underset{\sim}{x}',\underset{\sim}{x}_s^{(p)})\right]_{mij} G_{in}(\underset{\sim}{x}_s^{(p)},\underset{\sim}{x}'') \quad , \qquad (3.52)$$

so that (3.50) reduces to

$$G_{jn}(\underset{\sim}{x}',\underset{\sim}{x}'') = G^o_{jn}(\underset{\sim}{x}',\underset{\sim}{x}'')$$
$$+ \iint n_m^{(2)}(\underset{\sim}{x}_t)\left[TG^o(\underset{\sim}{x}',\underset{\sim}{x}^{(2)})\right]_{mij} G_{in}(\underset{\sim}{x}_s^{(2)},\underset{\sim}{x}'')d\underset{\sim}{x}_t$$
$$- \iint n_m^{(1)}(\underset{\sim}{x}_t)\left[TG^o(\underset{\sim}{x}',\underset{\sim}{x}_s^{(1)})\right]_{mij} G_{in}(\underset{\sim}{x}_s^{(1)},\underset{\sim}{x}'')d\underset{\sim}{x}_t \quad , \qquad (3.53)$$

which expresses the value of the field in the waveguide in terms of its values on the two surfaces.

To generate the two coupled integral equations for these surface values, first let $\underset{\sim}{x}' \to \underset{\sim}{x}_s'^{(2)}$. Since this approach is from <u>above</u> the surface $(z' > h_2(\underset{\sim}{x}_t'))$ the signum function in the regularized term (A.31) is positive, and the result is

$$Q_{jp}^{(2)}(\underset{\sim}{x}_t')G_{pn}(\underset{\sim}{x}_s'^{(2)},\underset{\sim}{x}'')$$

$$= G^o_{jn}(\underset{\sim}{x}_s'^{(2)},\underset{\sim}{x}'')$$

$$+ \iint K_{ji}(\underset{\sim}{x}_s'^{(2)},\underset{\sim}{x}_s^{(2)}) G_{in}(\underset{\sim}{x}_s^{(2)},\underset{\sim}{x}'')d\underset{\sim}{x}_t$$

$$- \iint n_m^{(1)}(\underset{\sim}{x}_t)\left[TG^o(\underset{\sim}{x}_s'^{(2)},\underset{\sim}{x}_s^{(1)})\right]_{mij} G_{in}(\underset{\sim}{x}_s^{(1)},\underset{\sim}{x}'')d\underset{\sim}{x}_t \quad , \qquad (3.54)$$

which is a surface integral equation for the two surface values. Note that the second integral in (3.54) does not have a singular kernel since $h_1 \neq h_2$.

To form the second integral equation, let $\underset{\sim}{x}' \to \underset{\sim}{x}_s'^{(1)}$. This approach is from below the surface ($z'<h_1(\underset{\sim}{x}_t')$) so the signum function in the regularized term (A.31) is negative, and the result is

$$Q_{jp}^{(1)}(\underset{\sim}{x}_t')G_{pn}(\underset{\sim}{x}_s'^{(1)},\underset{\sim}{x}'')$$

$$= G_{jn}^{o}(\underset{\sim}{x}_s'^{(1)},\underset{\sim}{x}'')$$

$$- \iint K_{ji}(\underset{\sim}{x}_s'^{(1)},\underset{\sim}{x}_s^{(1)})G_{in}(\underset{\sim}{x}_s^{(1)},\underset{\sim}{x}'')d\underset{\sim}{x}_t$$

$$+ \iint n_m^{(2)}(\underset{\sim}{x}_t)\left[TG^{o}(\underset{\sim}{x}_s'^{(1)},\underset{\sim}{x}_s^{(2)})\right]_{mij}G_{in}(\underset{\sim}{x}_s^{(2)},\underset{\sim}{x}'')d\underset{\sim}{x}_t \quad , \qquad (3.55)$$

which is our second integral equation for the two surface values. The procedure is thus to solve (3.54) and (3.55) for the surface values, and then to substitute the results into (3.53) to find the value of the field in the waveguide. In addition, if desired, the matrices Q can be inverted, and the results written in terms of what we called the surface values in Eg. 1.

Eg. 5. FLAT SURFACE AT ZERO TRACTION

From example 2 we have from (3.39) the integral equation for the vector displacement values on the arbitrary surface $h(\underset{\sim}{x}_t)$. For h=0 we have that

$$Q_{jm}(\underset{\sim}{x}_t') = \frac{1}{2}\delta_{jm} \quad , \qquad (3.56)$$

$$n_p(\underset{\sim}{x}_t) = \delta_{p3} \quad , \qquad (3.57)$$

and

$$K_{ji}(\underset{\sim}{x}_s',\underset{\sim}{x}_s) \rightarrow K_{ji}^{o}(\underset{\sim}{x}_t',\underset{\sim}{x}_t) \quad , \qquad (3.58)$$

where from (A.32) and (A.29) we get

$$K^{o}_{ji}(\underset{\sim}{x}'_t,\underset{\sim}{x}_t) = n_p(\underset{\sim}{x}_t)K_{pij}(\underset{\sim}{x}'_t,\underset{\sim}{x}_t)$$

$$= K_{3ij}(\underset{\sim}{x}'_t-\underset{\sim}{x}_t)$$

$$= k_T^{-2}\left[R^T_{3ij}(\underset{\sim}{x}'_t-\underset{\sim}{x}_t) - R^L_{3ij}(\underset{\sim}{x}'_t-\underset{\sim}{x}_t)\right]$$

$$-\frac{1}{2}\left[\delta_{ij}R^T_3(\underset{\sim}{x}'_t-\underset{\sim}{x}_t) + \delta_{3j}R^T_i(\underset{\sim}{x}'_t-\underset{\sim}{x}_t)\right.$$

$$\left. + \delta_{3i}[\lambda/(\lambda + 2\mu)]\, R^L_j(\underset{\sim}{x}'_t-\underset{\sim}{x}_t)\right] , \tag{3.59}$$

and from (A.23) and (A.21) we have that

$$R^{T,L}_{3ij}(\underset{\sim}{x}'_t-\underset{\sim}{x}_t) = \frac{1}{(2\pi)^3}\iiint d\underset{\sim}{k}\; e^{i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}'_t-\underset{\sim}{x}_t)}\,\tilde{G}^{T,L}(k)P^{T,L}_{3ij}(\underset{\sim}{k}) , \tag{3.60}$$

and

$$P^{T,L}_{3ij}(\underset{\sim}{k}) = 2i\left[\left[(k_{it}k_{jt}K^2_{T,L} + \delta_{i3}\delta_{j3}K^4_{T,L}\right]P\left[\frac{1}{k_z}\right]\right.$$

$$\left. + \left[\delta_{i3}k_{jt} + \delta_{j3}k_{it}\right]K^2_{T,L}\right] . \tag{3.61}$$

The exponential in (3.60) is independent of k_z because of the flat surface limit and $G^{T,L}(k)$ are even functions of k_z, so the principal value term in (3.61) vanishes because it is an odd function of k_z. We have that

$$R_{3ij}^{T,L}(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) = \frac{2i}{(2\pi)^3}\iiint d\underset{\sim}{k}\; e^{i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)} K_{T,L}^2 \;\cdot$$

$$\cdot\; \tilde{G}^{T,L}(k)\left[\delta_{i3}k_{jt} + \delta_{j3}k_{it}\right] \;. \tag{3.62}$$

The k_z -integral can be evaluated

$$\int dk_z \tilde{G}^{T,L}(k) = \pi i/K_{T,L} \;, \tag{3.63}$$

so that we have

$$R_{3ij}^{T,L}(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) = \frac{-1}{(2\pi)^2}\iint d\underset{\sim}{k}_t\; e^{i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)} K_{T,L}(\delta_{i3}k_{jt}+\delta_{j3}K_{it}) \;. \tag{3.64}$$

From (A.7) and (A.8) we have that

$$R_j^{T,L}(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) = \frac{1}{(2\pi)^3}\iiint d\underset{\sim}{k}\; e^{i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)} G^{T,L}(k) P_j^{T,L}(\underset{\sim}{k}) \;, \tag{3.65}$$

where

$$P_j^{T,L}(\underset{\sim}{k}) = 2i\left[k_{jt} + \delta_{j3}K_{T,L}^2\; P\left[\frac{1}{k_z}\right]\right] \;. \tag{3.66}$$

Again we can evaluate the k_z-integral since terms odd in k_z vanish. The result is

$$R_j^{T,L}(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) = \frac{-1}{(2\pi)^2}\iint d\underset{\sim}{k}_t\; e^{i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)} k_{jt}/K_{T,L} \;. \tag{3.67}$$

Using (3.64) and (3.67) in (3.59) we get the result

$$K_{ji}^{(o)}(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) = \frac{-1}{(2\pi)^2}\iint d\underset{\sim}{k}_t \; e^{i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)} M_{ji}^{(o)}(\underset{\sim}{k}_t) \quad , \tag{3.68}$$

where

$$M_{ji}^{(o)}(\underset{\sim}{k}_t) = \delta_{j3}k_{it}\left[k_T^{-2}(K_T-K_L) - \frac{1}{2}K_T^{-1}\right] + \delta_{i3}k_{jt}\left[k_T^{-2}(K_T-K_L) - \frac{1}{2}K_L^{-1}(\lambda/(\lambda+2\mu))\right] \quad . \tag{3.69}$$

This form appears somewhat unsymmetric. It can be rewritten. The coefficient in the first term is

$$k_T^{-2}(K_T-K_L) - \frac{1}{2}K_T = (K_T^2 - 2K_TK_L - k_t^2)/2k_T^2K_T \quad , \tag{3.70}$$

and, using the fact that

$$\lambda/(\lambda+2\mu) = (k_T^2 - 2k_L^2)/k_T^2 \quad , \tag{3.71}$$

the coefficient of the second term is

$$k_T^{-2}(K_T - K_L) - \frac{1}{2}K_L^{-1}(\lambda/(\lambda+2\mu)) = (2K_LK_T - K_T^2 + k_t^2)/2k_T^2K_L \quad , \tag{3.72}$$

and (3.69) becomes

$$M_{ji}^{(o)}(\underset{\sim}{k}_t) = (\delta_{j3}k_{it}K_T^{-1} - \delta_{i3}k_{jt}K_L^{-1})(K_T^2 - 2K_TK_L - k_t^2)/2k_T^2 \quad . \tag{3.73}$$

The resulting integral equations for the displacement on the surface is from (3.39)

$$\frac{1}{2}\, u_j(\underset{\sim}{x}'_t) = u_j^{in}(\underset{\sim}{x}'_t) + \iint d\underset{\sim}{x}_t\, K_{ji}^{(o)}(\underset{\sim}{x}'_t - \underset{\sim}{x}_t)\, u_i(\underset{\sim}{x}_t) \quad , \tag{3.74}$$

which are coupled convolution equations. They can be solved exactly. This is done in Sec. 13, after we first discuss the flat surface examples by more conventional means in Secs. 4-10. Once they are solved, the displacements in the field (i.e. off the surface) follow from (3.38)

$$u_j(\underset{\sim}{x}'_t) = u_j^{in}(\underset{\sim}{x}'_t) + \iint d\underset{\sim}{x}_t \left[TG^o(\underset{\sim}{x}'_t, \underset{\sim}{x}_t)\right]_{3ij} u_i(\underset{\sim}{x}_t) \quad , \tag{3.75}$$

where the traction operator on the free-space Green's tensor follows from (3.12)

$$\left[TG^o(\underset{\sim}{x}', \underset{\sim}{x}_t)\right]_{3ij} = \left[\mu \frac{\partial}{\partial z} G^o_{ij}(\underset{\sim}{x}', \underset{\sim}{x}) + \mu\, \partial_i G^o_{3j}(\underset{\sim}{x}', \underset{\sim}{x}) + \lambda\, \delta_{i3} \partial_m G^o_{mj}(\underset{\sim}{x}', \underset{\sim}{x})\right]_{z=0} \quad . \tag{3.76}$$

3.4 POTENTIALS AND PLANE WAVES

We have been solving for either the Green's tensor G_{ij} or the vector displacement

$$u_i(\underset{\sim}{x}) = \iiint G_{ij}(\underset{\sim}{x}-\underset{\sim}{x}')S_j(\underset{\sim}{x}')d\underset{\sim}{x}' \quad . \tag{4.1}$$

We now return to expressing this vector displacement in terms of potentials proviously mentioned in (1.19)-(1.22). We have that

$$u_i(\underset{\sim}{x}) = u_i^L(\underset{\sim}{x}) + u_i^T(\underset{\sim}{x}) \tag{4.2}$$

$$= \partial_i\phi + \varepsilon_{ijm}\partial_j A_m \quad , \tag{4.3}$$

in terms of the scalar potential ϕ for the longitudinal waves (or P-waves) and the vector potential A_m for the transverse waves (shear, or S-waves). The vector displacement has three independent components, and the scalar and vector potentials have four. We thus require a constraint on these potentials and it is usually written as

$$\partial_m A_m = 0 \quad , \tag{4.4}$$

in analogy to the gauge condition in electromagnetic theory. As we noted in (1.25) and (1.27) we have that

$$\Delta u_i^{L,T}(\underset{\sim}{x}) + k_{L,T}^2 u_i^{L,T}(\underset{\sim}{x}) = 0_i \quad , \tag{4.5}$$

and we can thus choose equations on the potentials as

$$\Delta\, \varphi(\underset{\sim}{x}) + k_L^2\, \phi(\underset{\sim}{x}) = 0 \quad , \tag{4.6}$$

and

$$\Delta\, A_m(\underset{\sim}{x}) + k_T^2\, A_m(\underset{\sim}{x}) = 0_m \quad , \tag{4.7}$$

Both the potentials thus satisfy Helmholtz equations with the corresponding longitudinal and transverse wavenumbers and wave speeds c given by

$$k_L = \omega/c_L \quad , \quad c_L = [(\lambda + 2\mu)/\rho]^{1/2} \quad , \tag{4.8}$$

and

$$k_T = \omega/c_T \quad , \quad c_T = [\mu/\rho]^{1/2} \quad . \tag{4.9}$$

Far enough away from any source, a wave front from that source can be (at least locally) approximated as a plane wave. We thus study plane wave sollutions of (4.6) and (4.7) given by

$$\phi(\underset{\sim}{x}) = \exp[i\ \underset{\sim}{k}^L \cdot \underset{\sim}{x}] = \exp[i\ k_m^L\, x_m] \quad , \tag{4.10}$$

and

$$A_m(\underset{\sim}{x}) = a_m \exp[i\ \underset{\sim}{k}^T \cdot \underset{\sim}{x}] = a_m \exp(i\ k_m^T\, x_m) \quad , \tag{4.11}$$

where

$$|\underset{\sim}{k}^L| = k_L \quad \text{and} \quad |\underset{\sim}{k}^T| = k_T \quad . \tag{4.12}$$

In each case the <u>direction of propagation</u> of the wave is $\underset{\sim}{k}^L$ and $\underset{\sim}{k}^T$ respectively. The <u>direction of the longitudinal displacement</u> is given by the gradient of (4.10), i.e.

$$u_i^L(\underset{\sim}{x}) = i\ k_i^L\, \varphi(\underset{\sim}{x}) \quad , \tag{4.13}$$

and thus the <u>longitudinal displacement is in the same direction as the</u>

propagation of the wave. The displacement of the shear potential is in the direction given by the vector a_m. Because of the gauge condition (4.4) we thus have

$$k_m^T a_m = 0 \quad , \tag{4.14}$$

so that for the transverse wave the direction of propagation of the wave is orthogonal to the direction of its displacement. We thus have three independent potentials, ϕ for P-waves, and A_m (with $\partial_m A_m=0$) for two independent components of shear waves, called SV and SH waves, where the V and H notations are referred to as polarizations of the shear waves, again in analogy with electromagnetic theory. We thus have to choose these potentials to yield three wave shapes. It will turn out that we can choose two of them in a plane (but not orthogonal). These are the P and SV waves. The third one, the SH-wave, will be orthogonal to this plane. The wave number vectors $\underset{\sim}{k}^L$ and $\underset{\sim}{k}^T$ indicate the direction of propagation of the waves. Their components are the direction cosines. An alternative teminology often used in the seismic literature are the slowness vectors, related to the wavenumbers vectors as

$$\underset{\sim}{S}^{L,T} = \hat{k}^{L,T}/c_{L,T} \quad .$$

The x-component of either of these vectors is the same. It is just the ray parameter

$$p = \sin \theta_L / c_L = \sin \theta_T / c_T \quad , \tag{4.15}$$

and the equality on the rhs is just an expression of Snell's Law.

For S-waves we choose the propagation direction by choosing the vector

$\underset{\sim}{k}^T$ in the x-z plane

$$\underset{\sim}{k}^T = k_x^T \hat{i} + k_z^T \hat{k} \quad , \tag{4.16}$$

where $\hat{i}$ and $\hat{k}$ are unit vectors along the x and z directions. The vector potential is thus

$$A_m(\underset{\sim}{x}) = a_m \exp(i\ k_x^T x + i\ k_z^T z) = A_m(x,z) \quad , \tag{4.17}$$

which is only a function of x and z. The gauge condition (4.4) yields

$$\frac{\partial}{\partial x} A_1 + \frac{\partial}{\partial z} A_3 = 0 \quad . \tag{4.18}$$

For the <u>SV-polarization</u> we assume in addition that the y-component of displacement vanishes, i.e. that

$$u_2^T = \varepsilon_{2jm} \partial_j A_m = \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} = 0 \quad . \tag{4.19}$$

The result of (4.18) and (4.19) is that A_1 andf A_3 satisfy Cauchy-Riemann equations, and thus that A_3+iA_1 is an analytic function of x+iz which is analytic everywhere. In addition, for A_m expressible as a plane wave, the complex function A_3+iA_1 is bounded. Liouville's theorem states that a function which is everywhere analytic and bounded is a constant. Since only gradients of A_3 and A_1 are used to calculate a physically measurable quantity such as displacement, the constant doesn't matter, and we can choose it to be zero. Thus $A_3=A_1=0$ and the vector potential has only one component

$$\underset{\sim}{A} = (0, A, 0) \quad , \tag{4.20}$$

whose divergence vanishes by (4.4). This latter is true if A is not a function of y. Thus the displacement for SV-waves can be written as

$$\begin{aligned} u_i^{T,SV}(x,z) &= \varepsilon_{ijm}\partial_j A_m(x,z) \\ &= (-\frac{\partial}{\partial z} A(x,z),\ 0\ ,\ \frac{\partial}{\partial x} A(x,z)) \quad , \end{aligned} \tag{4.21}$$

which has components in x and z directions only. By (4.7) A satisfies the differential equation

$$\Delta\, A(x,z) + k_T^2\, A(x,z) = 0 \quad , \tag{4.22}$$

where the Laplacian is only in x and z. Note that we not only have $\underset{\sim}{k}^T \cdot \underset{\sim}{A} = 0$ but also that

$$\underset{\sim}{k}^T \cdot \underset{\sim}{u}^{T,SV} = 0 \quad , \tag{4.23}$$

explicitly illustrating that SV shear displacements are orthogonal to the propagation direction.

For the SH-polarization, we assume that the displacement is orthogonal to SV, i.e. that it has no components in the x-z plane. We can write this displacement as

$$u_i^{T,SH}(x,z) = (0,\ v(x,z), 0) \quad , \tag{4.24}$$

which thus only has a y-component. The reason that v is not a function of y is that we must have

$$\partial_i u_i^{T,SH} = 0 \quad , \tag{4.25}$$

as we previously noted. This implies $\partial v/\partial y = 0$ and again we drop any constant, or absorb it into the other functional dependence. In addition from (4.5) we have that

$$(\Delta + k_T^2)v(x,z) = 0 \quad . \tag{4.26}$$

For the P-wave, in the x-z plane, there is no displacement in the y-direction so that

$$u_2^{L,P} = \frac{\partial \phi}{\partial y} = 0 \text{ and } \phi = \phi(x,z) \quad . \tag{4.27}$$

We thus have that

$$u_j^L(x,z) = (\frac{\partial}{\partial x}\phi(x,z),\ 0,\ \frac{\partial}{\partial z}\phi(x,z)) \quad . \tag{4.28}$$

Notice from (4.21), (4.24) and (4.28), P and SV waves decouple from SH waves.

For the Green's tensor we defined the operator T as in (3.11). Dotting this with the normal we get the traction

$$\begin{aligned} n_p(\underset{\sim}{x}_t)\left[TG(\underset{\sim}{x},\underset{\sim}{x}'')\right]_{pij} &= \mu\, n_m(\underset{\sim}{x}_t)\partial_m G_{ij}(\underset{\sim}{x},\underset{\sim}{x}'') \\ &+ \mu\, n_m(\underset{\sim}{x}_t)\partial_i G_{mj}(\underset{\sim}{x},\underset{\sim}{x}'') \\ &+ \lambda\, n_i(\underset{\sim}{x}_t)\partial_m G_{mj}(\underset{\sim}{x},\underset{\sim}{x}'') \quad . \end{aligned} \tag{4.29}$$

If we integrate this equation over $\underset{\sim}{x}''$ and a vector source $S_j(\underset{\sim}{x}'')$, we get that the Green's function becomes the displacement and we can define the (vector) stress as

$$\tau_{zi} = n_p(\underset{\sim}{x}_t)\left[Tu(\underset{\sim}{x})\right]_{pi}$$

$$= \mu\ n_m(\underset{\sim}{x}_t)\partial_m u_i(\underset{\sim}{x}) + \mu\ n_m(\underset{\sim}{x}_t)\partial_i u_m(\underset{\sim}{x}) + \lambda\ n_i(\underset{\sim}{x}_t)\partial_m u_m(\underset{\sim}{x}) \quad , \qquad (4.30)$$

the first term of which is a normal derivative, and the third term is a divergence. Its three components correspond to normal $(i=3=z)$ and tangential $(i=1,2=x,y)$ stresses. (The additional stress components which arise by replacing z on the lhs of (4.30) by x or y only act in the plane, and do not act across a boundary either normally or tangentially.) We can explicitly write the displacement as

$$u_i(x,z) = u_i^L(x,z) + u_i^T(x,z)$$

$$= \delta_{i1}u_1(x,z) + \delta_{i2}v(x,z) + \delta_{i3}u_3(x,z) \quad , \qquad (4.31)$$

where, from (4.21) and (4.28) we have that

$$u_1(x,z) = \frac{\partial\phi}{\partial x} - \frac{\partial A}{\partial z} \quad , \qquad (4.32)$$

and

$$u_3(x,z) = \frac{\partial\phi}{\partial z} + \frac{\partial A}{\partial x} \quad . \qquad (4.33)$$

From (4.30) the stress becomes

$$\tau_{zi} = \mu\, n_m(\underset{\sim}{x}_t)\partial_m\left[\delta_{i1}u_1(x,z) + \delta_{i2}v(x,z) + \delta_{i3}u_3(x,z)\right]$$

$$+ \mu\, n_m(\underset{\sim}{x}_t)\partial_i\left[\delta_{m1}u_1(x,z) + \delta_{m2}v(x,z) + \delta_{m3}u_3(x,z)\right]$$

$$+ \lambda\, n_i(\underset{\sim}{x}_t)\partial_m\left[\delta_{m1}u_1(x,z) + \delta_{m2}v(x,z) + \delta_{m3}u_3(x,z)\right] \quad . \tag{4.34}$$

Since we have discussed plane waves by themselves (i.e. no superposition of plane waves) and these are appropriate for planar iterface problems, we write the stress components for a flat interface (0-superscript) where $n_m(\underset{\sim}{x}_t) = \delta_{m3}$. These are easily seen to be from (4.34)

$$\tau^o_{z1} = \mu\left[\frac{\partial^2 A}{\partial x^2} - \frac{\partial^2 A}{\partial z^2}\right] + 2\mu\,\frac{\partial^2 \phi}{\partial x \partial z} \quad , \tag{4.35}$$

$$\tau^o_{z2} = \mu\,\frac{\partial v}{\partial z} \quad , \tag{4.36}$$

and

$$\tau^o_{z3} = 2\mu\left[\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 A}{\partial z \partial x}\right] - \lambda\, k_L^2 \phi \quad . \tag{4.37}$$

We can further reduce these stresses by quoting the results for the various polarizations separately as

P-wave

$$\tau_{z1} = 2\mu\,\frac{\partial^2 \varphi}{\partial x \partial z} \quad , \quad \tau_{z2} = 0 \quad , \quad \tau_{z3} = 2\mu\,\frac{\partial^2 \phi}{\partial z^2} - \lambda\, k_L^2\, \phi \quad , \tag{4.38}$$

SV-wave

$$\tau_{z1} = \mu\left[\frac{\partial^2 A}{\partial x^2} - \frac{\partial^2 A}{\partial z^2}\right] \quad , \quad \tau_{z2} = 0 \quad , \quad \tau_{z3} = 2\mu\frac{\partial^2 A}{\partial z \partial x} \quad , \tag{4.39}$$

and

SH-wave

$$\tau_{z1} = \tau_{z3} = 0 \quad , \quad \tau_{z2} = \mu\,\frac{\partial v}{\partial z} \quad . \tag{4.40}$$

These latter three equations are only useful provided the waves do not mix.

