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SOME ALGEBRAIC ASPECTS OF THE LAMB PROBLEM

by

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ABSTRACT

This paper discusses the classical Lamb problem for the elastic wave equation. The motivation, for the authors, is to be able to conveniently construct Green's functions (matrices) for later use in formulating and solving various inverse problems. For example, we will want to be able to solve for perturbations from constant reference densities and Lamé parameters. Hence, Green's function for the homogeneous isotropic equation is discussed. Due to the scattering taking place in inverse problems it is usually impossible to retain the P-SV and SH decoupling; hence, we do not pursue this decoupling in the formation of the Green's functions herein.

While nothing conceptually new is presented here, the approach is a bit different and, we believe, is helpful in isolating some important issues. The approach is algebraic in nature and makes heavy use of several simple facts from linear algebra; for example, the spectral decomposition of special matrices. This approach facilitates some helpful decoupling, particularly in solving for reflection coefficients.

GLOSSARY

$\underline{a}$, $\underline{b}$, $\underline{c}$	generic vectors in R^3.
A	(various) 3 by 3 matrices.
$B(\cdot)$	boundary operator at $x_3 = 0$ surface; equation (17).
c_p, c_s	pressure and shear speeds; above equation (12).
C	3 by 3 matrix used in constructing R; equation (28).
D, D′	determinants associated with reflection coefficients; equation (36).
$\underline{f}$	constant body force; equation (2).
F	free space portion of Lamb Green's function; equation (18).
$\underline{g}$	arbitrary body force; equation (1).
G	free space Green's function; equation (11).
G_1, G_2	portions of G; equation (12).
$\underline{h}$	vector associated with horizontal shear wave; equation (8b).
H	Green's function for Lamb problem; equation (37).
I	3 by 3 identity matrix; equation (5).
$\underline{k}$	$= (k_1, k_2, k_3)$ spatial Fourier transform variables; equation (5).
k	$= \lvert k \rvert$; below equation (5).
$\hat{k}$	$= \underline{k}/k$; below equation (5).
$\underline{k}'$	$= (k_1, k_2)$; equation (18).
$L(\cdot)$	basic differential operator in matrix form; equation (5).

$M(\cdot)$	operator $L(\cdot) - \rho\omega^2 I$; equation (9).
N_1, N_2	portions of M^{-1}; below equation (11).
$\underline{p}$	vector associated with pressure wave; equation (8a).
r	reflection coefficients, subscripted with p, h, v; equation (32).
R	reflection portion of the Lamb Green's function; equation (28).
T	transpose of vector or matrix.
$\underline{u}$	displacement vector; equation (1).
$\underline{v}$	vector associated with vertical shear wave; equation (8c).
$\underline{x}$	$= (x_1, x_2, x_3)$ spatial coordinate; equation (1).
$\underline{x}'$	$= (x_1, x_2)$; equation (18).
$\delta(\cdot)$	Dirac delta function; equation (2).
∇	gradient operator; equation (2).
Δ	Laplacian; equation (3).
ε	small positive number.
λ, μ	Lamé parameters; equation (1).
ρ	density; equation (1).
σ_i	eigenvalues; equation (7).
ω	frequency: equation (2).

INTRODUCTION

The purpose here is to present a somewhat different point of view toward constructing Green's functions (matrices) for a variety of problems associated with the elastic wave equation. The primary interest of the authors is inverse problems for various acoustics and elastic experiments (e.g. Cohen et al [4] and Boyse and Keller [2]). In this regard a clear view of certain basic principles and a systematic approach to constructing Green's functions is particularly important. A convenient setting for such studies is the frequency domain, hence the emphasis below is in obtaining the corresponding Green's functions. However, some mention is made regarding techniques for inverting to the time domain.