3.5 BOUNDARY CONDITIONS

At the interface between two elastic media (illustrated in Fig. 3.3 below)

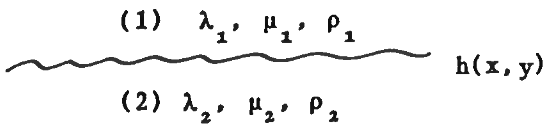


Fig. 3.3

we can have several types of boundary conditions.

(a) Rigid Contact

For this case both displacements u_j and stresses τ_{zj} are continuous. That is we have six conditions ($j = 1, 2, 3$)

$$u_j^{(1)}(\underset{\sim}{x}_s) = u_j^{(2)}(\underset{\sim}{x}_s) \quad , \tag{5.1}$$

and

$$\tau_{zj}^{(1)}(\underset{\sim}{x}_s) = \tau_{zj}^{(2)}(\underset{\sim}{x}_s) \quad , \tag{5.2}$$

where $\underset{\sim}{x}_s = (x, y, h)$ is a position vector on the surface, and the superscript symbols (1) and (2) indicate the displacement or stress in the particular region as evaluated on the boundary.

(b) Free Contact

For this case the stresses are continuous, but the displacements are discontinuous.

(c) Free Surface

For this case we have that the stresses vanish on the surface

$$\tau_{zj}(\underset{\sim}{x}_s) = 0_j \quad , \tag{5.3}$$

and the displacements are not specified. The latter can be easily computed for a flat interface and we do so in the next several sections. This boundary condition is also called the zero stress or zero traction or traction free condition.

(d) Infinitely Rigid Surface

For this case, the displacements vanish on the surface, i.e.

$$u_j(\underset{\sim}{x}_s) = 0_j \quad . \tag{5.4}$$

The stresses are not specified a priori, but can be computed.

FLUID-ELASTIC BOUNDARY

At a fluid-fluid interface we know that we have the continuity of pressure and normal velocity. The question is how do we pass from the continuity conditions at an elastic-elastic interface to those at a fluid-elastic interface.

(a) Continuity of Normal Velocity

For an elastic-elastic interface we have continuity of displacements as in (5.1). In general the displacement is time dependent, and the time derivative of displacement is velocity. Since we have time-harmonic problems we can work directly with displacements. In a fluid, the tangential displacements are zero, i.e.

$$u_1(\underset{\sim}{x}_s) = u_2(\underset{\sim}{x}_s) = 0 \quad , \tag{5.5}$$

and the normal displacements (equivalently normal velocities) are continuous

$$u_3^{(1)}(\underset{\sim}{x}_s) = u_3^{(2)}(\underset{\sim}{x}_s) \quad . \tag{5.6}$$

(b) Continuity of Pressure

In a fluid the shear modulus vanishes, $\mu = 0$. The shear wave speed also vanishes, $c_T = 0$, and the Lame' modulus is expressible in terms of the longitudinal (compressional) speed c_L as $\lambda = \rho c_L^2$ where ρ is density. We have continuity of stress, Eq. (5.2), and in a fluid the stress is using (4.30) and $\mu = 0$

$$\tau_{zj} = \lambda\, n_j \partial_m u_m \quad . \tag{5.7}$$

In a fluid, the time-derivative of the time-dependent displacement is just the velocity, which can be represented as the gradient of the scalar velocity potential Φ, i.e.

$$\frac{\partial}{\partial t} u_j(\underset{\sim}{x}, t) = \partial_j \Phi \quad . \tag{5.8}$$

For harmonic time-dependence $\exp(-i\omega t)$ we have

$$u_j(\underset{\sim}{x}) = (i/\omega)\, \partial_j \Phi(\underset{\sim}{x}) \quad . \tag{5.9}$$

The divergence of this can be written as

$$\partial_j u_j(\underset{\sim}{x}) = (i/\omega)\partial_j \partial_j \Phi(\underset{\sim}{x}) = -(i/\omega)k_o^2 \Phi \quad , \tag{5.10}$$

where $k_o = \omega/c_L$, and where the latter term on the rhs of (5.10) follows since Φ satisfies the acoustic Helmholtz equation. Substituting (5.10) into (5.7) using k_o and λ definitions we get

$$\tau_{zj} = n_j(-i\omega\rho\Phi) \quad . \tag{5.11}$$

The pressure can be written in terms of the velocity potentials

$$p = -\rho \frac{\partial}{\partial t} \Phi = i\omega\rho\Phi \quad , \tag{5.12}$$

so that, in a fluid,

$$\tau_{zj} = -n_j \; p \quad . \tag{5.13}$$

For a flat interface, $n_j = \delta_{j3}$ and the stress results are

$$p = -\tau_{zz} \quad , \quad 0 = \tau_{zx} \quad , \quad 0 = \tau_{zy} \quad . \tag{5.14}$$

In general for an arbitrary surface the boundary conditions are expressed as the continuity of normal velocity

$$n_j v_j = \frac{\partial}{\partial t} (n_j u_j) \quad , \tag{5.15}$$

the continuity of normal stress

$$-p = \tau_{ij} n_i n_j \quad , \tag{5.16}$$

and the vanishing of the elastic shear stresses

$$0 = \tau_{ij} n_i L_j = \tau_{ij} n_i v_j \quad , \tag{5.17}$$

where L_j and v_j are orthogonal vectors in the local tangent plane.

ELECTROMAGNETIC THEORY

As an aside we note that we can recover the results of electromagnetic theory from those of elasticity. The equation for dispalcement is (using vector analysis notation) from (1.16)

$$\mu \, \Delta \, \underset{\sim}{u} + (\lambda + \mu) \, \mathrm{grad}(\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) + K^2 \underset{\sim}{u} = \underset{\sim}{0} \quad , \tag{1.16}$$

where Δ = grad div-curlcurl. From the Maxwell equations on the electric $\underset{\sim}{E}$ and magnetic $\underset{\sim}{H}$ fields

$$\underset{\sim}{\nabla} x \underset{\sim}{E} = ik\underset{\sim}{H} \quad ; \quad \underset{\sim}{\nabla} x \underset{\sim}{H} = - ik\underset{\sim}{E} \quad , \tag{5.18}$$

we derive, by taking the curl of the first equation, an equation on $\underset{\sim}{E}$

$$\underset{\sim}{\nabla} x \underset{\sim}{\nabla} x \underset{\sim}{E} - k^2 \underset{\sim}{E} = \underset{\sim}{0} \quad , \tag{5.19}$$

or

$$\Delta \underset{\sim}{E} + k^2 \underset{\sim}{E} - \text{grad}(\underset{\sim}{\nabla} \cdot \underset{\sim}{E}) = \underset{\sim}{0} \quad , \tag{5.20}$$

If we make the make the formal interchanges

$$\underset{\sim}{E} \rightarrow \underset{\sim}{u} \quad ; \quad k^2 = K^2/\mu = k_T^2 \quad , \tag{5.21}$$

and set $\lambda + 2\mu = 0$ ($c_L = 0$), then (5.20 is just (1.16). Note that the electromagnetic wave number k becomes the transverse (shear) wavenumber k_T, whereas in the fluid the acoustic wavenumber k_0 became k_L, the longitudinal (compressional) wavenumber.

3.6 P-WAVE INCIDENCE ON A FREE SURFACE

We noted in Sec. 4 that P and SV waves couple and SH waves decouple from these. A flat interface does not alter this property, and here we consider P-wave incidence on a free (zero-traction) flat boundary located at z = 0. From (5.3) we thus have three boundary conditions

$$\tau_{zj} = 0 \qquad j = 1, 2, 3 \ , \tag{6.1}$$

but from (4.36) the j = 2 condition is automatically satisfied since we have no SH waves. From (4.35) and (4.37) we can thus write the two boundary conditions as

$$\frac{\partial^2 A}{\partial x^2} - \frac{\partial^2 A}{\partial z^2} + 2\,\frac{\partial^2 \phi}{\partial x \partial z} = 0 \ , \tag{6.2}$$

and

$$2\mu \left[\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 A}{\partial x \partial z}\right] - \lambda\, k_L^2 \phi = 0 \ , \tag{6.3}$$

where the potentials ϕ and A are evaluated at z = 0.

Assume the P-wave has incident (A_0, i) and reflected (B, r) plane wave components

$$\phi(x, z) = A_0 \exp\left[i(k_x^{L,i} x - k_z^{L,i} z)\right] + B \exp\left[i(k_x^{L,r} x + k_z^{L,r} z)\right] \ , \tag{6.4}$$

and the SV wave only a reflected component

$$A(x,z) \quad = \quad C \exp\left[i(k_x^{T,r}x + k_z^{T,r}z)\right] \quad , \tag{6.5}$$

where the wavenumber components satisfy the dispersion relations from (4.6), (4.7) and (4.20)

$$(k_x^{L,i})^2 + (k_z^{L,i})^2 \quad = \quad (k_x^{L,r})^2 + (k_z^{L,r})^2 \quad = \quad k_L^2 \quad , \tag{6.6}$$

and

$$(k_x^{T,r})^2 + (k_z^{T,r})^2 \quad = \quad k_T^2 \quad . \tag{6.7}$$

We have two boundary conditions but three unknown constants A_0, B, and C and we solve for the ratios $R_1 = B/A_0 = R_{P \to P}$, the reflection coefficient (amplitude ratio) for scattering to P-waves with a P-wave incident field, and $R_2 = C/A_0 = R_{P \to SV}$, the reflection coefficient for scattering from P to SV waves. Note these are reflection coefficients for potentials. Substituting (6.4) and (6.5) into (6.2) and (6.3) we get the equations

$$a_{11}R_2\, e^{ik_x^{T,r}x} - a_{12}R_1\, e^{ik_x^{L,r}x} = -a_{12}\, e^{ik_x^{L,i}x} \quad , \tag{6.8}$$

and

$$a_{21}R_2\, e^{ik_x^{T,r}x} + a_{22}\, R_1\, e^{ik_x^{L,r}x} = -a_{22}\, e^{ik_x^{L,i}x} \quad , \tag{6.9}$$

where

$$a_{11} = (k_z^{T,r})^2 - (k_x^{T,r})^2 \quad , \tag{6.10}$$

$$a_{12} = 2\, k_x^{L,r}\, k_z^{L,r} \quad , \tag{6.11}$$

$$a_{21} = 2\mu\, k_x^{T,r}\, k_z^{T,r} \quad , \tag{6.12}$$

and

$$a_{22} = 2\mu\, (k_z^{L,r})^2 + \lambda\, k_L^2 \quad . \tag{6.13}$$

Since we have plane wave incidence, the solution of (6.8) and (6.9) should be independent of x. Any point of incidence will do, no preferred value of x is possible, or equivalently, the equations must be translationally invariant in x. This is true in (6.8) and (6.9) provided

$$k_x^{L,i} = k_x^{L,r} = k_x^{T,r} \quad . \tag{6.14}$$

The lhs of this equation is true if

$$k_L \sin\theta_{Li} = k_L \sin\theta_{Lr} \quad , \tag{6.15}$$

where the angles are defined in Fig. 3.4 later in this section.. The latter is true if the angle of incidence and the angle of reflection (of the P-wave) are equal, $\theta_{Li} = \theta_{Lr}$. The right hand equality in (6.14) is true provided

$$k_L \sin\theta_{Li} = k_T \sin\theta_{Tr} \quad , \tag{6.16}$$

so that the sin of the angle of the reflected shear wave is

$$\sin\theta_{Tr} = (k_L/k_T)\sin\theta_{Li} = (c_T/c_L)\sin\theta_{Li} \quad . \qquad (6.17)$$

This is just Snell's Law (for reflection of P- to SV-waves) and can be expressed in terms of the ray parameter (4.15). Using these results the equations (6.8) and (6.9) reduce to

$$a_{11}R_2 - a_{12}R_1 = -a_{12} \quad , \qquad (6.18)$$

and

$$a_{21}R_2 + a_{22}R_1 = -a_{22} \quad , \qquad (6.19)$$

which have the solution

$$R_1 = R_{P\to P} = (a_{12}a_{21} - a_{11}a_{22})/\Delta \quad , \qquad (6.20)$$

and

$$R_2 = R_{P\to SV} = -2\, a_{12}a_{22}/\Delta \quad , \qquad (6.21)$$

where

$$\Delta = a_{12}a_{21} + a_{11}a_{22} \quad . \qquad (6.22)$$

Noting that

$$k_z^{T,r} = k_T\cos\theta_{Tr} \quad ; \quad k_x^{T,r} = k_T\sin\theta_{Tr} \quad , \qquad (6.23)$$

and

$$k_x^{L,r} = k_L\cos\theta_{Li} \quad ; \quad k_z^{L,r} = k_L\sin\theta_{Li} \quad , \qquad (6.24)$$

we can write the a_{ij} coefficients in many ways, viz.

$$a_{11} = k_T^2 (\cos^2\theta_{Tr} - \sin^2 \theta_{Tr})$$

$$= k_T^2(1-2 \sin^2\theta_{Tr})$$

$$= k_T^2 (1-2(c_T/c_L)^2 \sin^2\theta_{Li})$$

$$= k_T^2 (1-2p^2 c_T^2) \quad , \tag{6.25}$$

where p is the ray parameter from (4.15) and

$$a_{12} = 2 k_L^2 \sin \theta_{Li} \cos \theta_{Li}$$

$$= 2 k_L^2 p c_L(1-p^2 c_L^2)^{1/2} \tag{6.26}$$

$$a_{21} = 2\mu k_T^2 \sin \theta_{Tr} \cos \theta_{Tr}$$

$$= 2\mu k_T^2 (c_T/c_L) \sin \theta_{Li} \left[1-(c_T/c_L)^2 \sin^2\theta_{Li}\right]^{1/2}$$

$$= 2\mu k_T^2 p c_T (1-p^2 c_T^2)^{1/2}$$

$$= 2 \rho k_T^2 p c_T^3 (1-p^2 c_T^2)^{1/2} \quad , \tag{6.27}$$

and

$$a_{22} = k_L^2(2\mu \cos^2\theta_{Li} + \lambda)$$

$$= k_L^2 \left[(2\mu + \lambda) - 2\mu \sin^2\theta_{Li}\right]$$

$$= k_L^2 \rho(c_L^2 - 2c_T^2 \sin^2\theta_{Li})$$

$$= \rho k_L^2 c_L^2 (1-2p^2 c_T^2) \quad . \tag{6.28}$$

Using these forms we can write the reflection coefficients in various ways. We list two for each

$$\frac{B}{A_0} = R_{P\to P} = \frac{4p^2c_t^3(1-p^2c_L^2)^{1/2}(1-p^2c_t^2)^{1/2} - c_L(1-2p^2c_t^2)^2}{4p^2c_t^3(1-p^2c_L^2)^{1/2}(1-p^2c_t^2)^{1/2} + c_L(1-2p^2c_t^2)^2} , \qquad (6.29)$$

or, using the fact that

$$\cos\theta_{Li} = (1-p^2c_L^2)^{1/2} \quad ; \quad \cos\theta_{tr} = (1-p^2c_L^2)^{1/2} , \qquad (6.30)$$

we get

$$R_{P\to P} = \frac{4p^2(\cos\theta_{Li}/c_L)(\cos\theta_{Tr}/c_T) - (c_T^{-2} - 2p^2)^2}{4p^2(\cos\theta_{Li}/c_L)(\cos\theta_{Tr}/c_T) + (c_T^{-2} - 2p^2)^2} . \qquad (6.31)$$

Similarly we can derive

$$\frac{C}{A_0} = R_{P\to SV} = \frac{-4p\ c_T^2(1-p^2c_L^2)^{1/2}(1-2p^2c_T^2)}{4p^2c_T^3(1-p^2c_L^2)^{1/2}(1-p^2c_T^2) + c_L(1-2p^2c_T^2)^2} , \qquad (6.32)$$

or

$$R_{P\to SV} = \frac{-4p(\cos\theta_{Li}/c_L)(c_T^{-2} - 2p^2)}{4p^2(\cos\theta_{Li}/c_L)(\cos\theta_{tr}/c_T) + (c_T^{-2} - 2p^2)^2} . \qquad (6.33)$$

The conventional representation of this scattering process is illustrated in Fig. 3.4.

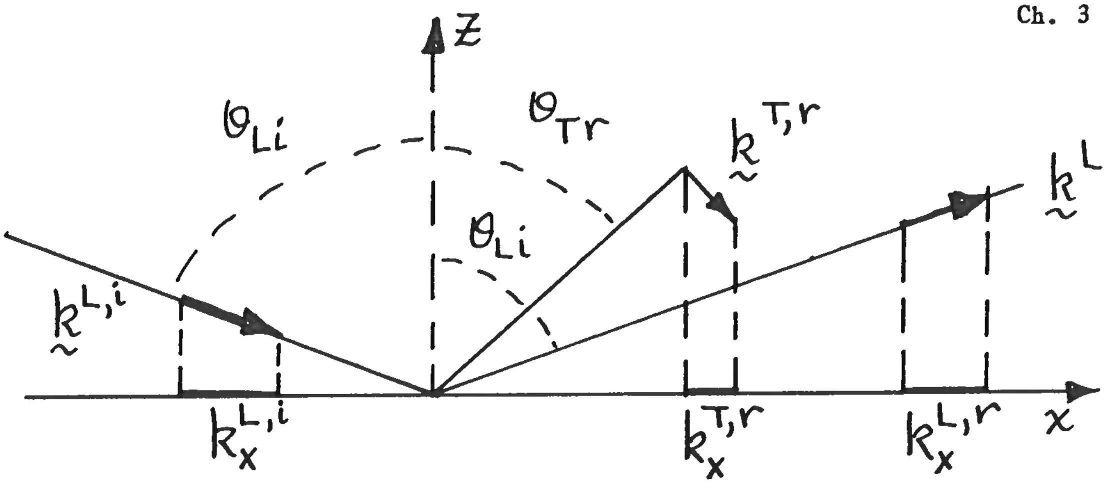


Fig. 3.4

The straight lines indicate the direction of propagation and the arrows indicate the direction of displacement of the wave. All the x-components of the wave vectors are equal from translational invariance in the x-direction. Note that the P-wave displacements

$$u_j^L = \partial_j \phi \quad , \tag{6.34}$$

are along the direction of propagation, whereas the SV-wave displacements

$$u_j^T = \varepsilon_{jmp} \partial_m A_p = -\delta_{ji} \frac{\partial A}{\partial z} + \delta_{j3} \frac{\partial A}{\partial x} \quad , \tag{6.35}$$

are orthogonal to the propagation direction.

~~~~~~~~~~

(proof) Find the displacements of the SV-wave

$$A(x,z) = C \exp\left[i(k_x^{T,r} x + k_z^{T,r} z) \right] \quad .$$

The z-component of displacement on the z = 0 surface is
~~~~~~~~~~

$$\frac{\partial}{\partial x} A(x,0) = i\, k_x^{T,r}\, C \exp(i\, k_x^{T,r} x) \quad .$$

Since we are using phasors we have to take the real part of this to get the physical displacement

$$\mathrm{Re}\, \frac{\partial}{\partial x} A(x,0) = -\, k_x^{T,r}\, \sin(k_x^{T,r} x)\, C \quad .$$

The x-component of displacement is

$$\mathrm{Re}\left[-\frac{\partial}{\partial z} A(x,0)\right] = k_x^{t,r}\, \sin(k_x^{t,r} x)\, C \quad ,$$

which is positive. Thus the z-component of displacement is negative, as we have illustrated in the figure.

~~~~~~~~~~~~~~

CONVENTIONS

As we remarked, the terms $R_{P\to P}$ and $R_{P\to SV}$ we have calculated are the reflection coefficients for the potentials. What are often quoted are reflection coefficients for displacements. There are several possible definitions for these. We have that the displacement for the incident P-wave is $\partial_j \phi^{in}$, for the scattered P-wave $\partial_j \phi^{SC}$, and for the SV waves given by (6.35). The reflection coefficient for the z-component of P-wave displacement is
~~~~~~~~~~~~~~

$$R^z_{P\rightarrow P} = \left.\frac{\partial\phi^{SC}/\partial z}{\partial\phi^{in}/\partial z}\right|_{z=0} = -\frac{B}{A_o} ,$$

and for the x-component of P-wave displacement

$$R^x_{P\rightarrow P} = \left.\frac{\partial\phi^{SC}/\partial x}{\partial\phi^{in}/\partial x}\right|_{z=0} = \frac{B}{A_o} .$$

The reflection coefficient for the z-componemt of SV-waves due to P-wave incidence is

$$R^z_{P\rightarrow SV} = \left.\frac{\partial A/\partial x}{\partial\phi^{in}/\partial z}\right|_{z=0} = \frac{k_x^{T,r}}{k_z^{L,i}} \frac{C}{A_o} = \frac{k_x^{L,i}}{k_z^{L,i}} \frac{C}{A_o} = \tan\theta_{Li} \frac{C}{A_o} ,$$

and for the x-component

$$R^x_{P\rightarrow SV} = \left.\frac{-\partial A/\partial z}{\partial\phi^{in}/\partial x}\right|_{z=0} = -\frac{k_z^{T,r}}{k_x^{L,i}} \frac{C}{A_o} = \tan\theta_{tr} \frac{C}{A_o} .$$

In addition, if one deals with displacement amplitudes which are angle independent, the way to find the reflection coefficients is to replace A_o, B, and C by

$$A_o \rightarrow k_L A_o , \quad B \rightarrow k_L B , \quad C \rightarrow k_T C ,$$

which define the reflection coefficients

$$\bar{R}_{P\to SV} = \frac{k_L B}{k_L A_o} = \frac{B}{A_o} ,$$

and

$$\bar{R}_{P\to SV} = \frac{k_L C}{k_L A_o} = \frac{c_L}{c_T} \frac{C}{A_o} .$$

In addition Aki and Richards (Ref. 3.3) use a convention whereby

$$\bar{R}_{P\to SV} = -\frac{C_L}{C_T} \frac{C}{A_o} ,$$

presumably because the z-component of displacement of the SV wave on the boundary is in the negative z-direction.

DISPLACEMENTS

Since we have calculated the reflection coefficients for this zero-traction surface it is useful to evaluate the displacements on the boundary also. Since v=0, u_2=0 identically. The other displacements can be found from (4.32) and (4.33) to be

$$u_1(x,0) = \frac{\partial\phi}{\partial x}(x,0) - \frac{\partial A}{\partial z}(x,0) , \tag{6.36}$$

and

$$u_3(x,0) = \frac{\partial\phi}{\partial z}(x,0) + \frac{\partial A}{\partial x}(x,0) . \tag{6.37}$$

Substituting (6.4) and (6.5) into (6.36) and (6.37) we get

$$u_1(x,0) = i\, A_0 \left[k_x^{L,i} \left(1 + \frac{B}{A_0}\right) - k_z^{T,r} \frac{C}{A_0} \right] e^{ik_x^{L,i}x} , \qquad (6.38)$$

and

$$u_3(x,0) = i\, A_0 \left[k_z^{L,i} \left(1 - \frac{B}{A_0}\right) - k_x^{T,r} \frac{C}{A_0} \right] e^{ik_x^{L,i}x} , \qquad (6.39)$$

which can be further evaluated using (6.29), (6.31), (6.32) or (6.33).

ZEROES OF $R_{P\to SV}$

There is no conversion of P to SV waves when $R_{P\to SV}$ vanishes. From (6.32) or (6.33) this can be seem to occur for three cases. The first is when p=0 or $\theta_{Li}=0$, i.e. for normal incidence. The second is when $p=1/c_L$ or $\theta_{Li}=\pi/2$, i.e. for grazing incidence, and the third is when $p=2^{-1/2}c_T^{-1}$ or $\theta_{Tr}=\pi/4$. In each case energy is conserved since $R_{P\to P}$ equals -1, -1, and +1 for the respective cases.

3.7 SV-WAVE INCIDENCE ON A FREE SURFACE

The additional coupled case to the problem in Sec. 6 is that of SV-wave incidence on a flat, free surface. The scattered field has both P and SV components as illustrated in Fig. 3.5.

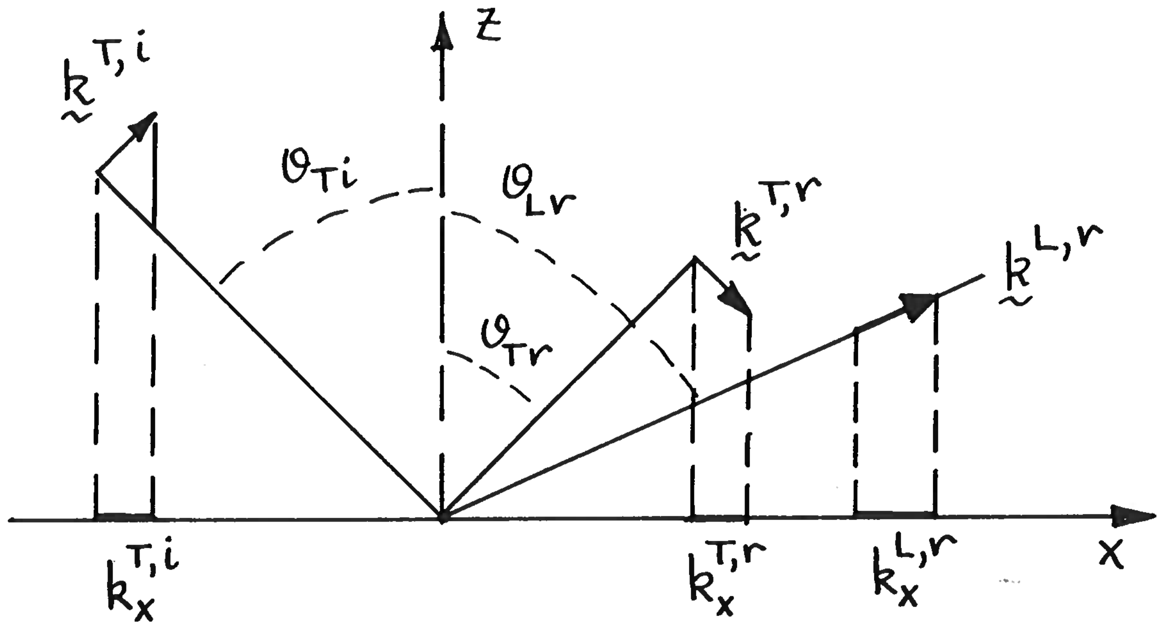


Fig. 3.5

The potentials ϕ and A satisfy the same boundary conditions as (6.2) and (6.3) but are here given by

$$\phi(x,z) = B \exp\left[i(k_x^{L,r} x + k_z^{L,r} z)\right] , \tag{7.1}$$

which contains only a scattered field, and

$$A(x,z) = A_0 \exp\left[i(k_x^{T,i} x - k_z^{T,i} z)\right] + C \exp\left[i(k_x^{T,r} x + k_z^{T,r} z)\right] , \tag{7.2}$$

containing both incident (A_0) and scattered (C) fields. Substituting these results into (6.2) and (6.3) and again using the translational invariance in x given by (6.14) we get the set of equations for $R_1=B/A_0=R_{SV\to P}$ and $R_2=C/A_0=R_{SV\to SV}$ given by

$$a_{11}R_2 - a_{12}R_1 = -a_{11} \quad , \tag{7.3}$$

$$a_{21}R_2 + a_{22}R_1 = a_{21} \quad , \tag{7.4}$$

where the a_{ij} are given by (6.10)-(6.13). Note the form of the equations (7.3) and (7.4) is the same as (6.18) and (6.19) except for the new interpretations of R_1 and R_2 and the fact that the right hand sides of the two equations are different. The latter occurs because the incident wave has changed from P in Sec. 6 to SV here.

The equations have the solutions

$$R_{SV\to SV} = (a_{12}a_{21} - a_{11}a_{22})/\Delta \quad , \tag{7.5}$$

and

$$R_{SV\to P} = 2\, a_{11}a_{21}/\Delta \quad , \tag{7.6}$$

where Δ is the same denominator as in Sec. 6, given by (6.22). Using the properties of the a_{ij} given in (6.25)-(6.28) we get

$$R_{SV\to SV} = \frac{4p^2 c_T^3(1-p^2c_L^2)^{1/2}\,(1-p^2c_T^2)^{1/2} - c_L(1-2p^2c_T^2)^2}{4p^2c_T^3(1-p^2c_L^2)^{1/2}(1-p^2c_T^2)^{1/2} + c_L(1-p^2c_T^2)^2} \quad , \tag{7.7}$$

which appears similar to the result in (6.29) but here note that

$$(1-p^2c_L^2)^{1/2} = \cos\theta_{Lr} \quad , \tag{7.8}$$

and

$$(1-p^2c_T^2)^{1/2} = \cos\theta_{Ti} \quad . \tag{7.9}$$

Using these results we can also write (7.7) as

$$R_{SV\rightarrow SV} = \frac{4p^2(\cos\theta_{Lr}/c_L)(\cos\theta_{Ti}/c_T) - (c_T^{-2} - 2p^2)^2}{4p^2(\cos\theta_{Lr}/c_L)(\cos\theta_{Ti}/c_T) + (c_T^{-2} - 2p^2)^2} \quad . \tag{7.10}$$

We can also write the other reflection coefficient as

$$R_{SV\rightarrow P} = \frac{4p\ c_Lc_T(1-p^2c_T^2)^{1/2}(1-2p^2c_T^2)}{4p^2c_T^3(1-p^2c_L^2)^{1/2}(1-p^2c_T^2)^{1/2} + c_L(1-2p^2c_T^2)^2} \quad , \tag{7.11}$$

or as

$$R_{SV\rightarrow P} = \frac{4p(\cos\theta_{Ti}/c_T)(c_T^{-2}-2p^2)}{4p^2(\cos\theta_{Lr}/c_L)(\cos\theta_{Ti}/c_T) + (c_T^{-2} - 2p^2)^2} \quad . \tag{7.12}$$

Note that $R_{SV\rightarrow P}$ vanishes when

(a) $p = 0$ $(\theta_{Ti} = 0$, normal incidence)

(b) $p = c_T^{-1}$ $(\theta_{Ti} = \pi/2$, grazing incidence)

and

(c) $p = 2^{-1/2}c_T^{-1}$ $(\theta_{Ti} = \pi/4)$,

and in these cases we get no conversion of SV to P waves. Again also note, as in Sec. 6., these are reflection coefficients for the potentials.

In addition we can also compute the values of displacement on the z=0 surface from (6.35) and (6.36). These are given by

$$u_1(x,0) = i\ A_o \left[k_x^{L,r} \frac{B}{A_o} + k_z^{T,i} \left(1 - \frac{C}{A_o}\right) \right] e^{ik_x x} \ , \qquad (7.13)$$

and

$$u_3(x,0) = i\ A_o \left[k_z^{T,r} \frac{B}{A_o} + k_x^{T,i} \left(1 + \frac{C}{A_o}\right) \right] e^{ik_x x} \ , \qquad (7.14)$$

where k_x stands for any of the x components of wave number.

3.8 SH-WAVE INCIDENCE

We consider the three possible cases of SH-wave incidence on a flat boundary. As we have already noted for this type of problem, the SH waves decouple from the P-SV waves. The first two cases are illustrated in Fig. 3.6 below.

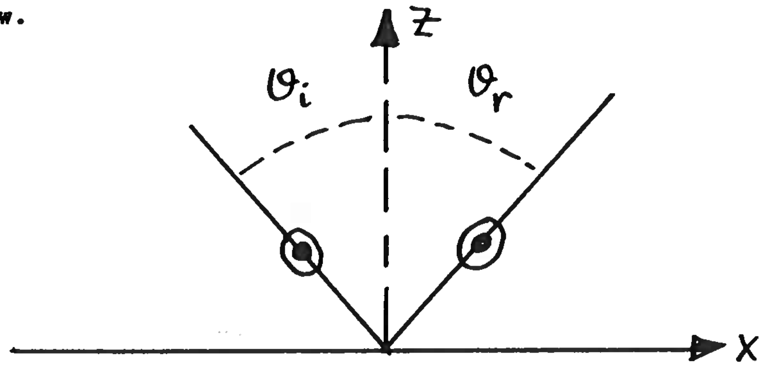


Fig. 3.6

Here the lines indicate the direction of propagation of the waves, and the circles with dots (the tips of arrows pointing out from the paper) indicate the direction of the displacements of the waves.

CASE (a): Free Flat Surface

For this case the displacement is

$$v(x,z) = A_o \exp\left[i(k_x^{T,i}x - k_z^{T,i}z)\right] + B \exp\left[i(k_x^{T,r}x + k_z^{T,r}z)\right] , \quad (8.1)$$

written in terms of incident (amplitude A) and reflected (amplitude B) waves. For both cases we have that

$$(k_x^{T,i})^2 + (k_z^{T,i})^2 = (k_x^{T,r})^2 + (k_z^{T,r})^2 = k_T^2 \quad . \tag{8.2}$$

The boundary condition at z=0 is

$$\tau_{z2} = \mu \frac{\partial v}{\partial z} = 0 \quad . \tag{8.3}$$

Since this must remain invariant for any value of x

$$k_x^{T,i} = k^T \sin\theta_i = k_x^{T,r} = k^T \sin\theta_r \quad , \tag{8.4}$$

so that the angle of reflection equals the angle of incidence. Further to satisfy the boundary condition, we require $B/A_0=1$.

CASE (b): Rigid Flat Surface

For this case the displacement is again given by (8.1), and (8.2) is satisfied. The boundary condition is now

$$v(x,0) = 0 \quad . \tag{8.5}$$

Again we get $\theta_i=\theta_r$ but now $B/A_0=-1$.

CASE (c): Elastic Interface

For this case we have a flat surface separating two elastic media, (1), with elastic parameters μ_1, λ_1, k_{T1}, k_{L1}, ρ_1, and (2), with parameters μ_2, λ_2, k_{T2}, k_{L2}, and ρ_2. Part of this case corresponding to incidence in region

(1) is illustrated in Fig. 3.7 below.

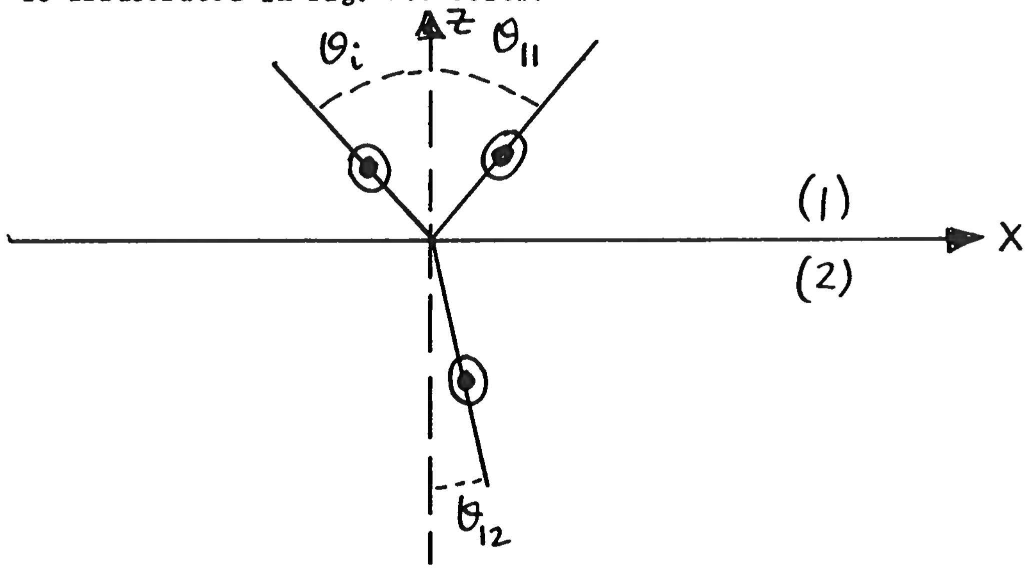


Fig. 3.7

The angles θ_{ij} correspond to scattering from region i to region j. The displacements for both regions can be written as

$$v_1(x,z) = A_0 \exp\left[i(k_x^{T,i}x - k_z^{T,i}z)\right] + B \exp\left[i(k_x^{T,r}x + k_z^{T,r}z)\right] \quad , \tag{8.6}$$

and

$$v_2(x,z) = C \exp\left[i(k_x^{T,t2}x - k_z^{T,t2}z)\right] \quad , \tag{8.7}$$

in terms of incident (i) and reflected (r) fields in region (1), and transmitted (t₂) field in region (2). We have the dispersion relations

$$(k_x^{T,i})^2 + (k_z^{T,i})^2 = (k_x^{T,r})^2 + (k_z^{T,r})^2 = (k_{T1})^2 \quad , \tag{8.8}$$

and

$$(k_x^{T,t2})^2 + (k_z^{T,t2})^2 = (k_{T2})^2 \quad . \tag{8.9}$$

The continuity conditions at z=0 in terms of stress and displacement are

$$\tau_{z2}^{(1)}(x,0) = \tau_{z2}^{(2)}(x,0) \quad , \tag{8.10}$$

and

$$u_2^{(1)}(x,0) = u_2^{(2)}(x,0) \quad . \tag{8.11}$$

Written in terms of v these become respectively

$$\mu_1 \frac{\partial v_1}{\partial z}(x,0) = \mu_2 \frac{\partial v_2}{\partial z}(x,0) \quad , \tag{8.12}$$

and

$$v_1(x,0) = v_2(x,0) \quad . \tag{8.13}$$

Substituting (8.6) and (8.7) in (8.12), and using the translational invariance in x given by

$$k_x^{T,i} = k_x^{T,r} = k_x^{T,t2} \quad , \tag{8.14}$$

which imply

$$\theta_i = \theta_{11} \quad , \quad k_{T1} \sin\theta_{11} = k_{T2} \sin\theta_{12} \quad , \tag{8.15}$$

(angle of incidence equals angle of reflection, and Snell's Law) yields the result

$$A - B = \alpha C \quad , \tag{8.16}$$

where

$$\alpha = \mu_2 k_z^{T,t2} / \mu_1 \, k_z^{T,i} \quad , \tag{8.17}$$

with $k_z^{T,i} = k_z^{T,r}$ from (8.14). Using the results

$$k_z^{T,t2} = k_{T2} \cos \theta_{12} \quad ; \quad k_z^{T,i} = k_{T1} \cos \theta_{11} \quad , \tag{8.18}$$

and the definitions

$$\mu = \mu_2 / \mu_1 \quad ; \quad K = k_{t2} / k_{t1} \quad , \tag{8.19}$$

we can write α, using Snell's Law as

$$\alpha = \mu (K^2 - \sin^2 \theta_{11})^{1/2} / \cos \theta_{11} \quad . \tag{8.20}$$

Substituting (8.6) and (8.7) in the second equation, (8.13), yields the result

$$A + B = C \quad , \tag{8.21}$$

where we have used (8.14). Simultaneous solution of (8.16) and (8.21) yields two components of our scattering matrix S_{ij}, $S_{11}=B/A_0$, the reflection coefficient from region (1) to region (1′), and $S_{12}=C/A_0$, the scattering (transmission) coefficient from region (1) to region (2). They are

$$S_{11} = \frac{\cos \theta_{11} - \mu (K^2 - \sin^2 \theta_{11})^{1/2}}{\cos \theta_{11} + \mu (K^2 - \sin^2 \theta_{11})^{1/2}} \quad , \tag{8.22}$$

and

$$S_{12} = \frac{2 \cos \theta_{11}}{\cos \theta_{11} + \mu(K^2 - \sin^2\theta_{11})^{1/2}} \quad . \tag{8.23}$$

Using Snell's Law these can also be written as

$$S_{11} = (\cos \theta_{11} - \mu K \cos \theta_{12})(\cos \theta_{11} + \mu K \cos \theta_{12})^{-1} \quad , \tag{8.24}$$

and

$$S_{12} = 2 \cos \theta_{11}(\cos \theta_{11} + \mu K \cos \theta_{12})^{-1} \quad , \tag{8.25}$$

or, expressing everything in terms of θ_{12} as

$$S_{11} = \frac{(1 - K^2\sin^2\theta_{12})^{1/2} - \mu K \cos \theta_{12}}{(1 - K^2\sin^2\theta_{12})^{1/2} + \mu K \cos \theta_{12}} \quad , \tag{8.26}$$

and

$$S_{12} = \frac{2(1 - K^2\sin^2\theta_{12})^{1/2}}{(1 - K^2\sin^2\theta_{12})^{1/2} + \mu K \cos \theta_{12}} \quad . \tag{8.27}$$

The other two components of the scattering matrix S_{22} and S_{21} correspond to incidence from region (2) as illustrated in Fig. 3.8 below.

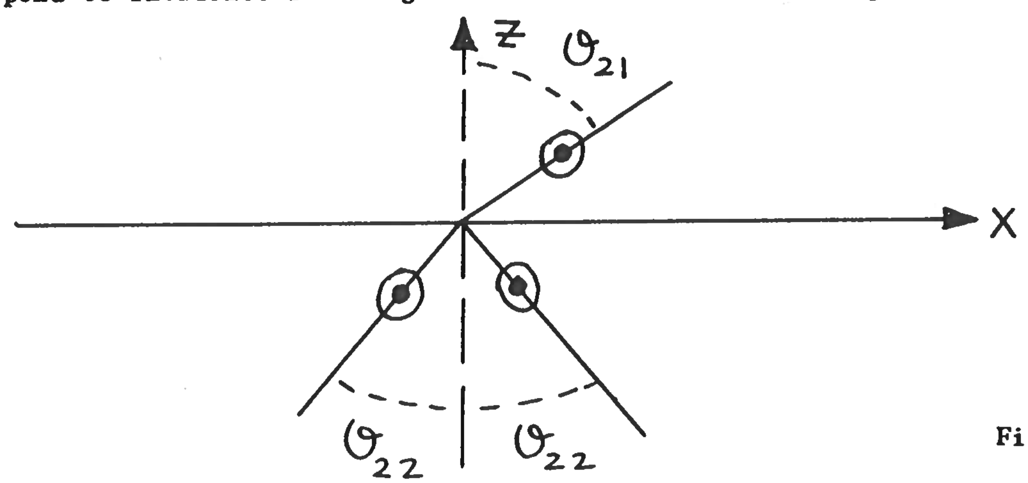


Fig. 3.8

We can find these components from (8.26) and (8.27) using the following interchange

$$\mu_1 \leftrightarrow \mu_2 \quad , \quad k_1 \leftrightarrow k_2 \quad , \quad \theta_{12} \rightarrow \theta_{21} \quad , \tag{8.28}$$

or equivalently

$$\mu \rightarrow 1/\mu \quad , \quad K \rightarrow 1/K \quad , \quad \theta_{12} \rightarrow \theta_{21} \quad . \tag{8.29}$$

Using (8.29) in (8.26) we find S_{22} given by

$$S_{22} = \frac{\mu\,(K^2 - \sin^2\theta_{21})^{1/2} - \cos\theta_{21}}{\mu\,(K^2 - \sin^2\theta_{21})^{1/2} + \cos\theta_{21}} \quad , \tag{8.30}$$

and using (8.29) in (8.27) we find

$$S_{21} = \frac{2\mu(K^2 - \sin^2\theta_{21})^{1/2}}{\mu(K^2 - \sin^2\theta_{21})^{1/2} + \cos\theta_{21}} \quad . \tag{8.31}$$

By Snell's Law, we have the same set of angles in Figs. (3.7) and (3.8), so $\theta_{21}=\theta_{11}$. Comparing (8.30) with (8.22) we see that

$$S_{22} = -S_{11} \quad . \tag{8.32}$$

Other representations for S_{22} and S_{21} can be found by doing this $2 \leftrightarrow 1$ interchange in the pairs of equations (8.24) and (8.25), and in (8.22) and (8.23).

3.9 ELASTIC INTERFACE (RIGID CONTACT)

In Sec. 8, Case (c), we treated the case of an SH-wave incident on a flat surface separating two elastic media with different elastic parameters and in rigid contact. Here we treat the same geometry for the remaining two cases, those of P-wave incidence and SV-wave incidence.

CASE (a): P-wave Incidence

The geometry for this case is illustrated in Fig. 3.9 below.

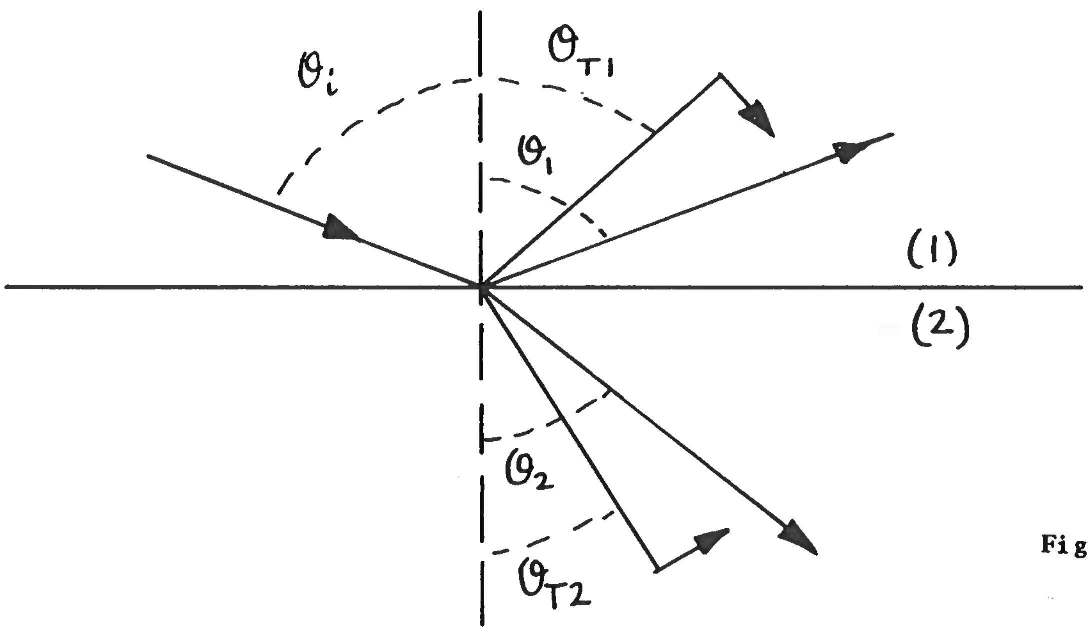


Fig. 3.9

We have a P-wave incident at angle θ_i, reflected and transmitted P-waves at angles θ_1 and θ_2 respectively, and reflected and transmitted SV-waves at angles θ_{T1} and θ_{T2} respectively. Our potentials can be written in Region 1 as:

$$\phi^{(1)}(x,z) = \phi^{in}(x,z) + \phi^{SC}(x,z) \quad , \tag{9.1}$$

where

$$\phi^{in}(x,z) = A_0^P \exp\left[i\,(k_x^{L,i}x - k_z^{L,i}z)\right] \quad , \tag{9.2}$$

and

$$\phi^{SC}(x,z) = B \exp\left[i(k_x^{L,r}x + k_z^{L,r}z)\right] \quad , \tag{9.3}$$

and for the A potential

$$A^{(1)}(x,z) = C \exp\left[i(k_x^{T1}x + k_z^{T1}z)\right] \quad . \tag{9.4}$$

Also in Region 1 the displacement and stress components are written on the z=0 surface as

$$u_1^{(1)}(x,0) = \frac{\partial\phi^{SC}}{\partial x}(x,0) - \frac{\partial A^{(1)}}{\partial z}(x,0) + \frac{\partial\phi^{in}}{\partial x}(x,0) \quad , \tag{9.5}$$

$$u_3^{(1)}(x,0) = \frac{\partial\phi^{SC}}{\partial z}(x,0) + \frac{\partial A^{(1)}}{\partial x}(x,0) + \frac{\partial\phi^{in}}{\partial z}(x,0) \quad , \tag{9.6}$$

$$\tau_{z1}^{(1)}(x,0) = \mu_1\left[\frac{\partial^2 A^{(1)}}{\partial x^2}(x,0) - \frac{\partial^2 A^{(1)}}{\partial z^2}(x,0) + 2\,\frac{\partial^2\phi^{SC}}{\partial x\partial z}(x,0)\right] + 2\mu_1\,\frac{\partial^2\phi^{in}}{\partial x\partial z}(x,0) \quad , \tag{9.7}$$

and

$$\tau_{z3}^{(1)}(x,0) = 2\mu_1\left[\frac{\partial^2\phi^{SC}}{\partial z^2}(x,0) + \frac{\partial^2 A^{(1)}}{\partial x\partial z}(x,0)\right] - \lambda_1\, k_{L1}^2\phi^{SC}(x,0) + 2\mu_1\,\frac{\partial^2\phi^{in}}{\partial z^2}(x,0) - \lambda_1 k_{L1}^2\phi^{in}(x,0) \quad . \tag{9.8}$$