The approach taken here is algebraic in nature, making repeated use of several very elementary properties of square matrices. Consequently, it is a bit more contemporary in style than that found in most sources and, we believe, a little more natural. There are of course many high quality studies of the classical problems discussed here (e.g. Aki and Richards [1], Červený and Ravindra [3] and Johnson [7]) and the authors lay no claim to the discovery of heretofore unknown phenomena, or even methods. The hope is that the general approach will be useful in attacking new problems in a more straight-forward fashion.

Rather than attempt to discuss things in great generality, we choose to illustrate the approach on a well-known problem. In particular, our goal here will be the construction of the Green's function for (one statement of) the Lamb problem; a single traction-free reflecting surface at $z = 0$. Along the way the "free space" Green's function will be obtained. In order to exploit the convenient algebraic properties we tend to Fourier transform as much as possible at each stage. By making use of a simple algebraic decomposition of a constant matrix (associated with P, SV and SH modes) valuable insight is gained and a certain amount of desirable decoupling is facilitated. For example, the nine reflection coefficients are rather naturally found by solving three 3 by 3 linear systems (instead of one 9 by 9 system).

SOME PRELIMINARIES

The linear equations of elasticity in an isotropic, inhomogeneous media are given in terms of the displacement vector $\underline{u} = \underline{u}(\underline{x},t)$ as

$$\rho\partial_t^2 \underline{u} = \nabla(\lambda\nabla\cdot\underline{u}) + \nabla\times(\mu\nabla\times\underline{u}) + 2(\nabla\cdot\mu\nabla)\underline{u} + \underline{g}(\underline{x},t) \quad ,$$
$$\underline{x} = (x_1,x_2,x_3) \quad , \quad t \geq 0 \quad . \tag{1}$$

In particular, we seek various solutions when $\underline{g}(\underline{x},t) = \delta(\underline{x}-\underline{y})\delta(t)\underline{f}$ where $\underline{f}$ is a constant vector or, later, a constant matrix. Hence, we seek various Green's functions for this equation. We Fourier transform in time ($t \rightarrow \omega$, $\underline{u}(\underline{x},t) \rightarrow \underline{u}(\underline{x},\omega)$) producing

$$\nabla(\lambda\nabla\cdot\underline{u}) + \nabla\times(\mu\nabla\times\underline{u}) + 2(\nabla\cdot\mu\nabla)\underline{u} + \rho\omega^2 u = -\delta(\underline{x}-\underline{y})\underline{f} \quad . \tag{2}$$

Our attention will focus on the homogeneous, isotropic case in which λ, μ and ρ are constants. Then (2) becomes

$$(\lambda+\mu)\nabla(\nabla\cdot\underline{u}) + \mu\Delta\underline{u} + \rho\omega^2\underline{u} = -\delta(\underline{x}-\underline{y})f \quad . \tag{3}$$

We use the following notation:

1. $\underline{u}$, $\underline{x}$, etc. will denote column vectors. When a row vector is needed, we write e.g. $\underline{u}^T$, where T denotes transpose.
2. The inner product of two vectors is denoted by $\underline{u}^T\underline{v}$ or $\underline{u}\cdot\underline{v}$. The outer (or tensor) product is denoted by $\underline{u}\underline{v}^T$ and is of course a 3 by 3 matrix.
3. $\hat{\underline{k}}$, etc. will denote unit vectors: $\hat{\underline{k}} = \underline{k}/|\underline{k}|$. Moreover, k denotes $|\underline{k}|$.

4. $\underline{k}'$, $\underline{x}'$, etc. will denote the first two components of $\underline{k}$, $\underline{x}$; e.g. $\underline{k}' = (k_1, k_2)$.

5. Fourier transform is defined by

$$f(t) \to \int_{-\infty}^{\infty} dt\ e^{i\omega t} f(t) \quad ,$$

$$g(\underline{x}) \to \int_{-\infty}^{\infty} d\underline{x}\ e^{-i\underline{k}\cdot\underline{x}} g(\underline{x}) \quad , \quad \underline{x} \in R^2 \text{ or } R^3 \quad .$$

(Note the difference in definition for time and spatial transforms. This facilitates the notion of "outgoing" below.)