In Region 2 the scalar potentials for the transmitted P and SV waves are

$$\phi^{(2)}(x,z) = D \exp\left[i(k_x^{L2} x - k_z^{L2} z)\right] , \tag{9.9}$$

and

$$A^{(2)}(x,z) = E \exp\left[i\ (k_x^{T2} x - k_z^{T2} z)\right] . \tag{9.10}$$

Simarly the displacement and stress components expressed in terms of these potentials are given by (on the z=0 surface)

$$u_1^{(2)}(x,0) = \frac{\partial}{\partial x} \phi^{(2)}(x,0) - \frac{\partial}{\partial z} A^{(2)}(x,0) , \tag{9.11}$$

$$u_3^{(2)}(x,0) = \frac{\partial}{\partial z} \phi^{(2)}(x,0) + \frac{\partial}{\partial x} A^{(2)}(x,0) , \tag{9.12}$$

$$\tau_{z1}^{(2)}(x,0) = \mu_2 \left[\frac{\partial^2}{\partial x^2} A^{(2)}(x,0) - \frac{\partial^2}{\partial z^2} A^{(2)}(x,0)\right] + 2\mu_2 \frac{\partial^2}{\partial x \partial z} \phi^{(2)}(x,0) , \tag{9.13}$$

and

$$\tau_{z3}^{(2)}(x,0) = 2\mu_2 \left[\frac{\partial^2}{\partial z^2} \phi^{(2)}(x,0) + \frac{\partial^2}{\partial x \partial z} A^{(2)}(x,0)\right] - \lambda_2 k_{L2}^2 \phi^{(2)}(x,0) . \tag{9.14}$$

The boundary conditions for two elastic media in rigid contact are the continuity of displacements and stresses at the z=0 interface. The conditions are given by

$$u_1^{(1)}(x,0) = u_1^{(2)}(x,0) \quad , \tag{9.15}$$

$$u_3^{(1)}(x,0) = u_3^{(2)}(x,0) \quad , \tag{9.16}$$

$$\tau_{z1}^{(1)}(x,0) = \tau_{z1}^{(2)}(x,0) \quad , \tag{9.17}$$

and

$$\tau_{z3}^{(1)}(x,0) = \tau_{z3}^{(2)}(x,0) \quad . \tag{9.18}$$

Substituting (9.5)-(9.8) and (9.11)-(9.14) into (9.15)-(9.18) and writing the results in terms of the incident field on the right hand side we get (all fields are evaluated at (x,0))

$$\frac{\partial \phi^{SC}}{\partial x} - \frac{\partial A^{(1)}}{\partial z} - \frac{\partial \phi^{(2)}}{\partial x} + \frac{\partial A^{(2)}}{\partial z} = - \frac{\partial \phi^{in}}{\partial x} \quad , \tag{9.19}$$

$$\frac{\partial \phi^{SC}}{\partial z} + \frac{\partial A^{(1)}}{\partial x} - \frac{\partial \phi^{(2)}}{\partial z} - \frac{\partial A^{(2)}}{\partial x} = - \frac{\partial \phi^{in}}{\partial z} \quad , \tag{9.20}$$

$$2\mu_1 \frac{\partial^2 \phi^{SC}}{\partial x \partial z} + \mu_1 \left[\frac{\partial^2 A^{(1)}}{\partial x^2} - \frac{\partial^2 A^{(1)}}{\partial z^2}\right] - 2\mu_2 \frac{\partial^2 \phi^{(2)}}{\partial x \partial z} - \mu_2 \left[\frac{\partial^2 A^{(2)}}{\partial x^2} - \frac{\partial^2 A^{(2)}}{\partial z^2}\right] = -2\mu_1 \frac{\partial^2 \phi^{in}}{\partial x \partial z} \quad , \tag{9.21}$$

and

$$2\mu_1 \frac{\partial^2 \phi^{SC}}{\partial z^2} - \lambda_1 k_{L1}^2 \phi^{SC} + 2\mu_1 \frac{\partial^2 A^{(1)}}{\partial x \partial z}$$

$$- 2\mu_2 \frac{\partial^2 \phi^{(2)}}{\partial z^2} + \lambda_2 k_{L2}^2 \phi^{(2)} - 2\mu_2 \frac{\partial^2 A^{(2)}}{\partial x \partial z}$$

$$= -2\mu_1 \frac{\partial^2 \phi^{in}}{\partial z^2} + \lambda_1 k_{L1}^2 \phi^{in} \quad . \tag{9.22}$$

All the z-components of the phases in each of the waves vanish since we evaluate at z=0. By translational invariance all the x components of the wave numbers are equal. Expressing these in terms of the angles in Fig. 9.1 we have

$$k_x^{L,i} = k_{L1} \sin \theta_i \quad , \quad k_x^{L,r} = k_{L1} \sin \theta_1 \quad , \quad k_x^{T1} = k_{T1} \sin \theta_{T1}$$

$$k_x^{L2} = k_{L2} \sin \theta_2 \quad , \quad k_x^{T2} = k_{T2} \sin \theta_{T2} \quad . \tag{9.23}$$

Equating these we get the laws of reflection $(\theta_1 = \theta_i)$ for P-waves and $k_{L1} \sin \theta_i = k_{T1} \sin \theta_{T1}$ for SV-waves, and the laws of refraction for P-waves $(k_{L1} \sin \theta_i = k_{L2} \sin \theta_2)$ and SV-waves $(k_{T1} \sin \theta_{T1} = k_{T2} \sin \theta_{T2})$. In addition we could express all these equal k_x-components in terms of the ray parameter, i.e. $k_x = \omega p$. Further the z-components of wave number can be written as

$$k_z^{L,i} = k_z^{L,r} = K_{L1} = k_{L1} \cos \theta_i \quad , \tag{9.24}$$

$$k_z^{T1} = K_{T1} = k_{T1} \cos \theta_{T1} \quad , \tag{9.25}$$

$$k_z^{L2} = K_{L2} = k_{L2} \cos \theta_2 \quad , \tag{9.26}$$

and

$$k_z^{T_2} = K_{T_2} = k_{T_2} \cos \theta_{T_2} \quad . \tag{9.27}$$

Substituting the wave forms and the above results into (9.19)-(9.22) we get the linear equations (Zoeppritz Equations)

$$m_{11}S_1 + m_{12}S_2 + m_{13}T_1 + m_{14}T_2 = P_1 \quad , \tag{9.28}$$

$$m_{21}S_1 + m_{22}S_2 + m_{23}T_1 + m_{24}T_2 = P_2 \quad , \tag{9.29}$$

$$m_{31}S_1 + m_{32}S_2 + m_{33}T_1 + m_{34}T_2 = P_3 \quad , \tag{9.30}$$

$$m_{41}S_1 + m_{42}S_2 + m_{43}T_1 + m_{44}T_2 = P_4 \quad , \tag{9.31}$$

in terms of the two components of the scattering and transmission matrices

$$S_1 = B/A_0^P \quad , \quad S_2 = C/A_0^P \quad , \tag{9.32}$$

$$T_1 = D/A_0^P \quad , \quad T_2 = E/A_0^P \quad , \tag{9.33}$$

and the four terms P_j related to the incident P-wave field occurring on the right hand sides of (9.19)-(9.22). The matrix elements m_{ij} are given by

$$\begin{bmatrix} ik_x & -iK_{T_1} & -ik_x & -iK_{T_2} \\ iK_{L_1} & ik_x & iK_{L_2} & -ik_x \\ -2\mu_1 k_x K_{L_1} & \mu_1 (K_{T_1}^2 - k_x^2) & -2\mu_2 k_x K_{L_2} & \mu_2 (k_x^2 - K_{T_2}^2) \\ -2\mu_1 K_{L_1}^2 - \lambda_1 k_{L_1}^2 & -2\mu_1 k_x K_{T_1} & 2\mu_2 K_{L_2}^2 + \lambda_2 k_{L_2}^2 & -2\mu_2 k_x K_{T_2} \end{bmatrix} \quad , \tag{9.34}$$

and the elements for the incident field are

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} -m_{11} \\ m_{21} \\ m_{31} \\ -m_{41} \end{bmatrix} = \begin{bmatrix} -ik_x \\ iK_{L1} \\ -2\mu_1 k_x K_{L1} \\ 2\mu_1 K_{L1}^2 + \lambda_1 k_{L1}^2 \end{bmatrix} . \qquad (9.35)$$

Note how the incident field elements are related to the first column of the m-matrix, i.e. that column corresponding to the P-waves.

The solution is found once the inverse of the matrix m is known, i.e.

$$\begin{bmatrix} S_1 \\ S_2 \\ T_1 \\ T_2 \end{bmatrix} = m^{-1} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} . \qquad (9.36)$$

The matrix elements of the inverse can be written as

$$\left[m^{-1}\right]_{ij} = D_{ji}/\Delta \ , \qquad (9.37)$$

where the D_{ji} are the cofactors of the matrix m and Δ is its determinant. For example, the reflection coefficients can be explicitly written as

$$S_1 = \Delta^{-1} \sum_{j=1}^{4} D_{j1} P_j \ , \qquad (9.38)$$

and

$$S_2 = \Delta^{-1} \sum_{j=1}^{4} D_{j2}P_j \quad . \tag{9.39}$$

There are thus a total of eight cofactors to evaluate, and the determinant can be written in terms of these as

$$\Delta = m_{11}D_{11} + m_{21}D_{21} + m_{31}D_{31} + m_{41}D_{41} \quad , \tag{9.40}$$

as an expansion in the first column of m. The specific form of the matrix elements in (9.34) enable us to write

$$m = \begin{bmatrix} m_{11} & m_{12} & -m_{11} & m_{14} \\ m_{21} & m_{11} & m_{23} & -m_{11} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \quad . \tag{9.41}$$

The cofactors are

$$D_{11} = m_{11}\left[m_{33}m_{44} + m_{33}m_{42} - m_{43}m_{32} - m_{43}m_{34}\right] + m_{23}\left[m_{42}m_{34} - m_{44}m_{32}\right] , \tag{9.42}$$

$$D_{21} = m_{12}\left[m_{34}m_{43} - m_{33}m_{44}\right] + m_{11}\left[m_{34}m_{42} - m_{32}m_{44}\right] + m_{14}\left[m_{33}m_{42} - m_{32}m_{43}\right] , \tag{9.43}$$

$$D_{31} = m_{11}^{2}\left[m_{42} + m_{44}\right] + m_{11}\left[m_{43}m_{14} + m_{42}m_{12}\right] + m_{23}\left[m_{12}m_{44} - m_{14}m_{42}\right] , \tag{9.44}$$

$$D_{41} = -m_{11}^{2}\left[m_{32} + m_{34}\right] - m_{11}\left[m_{14} + m_{12}\right]m_{33} + m_{23}\left[m_{14}m_{32} - m_{12}m_{34}\right] , \tag{9.45}$$

$$D_{12} = m_{11}\left[m_{31}m_{43} - m_{33}m_{41}\right] + m_{21}\left[m_{34}m_{43} - m_{33}m_{44}\right] + m_{23}\left[m_{44}m_{31} - m_{41}m_{34}\right] , \tag{9.46}$$

$$D_{22} = m_{11}\left[m_{33}m_{44} + m_{31}m_{44} - m_{34}m_{43} - m_{34}m_{41}\right] + m_{14}\left[m_{31}m_{43} - m_{33}m_{41}\right] , \tag{9.47}$$

$$D_{32} = m_{14}\left[m_{23}m_{41} - m_{21}m_{43}\right] - m_{11}\left[m_{23}m_{44} + m_{11}m_{43} + m_{21}m_{44} + m_{11}m_{41}\right] , \tag{9.48}$$

and

$$D_{42} = m_{14}\left[m_{21}m_{33} - m_{23}m_{31}\right] + m_{11}\left[m_{23}m_{34} + m_{21}m_{34} + m_{11}m_{33} + m_{11}m_{31}\right] , \tag{9.49}$$

and these can be explicitly evaluated using (9.34).

CASE (b): SV-wave incidence

The geometry for this case is illustrated in Fig. 3.10.

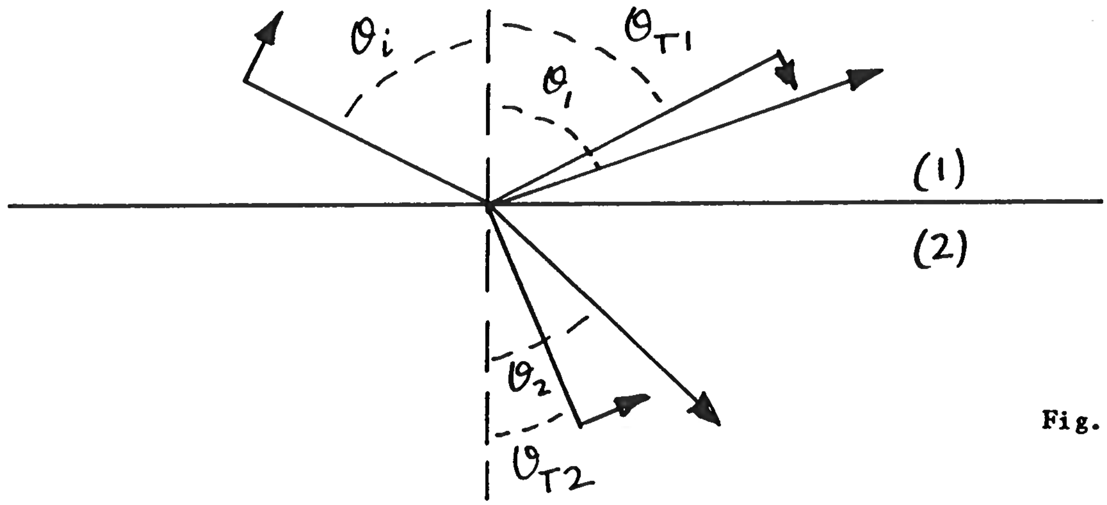


Fig. 3.10

We have an SV-wave incident at angle θ_i, reflected and transmitted P-waves at angles θ_1 and θ_2 respectively, and reflected and transmitted SV-waves at angles θ_{T1} and θ_{T2} respectively. We retain much of the notation of CASE (a). The potentials in Region 1 are

$$\varphi^{(1)}(x,z) = B \exp\left[i(k_x^{L,r} x + k_z^{L,r} z)\right] , \tag{9.50}$$

which is the same as the scattered field (9.3). The B amplitude will now represent a different reflection coefficient but it is convenient to keep the same notation. The A-potential now has incident and scattered parts

$$A^{(1)}(x,z) = A^{in}(x,z) + A^{SC}(x,z) \tag{9.51}$$

where

$$A^{in}(x,z) = A_o^v \exp\left[i(k_x^{T,i} x - k_z^{T,i} z)\right] , \tag{9.52}$$

and

$$A^{sc}(x,z) = C\exp\left[i(k_x^{T,i}x + k_z^{T,i}z)\right] \quad , \tag{9.53}$$

the latter of which agrees with (9.4) with the same proviso on the reflection coefficient C.

The displacement and stress components on the z=0 surface are given by

$$u_1^{(1)}(x,0) = \frac{\partial}{\partial x}\phi^{(1)}(x,0) - \frac{\partial}{\partial z}A^{sc}(x,0) - \frac{\partial}{\partial z}A^{in}(x,0) \quad , \tag{9.54}$$

$$u_3^{(1)}(x,0) = \frac{\partial}{\partial z}\phi^{(1)}(x,0) + \frac{\partial}{\partial x}A^{sc}(x,0) + \frac{\partial}{\partial x}A^{in}(x,0) \quad , \tag{9.55}$$

$$\tau_{z1}^{(1)}(x,0) = \mu_1\left[\frac{\partial^2}{\partial x^2}A^{sc}(x,0) - \frac{\partial^2}{\partial z^2}A^{sc}(x,0) + 2\frac{\partial^2}{\partial x\partial z}\phi^{(1)}(x,0)\right]$$

$$+ \mu_1\left[\frac{\partial^2}{\partial x^2}A^{in}(x,0) - \frac{\partial^2}{\partial z^2}A^{in}(x,0)\right] \quad , \tag{9.56}$$

and

$$\tau_{z3}^{(1)}(x,0) = 2\mu_1\left[\frac{\partial^2}{\partial z^2}\phi^{(1)}(x,0) + \frac{\partial^2}{\partial z\partial x}A^{sc}(x,0)\right] - \lambda_1 k_{L1}^2\phi^{(1)}(x,0)$$

$$+ 2\mu_1\frac{\partial^2}{\partial z\partial x}A^{in}(x,0) \quad . \tag{9.57}$$

In Region 2 the potentials for the transmitted P- and SV-waves are analogous to (9.9) and (9.10)

$$\varphi^{(2)}(x,z) = D\exp\left[i(k_x^{L2}x - k_z^{L2}z)\right] \quad , \tag{9.58}$$

and

$$A^{(2)}(x,z) = E\exp\left[i(k_x^{T2}x - k_z^{T2}z)\right] \quad , \tag{9.59}$$

and the displacement and stress components at z=0 given by

$$u_1^{(2)}(x,0) = \frac{\partial}{\partial x}\phi^{(2)}(x,0) - \frac{\partial}{\partial z}A^{(2)}(x,0) \quad , \tag{9.60}$$

$$u_3^{(2)}(x,0) = \frac{\partial}{\partial z}\phi^{(2)}(x,0) + \frac{\partial}{\partial z}A^{(2)}(x,0) \quad , \tag{9.61}$$

$$\tau_{z1}^{(2)}(x,0) = \mu_2\left[\frac{\partial^2}{\partial x^2}A^{(2)}(x,0) - \frac{\partial^2}{\partial z^2}A^{(2)}(x,0)\right]$$

$$+ 2\mu_2\frac{\partial^2}{\partial x\partial z}\phi^{(2)} \quad , \tag{9.62}$$

and

$$\tau_{z3}^{(2)}(x,0) = 2\mu_2\left[\frac{\partial^2}{\partial z^2}\phi^{(2)}(x,0) + \frac{\partial^2}{\partial x\partial z}A^{(2)}(x,0)\right]$$

$$- \lambda_2 k_{L2}^2\phi^{(2)}(x,0) \quad , \tag{9.63}$$

analogous to (9.11)-(9.14). The boundary conditions of continuity of displacements and stresses are given by (9.15)-(9.18). Substituting (9.54)-(9.57) and (9.60)-(9.63) into these equations and writing the incident field on the rhs as in (9.19)-(9.22) we get (all fields evaluated at (x,0))

$$\frac{\partial\phi^{(1)}}{\partial x} - \frac{\partial A^{sc}}{\partial z} - \frac{\partial\phi^{(2)}}{\partial x} + \frac{\partial A^{(2)}}{\partial z} = \frac{\partial A^{in}}{\partial z} \quad , \tag{9.64}$$

$$\frac{\partial\phi^{(1)}}{\partial z} + \frac{\partial A^{sc}}{\partial x} - \frac{\partial\phi^{(2)}}{\partial z} - \frac{\partial A^{(2)}}{\partial x} = -\frac{\partial A^{in}}{\partial x} \quad , \tag{9.65}$$

$$2\mu_1 \frac{\partial^2 \phi^{(1)}}{\partial x \partial z} + \mu_1 \left[\frac{\partial^2 A^{sc}}{\partial x^2} - \frac{\partial^2 A^{sc}}{\partial z^2} \right]$$

$$- 2\mu_2 \frac{\partial^2 \phi^{(2)}}{\partial x \partial z} - \mu_2 \left[\frac{\partial^2 A^{(2)}}{\partial x^2} - \frac{\partial^2 A^{(2)}}{\partial z^2} \right] = -\mu_1 \left[\frac{\partial^2 A^{in}}{\partial x^2} - \frac{\partial^2 A^{in}}{\partial z^2} \right] , \tag{9.66}$$

and

$$2\mu_1 \frac{\partial^2 \phi^{(1)}}{\partial z^2} - \lambda_1 k_{L1}^2 \phi^{(1)} + 2\mu_1 \frac{\partial^2 A^{sc}}{\partial x \partial z}$$

$$- 2\mu_2 \frac{\partial^2 \phi^{(2)}}{\partial z^2} + \lambda_2 k_{L2}^2 \phi^{(2)} - 2\mu_2 \frac{\partial^2 A^{(2)}}{\partial x \partial z} = -2\mu_1 \frac{\partial^2 A^{in}}{\partial x \partial z} . \tag{9.67}$$

Again all the z-components of the phases of the waveforms in these equations vanish, and, by translational invariance all the x-components are equal. The only difference in the forms from (9.23) is that the incident P-wave $k_x^{L,i}$ component is replaced by

$$k_x^{T,i} = k_{T1} \sin \theta_i . \tag{9.68}$$

Equating this to the appropriate remaining components in (9.23) we get the usual laws of reflection and refraction. Substituting the wave forms into (9.64) to (9.67) we get the Zoeppritz equations for this problem

$$m_{11}S_3 + m_{12}S_4 + m_{13}T_3 + m_{14}T_4 = V_1 , \tag{9.69}$$

$$m_{21}S_3 + m_{22}S_4 + m_{23}T_3 + m_{24}T_4 = V_2 , \tag{9.70}$$

$$m_{31}S_3 + m_{32}S_4 + m_{33}T_3 + m_{34}T_4 = V_3 , \tag{9.71}$$

$$m_{41}S_3 + m_{42}S_4 + m_{43}T_3 + m_{44}T_4 = V_4 , \tag{9.72}$$

in terms of the two components of the scattering and transmission matrices for the SV-incidence problem

$$S_3 = B/A_0^V \quad , \quad S_4 = C/A_0^V \quad , \tag{9.73}$$

$$T_3 = D/A_0^V \quad , \quad T_4 = E/A_0^V \quad , \tag{9.74}$$

and the four terms V_j related to the incident SV-wave field. The matrix components m_{ij} are the same as those for the P-wave incidence problem and are defined in (9.34). The V_j terms replace the P_j terms for the P-wave incidence and are given by

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} -i\ k_{T1} \\ -i\ k_x \\ \mu_1 (K_{T1}^2 - k_x^2) \\ -2\mu_1 k_x K_{T1} \end{bmatrix} = \begin{bmatrix} m_{12} \\ -m_{22} \\ m_{32} \\ m_{42} \end{bmatrix} , \tag{9.75}$$

which are related to the second column of the m-matrix in (9.34) corresponding to SV-waves.

The set of equations (9.69)-(9.72) is the same as (9.18)-(9.31) and the solution is in terms of m^{-1} which was previously treated. It is, for the reflection coefficients given by

$$S_3 = \Delta^{-1} \sum_{j=1}^{4} D_{j1} V_j \quad , \tag{9.76}$$

and

$$S_4 = \Delta^{-1} \sum_{j=1}^{4} D_{j2} V_j \quad , \tag{9.77}$$

in analogy with (9.38) and (9.39). In general we have

$$\begin{bmatrix} S_3 \\ S_4 \\ T_3 \\ T_4 \end{bmatrix} = m^{-1} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} . \tag{9.78}$$

The cofactors are defined in (9.42)-(9.49).

Note finally that all the reflection and transmission coefficients for both P- and SV-wave incidence have the same denominator Δ defined in (9.40).

SURFACE DISPLACEMENTS AND STRESSES

We can easily compute the surface values of the displacements and stresses for P- and SV-wave problems using results in this section, and for SH-incidence using the results in Sec. 8.