Some elementary facts from linear algebra are now stated. These facts are established in about any standard text (e.g. Strang [9], Hoffman and Kunze [6], or Mostow and Sampson [8]), and can be easily verified by the reader.

1. Let $\underline{a}$ be a unit vector. The matrix $P = \underline{a}\underline{a}^T$ is a (orthogonal) projection matrix (i.e. $P^2\underline{x} = P(P\underline{x}) = P\underline{x}$; moreover, if $\underline{x}$ is orthogonal to $\underline{a}$ then $P\underline{x} = 0$). Clearly $P\underline{x} = \underline{a}\underline{a}^T\underline{x} = c\underline{a}$, $c = \underline{a}\cdot\underline{x}$; so P projects R^3 orthogonally onto the line spanned by $\underline{a}$.
2. Suppose $\underline{a}$, $\underline{b}$ and $\underline{c}$ are unit vectors and mutually orthogonal. Then the identity matrix can be decomposed into the three projections; $I = \underline{a}\underline{a}^T + \underline{b}\underline{b}^T + \underline{c}\underline{c}^T$. (This is easily verified by applying both sides to an arbitrary vector $\underline{x}$).
3. Suppose A is a 3 by 3 matrix with orthonormal eigenvectors $\underline{a}$, $\underline{b}$, $\underline{c}$ and eigenvalues σ_1, σ_2, σ_3, respectively. Then A has the spectral decomposition

$$A = \sigma_1 \underline{a}\underline{a}^T + \sigma_2 \underline{b}\underline{b}^T + \sigma_3 \underline{c}\underline{c}^T \ .$$

(This is verified by applying both sides to an arbitrary vector $\underline{x} = \alpha_1\underline{a} + \alpha_2\underline{b} + \alpha_3\underline{c}$). Finally, if the $\sigma_j \neq 0$ then A is invertible and A^{-1} has the convenient expression

$$A^{-1} = \frac{1}{\sigma_1} \underline{a}\underline{a}^T + \frac{1}{\sigma_2} \underline{b}\underline{b}^T + \frac{1}{\sigma_3} \underline{c}\underline{c}^T \ . \tag{4}$$

(This is verified by multiplying AA^{-1} and using fact 2. above).

4. For vectors in R^3, the following is easily verified:

$$[\underline{u},\underline{v},\underline{w}] \begin{bmatrix} \underline{a}^T \\ \underline{b}^T \\ \underline{c}^T \end{bmatrix} = \underline{u}\underline{a}^T + \underline{v}\underline{b}^T + \underline{w}\underline{c}^T \ .$$

5. Suppose $A\underline{b} = \sigma\underline{b}$, then observe that

$$(A + \varepsilon I)\underline{b} = (\sigma+\varepsilon)\underline{b} \quad ,$$

i.e. the eigenvalues of $A + \varepsilon I$ are those of A shifted by ε, and the eigenvectors are unchanged.

Returning to the elastic equations (3), we next Fourier transform the spatial variables ($\underline{x} \rightarrow \underline{k}$) thus reducing our problem to an algebraic one (at least temporarily). The resulting equation can be written

$$(L(\underline{k})-\rho\omega^2 I) \ \underline{u}(\underline{k},\omega) = e^{-i\underline{k}\cdot\underline{Y}} \ \underline{f} \quad , \tag{5}$$

where

$$L(\underline{k}) = (\lambda+\mu)\ \underline{k}\underline{k}^T + \mu k^2 I$$

$$= (\lambda+\mu)\ k^2\hat{\underline{k}}\hat{\underline{k}}^T + \mu k^2\ I$$

$$= A + \mu k^2 I\quad .$$

Formally, therefore, if no boundary conditions are imposed, the "free" solution in the transform variables can be expressed

$$\underline{u}(\underline{k},\omega) = (L(\underline{k})-\rho\omega^2 I)^{-1}\ e^{-i\underline{k}\cdot\underline{y}_f} \tag{6}$$

In the next section we invert this ($\underline{k} \rightarrow \underline{x}$) in the simplist possible context, obtaining the "free space" Green's function. This will make good use of the above algebraic observations and will set the stage for the subsequent section where the Lamb problem is considered.