P-WAVES

From (9.5)-(9.8) we get (dropping the superscript because of the continuity conditions)

$$u_1(x,0) = i\left[k_x(S_1 + 1) - K_{T1}S_2\right]A_0^P \exp(ik_x x) \quad , \tag{9.79}$$

$$u_3(x,0) = i\left[K_{L1}(S_1 - 1) + k_x S_2\right]A_0^P \exp(ik_x x) \quad , \tag{9.80}$$

$$\tau_{z1}(x,0) = \mu_1\left[2k_x K_{L1}(S_1 - 1) + (K_{T1}^2 - k_x^2)S_2\right]A_0^P \exp(ik_x x) \quad , \tag{9.81}$$

and

$$\tau_{z3}(x,0) = -\left[(2\mu_1 K_{L1}^2 + \lambda_1 k_{L1}^2)(S_1 + 1) + 2\mu_1 k_x K_{T1} S_2\right] A_0^P \exp(ik_x x) \ . \tag{9.82}$$

<u>**SV-WAVES**</u>

From (9.54)-(9.57) we get

$$u_1(x,0) = i\left[k_x S_3 + K_{T1}(1 - S_4)\right] A_0^V \exp(ik_x x) \ , \tag{9.83}$$

$$u_3(x,0) = i\left[K_{L1} S_3 + k_x(1 + S_4)\right] A_0^V \exp(ik_x x) \ , \tag{9.84}$$

$$\tau_{z1}(x,0) = \mu_1\left[-2k_x K_{L1} S_3 + (K_{T1}^2 - k_x^2)(S_4 + 1)\right] A_0^V \exp(ik_x x) \ , \tag{9.85}$$

and

$$\tau_{z3}(x,0) = -\left[(2\mu_1 K_{L1}^2 + \lambda_1 k_{L1}^2) S_3 + 2\mu_1 k_x K_{T1}(S_4 - 1)\right] A_0^V \exp(ik_x x) \ . \tag{9.86}$$

<u>**SH-WAVES**</u>

From the results in (8.6) and (8.22) (or (8.24) or (8.26))

$$u_2(x,0) = v^{(1)}(x,0) = (1 + S_{11}) A_0^H \exp(ik_x x) \ , \tag{9.87}$$

and

$$\tau_{z2}(x,0) = \mu_1 \frac{\partial v^{(1)}}{\partial z}(x,0) = -iK_{T1}(1 - S_{11}) A_0^H \exp(ik_x x) \ . \tag{9.88}$$

The result is that each of the displacements and stresses evaluated on the surface can be related to its incident plane wave field on the surface times factors involving the five plane wave reflection coefficients S_j, $j=1,\ldots,4$, and S_{11} from (9.32) or (9.38) and (9.39), (9.73) or (9.76) and (9.77), and (8.22) or (8.24) or (8.26) respectively.

3.10 ELASTIC LAYER - P-WAVE INCIDENCE

In this section we briefly discuss the problem of P-wave incidence on a flat elastic layer, a three region problem. Each of the regions is assumed to have different elastic properties. The geometry is illustrated in Fig. 3.11.

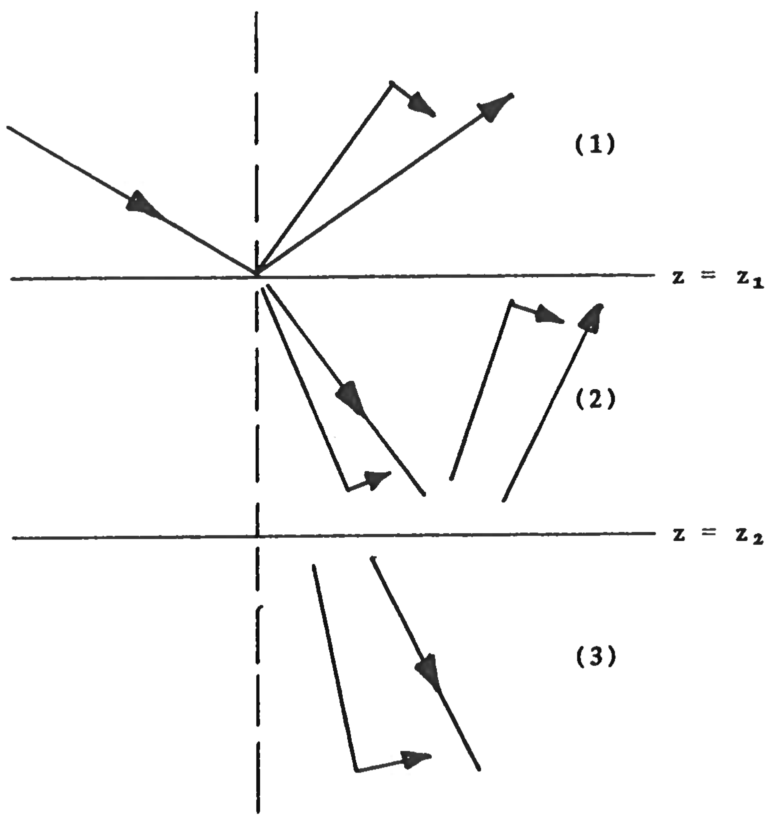


Fig. 3.11

Region (1) has incident and reflected P-waves (with the angles of incidence and reflection equal, and a reflected SV-wave at a relected angle found via (6.17). Region (2) has both up-and down-going P- and SV-waves (or equivalently standing waves). Region (3) has down going P- and SV-waves. No coupling to SH-waves occurs because of the planar P-wave incidence and the flat geometries.

We thus have eight wave coefficents to solve for, all expressed as a ratio with the amplitude of the incident field. They are the reflection coefficients:

$$\text{Region (1):} \quad R_{P\to P}\ ,\quad R_{P\to SV}\ , \tag{10.1}$$

$$\text{Region (2):} \quad P- \ ;\ SV- \ ;\ P+ \ ;\ SV+ \ , \tag{10.2}$$

$$\text{Region (3):} \quad P- \ ;\ SV- \ , \tag{10.3}$$

where the + and - symbols refer to up- and down-going waves. There are a total of eight continuity conditions. For i=1,3 they are

$$u_i^{(1)} = u_i^{(2)} \qquad (z = z_1)\ , \tag{10.4}$$

$$\tau_{zi}^{(2)} = \tau_{zi}^{(2)} \qquad (z = z_1)\ , \tag{10.5}$$

$$u_i^{(2)} = u_i^{(3)} \qquad (z = z_2)\ , \tag{10.6}$$

and

$$\tau_{zi}^{(2)} = \tau_{zi}^{(3)} \qquad (z = z_2)\ . \tag{10.7}$$

In each region, we have the representations

$$u_1 = \frac{\partial\phi}{\partial x} - \frac{\partial A}{\partial z}\ , \tag{10.8}$$

$$u_3 = \frac{\partial\phi}{\partial z} + \frac{\partial A}{\partial x}\ , \tag{10.9}$$

$$\tau_{z1} = \mu\left[\frac{\partial^2 A}{\partial x^2} - \frac{\partial^2 A}{\partial z^2}\right] + 2\mu\,\frac{\partial^2\phi}{\partial x\partial z}\ , \tag{10.10}$$

and

$$\tau_{z3} = 2\mu\left[\frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2 A}{\partial x\partial z}\right] - \lambda k_L^2\phi\ . \tag{10.11}$$

Again, we are able to retain the potential representation for each of these quantities since we have a planar problem in 2 dimensions, x and z. Applying the boundary conditions we are also able to retain the overall coservation of horizontal wave number k_x. The translational invariance is maintained. We are thus able to write the potential wave shapes in each region as

REGION (1):

$$\varphi^{(1)}(x,z) = A_0^P \exp\left[i(k_x x - K_{L1} z)\right] + B \exp\left[i(k_x x + K_{L1} z)\right] \quad , \quad (10.12)$$

and

$$A^{(1)}(x,z) = C \exp\left[i(k_x x + K_{T1} z)\right] \quad , \quad (10.13)$$

and to apply the boundary conditions we need to find

$$u_1^{(1)}(x,z_1), \quad u_3^{(1)}(x,z_1), \quad \tau_{z1}(x,z_1) \text{ and } \tau_{z3}(x,z_1) \quad .$$

REGION (2):

$$\varphi^{(2)}(x,z) = \phi_- \exp\left[i(k_x x - K_{L1} z)\right] + \phi_+ \exp\left[i(k_x x + K_{L2} z)\right] \quad , \quad (10.14)$$

and

$$A^{(2)}(x,z) = V_- \exp\left[i(k_x x - K_{T2} z)\right] + V_+ \exp\left[i(k_x x + K_{T2} z)\right] \quad , \quad (10.15)$$

and to apply the boundary conditions we need to find $u_1^{(2)}(x,z_1)$, $u_3^{(2)}(x,z_1)$, $\tau_{z1}^{(2)}(x,z_1)$ and $\tau_{z3}^{(2)}(x,z_1)$ at the upper boundary, and $u_1^{(2)}(x,z_2)$, $u_3^{(2)}(x,z_2)$, $\tau_{z1}^{(2)}(x,z_2)$ and $\tau_{z3}^{(2)}(x,z_2)$ at the lower boundary.

REGION (3):

$$\varphi^{(3)}(x,z) = \phi_3 \exp\left[i(k_x x - K_{L3} z)\right] \quad , \quad (10.16)$$

and

$$A^{(3)}(x,z) = A_3 \exp\left[i(k_x x - K_{T3} z)\right] \quad , \quad (10.17)$$

and to use the boundary conditions we need

$$u_1^{(3)}(x, z_2), \quad u_3^{(3)}(x, z_2), \quad \tau_{z1}^{(3)}(x, z_2) \text{ and } \tau_{z3}^{(3)}(x, z_2) \ .$$

Note that in general we will be left with exponentials involving both z_1 and z_2. We can set one interface at zero, but not both.

3.11 MULTILAYERS

In this section we briefly describe the systematics of notation, etc., involved in setting up the scattering problem for a planer multilayer structure with no variability in the y-direction. We thus maintain the decoupling between P-SV-waves and SH-waves. The displacements and stresses are written as usual for any layer in terms of ϕ and A potentials as (i = 1,3)

$$u_i(x,z) = \delta_{i1}\left[\frac{\partial\phi}{\partial x} - \frac{\partial A}{\partial z}\right] + \delta_{i3}\left[\frac{\partial\phi}{\partial z} + \frac{\partial A}{\partial x}\right] \quad , \tag{11.1}$$

and

$$\tau_{zi}(x,z) = \mu\delta_{i1}\left[\frac{\partial^2 A}{\partial x^2} - \frac{\partial^2 A}{\partial z^2} + 2\,\frac{\partial^2\phi}{\partial x\partial z}\right]$$

$$+ \delta_{i3}\left[2\mu\,\frac{\partial^2\phi}{\partial z^2} - \lambda\, k_L^2\phi + 2\mu\,\frac{\partial^2 A}{\partial x\partial z}\right] \quad . \tag{11.2}$$

At the mth interface, $z = z_m$, between layers m and m+1, we have the continuity conditions

$$u_i^{(m+1)}(x,z_m) = u_i^{(m)}(x,z_m) \quad , \tag{11.3}$$

and

$$\tau_{zi}^{(m+1)}(x,z_m) = \tau_{zi}^{(m)}(x,z_m) \quad . \tag{11.4}$$

Also at each interface we have up- and down- going waves from each layer. The geometry is illustrated in Fig. 3.12. The potentials can be decomposed

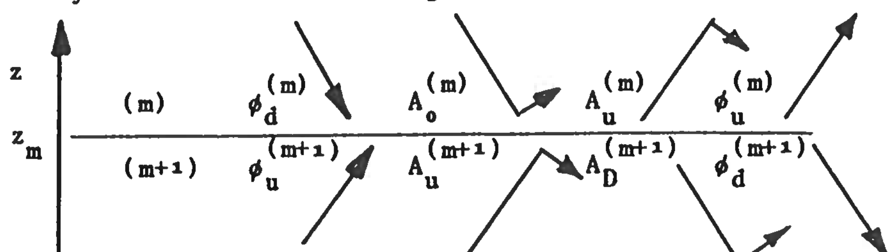


Fig. 3.12

in terms of down- and up- going waves in each layer. In the (m)th layer we can write

$$\phi^{(m)}(x,z) = d^{(m)}\phi_d^{(m)}(x,z) + u^{(m)}\phi_u^{(m)}(x,z) \quad , \tag{11.5}$$

where $\phi_d^{(m)}$ is a downward traveling plane wave with amplitude $d^{(m)}$ and $\phi_u^{(m)}$ upward traveling with amplitude $u^{(m)}$. For the A-potential we have an analogous expression

$$A^{(m)}(x,z) = D^{(m)}A_D^{(m)}(x,z) + V^{(m)}A_U^{(m)}(x,z) \quad . \tag{11.6}$$

The wave shapes have the general form

$$\phi_{\substack{d\\u}}^{(m)}(x,z) = \exp\left[i(k_x x \mp K_L^{(m)} z)\right] \quad , \tag{11.7}$$

and

$$A_{\substack{d\\u}}^{(m)}(x,z) = \exp\left[i(k_x x \mp K_T^{(m)} z)\right] \quad . \tag{11.8}$$

Using these up- and down- going wave representations we can write the displacements and stresses in each region in terms of an up- and down- going wave decomposition. In the m^{th} region we have

$$u_i^{(m)}(x,z) = u_i^{(m,u)}(x,z) + u_i^{(m,d)}(x,z) \quad , \tag{11.9}$$

and

$$\tau_{zi}^{(m)}(x,z) = \tau_{zi}^{(m,u)}(x,z) + \tau_{zi}^{(m,d)}(x,z) \quad . \tag{11.10}$$

At $z = z_m$ the displacement continuity condition is

$$u_i^{(m,u)}(x,z_m) + u_i^{(m,d)}(x,z_m) = u_i^{(m+1,u)}(x,z_m) + u_i^{(m+1,d)}(x,z_m) \quad , \tag{11.11}$$

and the stress continuity is

$$\tau_{zi}^{(m,u)}(x,z_m) + \tau_{zi}^{(m,d)}(x,z_m) = \tau_{zi}^{(m+1,u)}(x,z_m) + \tau_{zi}^{(m+1,d)}(x,z_m) \quad . \tag{11.12}$$

There are various way to rewrite the above equations. One possible way is to treat them in analogy with the incident and scattered wave interpretation in Secs. 6-9. Here the "incident' waves are those whose propagation is directed towards the interface, and the "scattered" waves those directed away from it. For example, the displacements in (11.11) would be written as

$$u_i^{(m,u)}(x,z_m) - u_i^{(m+1,d)}(x,z_m) = u_i^{(m+1,u)}(x,z_m) - u_i^{(m,d)}(x,z_m) \quad , \tag{11.13}$$

where the "incident" waves are on the rhs, and the "scattered" waves on the lhs. An analogous equation can be written for stresses, and from the resulting four equations we can write a matrix equation for the unknown (here the "scattered") coefficients in terms of the known ("incident") coefficients as

$$M^{(m)} \begin{bmatrix} u^{(m)} \\ U^{(m)} \\ d^{(m+1)} \\ D^{(m+1)} \end{bmatrix} = N^{(m)} \begin{bmatrix} u^{(m+1)} \\ U^{(m+1)} \\ d^{(m)} \\ D^{(m)} \end{bmatrix} \quad . \tag{11.14}$$

in terms of matrices M and N for each interface. These involve the wave shapes evaluated on the interface. The x-variability cancels due to translational invariance.

Alternatively the coefficients of the m^{th} region may be known, and it is desired to find the coefficients in the next layer. These can be determined from (11.11) and (11.12) in terms of a propagation matrix $P^{(m)}$ which "propagates" the coefficients from one layer to the next, viz.

$$\begin{bmatrix} u^{(m+1)} \\ U^{(m+1)} \\ d^{(m+1)} \\ D^{(m+1)} \end{bmatrix} = P^{(m)} \begin{bmatrix} u^{(m)} \\ U^{(m)} \\ d^{(m)} \\ D^{(m)} \end{bmatrix} . \tag{11.15}$$

3.12 SURFACE WAVES

In this section we give a brief description of the types of waves which can arise at a free surface or a fluid-elastic interface and which in general propagate along the interface and decay away from it. The results are related to the free surface problems in Secs. 6 and 7. We consider only two-dimensional problems.

(a) FREE SURFACE

We search for zeroes of the denominator of any of the reflection coefficients in Secs. 6 or 7, for example the denominator appearing in (6.28). It is convenient to multiply this denominator by $(c_L c_T)^{-1}$ and to parameterize the ray parameter in terms of a wave speed as $p = -1/c$. This Rayleigh denominator then becomes

$$D_R = 4\left[\frac{c_T}{c}\right]^2 \left[1 - \left[\frac{c_T}{c}\right]^2\right]^{1/2} \left[\left[\frac{c_T}{c_L}\right]^2 - \left[\frac{c_T}{c}\right]^2\right]^{1/2} + \left[1 - 2\left[\frac{c_T}{c}\right]^2\right]^2 , \quad (12.1)$$

and we want solutions of the secular equation

$$D_R = 0 \quad , \quad (12.2)$$

for the unknown wave speed c (or the ray parameter). It turns out that there is a real root $c = c_R$ where

$$0 < c_R < c_T < c_L \quad . \quad (12.3)$$

At this value of c, the vertical components of wave number K_L and K_T become

$$K_L = i\gamma_L \quad ,$$
$$K_T = i\gamma_T \quad , \tag{12.4}$$

where

$$\gamma_L^2 = \omega^2 \left[c_R^{-2} - c_L^{-2}\right] \quad , \tag{12.5}$$

and

$$\gamma_T^2 = \omega^2 \left[c_R^{-2} - c_T^{-2}\right] \quad , \tag{12.6}$$

so that the full time-dependent wave shapes for the potentials in the elastic material are

$$\phi(x,z) = \phi_0 \exp\left[-\gamma_L z\right] \exp\left[i\left[k_x x - \omega t\right]\right] \quad , \tag{12.7}$$

and

$$A(x,z) = A_0 \exp\left[-\gamma_T z\right] \exp\left[i\left[k_x x - \omega t\right]\right] \quad . \tag{12.8}$$

These waves are:

(a) non-dispersive (independent of frequency) since (12.2) is independent of frequency

(b) undamped in the direction of propagation (x)

(c) damped normal to the boundary (z)

(d) a coupled compressional-shear system

The wave is called a Rayleigh wave. It is schematically illustrated in Fig. 3.13. The solid lines represent the decay of intensity away from the boundary in the positive z-direction.

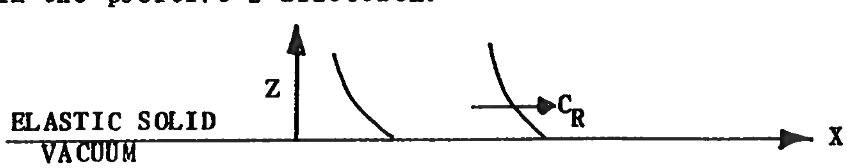


Fig. 3.13

(b) LIQUID-SOLID INTERFACE

1. GENERALIZED RAYLEIGH WAVES

For this case we also have a Rayleigh wave, but in addition energy leaks into the liquid, so the Rayleigh wave becomes damped in the direction of motion, and is called a generalized Rayleigh wave. The secular equation is given by

$$D_R = -\left[\frac{\rho_L}{\rho_s}\right]\left[\frac{[c_T/c_L]^2 - [c_T/c]^2}{[c_T/c_L]^2 - [c_T/c]^2}\right]^{1/2} , \qquad (12.9)$$

where ρ_L and ρ_s are the densities of the liquid and solid respectively, and c_L is the sound speed in the liquid. Note that as $(\rho_L/\rho_s) \to 0$, the root $c \to c_R$, the Rayleigh root. The root of (12.9) is complex and the wave decays as it travels along the surface. If $c_{R'}$ is the real part of the velocity then

$$c_R < c'_R < c_T < c_L \quad . \qquad (12.10)$$

Its decay into the liquid categorizes it as a <u>leaky wave</u>. It also turns out that most of the energy in the wave is in the solid, and it is launched at a specific angle given by $\sin \theta_{R'} = c_L/c_{R'}$.

2. STONELEY WAVE

There also exists a real root of (12.9) given by

$$c = c_S < c_L \quad , \qquad (12.11)$$

called a Stoneley wave. It propagates paralled to the boundary without attenuation along x. It is exponentially damped in both directions away from the surface, and doesn't always exist at a solid-solid interface. For $\rho_L/\rho_s \ll 1$, most of the energy is in the liquid.

3.13 FREE FLAT SURFACE USING INTEGRAL EQUATIONS

In Secs. 6-8 we studied the scattering from a free flat elastic surface using conventional methods involving potentials and plane waves. In particular we noted that the P- and SV-waves decoupled from the SH-waves. We also calculated in Sec. 3 the coupled integral equations for the displacement components for scattering from a free surface which however was not flat, but arbitrarily rough. We compute in Sec. 14, using perturbation theory, that this roughness induces a coupling between the P-, SV-, and SH-waves, i.e. a polarization change occurs in the scattering from a rough surface. Our perturbation theory will be about the flat surface limit of the equations in Sec. 3 and we must show that this flat surface limit yields the same results for the total displacement on the surface as those found conventionally in Secs. 6-8. We begin by solving the convolution equation (3.74) and projecting the results on the x-z plane in order to compare with the conventional results (Ref. 3.8).

Equation (3.74) for the (flat) surface values of displacement on a free surface is a convolution equation and can be solved using Fourier transforms. Introduce the two-dimensional transform

$$u_j(\underset{\sim}{x}_t') = (2\pi)^{-2} \iint \exp(i\underset{\sim}{k}_t' \cdot \underset{\sim}{x}_t')\, \tilde{u}_j(\underset{\sim}{k}_t')\, d\underset{\sim}{k}_t' \quad . \tag{13.1}$$

Using (13.1) and (3.68) in (3.74) we get the result

$$1/2\ \tilde{u}_j(\underset{\sim}{k}_t') = \tilde{u}_j^{in}(\underset{\sim}{k}_t') - M_{ji}^{(o)}(\underset{\sim}{k}_t')\tilde{u}_i(\underset{\sim}{k}_t') \quad , \tag{13.2}$$

which can also be written as

$$\left[\delta_{ji} + 2\, M_{ji}^{(o)}(\underset{\sim}{k}_t')\right] \tilde{u}_i(\underset{\sim}{k}_t') = 2\, \tilde{u}_j^{in}(\underset{\sim}{k}_t') \quad . \tag{13.3}$$

Note that the matrix on the lhs of (13.3) has components

$$I + 2M^{(o)} = \begin{bmatrix} 1 & 0 & 2M_{13}^{(o)} \\ 0 & 1 & 2M_{23}^{(o)} \\ 2M_{31}^{(o)} & 2M_{32}^{(o)} & 1 \end{bmatrix} . \tag{13.4}$$

Define a matrix L as the following inverse

$$L_{mj}(\underset{\sim}{k}_t')\left[\delta_{ji} + 2M_{ji}^{(o)}(\underset{\sim}{k}_t')\right] = \delta_{mi} \quad , \tag{13.5}$$

which is explicity given by

$$L = \Delta^{-1}\begin{bmatrix} 1-4\, M_{32}^{(o)} M_{23}^{(o)} & 4\, M_{13}^{(o)}\, M_{32}^{(o)} & -2\, M_{13}^{(o)} \\ 4\, M_{31}^{(o)}\, M_{23}^{(o)} & 1-4\, M_{13}^{(o)}\, M_{31}^{(o)} & -2\, M_{23}^{(o)} \\ -2\, M_{31}^{(o)} & -2\, M_{32}^{(o)} & 1 \end{bmatrix} , \tag{13.6}$$

where the denominator determinant is given by

$$\Delta = 1 - 4M_{13}^{(o)}\, M_{31}^{(o)} - 4M_{32}^{(o)}\, M_{23}^{(o)} \quad . \tag{13.7}$$

Multiply (13.3) by L_{mj} and the result is the solution

$$\tilde{u}_m(\underset{\sim}{k}_t') = 2\, L_{mj}(\underset{\sim}{k}_t')\, \tilde{u}_j^{in}(\underset{\sim}{k}_t') \quad , \tag{13.8}$$

for the Fourier transform of the total surface field values in terms of the

Fourier transform of the incident field. The spatial value is found by Fourier transforming (13.8) to yield

$$u_m(\underset{\sim}{x}_t') = 2\iint \tilde{L}_{mj}(\underset{\sim}{x}_t' - \underset{\sim}{x}_t)u_j^{in}(\underset{\sim}{x}_t)d\underset{\sim}{x}_t \quad , \tag{13.9}$$

where $\tilde{L}$ is the Fourier transform of (13.6) given by

$$\tilde{L}_{mj}(\underset{\sim}{x}_t' - \underset{\sim}{x}_t) = (2\pi)^{-2}\iint e^{i\underset{\sim}{k}_t'\cdot(\underset{\sim}{x}_t' - \underset{\sim}{x}_t)} L_{mj}(\underset{\sim}{k}_t')d\underset{\sim}{k}_t' \quad , \tag{13.10}$$

with L_{mj} given by (13.6). Since no components of L vanish, (13.9) illustrates that <u>for a flat surface in two dimensions, all the displacement components are related to all the incident field components</u>. For example, an SH-incident component u_2^{in} couples to the P-SV components u_1 and u_3 on the surface. Equation (13.9) is the limit of arbitrary incidence (a three-dimensional plane-wave for example) so all displacement components couple. This is not yet the cases we discussed in Secs. 6-8 since we have not projected our results onto the x-z plane. We do this now.

Assume the incident displacement field on the surface is independent of y, i.e. let

$$u_j^{in}(\underset{\sim}{x}_t) = u_j^{in}(x) \quad . \tag{13.11}$$

This now corresponds to the incident fields evaluated on the surface z=0 in Secs. 6-8. Here what it means is that we can carry out the y-integration in (13.9). That is using (13.10) we get

$$\int dy\ \tilde{L}_{mj}(\underset{\sim}{x}_t' - \underset{\sim}{x}_t) = (2\pi)^{-1}\int dk_x'\ L_{mj}(k_x',0)\ e^{ik_x'(x'-x)} \quad , \tag{13.12}$$

which is independent of y'. Further we note from (3.73) that for $k_y'=0$ we have that

$$M^{(o)}_{23}(k'_x,0) \quad = \quad M^{(o)}_{32}(k'_x,0) \quad = \quad 0 \quad , \tag{13.13}$$

$$M^{(o)}_{13}(k'_x,0) \quad = \quad -k'_x(k^2_T - 2K'_TK'_L - 2k'^2_x)/2k^2_TK'_L \quad , \tag{13.14}$$

and

$$M^{(o)}_{31}(k'_x,0) \quad = \quad -M^{(o)}_{13}(k'_x,0)(K'_L/K'_T\ ) \quad , \tag{13.15}$$

where here $K'_L = (k^2_L - k'^2_x)^{1/2}$ and $K'_T = (k^2_T - k'^2_x)^{1/2}$, and from (13.7)

$$\Delta = 1 - 4M^{(o)}_{13}(k'_x,0)\ M^{(o)}_{31}(k'_x,0) \quad . \tag{13.16}$$

The resulting matrix L is given from (13.6) and (13.13)

$$L(k'_x,0) \quad = \quad \Delta^{-1}\begin{bmatrix} 1 & 0 & -2M^{(o)}_{13}(k'_x,0) \\ 0 & \Delta & 0 \\ -2M^{(o)}_{31}(k'_x,0) & 0 & 1 \end{bmatrix} \tag{13.17}$$

with Δ from (13.16). The resulting surface field values can be written using (13.9) and (13.12) as

$$u_m(x',0) = \pi^{-1}\int dk_x'\, L_{mj}(k_x',0)\, e^{ik_x'x'}\, \tilde{u}_j^{in}(k_x',0) \quad , \qquad (13.18)$$

in terms of the Fourier transform of the incident displacement on the surface. Now because of the form of the projected value of L in (13.17), (13.18) illustrates the fact that the 1- and 3- components of total displacement P and SV only couple to the 1- and 3- components of incident displacement, and both decouple from the 2- component (SH) of displacement. This corresponds to our examples in Secs. 6-8.

Note that the cross-coupling between P-SV and SH waves was destroyed by projecting onto the y=0 plane (x,z plane). This was accomplished by choosing the incident field to be a function of only x and z, with the corresponding value of the incident field on the z=0 surface to be only a function of x. We must still show however that the results in (13.18) reduce to our results in Secs. 6-8. We do two cases, the first for SH waves which is easy, and the second for P-wave incidence which is more involved.

CASE 1 - SH-WAVE INCIDENCE

From Sec. 8 we know that on the surface z=0 the incident displacement is

$$u_2^{in}(x,0) = A_0^H \exp(ik_x^{T,i}x) \quad . \qquad (13.19)$$

Its Fourier transform is

$$\tilde{u}_2^{in}(k_x',0) = 2\pi\, A_0^H\, \delta(k_x' - k_x^{T,i}) \quad , \qquad (13.20)$$

and the total displacement on the surface is with B=A

$$u_2(x',0) = v(x',0) = 2\, A_0^H \exp(ik_x^{T,i}x') \quad . \tag{13.21}$$

From the integral equation (13.18) we get that

$$u_2(x',0) = \pi^{-1}\int dk_x' \exp(ik_x'x')\, \tilde{u}_j^{in}(k_x',0) \quad , \tag{13.22}$$

since L_{22}=1 from (13.17). Substituting (13.20) in (13.22) we again recover (13.21). Thus for SH-wave incidence on a free flat surface (in one dimension), the integral equation produces the result found in Sec. 8.

CASE 2 - P-WAVE INCIDENCE

From (6.37) the first component of displacement on the surface is given by

$$u_1(x,0) = i\, A_0\left[k_x(1 + B/A_0) - K_T(C/A_0)\right]e^{ik_x x} \quad , \tag{13.23}$$

where all k_x components are equal and where $k_z^{T,r}$=K_T. From (6.30) and (6.32) it is possible to write

$$B/A_0 = (L - R)/(L + R) \quad ; \quad C/A_0 = Q/(L + R) \quad , \tag{13.24}$$

where

$$L = 4\, p^2(\cos\theta_{Li}/c_L)(\cos\theta_{Tr}/c_T) \quad , \tag{13.25}$$

$$R = (c_T^{-2} - 2p^2)^2 \quad , \tag{13.26}$$

and

$$Q = -4p(\cos\theta_{Li}/c_L)(c_T^{-2} - 2p^2) \quad . \tag{13.27}$$

Note the additional symmetry restriction

$$Q/2R = (2L/Q)(K_L/K_T) \quad . \tag{13.28}$$

From (13.23) we can thus define

$$\alpha = k_x(1 + B/A_0) - K_T(C/A_0) \quad . \tag{13.29}$$

Using (13.24) it becomes

$$\alpha = (2R\, k_x - Q\, K_T)/(L + R) \quad , \tag{13.30}$$

or

$$\alpha = (L + R)^{-1}\left[k_x 8p^2(\cos\theta_{Li}/c_L)(\cos\theta_{Tr}/c_T) + 4p\, K_T(\cos\theta_{Li}/c_L)(c_T^{-2} - 2p^2)\right] \quad . \tag{13.31}$$

Similarly from (6.38) the third component of displacement on the surface is given by

$$u_3(x,0) = -iA_0\left[K_L(1 - B/A_0) - k_x\, C/A_0\right]e^{ik_x x} \quad , \tag{13.32}$$

and if we define

$$\beta = K_L(1 - B/A_0) - k_x C/A_0 \quad , \tag{13.33}$$

we can write it using (13.24) as

$$\beta = (2R\, K_L - Qk_x)/(L + R) \quad . \tag{13.34}$$

Similarly we can write

$$\beta - 2K_L = (-2L\,K_L - Qk_x)/(L + R) \quad , \tag{13.35}$$

or substituting for the numerator from (13.25) and (13.27)

$$\beta - 2K_L = (L + R)^{-1}\Big[-K_L 8p^2(\cos\theta_{Li}/c_1)(\cos\theta_{Tr}/c_T) + k_x 4p(\cos\theta_{Li}/c_L)(c_T^{-2} - 2p^2)\Big] \quad . \tag{13.36}$$

Dividing (13.36) by (13.31) and using

$$k_x = \omega \sin\theta_{Li}/c_L \quad , \quad K_L = \omega\cos\theta_{Li}/c_L \quad , \quad K_T = \omega\cos\theta_{Tr}/c_T \quad , \tag{13.37}$$

we get

$$\frac{(\beta - 2K_L)}{\alpha} = \frac{4p(c_T^{-2} - 2p^2)(\sin\theta_{Li}/c_L) - 8p^2(\cos\theta_{Tr}/c_T)(\cos\theta_{Li}/c_L)}{4p(c_T^{-2} - 2p^2)(\cos\theta_{Tr}/c_T) + 8p^2(\cos\theta_{Tr}/c_T)(\sin\theta_{Li}/c_L)} \, . \tag{13.38}$$

We will use these results to simplify the matrix elements from the integral equation approach below.

We now compute the values $u_1(x,0)$ and $u_3(x,0)$ from the integral equation approach. For P-wave incidence we have that on the surface the incident displacement is

$$u_j^{in}(x,0) = i\,A_0(k_x\delta_{j1} - K_L\delta_{j3})\,e^{ik_x x} \quad , \tag{13.39}$$

and its Fourier transform is

$$\tilde{u}_j^{in}(k_x',0) = 2\pi i\ A_o(k_x\delta_{j1} - K_L\delta_{j3})\delta(k_x' - k_x) \quad . \tag{13.40}$$

Substituting this in (13.18) we have that

$$u_m(x,0) = 2i\ A_o\left[k_x L_{m1}(k_x,0) - K_L L_{m3}(k_{x1},0)\right]e^{ik_x x} \quad . \tag{13.41}$$

Evaluating the matrix elements from (13.17) and defining

$$U = M_{13}^{(o)}(k_x,0) \quad , \tag{13.42}$$

so that from (13.15)

$$M_{31}^{(o)}(k_x,0) = -U\ K_L/K_T \quad , \tag{13.43}$$

we get for m=1

$$u_1(x,0) = 2i\ A_o(k_x + 3K_L U)(1 + 4U^2K_L/K_T)^{-1}e^{ik_x x} \quad , \tag{13.44}$$

and for m=3

$$u_3(x,0) = -2i\ A_o K_L(1 - 2k_x U/K_T)(1 + 4U^2K_L/K_T)^{-1}e^{ik_x x} \quad . \tag{13.45}$$

Comparing these results to (13.23) with (13.29) and (13.22) with (13.33) we get agreement of the displacement components provided

$$2(k_x + 2K_L U)(1 + 4U^2K_L/K_T)^{-1} = \alpha \quad , \tag{13.46}$$

and

$$2K_L(1 - 2k_x U/K_T)(1 + 4U^2K_L/K_T)^{-1} = \beta \quad . \tag{13.47}$$

Dividing (13.46) and (13.47) and solving the result for V we get

$$U = -\frac{K_T}{2K_L}\frac{\beta k_x - \alpha K_L}{\alpha k_x + \beta K_T} \quad . \tag{13.48}$$

We next use the definition of U to write it in another form involving α and β, and show the comparison of the result with (13.48) is an identity.

From (13.42) and the definition (13.14) we have

$$U = -(k_x/2K_L)(1 - 2k_x^2 k_T^{-2} - 2K_T K_L k_T^{-2}) \quad . \tag{13.49}$$

Using the definition of the ray parameter we can write $k_x^2 k_T^{-2} = p^2 c_T^2$ and using (13.37) U becomes

$$U = -\frac{k_x}{2K_L}\frac{(c_T^{-2} - 2p^2) - 2(\cos\theta_{Tr}/c_T)(\cos\theta_{Li}/c_L)}{(c_T^{-2} - 2p^2) + 2p^2} \quad , \tag{13.50}$$

where we have divided by c_T^2 and added and subtracted a term in the denominator. Multiplying numerator and denominator using the identity

$$\frac{4p^2}{4p^2}\;\frac{\cos\theta_{Tr}/c_T}{\cos\theta_{Tr}/c_T} = \frac{\cos\theta_{Tr}}{pc_T}\;\frac{4p^2}{4p\cos\theta_{Tr}/c_T} \quad , \tag{13.51}$$

we get

$$U = -\frac{k_x}{2K_L}\;\frac{\cos\theta_{Tr}}{pc_T}\left[\frac{4p^2(c_T^{-2} - 2p^2) - 8p^2\cos\theta_{Tr}\cos\theta_{Li}/c_T c_L}{4p(c_T^{-2} - 2p^2)(\cos\theta_{Tr}/c_T) + 8p^3(\cos\theta_{Tr}/c_T)}\right] \quad ,$$

Using $p = \sin\theta_{Li}/c_L$ in one power of p in the left hand term in the numerator of the bracket term, we see that the bracket term is just (13.38). Substituting for the remainder of the coefficients in V we get finally

$$U = -\frac{K_T}{2K_L}\,\frac{(\beta - 2k_L)}{\alpha} \quad . \tag{13.53}$$

This is the expression for U directly from the matrix element definition. Comparing it to (13.48) from the integral equation we must prove that

$$(\beta k_x - \alpha K_L)(\alpha k_x + \beta K_T)^{-1} = (\beta - 2K_L)\alpha^{-1} \quad , \tag{13.54}$$

is true. Cross multipication of (13.54) yields the relation

$$K_L\alpha^2 + K_T\beta^2 = 3K_L(k_x\alpha + K_T\beta) \quad .$$

This is easily proved to be an identity by substituting (13.30) for α and (13.34) for β.

We have thus shown that the value of the matrix element U necessary for the integral equation to agree with the standard result (13.48) is the same as its definition (13.53). The integral equation results are thus the same as the standard results for P-wave incidence.

3.14 HORIZONTAL VARIABILITY AND POLARIZATION CHANGE

In Sec. 3 we presented the surface integral equations and field representations for displacements u_j for the case of a rough surface h. These were exact but formal expressions for the coupling of the displacement components due to the horizontal variability induced by h. In particular it was at least formally illustrated how the incident displacement field polarization (i.e. its particular vector value) coupled to all the total surface displacement field values, and correspondingly to all the displacement values off the surface. We also computed the flat surface limit in Sec. 3 and showed in Sec. 13 that this flat surface limit agreed with our standard results using potentials in Secs. 6-8.

In this section we are more explicit. We show how the polarization changes for the case of a free surface (Sec. 3, Ex. 2) for a deterministic surface h which is small, i.e. in the sense that the problem can be treated in a perturbation theory expansion in powers of h. From (3.39) we had the integral equation for displacements on the surface given by

$$Q_{ji}(\underset{\sim}{x}_t')u_i(\underset{\sim}{x}_s') = u_j^{in}(\underset{\sim}{x}_s') + \iint K_{ji}(\underset{\sim}{x}_s',\underset{\sim}{x}_s)\ u_i(\underset{\sim}{x}_s)d\underset{\sim}{x}_t \quad , \tag{14.1}$$

where, from (3.23), we had

$$Q_{ji}(\underset{\sim}{x}') = \frac{1}{2}\,\delta_{ji} + Q_{ji}^{(1)}(\underset{\sim}{x}_t') \quad , \tag{14.2}$$

where

$$Q_{ji}^{(1)}(\underset{\sim}{x}_t') = \frac{1}{2}\left[\ \delta_{i3}\partial_{jt}'h(\underset{\sim}{x}_t') + \Lambda\delta_{j3}\partial_{it}'h(\underset{\sim}{x}_t')\ \right] \tag{14.3}$$

is first order in the height h. Further, from (A.32) we had

$$K_{ji}(\underset{\sim}{x}_s', \underset{\sim}{x}_s) = n_p(\underset{\sim}{x}_t)\, K_{pij}(\underset{\sim}{x}_s' - \underset{\sim}{x}_s) \quad , \tag{14.4}$$

where

$$n_p(\underset{\sim}{x}_t) = \delta_{p3} - \partial_{pt} h(\underset{\sim}{x}_t) \quad , \tag{14.5}$$

and, from (A.32),

$$K_{pij}(\underset{\sim}{x}_s' - \underset{\sim}{x}_s) = (2\pi)^{-3} \iiint \exp\left[i\underset{\sim}{k}\cdot(\underset{\sim}{x}_s' - \underset{\sim}{x}_s)\right] M_{pij}(\underset{\sim}{k})\, d\underset{\sim}{k} \quad , \tag{14.6}$$

with m_{pij} defined using (A.29), (A.23) and (A.7) as

$$M_{pij}(\underset{\sim}{k}) = k_T^{-2}\left[\tilde{G}^T(k)\, P^T_{pij}(\underset{\sim}{k}) - \tilde{G}^L(k)\, P^L_{pij}(\underset{\sim}{k})\right] - \frac{1}{2}\left[\delta_{ij}\tilde{G}^T(k)\, P^T_p(\underset{\sim}{k}) + \delta_{pj}\tilde{G}^T(k)\, P^T_i(\underset{\sim}{k}) + \delta_{ip}\Lambda\, \tilde{G}^L(k)\, P^L_j(\underset{\sim}{k})\right] . \tag{14.7}$$

Here the P^T_{pij} terms are defined in (A.21) with P^L_{pij} defined by (A.21) if k_T is replaced by k_L, and the P_j terms defined by (5.18) in Ch. 1 with the appropriate wavenumber k_T or k_L.

We next expand each of the terms in (14.1) to first order in h. We have

$$u_i(\underset{\sim}{x}_s') = u_i^{(0)}(\underset{\sim}{x}_t') + h(\underset{\sim}{x}_t')u_i^{(1)}(\underset{\sim}{x}_t') \quad , \tag{14.8}$$

$$u_j^{in}(\underset{\sim}{x}_s') = u_j^{in}(\underset{\sim}{x}_t') + h(\underset{\sim}{x}_t')v_j^{(1)}(\underset{\sim}{x}_t') \quad , \tag{14.9}$$

and to first order

$$K_{ji}(\underset{\sim}{x}_s',\underset{\sim}{x}_s) = \overset{\circ}{K}_{ji}(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) + \left[h(\underset{\sim}{x}_t')-h(\underset{\sim}{x}_t) \right] B_{ji}(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)$$

$$- \partial_{pt} h(\underset{\sim}{x}_t) K_{pij}(\underset{\sim}{x}_t' - \underset{\sim}{x}_t) \quad , \qquad (14.10)$$

where K°_{ji} is defined in (3.59) and

$$B_{ji}(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) = (2\pi)^{-3} \iiint \exp\left[i\underset{\sim}{k}_t\cdot(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) \right] ik_z M_{3ij}(\underset{\sim}{k})d\underset{\sim}{k} \quad . \qquad (14.11)$$

The terms zeroth-order in h on the right and left hand sides of (14.1) form the flat surface limit discussed in Secs. 3 and 13. Equating the first order terms, we can write a convolution equation for the quantity

$$W_j(\underset{\sim}{x}_t) = h(\underset{\sim}{x}_t) u_j^{(1)}(\underset{\sim}{x}_t) \quad , \qquad (14.12)$$

which is given by

$$\frac{1}{2} W_j(\underset{\sim}{x}_t') = b_j(\underset{\sim}{x}_t') + \iint \overset{\circ}{K}_{ji}(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) W_i(\underset{\sim}{x}_t)d\underset{\sim}{x}_t \quad , \qquad (14.13)$$

where the Born term is known and given by

$$b_j(\underset{\sim}{x}_t') = h(\underset{\sim}{x}_t') v_j^{(1)}(\underset{\sim}{x}_t') - Q_{ji}^{(1)}(\underset{\sim}{x}_t') u_i^{in}(\underset{\sim}{x}_t')$$

$$+ \iint \left[h(\underset{\sim}{x}_t')-h(\underset{\sim}{x}_t) \right] B_{ji}(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) u_i^{in}(\underset{\sim}{x}_t)d\underset{\sim}{x}_t$$

$$- \iint \partial_{\rho t} h(\underset{\sim}{x}_t) K_{pij}(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) u_i^{in}(\underset{\sim}{x}_t)d\underset{\sim}{x}_t \quad . \qquad (14.14)$$

Equation (14.13) can be solved by Fourier transform techniques. In fact we have already solved the equation since it is the same as (3.74) if we replace the incident displacement on the surface by b_j. The solution of (14.13) is given in analogy with (13.9) as

$$W_i(x_t') = 2 \iint \tilde{L}_{ij}(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)\; b_j(\underset{\sim}{x}_t)d\underset{\sim}{x}_t \quad , \tag{14.15}$$

where the matrix L is defined in (13.6). To first order the full solution of the surface displacement is, using (14.8) and (14.12)

$$u_i(\underset{\sim}{x}_s') = u_i^{(o)}(\underset{\sim}{x}_t') + W_i(\underset{\sim}{x}_t') \quad , \tag{14.16}$$

where $u_i^{(o)}$ is given by (13.9), i.e. the total displacement field at a flat surface where we have as yet not projected the incident field onto the x-z plane. We explicitly showed how this projection reduced $u_i^{(o)}$ to the standard results for SH-wave incidence in (13.22) and for P-wave incidence from (13.44) and (13.45) (with the subsequent proof of an identity.) As we remarked, the coupling of all P-, SV-, and SH-waves for this case was due solely to the non-planar nature of the incident displacement. The coupling in the W_i term however arises both from this non-planar incident displacement as well as the height and slope variability of the surface. Combining (13.9) and (14.15) we can write to first order approximation in surface height that the total displacement field as the surface is

$$u_i(\underset{\sim}{x}_s') = 2 \iint \tilde{L}_{ij}(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) \left[u_j^{in}(\underset{\sim}{x}_t) + b_j(\underset{\sim}{x}_t) \right] d\underset{\sim}{x}_t \quad . \tag{14.17}$$

The displacement components off the surface (i.e. at a field point) can be evaluated using (14.17) in (3.38).

3.15 KIRCHHOFF APPROXIMATION

For the scalar case, Ch. 2 Sec. 6, we found approximate expressions for the surface values of the total (velocity potential) field ϕ and its normal derivative N in terms of the reflection coefficient R and the incident field ϕ^{in}. These were

$$\phi(\underset{\sim}{x}_s) = (1 + R)\phi^{in}(\underset{\sim}{x}_s) \quad , \tag{15.1}$$

and

$$N(\underset{\sim}{x}_s) = (1 - R)n_m(\underset{\sim}{x}_t)\partial_m\phi^{in}(\underset{\sim}{x}_s) \quad . \tag{15.2}$$

These were three-dimensional approximations in the sense that the evaluation was on a surface $h(\underset{\sim}{x}_t)$ which was a function of two variables. The incident field in general had a component out of the x-z plane. Surface slope terms appeared in the normal n_m. Also, of course, the reflection coefficient R was given for the full transmission problem into another medium with different parameters. The results also reduced to the cases where we had perfect reflection, R=1 (Neumann boundary condition) and R=-1 (Dirichlet boundary condition). Only a single scalar field was incident on the surface, and the terms involving the reflection coefficients $(1 \pm R)$ were coordinate-independent.

For the elastic case we have three possible incident fields, P, SV, and SH, so the factoring out of the incident field becomes a problem. We also must use the reflection coefficients for the full transmission problem as in Secs. 8 and 9. It is not enough to choose the reflection coefficients for the free or perfectly rigid surface since these do not include energy loss in the lower medium. In addition there are a total of five reflection coefficients to consider, S_j, j=1, ..., 4, from Sec. 9 and S_{11} from Sec. 8. We computed the flat surface values of total displacement and stress at the end of Sec. 9.

Before discussing the Kirchhoff approximation we consider some remarks:

1. We have expressed the displacements and stresses in terms of potentials ϕ, A, and v. All were assumed independent of y, and this is why the P and SV potentials decoupled from the SH potential. Indeed, this is why we were able to make the Cauchy-Riemann argument in Sec. 4 which led to only a single component for the vector potential $\underset{\sim}{A}$. The development of these potential arguments was fundamentally based not only on flat surface but also on two-dimensional (x and z) behavior. Nevertheless, we argue in this section that we can maintain the specific potential forms to find displacements and stresses on the boundary.

2. Flat surface arguments also obscure the resulting wave shapes evaluated on the surface. If the z-dependent parts of the incident and scattered fields are exp(-iKz) and exp(iKz) respectively, setting z=0 makes them both equal one. However setting z=h makes them complex conjugate pairs (for real K), which are different wave shapes. This is actually true even in the scalar Kirchhoff approximation and is ignored. Note that this doesn't affect the wave shape of the scattered field since it's an integral over the surface values and is an outgoing wave. But its surface approximation arises from an incident (incoming) wave shape.

3. Factorization of any one of the three incident wave shapes also skews the remaining wave shapes, and in general leads to overall coordinate dependent reflection coefficients as we

show.

4. Lowest order (i.e. two-dimensional flat surface) coupling occurs between P- SV- and SH-waves if the incident field is assumed to have a component out of the x-z plane, or, equivalently, if it is a function of y. We discussed This in Sec. 13.

Here we develop the Kirchhoff approximation by retaining the potential formalism. The displacements and stresses are defined in terms of the potentials ϕ, A, and v as

$$u_1 = \frac{\partial\phi}{\partial x} - \frac{\partial A}{\partial z} , \tag{15.3}$$

$$u_2 = v , \tag{15.4}$$

$$u_3 = \frac{\partial\phi}{\partial z} + \frac{\partial A}{\partial x} , \tag{15.5}$$

$$\tau_{z1} = \mu\left[\frac{\partial^2 A}{\partial x^2} - \frac{\partial^2 A}{\partial z^2}\right] + 2\mu\,\frac{\partial^2 \phi}{\partial x\partial z} , \tag{15.6}$$

$$\tau_{z2} = \mu\,\frac{\partial v}{\partial z} , \tag{15.7}$$

and

$$\tau_{z3} = 2\mu\left[\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 A}{\partial x\partial z}\right] - \lambda\, k_L^2 \phi . \tag{15.8}$$

These follow from Sec. 4.