THE FREE SPACE GREEN'S FUNCTION

By the free space Green's function we mean the matrix solution to (3) with f = I. Perhaps the easiest way to invert ($\underline{k} \rightarrow \underline{x}$) the expression for $\underline{u}(\underline{k},\omega)$ in (6) is to first perform the spectral decomposition of the matrix in (5). This turns out to be surprisingly easy and generally informative.

First, note that it follows immediately from the facts noted above that the matrix $A = L(\underline{k})-\mu k^2\underline{I} = (\lambda+\mu)\underline{k}\underline{k}^T$ above is a projection operator (onto $\underline{k}$); hence $\underline{k}$ is an eigenvector with associated eigenvalue $(\lambda+\mu)k^2$. Moreover, the only other eigenvalue is zero and the associated (2-dim) eigenspace must be orthogonal to $\underline{k}$. It follows that $L(\underline{k}) = A + \mu k^2 I$ has the same eigenvectors and (shifted) eigenvalues

$$\begin{aligned} \sigma_1 &= (\lambda+\mu)k^2 + \mu k^2 = (\lambda+2\mu)k^2 \quad , \\ \sigma_2 &= \mu k^2 \quad . \end{aligned} \tag{7}$$

For convenience we select and denote the eigenvectors as follows. Associated with σ_1 we take

$$\underline{p} = \hat{\underline{k}} = \underline{k}/\ |\underline{k}| \tag{8a}$$

where $\underline{p}$ suggests pressure (or longitudinal) wave. For σ_2 we can choose any two vectors orthogonal to $\underline{p}$. We select

$$\underline{h} = \frac{1}{\sqrt{k_1^2 + k_2^2}}(k_2,-k_1,0) = \frac{1}{k'}(k_2,-k_1,0) \tag{8b}$$

$$\underline{v} = \frac{1}{kk'}(k_1k_3,k_2k_3,-|k'|^2) \quad . \tag{8c}$$

Note that $\underline{p}$, $\underline{h}$, $\underline{v}$ form an orthonormal set. The symbol $\underline{h}$ suggests the horizontal component of the shear (or transverse) wave and $\underline{v}$ suggests the vertical component of shear. In particular, $\underline{h}$ is horizontal in the sense $\underline{h}\cdot(0,0,1) = 0$. In the homogeneous case (ρ, λ and μ constant and no body forces), the corresponding waves will remain orthogonal until encountering some form of inhomogeniety or boundary.

We can now decompose the matrix $L(\underline{k})-\rho\omega^2 I$ using these eigenvectors and eigenvalues as follows

$$M(\underline{k}) = M(\underline{k};\omega) \equiv L(\underline{k}) - \rho\omega^2 I$$

$$= (\sigma_1 - \rho\omega^2)\,\underline{p}\underline{p}^T + (\sigma_2 - \rho\omega^2)\left[\underline{h}\underline{h}^T + \underline{v}\underline{v}^T\right] .$$

Note, once again, the eigenvalues for M have been shifted by $\rho\omega^2$ relative to those of L. Assuming $\sigma_1 - \rho\omega^2 \neq 0$ and $\sigma_2 - \rho\omega^2 \neq 0$, the inverse of $M(\underline{k})$ is

$$M(\underline{k})^{-1} = \frac{1}{\sigma_1 - \rho\omega^2}\,\underline{p}\underline{p}^T + \frac{1}{\sigma_2 - \rho\omega^2}\left[\underline{h}\underline{h}^T + \underline{v}\underline{v}^T\right] . \tag{9}$$

Hence, the formal "free" solution to (3) (in $(\underline{k},\omega)$ - space) is

$$\underline{u}(\underline{k},\omega) = M(\underline{k})^{-1}\, e^{-i\underline{k}\cdot\underline{y}}\,\underline{f} \tag{10}$$

where $M(\underline{k})^{-1}$ is given by (9). To complete the construction of the free space Green's matrix $G(\underline{x},t;\underline{y})$ it remains to invert transform the matrix in (10), i.e. matrix

$$G(\underline{k},\omega;\underline{y}) \equiv e^{-i\underline{k}\cdot\underline{y}}\, M(k)^{-1} . \tag{11}$$

This inversion follows familiar procedures (e.g. see Aki and Richards [1]), but is outlined here for sake of completeness.