P-WAVES

For P-wave incidence the ϕ and A potentials are found from (9.1)-(9.4). Substitute these forms into (15.3), (15.5), and (15.8), evaluate the results on the surface, and factor out the incident wave. For example for u_1 we have that

$$u_1(\underset{\sim}{x}_s) = ik_x\phi^{in}(\underset{\sim}{x}_s) + ik_x\phi^{sc}(\underset{\sim}{x}_s) - i\,K_T\,A(\underset{\sim}{x}_s) \quad , \tag{15.9}$$

which can be written as

$$\begin{aligned} u_1(\underset{\sim}{x}_s) &= i\left[k_x + k_x\phi^{sc}(\underset{\sim}{x}_s)/\phi^{in}(\underset{\sim}{x}_s) - K_T\,A(\underset{\sim}{x}_s)/\phi^{in}(\underset{\sim}{x}_s)\right]\phi^{in}(\underset{\sim}{x}_s) \\ &= i\left[k_x(1 + S_1u) - K_Tw\;S_2\right]\phi^{in}(\underset{\sim}{x}_s) \quad , \end{aligned} \tag{15.10}$$

where

$$u = \exp(2iK_1h) \quad ; \quad w = \exp\left[i(K_T + K_L)h\right] \quad , \tag{15.11}$$

and S_1 and S_2 are the reflection coefficients defined in (9.32). If we further write the incident P-wave on the surface as

$$I_1 = \phi^{in}(\underset{\sim}{x}_s) = A_0^P\,\exp\left[i(k_xx - K_Lh)\right] \quad , \tag{15.12}$$

we can write (15.10) as

$$u_1(\underset{\sim}{x}_s) = R_{11}I_1 \quad , \tag{15.13}$$

where

$$R_{11} = i\left[k_x(1 + S_1u) - K_Tw\;S_2\right] \quad , \tag{15.14}$$

is the overall spatially dependent reflection coefficient from a P-wave incident field to the first component of displacement. We also get that

$$u_3(\underset{\sim}{x}_s) = R_{31}\,I_1 \quad , \tag{15.15}$$

where

$$R_{31} = -i\left[K_L(1 - S_1 u) - k_x w\, S_2\right] \quad . \tag{15.16}$$

Both displacement terms (15.13) and (15.15) are the first and third components of total displacement resulting from an incident P-wave.

The stress components from (15.6) and (15.8) evaluated on the surface can be written as

$$T_1(\underset{\sim}{x}_s) = \tau_{z1}(\underset{\sim}{x}_s) = T_{11} I_1 \quad , \tag{15.17}$$

and

$$T_3(\underset{\sim}{x}_s) = \tau_{z3}(\underset{\sim}{x}_s) = T_{31} I_1 \quad , \tag{15.18}$$

where

$$T_{11} = \mu(K_T^2 - k_x^2) S_2\, w + 2\mu\, k_x K_L (1 - S_1 u) \quad , \tag{15.19}$$

and

$$T_{31} = -(2\mu K_L^2 + \lambda k_L^2)(1 + S_1 u) - 2\mu k_x K_T S_2 w \quad , \tag{15.20}$$

Here T_{11} and T_{31} are the overall traction reflection coefficients which take us from an incident P-wave field to the first and third components of traction on the surface which are due to the P-wave incidence.

<u>SV-WAVES</u>

For SV-wave incidence the ϕ and A potentials are found from (9.50)-(9.53). If we define the incident field on the surface as

$$I_3 = A^{in}(\underset{\sim}{x}_s) = A_0^V \exp\left[i(k_x x - K_T h)\right] , \tag{15.21}$$

and

$$t = \exp(2iK_T h) , \tag{15.22}$$

we can write the results using (15.3), (15.5), (15.6) and (15.8) as

$$u_1(\underset{\sim}{x}_s) = R_{13} I_3 , \tag{15.23}$$

$$u_3(\underset{\sim}{x}_s) = R_{33} I_3 , \tag{15.24}$$

$$T_1(\underset{\sim}{x}_s) = T_{13} I_3 , \tag{15.25}$$

and

$$T_3(\underset{\sim}{x}_s) = T_{33} I_3 , \tag{15.26}$$

where

$$R_{13} = i\left[K_T(1 - S_4 t) + k_x w S_3\right] , \tag{15.27}$$

$$R_{33} = i\left[k_x(1 + S_4 t) + K_1 w S_3\right] , \tag{15.28}$$

$$T_{13} = \mu(K_T^2 - k_x^2)(1 + S_4 t) - 2\mu k_x K_L w S_3 , \tag{15.29}$$

and

$$T_{33} = 2\mu k_x K_T(1 - S_4 t) - (2\mu K_L^2 + \lambda k_L^2) w S_3 , \tag{15.30}$$

in terms of the reflection coefficients S_3 and S_4 defined in (9.73).

SH-WAVES

For SH-incidence we define the incident SH-wave evaluated on the surface from (8.6) as

$$I_2 = A_0^H \exp\left[i(k_x x - K_T h)\right] , \tag{15.31}$$

so that the components of displacement and stress are from (15.4) and (15.7) written as

$$u_2(\underset{\sim}{x}_s) = R_{22} I_2 , \tag{15.32}$$

and

$$T_2(\underset{\sim}{x}_s) = T_{22} I_2 , \tag{15.33}$$

where

$$R_{22} = 1 + S_{11} t , \tag{15.34}$$

and

$$T_{22} = -\mu K_T (1 - S_{11} t) , \tag{15.35}$$

written in terms of the reflection coefficient S_{11} defined by (8.22).

All these results can be combined in the matrix representations as the Kirchhoff approximation for displacements

$$\begin{bmatrix} u_1(\underset{\sim}{x}_s) \\ u_2(\underset{\sim}{x}_s) \\ u_3(\underset{\sim}{x}_s) \end{bmatrix} = \begin{bmatrix} R_{11} & 0 & R_{13} \\ 0 & R_{22} & 0 \\ R_{31} & 0 & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} , \tag{15.36}$$

and the Kirchhoff approximation for stresses or tractions

$$\begin{bmatrix} T_1(\underset{\sim}{x}_s) \\ T_2(\underset{\sim}{x}_s) \\ T_3(\underset{\sim}{x}_s) \end{bmatrix} = \begin{bmatrix} T_{11} & 0 & T_{13} \\ 0 & T_{22} & 0 \\ T_{31} & 0 & T_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} , \qquad (15.37)$$

An additional version more closely corresponding to the scalar case would have u=w=t=1, in which case the R_{ij} and T_{ij} are spatially independent. Note also that this version of the Kirchhoff approximation doesn't couple P- and SV-waves to SH-wave <u>on the surface</u>. The field values, however, are coupled through the integral relations.

APPENDIX 3A. REGULARIZATION OF TG^0

TG^0 is defined by

$$\left[TG^{o}(\underset{\sim}{x}'-\underset{\sim}{x})\right]_{pij} = \mu\ \delta_{pm}\partial_m G^{o}_{ij}(\underset{\sim}{x}'-\underset{\sim}{x}) + \mu\ \delta_{pm}\partial_i G^{o}_{mj}(\underset{\sim}{x}'-\underset{\sim}{x})$$

$$+ \lambda\ \delta_{ip}\partial_m G^{o}_{mj}(\underset{\sim}{x}'-\underset{\sim}{x}) \quad . \tag{A.1}$$

G^0 is defined by

$$G^{o}_{mj}(\underset{\sim}{x}'-\underset{\sim}{x}) = \mu^{-1}\delta_{mj}G^{T}(\underset{\sim}{x}'-\underset{\sim}{x}) + K^{-2}\partial_m\partial_j\left[G^{T}(\underset{\sim}{x}'-\underset{\sim}{x}) - G^{L}(\underset{\sim}{x}'-\underset{\sim}{x})\right] \quad , \tag{A.2}$$

where $G^{T,L}$ are the scalar free space Green's functions defined by (we choose the retarded Green's functions)

$$G^{T,L}(\underset{\sim}{x}'-\underset{\sim}{x}) = \exp\left[ik_{T,L}|\underset{\sim}{x}'-\underset{\sim}{x}|\right]/4\pi|\underset{\sim}{x}'-\underset{\sim}{x}| \quad . \tag{A.3}$$

From (A.2) we can compute the third term in (A.1)

$$\partial_m G^{o}_{mj}(\underset{\sim}{x}'-\underset{\sim}{x}) = \mu^{-1}\partial_j G^{T}(\underset{\sim}{x}'-\underset{\sim}{x}) + K^{-2}\partial_j\left[-k_T^2 G^{T}(\underset{\sim}{x}'-\underset{\sim}{x}) + k_L^2 G^{L}(\underset{\sim}{x}'-\underset{\sim}{x})\right]$$

$$= (\lambda + 2\mu)^{-1}\partial_j G^{L}(\underset{\sim}{x}'-\underset{\sim}{x}) \quad , \tag{A.4}$$

where we have used the differential equation for the free-space Green's functions and the identity $k_L^2 = K^2/(\lambda+2\mu)$. Using (A.4) and (A.2) in (A.1) and the identity $k_T^2 = K^2/\mu$ we get

$$\left[TG^0(\underset{\sim}{x}'-\underset{\sim}{x})\right]_{pij} = \left[\delta_{ij}\partial_p + \delta_{pj}\partial_i\right]G^T(\underset{\sim}{x}'-\underset{\sim}{x})$$

$$+ \left[\lambda/(\lambda + 2\mu)\right]\delta_{ip}\partial_j G^L(\underset{\sim}{x}'-\underset{\sim}{x}) \tag{A.5}$$

$$+ (2/k_t^2)\partial_p\partial_i\partial_j\left[G^T(\underset{\sim}{x}'-\underset{\sim}{x}) - G^L(\underset{\sim}{x}'-\underset{\sim}{x})\right] .$$

We have already regularized the first derivative of the free-space retarded Green's function in Ch. 1, Sec. 5. Here we are differentiating on the source coordinate and we can write for example

$$\partial_j G^T(\underset{\sim}{x}'-\underset{\sim}{x}) = -\frac{1}{2} R_j^T(\underset{\sim}{x}'-\underset{\sim}{x}) + \frac{1}{2}\,\mathrm{sgn}(z'-z)\delta_{j3}\delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) , \tag{A.6}$$

where

$$R_j^T(\underset{\sim}{x}'-\underset{\sim}{x}) = (2\pi)^{-3}\iiint e^{i\underset{\sim}{k}\cdot\left[\underset{\sim}{x}'-\underset{\sim}{x}\right]}\tilde{G}^T(k)P_j^T(\underset{\sim}{k})d\underset{\sim}{k} , \tag{A.7}$$

and

$$P_j^T(\underset{\sim}{k}) = 2i\left[k_{jt} + \delta_{j3}\,P\left[\frac{K_T^2}{k_z}\right]\right] , \tag{A.8}$$

where $K_T^2 = k_T^2 - k_t^2$ and $\tilde{G}^T(k) = \left[k^2 - k_T^2\right]^{-1}$. The result for the longitudinal Green's function G^L requires the replacement $k_T \rightarrow k_L$. We can thus write (A.5) as

$$\left[TG^{o}(\underset{\sim}{x}'-\underset{\sim}{x})\right]_{pij} = -\frac{1}{2}\left[\delta_{ij}R_p^T(\underset{\sim}{x}'-\underset{\sim}{x}) + \delta_{pj}R_j^T(\underset{\sim}{x}'-\underset{\sim}{x}) + \delta_{ip}(\lambda/(\lambda+2\mu))R_j^L(\underset{\sim}{x}'-\underset{\sim}{x})\right]$$

$$+ \frac{1}{2}\,\mathrm{sgn}(z'-z)\delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)\ \cdot$$

$$\left[\delta_{ij}\delta_{p3} + \delta_{pj}\delta_{i3} + [(\lambda/(\lambda+2\mu)]\delta_{ip}\delta_{j3}\right]$$

$$+ \frac{2}{k_T^2}\,\partial_p\partial_i\partial_j\left[G^T(\underset{\sim}{x}'-\underset{\sim}{x}) - G^L(\underset{\sim}{x}'-\underset{\sim}{x})\right]\ . \tag{A.9}$$

Thus (A.5) is partially regularized in (A.9). It remains to regularize the triple derivative terms. One is sufficient, the other will follow with the replacement $k_T \leftrightarrow k_L$.

<u>REGULARIZATION OF</u> $\partial_p\partial_i\partial_j G^T$

We begin with the Weyl representation

$$G^T(\underset{\sim}{x}'-\underset{\sim}{x}) = \frac{\pi i}{[2\pi]^3}\iint \frac{d\underset{\sim}{k}_t}{K_T}\, e^{i\underset{\sim}{k}_t\cdot[\underset{\sim}{x}_t'-\underset{\sim}{x}_t]}\, e^{iK_T|z'-z|}\ . \tag{A.10}$$

We take the triple derivative $\partial_p\partial_i\partial_j$ of this. Break it up as

$$\partial_p\partial_i\partial_j = \left[\partial_{pt} + \delta_{p3}\partial_z\right]\left[\partial_{it} + \delta_{i3}\partial_z\right]\left[\partial_{jt} + \delta_{j3}\partial_z\right] =$$

$$= \partial_{pt}\partial_{it}\partial_{jt} + \left[\partial_{pt}\partial_{it}\delta_{j3} + \partial_{pt}\partial_{jt}\delta_{i3} + \partial_{it}\partial_{jt}\delta_{p3}\right]\partial_z$$

$$+ \left[\partial_{pt}\delta_{i3}\delta_{j3} + \partial_{it}\delta_{p3}\delta_{j3} + \partial_{jt}\delta_{p3}\delta_{i3}\right]\partial_z^2 + \delta_{i3}\delta_{j3}\delta_{p3}\partial_z^3\ . \tag{A.11}$$

For the z-derivatives we have

$$\partial_z\, e^{iK_T|z'-z|} = -i\, K_T\, \mathrm{sgn}(z'-z)\, e^{iK_T|z'-z|} \quad , \tag{A.12}$$

$$\partial_z^2\, e^{iK_T|z'-z|} = \left[2iK_T\delta(z'-z) - K_T^2\right] e^{iK_T|z'-z|}$$

$$= 2iK_T\delta(z'-z) - K_T^2\, e^{iK_T|z'-z|} \quad , \tag{A.13}$$

and

$$\partial_z^3\, e^{iK_T|z'-z|} = -2iK_T\delta'(z'-z) + iK_T^3\, \mathrm{sgn}(z'-z)\, e^{iK_T|z'-z|} \quad . \tag{A.14}$$

It is the z-derivative terms which determine whether or not we have a regular or singular representation. For example, (A.12) yields a singular result since it contains a single power of K_T. The first term in (A.13) and both terms in (A.14) also yield singular results.

We can thus write the full result as

$$\partial_p \partial_i \partial_j G^T(\underset{\approx}{x}' - \underset{\approx}{x})$$

$$= \frac{\pi i}{(2\pi)^3} \iint \frac{d\underset{\sim}{k}_t}{K_T} e^{i\underset{\sim}{k}_t \cdot \left[\underset{\sim}{x}'_t - \underset{\sim}{x}_t\right]} (-ik_{pt})(-ik_{it})(-ik_{jt})\, e^{iK_T |z'-z|}$$

$$+ \left[\partial_{pt}\partial_{it}\delta_{j3} + \partial_{pt}\partial_{jt}\delta_{i3} + \partial_{it}\delta_{p3}\right](-i)\,\mathrm{sgn}(z'-z) \;\cdot$$

$$\cdot \frac{\pi i}{(2\pi)^3} \iint d\underset{\sim}{k}_t\; e^{i\underset{\sim}{k}_t \cdot \left[\underset{\sim}{x}'_t - \underset{\sim}{x}_t\right]}\, e^{iK_T |z'-z|}$$

$$+ \left[\partial_{pt}\delta_{i3} + \partial_{it}\delta_{p3}\delta_{j3} + \partial_{jt}\delta_{i3}\delta_{p3}\right] 2i\, \delta(z'-z) \;\cdot$$

$$\cdot \frac{\pi i}{(2\pi)^3} \iint d\underset{\sim}{k}_t\; e^{i\underset{\sim}{k}_t \cdot \left[\underset{\sim}{x}'_t - \underset{\sim}{x}_t\right]}$$

$$+ \left[\partial_{pt}\delta_{i3}\delta_{j3} + \partial_{it}\delta_{p3}\delta_{j3} + \partial_{jt}\delta_{i3}\delta_{p3}\right](-1) \;\cdot$$

$$\cdot \frac{\pi i}{(2\pi)^3} \iint d\underset{\sim}{k}_t\; K_T\, e^{i\underset{\sim}{k}_t \cdot \left[\underset{\sim}{x}'_t - \underset{\sim}{x}_t\right]}\, e^{iK_T |z'-z|}$$

$$+ \delta_{i3}\delta_{j3}\delta_{p3}\; (-2i)\; \delta'(z'-z) \;\cdot$$

$$\cdot \frac{\pi i}{(2\pi)^3} \iint d\underset{\sim}{k}_t\; e^{i\underset{\sim}{k}_t \cdot \left[\underset{\sim}{x}'_t - \underset{\sim}{x}_t\right]}$$

$$+ \delta_{i3}\delta_{j3}\delta_{p3}\; i\; \mathrm{sgn}(z'-z)\; \frac{\pi i}{(2\pi)^3} \iint d\underset{\sim}{k}_t K_T^2\; e^{i\underset{\sim}{k}_t \cdot \left[\underset{\sim}{x}'_t - \underset{\sim}{x}_t\right]}\, e^{iK_T |z'-z|} \;. \qquad \text{(A.15)}$$

The first term in (A.15) is not singular and can be integrated up to a three-dimensional integral. The second term must be regularized. The third and fifth terms have the singularities directly. The fourth term is not singular and the sixth term contains singularities which we recover.

$$\partial_p \partial_i \partial_j G^T(\underset{\sim}{x}'-\underset{\sim}{x})$$

$$= \frac{1}{[2\pi]^3} \iiint d\underset{\sim}{k}\; e^{i\underset{\sim}{k}\cdot[\underset{\sim}{x}'-\underset{\sim}{x}]}\, \tilde{G}^T(k)\left[i\, k_{pt}k_{it}k_{jt}\right]$$

$$+ (-i)\,\mathrm{sgn}(z'-z)\left[\partial_{pt}\partial_{it}\delta_{j3} + \partial_{pt}\partial_{jt}\delta_{i3} + \partial_{it}\partial_{jt}\delta_{p3}\right]\cdot$$

$$\cdot\left[\frac{\pi i}{(2\pi)^3}\iint d\underset{\sim}{k}_t\; e^{i\underset{\sim}{k}_t\cdot[\underset{\sim}{x}_t'-\underset{\sim}{x}_t]}\left[e^{iK_T|z'-z|}-1\right] + \frac{i}{2}\,\delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)\right]$$

$$- \delta(z'-z)\left[\delta_{i3}\delta_{j3}\partial_{pt} + \delta_{p3}\delta_{j3}\partial_{it} + \delta_{i3}\delta_{p3}\partial_{jt}\right]\delta(x_t'-x_t)$$

$$+ (-1)\left[\delta_{i3}\delta_{j3}\partial_{pt} + \delta_{p3}\delta_{j3}\partial_{it} + \delta_{i3}\delta_{p3}\partial_{jt}\right]\cdot$$

$$\cdot\frac{1}{(2\pi)^3}\iiint d\underset{\sim}{k}\; e^{i\underset{\sim}{k}\cdot[\underset{\sim}{x}'-\underset{\sim}{x}]}\tilde{G}^T(k)K_T^2$$

$$+ \delta_{i3}\delta_{j3}\delta_{p3}\quad \delta'(z'-z)\,\delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t).$$

$$+ i\delta_{i3}\delta_{j3}\delta_{p3}\quad \mathrm{sgn}(z'-z)\,.$$

$$\cdot\left[\frac{\pi i}{(2\pi)^3}\iint d\underset{\sim}{k}_t\; K_T^2\, e^{i\underset{\sim}{k}_t\cdot[\underset{\sim}{x}_t'-\underset{\sim}{x}_t]}\left[e^{iK_T|z'-z|}-1\right]\right.$$

$$\left. + \frac{\pi i}{(2\pi)^3}\iint d\underset{\sim}{k}_t\; K_T^2\, e^{i\underset{\sim}{k}_t\cdot[\underset{\sim}{x}_t'-\underset{\sim}{x}_t]}\right]\,, \tag{A.16}$$

where we have used the relation

$$K_T\, e^{iK_T|z'-z|} = \frac{K_T^2}{\pi i}\int dk_z\; e^{ik_z(z'-z)}\tilde{G}^T(k)\quad, \tag{A.17}$$

in the fourth term. We also use

$$\operatorname{sgn}(z'-z)\ \left[e^{iK_T|z'-z|}-1\right] = \frac{1}{\pi i}\int dk_z\ e^{ik_z(z'-z)}\tilde{G}^T(k)P\left[\frac{K_T^2}{k_z}\right] \tag{A.18}$$

in the second and last terms in (A.16) to get

$$\partial_p\partial_i\partial_j\ G^T(\underset{\sim}{x}'-\underset{\sim}{x})$$

$$= \frac{1}{(2\pi)^3}\iiint d\underset{\sim}{k}\ e^{i\underset{\sim}{k}\cdot[\underset{\sim}{x}'-\underset{\sim}{x}]}\tilde{G}^T(k)\left[ik_{pt}k_{it}k_{jt}\right]$$

$$+ (-i)\left[\delta_{j3}\partial_{pt}\partial_{it} + \delta_{i3}\partial_{pt}\partial_{jt} + \delta_{p3}\partial_{it}\partial_{jt}\right]\cdot$$

$$\cdot\ \frac{1}{(2\pi)^3}\iiint d\underset{\sim}{k}\ e^{i\underset{\sim}{k}\cdot[\underset{\sim}{x}'-\underset{\sim}{x}]}\tilde{G}^T(k)P\left[\frac{K_T^2}{k_z}\right]$$

$$+ \frac{1}{2}\operatorname{sgn}(z'-z)\left[\delta_{j3}\partial_{pt}\partial_{it} + \delta_{i3}\partial_{pt}\partial_{jt} + \delta_{p3}\partial_{it}\partial_{jt}\right]\delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)$$

$$- \delta(z'-z)\left[\delta_{i3}\delta_{j3}\partial_{pt} + \delta_{p3}\delta_{j3}\partial_{it} + \delta_{i3}\delta_{p3}\partial_{jt}\right]\delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)$$

$$- \left[\delta_{i3}\delta_{j3}\partial_{pt} + \delta_{p3}\delta_{j3}\partial_{it} + \delta_{i3}\delta_{p3}\partial_{jt}\right]\cdot$$

$$\cdot\ \frac{1}{(2\pi)^3}\iiint d\underset{\sim}{k}\ e^{i\underset{\sim}{k}\cdot[\underset{\sim}{x}'-\underset{\sim}{x}]}\tilde{G}^T(k)K_T^2$$

$$+ \delta_{i3}\delta_{j3}\delta_{p3}\ \delta'(z'-z)\ \delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)$$

$$+ i\ \delta_{i3}\delta_{j3}\delta_{p3}\ \frac{1}{(2\pi)^3}\iiint d\underset{\sim}{k}\ e^{i\underset{\sim}{k}\cdot[\underset{\sim}{x}'-\underset{\sim}{x}]}\tilde{G}^T(k)K_T^2P\left[\frac{K_T^2}{k_z}\right]$$

$$- \frac{1}{2}\delta_{i3}\delta_{j3}\delta_{p3}\ k_T^2\ \operatorname{sgn}(z'-z)\ \delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)$$

$$- \frac{1}{2}\delta_{i3}\delta_{j3}\delta_{p3}\ \operatorname{sgn}(z'-z)\ \partial_t^2\delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)\ . \tag{A.19}$$

Carrying out the differentiations in the regular terms and combining the results we get

$$\partial_p \partial_i \partial_j G^T(\underset{\sim}{x}'-\underset{\sim}{x})$$

$$= \frac{1}{(2\pi)^3} \iiint d\underset{\sim}{k}\; e^{i\underset{\sim}{k}\cdot[\underset{\sim}{x}'-\underset{\sim}{x}]} \tilde{G}^T(k) \;.$$

$$\cdot \Big[i k_{pt} k_{it} k_{jt} + i\big[k_{pt} k_{it} \delta_{j3} + k_{pt} k_{jt} \delta_{i3} + k_{it} k_{jt} \delta_{p3}\big] P\left[\frac{K_T^2}{k_z}\right]$$

$$+ i\big[\delta_{i3}\delta_{j3} k_{pt} + \delta_{p3}\delta_{j3} k_{it} + \delta_{i3}\delta_{p3} k_{jt}\big] K_T^2$$

$$+ i\, \delta_{i3}\delta_{j3}\delta_{p3}\, K_T^4 P\left[\frac{1}{k_z}\right] \Big] \;.$$

$$+ \frac{1}{2}\,\mathrm{sgn}(z'-z)\big[\delta_{j3}\partial_{pt}\partial_{it} + \delta_{i3}\partial_{pt}\partial_{jt} + \delta_{p3}\partial_{it}\partial_{jt}\big]\delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)$$

$$- \delta(z'-z)\big[\delta_{i3}\delta_{j3}\partial_{pt} + \delta_{p3}\delta_{j3}\partial_{it} + \delta_{i3}\delta_{p3}\partial_{it}\big]\delta(x_t'-x_t)$$

$$+ \delta_{i3}\delta_{j3}\delta_{p3}\; \delta'(z'-z)\; \delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)$$

$$- \frac{1}{2}\,\delta_{i3}\delta_{j3}\delta_{p3} k_T^2 \mathrm{sgn}(z'-z)\delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)$$

$$- \frac{1}{2}\,\delta_{i3}\delta_{j3}\delta_{p3}\; \mathrm{sgn}(z'-z)\partial_t^2\; \delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) \quad ,$$

which can be rewritten as

$$\partial_p \partial_i \partial_j G^T(\underset{\sim}{x}'-\underset{\sim}{x})$$

$$= \frac{1}{2} \frac{1}{(2\pi)^3} \iiint d\underset{\sim}{k}\, e^{i\underset{\sim}{k}\cdot [\underset{\sim}{x}'-\underset{\sim}{x}]} \tilde{G}^T(k) P^T_{pij}(\underset{\sim}{k})$$

$$+ \frac{1}{2} \operatorname{sgn}(z'-z) \left[\delta_{j3}\partial_{pt}\partial_{it} + \delta_{i3}\partial_{pt}\partial_{jt} + \delta_{p3}\partial_{it}\partial_{jt}\right] \delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)$$

$$- \delta(z'-z) \left[\delta_{i3}\delta_{j3}\partial_{pt} + \delta_{p3}\delta_{j3}\partial_{it} + \delta_{i3}\delta_{p3}\partial_{jt}\right] \delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)$$

$$+ \delta_{i3}\delta_{j3}\delta_{p3}\delta'(z'-z)\delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)$$

$$- \frac{1}{2} \delta_{i3}\delta_{j3}\delta_{p3}\, k_T^2 \operatorname{sgn}(z'-z)\, \delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)$$

$$- \frac{1}{2} \delta_{j3}\delta_{j3}\delta_{p3} \operatorname{sgn}(z'-z) \partial_t^2 \delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) \quad . \tag{A.20}$$

where the regular part is given by

$$P^T_{pij}(\underset{\sim}{k}) = 2i \Bigg[k_{pt}k_{it}k_{jt}$$

$$+ \left[\delta_{j3}k_{pt}k_{it} + \delta_{i3}k_{pt}k_{jt} + \delta_{p3}k_{it}k_{jt}\right] P\left[\frac{K_T^2}{k_z}\right]$$

$$+ \left[\delta_{i3}\delta_{j3}k_{pt} + \delta_{p3}\delta_{j3}k_{it} + \delta_{i3}\delta_{p3}k_{jt}\right] K_T^2$$

$$+ \delta_{i3}\delta_{j3}\delta_{p3}\, K_T^4\, P\left[\frac{1}{k_z}\right] \Bigg] \quad . \tag{A.21}$$

Alternatively we can write (A.20) as the sum of a regular part plus singular terms where we can combine several singular terms

$$\partial_p \partial_i \partial_j G^T(\underset{\sim}{x}'-\underset{\sim}{x})$$

$$= \frac{1}{2} R^T_{pij}(\underset{\sim}{x}'-\underset{\sim}{x})$$

$$+ \frac{1}{2} \operatorname{sgn}(z'-z) \left[\delta_{j3}\partial_{pt}\partial_{it} + \delta_{i3}\partial_{pt}\partial_{jt} + \delta_{p3}\partial_{it}\partial_{jt} - \delta_{i3}\delta_{j3}\delta_{p3}\partial_t^2\right]\delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)$$

$$- \left[\delta_{i3}\delta_{j3}\partial_{pt} + \delta_{p3}\delta_{j3}\partial_{it} + \delta_{i3}\delta_{p3}\partial_{jt} + \delta_{i3}\delta_{j3}\delta_{p3}\partial_z\right]\delta(\underset{\sim}{x}'-\underset{\sim}{x})$$

$$- \frac{1}{2} k_T^2 \delta_{i3}\delta_{j3}\delta_{p3} \operatorname{sgn}(z'-z)\delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) \quad , \tag{A.22}$$

where the regular part is given by

$$R^T_{pij}(\underset{\sim}{x}'-\underset{\sim}{x}) = \frac{1}{(2\pi)^3} \iiint d\underset{\sim}{k}\, e^{i\underset{\sim}{k}\cdot\left[\underset{\sim}{x}'-\underset{\sim}{x}\right]} \tilde{G}^T(k) P^T_{pij}(\underset{\sim}{k}) \quad . \tag{A.23}$$

As a check, we note that if we set p=i in (A.22) and sum so that we have

$$\partial_p \partial_p \partial_j G^T(\underset{\sim}{x}'-\underset{\sim}{x}) = \frac{1}{2} R^T_{ppj}(\underset{\sim}{x}'-\underset{\sim}{x}) - \partial_j \delta(\underset{\sim}{x}'-\underset{\sim}{x})$$

$$- \frac{1}{2} k_T^2\, \delta_{j3} \operatorname{sgn}(z'-z)\delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) \quad , \tag{A.24}$$

and we use the result

$$P^T_{ppj}(\underset{\sim}{k}) = 2i\left[k_t^2 k_{jt} + \delta_{j3} k_t^2 P\left[\frac{K_T^2}{k_z}\right] + k_{jt} K_T^2 + \delta_{j3} K_T^4 P\left[\frac{1}{k_z}\right]\right]$$

$$= 2i\left[k_T^2 k_{jt} + \delta_{j3} P\left[\frac{K_T^2}{k_z}\right] k_T^2\right]$$

$$= k_T^2 P_j^T(\underset{\sim}{k}) \quad , \tag{A.25}$$

(where $P_j^T(\underset{\sim}{k})$ was defined in (5.18) of Ch. 1, so that

$$R_{ppj}^T(\underset{\sim}{x}'-\underset{\sim}{x}) = k_T^2 R_j^T(\underset{\sim}{x}'-\underset{\sim}{x}) \quad , \tag{A.26}$$

and we get the regularization of the first derivative of G^T as defined in Ch. 1. For example, we could write, using the results in Ch. 1,

$$\begin{aligned}
\partial_p\partial_p\partial_j G^T(\underset{\sim}{x}'-\underset{\sim}{x}) &= \partial_j\partial_p\partial_p G^T(\underset{\sim}{x}'-\underset{\sim}{x}) \\
&= \partial_j\left[-k_T^2 G^T(\underset{\sim}{x}'-\underset{\sim}{x}) - \delta(\underset{\sim}{x}'-\underset{\sim}{x})\right] \\
&= -k_T^2\left[-\frac{1}{2} R_j^T(\underset{\sim}{x}'-\underset{\sim}{x}) + \frac{1}{2}\delta_{j3}\,\mathrm{sgn}(z'-z)\delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)\right] \\
&\qquad - \partial_j\delta(\underset{\sim}{x}'-\underset{\sim}{x}) \\
&= \frac{1}{2} k_T^2 R_j^T(\underset{\sim}{x}'-\underset{\sim}{x}) - \partial_j\delta(\underset{\sim}{x}'-\underset{\sim}{x}) \\
&\qquad - \frac{1}{2} k_T^2\delta_{j3}\,\mathrm{sgn}(z'-z)\delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) \quad ,
\end{aligned}$$

which is the same as (A.24) using (A.26).

Using (A.22) we can thus write

$$\begin{aligned}
&\partial_p\partial_i\partial_j\left[G^T(\underset{\sim}{x}'-\underset{\sim}{x}) - G^L(\underset{\sim}{x}'-\underset{\sim}{x})\right] \\
&\quad = \frac{1}{2}\left[R_{pij}^T(\underset{\sim}{x}'-\underset{\sim}{x}) - R_{pij}^L(\underset{\sim}{x}'-\underset{\sim}{x})\right] \\
&\qquad - \frac{1}{2}(k_T^2 - k_L^2)\delta_{i3}\delta_{j3}\delta_{p3}\,\mathrm{sgn}(z'-z)\delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t) \quad ,
\end{aligned} \tag{A.27}$$

where R_{pij}^L is analogous to (A.23) with $\tilde{G}^T$ and P_{pij}^T replaced by $\tilde{G}^L$ and P_{pij}^L.

The latter follows from (A.21) by replacing K_T by $K_L=(k_L^2-k_t^2)^{1/2}$. We thus have from (A.9) and (A.27) that

$$\left[TG^0(\underset{\sim}{x}'-\underset{\sim}{x})\right]_{pij} = K_{pij}(\underset{\sim}{x}'-\underset{\sim}{x})$$

$$+ \frac{1}{2}\,\mathrm{sgn}(z'-z)\,\delta(x_t'-x_t)\;\cdot$$

$$\cdot\left[\delta_{ij}\delta_{p3} + \delta_{pj}\delta_{i3} + \left[\lambda/(\lambda+\mu)\right]\delta_{ip}\delta_{j3}\right.$$

$$\left. + 2\,(k_L^2-k_T^2)k_T^{-2}\delta_{i3}\delta_{j3}\delta_{p3}\right] , \qquad (A.28)$$

where

$$K_{pij}(\underset{\sim}{x}'-\underset{\sim}{x}) = k_T^{-2}\left[R^T_{pij}(\underset{\sim}{x}'-\underset{\sim}{x}) - R^L_{pij}(\underset{\sim}{x}'-\underset{\sim}{x})\right]$$

$$- \frac{1}{2}\left[\delta_{ij}R^T_p(\underset{\sim}{x}'-\underset{\sim}{x}) + \delta_{pj}R^T_i(\underset{\sim}{x}'-\underset{\sim}{x})\right.$$

$$\left. + \delta_{ip}\left[(\lambda/(\lambda+\mu)\right]R^L_j(\underset{\sim}{x}'-\underset{\sim}{x})\right] . \qquad (A.29)$$

Using the definitions of k_L and k_T we can rewrite the singular part of (A.28) to yield

$$\left[TG^0(\underset{\sim}{x}'-\underset{\sim}{x})\right]_{pij} = K_{pij}(\underset{\sim}{x}'-\underset{\sim}{x})$$

$$+ \frac{1}{2}\,\mathrm{sgn}(z'-z)\,\delta(x_t'-x_t)\;.$$

$$\cdot\left[\delta_{ij}\delta_{p3} + \delta_{pj}\delta_{i3} + \left[\lambda/\lambda + 2\mu\right]\delta_{ip}\delta_{j3}\right.$$

$$\left. -2\,\left[(\lambda+\mu)/(\lambda+2\mu)\right]\delta_{i3}\delta_{j3}\delta_{p3}\right] . \qquad (A.30)$$

In the integral equation it is (A.30) dotted into the normal vector $n_p(\underset{\sim}{x}_t)=\delta_{p3}-\partial_{pt}h(\underset{\sim}{x}_t)$ which is important. The result from (A.30) is

$$
\begin{aligned}
n_p(\underset{\sim}{x}_t)\ &\left[TG^{o}(\underset{\sim}{x}'-\underset{\sim}{x})\right]_{pij} \\
&= K_{ji}(\underset{\sim}{x}',\underset{\sim}{x}) + \\
&\quad + \frac{1}{2}\,\mathrm{sgn}(z'-z)\,\delta(\underset{\sim}{x}_t'-\underset{\sim}{x}_t)\ \cdot \\
&\quad \cdot \left[\delta_{ij} - \delta_{i3}\partial_{jt}h(\underset{\sim}{x}_t) - \left[\frac{\lambda}{\lambda+2\mu}\right]\delta_{j3}\delta_{it}h(\underset{\sim}{x}_t)\right] \quad , \qquad \text{(A.31)}
\end{aligned}
$$

where

$$
K_{ji}(\underset{\sim}{x}',\underset{\sim}{x}) = n_p(\underset{\sim}{x}_t)K_{pij}(\underset{\sim}{x}'-\underset{\sim}{x}) \quad , \qquad \text{(A.32)}
$$

which is not a function of the difference of coordinates.

APPENDIX 3B. DERIVATION OF THE FREE-SPACE ELASTIC GREEN'S FUNCTION

a. Coordinate-space derivation

The free-space elastic Green's function satisfies the differential equation

$$\left[\Delta^{*} G^{0}(\underset{\sim}{x},\underset{\sim}{x}')\right]_{ij} + K^{2} G^{0}_{ij}(\underset{\sim}{x},\underset{\sim}{x}') = - \delta_{ij}\delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad , \tag{B.1}$$

or, explicitly writing the operator Δ^{*} (from (1.17))

$$\mu \nabla^{2} G^{0}_{ij} + (\lambda + \mu)\partial_{i}\partial_{m} G^{0}_{mj} + K^{2} G^{0}_{ij} = - \delta_{ij}\delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad , \tag{B.2}$$

where we have suppressed the coordinate dependence. The scalar free-space Green's functions satisfy the equations

$$(\nabla^{2} + k_{L}^{2}) G^{L} = - \delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad , \tag{B.3}$$

and

$$(\nabla^{2} + k_{T}^{2}) G^{T} = - \delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad , \tag{B.4}$$

where

$$k_{L}^{2} = K^{2}/(\lambda + 2\mu) \; , \quad k_{T}^{2} = K^{2}/\mu \quad . \tag{B.5}$$

If the divergence term in (B.2) were absent we could solve the equation as a scalar equation, so we first solve for it by taking the divergence ∂_i of (B.2) to get

$$(\lambda + 2\mu)\,\nabla^2(\partial_i G^0_{ij}) + K^2(\partial_i G^0_{ij}) = -\,\partial_j \delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad , \tag{B.6}$$

which can be rewritten as

$$\nabla^2(\partial_i G^0_{ij}) + k_L^2(\partial_i G^0_{ij}) = -\,\frac{k_L^2}{K^2}\,\partial_j \delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad . \tag{B.7}$$

This has the solution

$$\partial_i G^0_{ij} = \frac{k_L^2}{K^2}\,\partial_j G^L \quad , \tag{B.8}$$

which follows from (B.3). Hence (B.2) can be rewritten as

$$\nabla^2 G^0_{ij} + k_T^2 G^0_{ij} = -\,\frac{1}{\mu}\,\delta_{ij}\delta(\underset{\sim}{x}-\underset{\sim}{x}') - \frac{(\lambda + \mu)}{\mu}\,\frac{k_L^2}{K^2}\,\partial_i\partial_j G^L \quad . \tag{B.9}$$

From (B.4) we note that

$$(\nabla^2 + k_T^2)\left[\frac{1}{\mu}\,\delta_{ij} G^T\right] = -\,\frac{1}{\mu}\,\delta_{ij}\delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad . \tag{B.10}$$

Subtract (B.10) from (B.9) to get

$$(\nabla^2 + k_T^2)\left[G^0_{ij} - \frac{1}{\mu}\,\delta_{ij} G^T\right] = -\,\frac{\lambda + \mu}{\mu}\,\frac{k_L^2}{K^2}\,\partial_i\partial_j G^L \quad . \tag{B.11}$$

Next rewrite the rhs of (B.11) using

$$\frac{\lambda + \mu}{\mu} = \frac{\lambda + 2\mu - \mu}{\mu}$$

$$= (k_T^2/k_L^2 - 1) \quad , \tag{B.12}$$

so that (B.11) becomes

$$(\nabla^2 + k_T^2) \left[G^0_{ij} - \frac{1}{\mu} \delta_{ij} G^T \right] = - \frac{k_T^2}{K^2} \partial_i \partial_j G^L + \frac{k_L^2}{K^2} \partial_i \partial_j G^L \quad . \tag{B.13}$$

For the second term on the rhs of (B.13) we substitute from (B.3)

$$k_L^2 G^L = - \nabla^2 G^L - \delta(\underset{\sim}{x} - \underset{\sim}{x}') \quad , \tag{B.14}$$

so that (B.13) becomes

$$(\nabla^2 + k^2_T) \left[G^0_{ij} - \frac{1}{\mu} \delta_{ij} G^T + \frac{1}{K^2} \partial_i \partial_j G^L \right] = - \frac{1}{K^2} \partial_i \partial_j \delta(\underset{\sim}{x} - \underset{\sim}{x}') \quad , \tag{B.15}$$

whose solution is

$$G^0_{ij} - \frac{1}{\mu} \delta_{ij} G^T_{ij} + \frac{1}{K^2} \partial_i \partial_j G^L = \frac{1}{K^2} \partial_i \partial_j G^T \quad , \tag{B.16}$$

which is just (2.2) quoted in the text.

b. Fourier transform-space derivation

The coefficients in (B.2) are constant so we introduce the Fourier transform

$$\tilde{G}^0_{ij}(\underset{\sim}{a}) = \iiint G^0_{ij}(\underset{\sim}{x} - \underset{\sim}{x}') \exp \left[-i \underset{\sim}{a} \cdot (\underset{\sim}{x} - \underset{\sim}{x}') \right] d\underset{\sim}{x} \quad , \tag{B.17}$$

where we use the homogeneity of the Green's functions. The transform of (B.2) is thus

$$\mu\alpha^2\tilde{G}^o_{ij} + (\lambda+\mu)\alpha_i\alpha_m \tilde{G}^o_{mj} - K^2\tilde{G}^o_{ij} = \delta_{ij} \quad . \tag{B.18}$$

Again we must first solve for the divergence term so multiply (B.18) by α_i so that the solution for

$$g_j = \alpha_m \tilde{G}^o_{mj} \tag{B.19}$$

is given by

$$g_j = \left[(\lambda+2\mu)\alpha^2-K^2 \right]^{-1}\alpha_j \quad . \tag{B.20}$$

Substituting (B.20) into the second term on the lhs of (B.18) yields

$$(\mu\alpha^2-K^2)\ \tilde{G}^o_{ij} = \delta_{ij} - (\lambda+\mu) \left[(\lambda+2\mu)\alpha^2-K^2 \right]^{-1}\alpha_i\alpha_j \tag{B.21}$$

This can be written using (B.5) and partial fractions as

$$\tilde{G}^o_{ij}(\alpha) = \delta_{ij}\mu^{-1}\tilde{G}^T(\alpha) + \alpha_i\alpha_j K^{-2} \left[\tilde{G}^L(\alpha) - \tilde{G}^T(\alpha) \right] \tag{B.22}$$

where

$$\tilde{G}^{T,L}(\alpha) = (\alpha^2-K^2_{T,L})^{-1} \tag{B.23}$$

It is easily seen that the Fourier inverse of (B.22) is just (2.2) in the text.

APPENDIX 3C. DIFFERENTIAL RELATIONS FOR INHOMOGENEOUS MEDIA

In Section 3 we derived differential relations for the Green's functions and surface displacements for homogeneous media. We specified the resulting integral equations to surfaces and this illustrated the coupling of displacement components due to surface variability. Here we derive these differential relations for inhomogeneous media and this illustrates the coupling of displacement components due to volume variability. We begin with the equations of motion of the stress tensor, (1.12), written as

$$\partial_k \tau_{jk}(\underset{\sim}{x}) + K^2 u_j(\underset{\sim}{x}) = 0_j \quad . \tag{C.1}$$

The stress is given by

$$\tau_{jk}(\underset{\sim}{x}) = \mu(\partial_j u_k + \partial_k u_j) + \lambda\delta_{jk}\partial_m u_m \quad , \tag{C.2}$$

where now μ and λ can be spatially dependent. We define the traction operator T as

$$\tau_{jk} = T_{jkp} u_p \quad , \tag{C.3}$$

so that it is explicitly given by

$$T_{jkp} = \mu\delta_{jp}\partial_k + \mu\delta_{kp}\partial_j + \lambda\delta_{jk}\partial_p \quad . \tag{C.4}$$

For the scattered displacement components (C.1) can be written using (C.3) as

$$\partial_k T_{jkp} u_p^{sc}(\underset{\sim}{x}) + K^2 u_j^{sc}(\underset{\sim}{x}) = 0 \quad . \tag{C.5}$$

The Green's function G for this equation is given by the solution of

$$\partial_k T_{jkp} G_{pr}(\underset{\sim}{x}, \underset{\sim}{x}') + K^2 G_{jr}(\underset{\sim}{x}, \underset{\sim}{x}') = -\delta_{jr}\delta(\underset{\sim}{R}) \quad , \tag{C.6}$$

where

$$\underset{\sim}{R} = \underset{\sim}{x} - x' \; ; \quad \underset{\sim}{x}' = \text{source position}$$

and where we have the same operator T, i.e. the same inhomogeneous λ and μ as in (C.5).

Next, cross multiply the solutions, i.e. form the quantity

$$G_{ij}(\underset{\sim}{x}, \underset{\sim}{x}') \left[\partial_k T_{ikp} u_p^{sc}(\underset{\sim}{x})\right] - \left[\partial_k T_{ikp} G_{pj}(\underset{\sim}{x}, \underset{\sim}{x}')\right] u_i^{sc} \quad . \tag{C.7}$$

Substituting in (C.5) and (C.6) as appropriate we can derive the relation

$$u_j^{sc}(\underset{\sim}{x})\delta(\underset{\sim}{R}) = \partial_k\left[G_{ij} T_{ikp} u_p^{sc} - (T_{ikp} G_{pj}) u_i^{sc}\right]$$

$$+ (T_{ikp} G_{pj}) \partial_k u_i^{sc} - (\partial_k G_{ij})(T_{ikp} u_p^{sc}) \quad . \tag{C.8}$$

Explicit evaluation of the latter two terms in (C.8) shows that they vanish. The result is that if the equations for G and u contain the same λ and μ we get a pure divergence

$$u_j^{sc}(\underset{\sim}{x})\delta(\underset{\sim}{R}) = \partial_k\left[G_{ij} T_{ikp} u_p^{sc} - (T_{ikp} G_{pj}) u_i^{sc}\right] \quad , \tag{C.9}$$

which is our first differential relation.

To derive the second differential relation we start with the equation for the homogeneous Green's function (2.1) written in terms of the traction operator T^0

$$\partial_k T^o_{jkp} G^o_{pr} + (K^o)^2 G^o_{jr}(\underset{\sim}{x}-\underset{\sim}{x}') = -\delta_{jr}\delta(\underset{\sim}{R}) \quad , \tag{C.10}$$

where T^o is given by

$$T^o_{jkp} = \mu^o \delta_{jp}\partial_k + \mu^o \delta_{kp}\partial_j + \lambda^o \delta_{jk}\partial_p \quad , \tag{C.11}$$

and we explicitly note that we have a constant background medium, i.e. where μ^o, λ^o, and ρ^o are constant. Next, cross multiply the solutions of (C.10) and (C.5) to form the quantity

$$G^o_{ij}(\partial_k T_{ikp} u^{sc}_p) - \partial_k\left[T^o_{ikp} G^o_{pj}\right] u^{sc}_i \quad . \tag{C.12}$$

Substituting the solution forms we get the result

$$\begin{aligned} u^{sc}_j(\underset{\sim}{x})\delta(\underset{\sim}{R}) &= \partial_k\left[G^o_{ij} T_{ikp} u^{sc}_p - \left[T^o_{ikp} G^o_{pj}\right] u^{sc}_i\right] \\ &\quad + \left[K^2 - (K^o)^2\right] G^o_{ij} u^{sc}_i \\ &\quad + (T^o_{ikp} G^o_{pj})\partial_k u^{sc}_i - (\partial_k G^o_{ij})(T_{ikp} u^{sc}_p) \quad . \end{aligned} \tag{C.13}$$

Evaluation of the latter three terms in (C.13) yields the second differential relation

$$\begin{aligned} u^{sc}_j(\underset{\sim}{x})\delta(\underset{\sim}{R}) &= \partial_k\left[G^o_{ij} T_{ikp} u^{sc}_p - (T^o_{ikp} G^o_{pj}) u^{sc}_i\right] \\ &\quad + \omega^2(\rho-\rho^o) G^o_{ij} u^{sc}_i \\ &\quad - (\mu-\mu^o)(\partial_k G^o_{ij})(\partial_k u^{sc}_i + \partial_i u^{sc}_k) \\ &\quad - (\lambda-\lambda^o)(\lambda^o+2\mu^o)^{-1}(\partial_j G^L)(\partial_p u^{sc}_p) \quad . \end{aligned} \tag{C.14}$$

In addition to a divergence term which, when integrated, yields surface integrals, we also get pure volume terms proportional to the differences between the homogeneous (constant background) parameters and the inhomogeneous ones.

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