In (9) we replace $\underline{h}\underline{h}^T + \underline{v}\underline{v}^T$ by $I - \underline{p}\underline{p}^T$ (algebra fact 2. above) and using (7) we have

$$M(\underline{k})^{-1} = \frac{1}{\sigma_1 - \rho\omega^2}\,\underline{p}\underline{p}^T + \frac{1}{\sigma_2 - \rho\omega^2}\left[I - \underline{p}\underline{p}^T\right]$$

$$= \frac{1}{\sigma_2 - \rho\omega^2}\, I + \left[\frac{1}{\sigma_1 - \rho\omega^2} - \frac{1}{\sigma_2 - \rho\omega^2}\right]\underline{p}\underline{p}^T$$

$$= \frac{1}{\rho(c_s^2 k^2 - \omega^2)}\, I + \frac{1}{\rho}\left[\frac{c_s^2 - c_p^2}{(c_p^2 k^2 - \omega^2)(c_s^2 k^2 - \omega^2)}\right]\underline{k}\underline{k}^T$$

$$\equiv N_1(\underline{k}) + N_2(\underline{k})$$

where

$$c_p^2 = \frac{\lambda + 2\mu}{\rho} \quad ,$$

$$c_s^2 = \frac{\mu}{\rho} \quad .$$

Hence in this notation,

$$G(\underline{k},\omega;\underline{y}) = e^{-\underline{k}\cdot\underline{y}}\left[N_1(\underline{k}) + N_2(\underline{k})\right]$$

$$\equiv G_1(\underline{k},\omega;\underline{y}) + G_2(\underline{k},\omega;\underline{y}) \quad . \qquad (12)$$

The inversion ($\underline{k} \rightarrow \underline{x}$) of G_1 is standard and proceeds or follows:

$$G_1(\underline{x},\omega,\underline{y}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d\underline{k}\; e^{i\underline{k}\cdot\underline{x}}\, e^{-i\underline{k}\cdot\underline{y}}\, \frac{1}{\rho(c_s^2k^2-\omega^2)}\, I$$

$$= \frac{1}{(2\pi)^3\rho c_s^2}\int_{-\infty}^{\infty} d\underline{k}\; e^{i\underline{k}\cdot(\underline{x}-\underline{y})}\, \frac{1}{k^2-\omega^2/c_s^2}\, I \; . \qquad (13)$$

This triple integral can be evaluated by going to spherical coordinates $\underline{k} \rightarrow (r,\theta,\phi)$. These steps are outlined below in discussing G_2. The result, after using $\mu = \rho c_s^2$, is

$$G_1(\underline{x},\omega;\underline{y}) = \frac{1}{4\pi\mu|\underline{x}-\underline{y}|}\exp\left[\frac{i\omega}{c_s}|\underline{x}-\underline{y}|\right] I \; . \qquad (14)$$

The more interesting step is the inversion of G_2 in (12), which we outline.

$$G_2(\underline{x},\omega;\underline{y}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d\underline{k}\; e^{i\underline{k}\cdot(\underline{x}-\underline{y})}\, \frac{c_s^2-c_p^2}{\rho(c_p^2k^2-\omega^2)(c_s^2k^2-\omega^2)}\, \underline{k}\underline{k}^T$$

$$= \frac{1}{8\pi^3}\,\frac{c_s^2-c_p^2}{\rho\, c_s^2c_p^2}\int_{-\infty}^{\infty} d\underline{k}\; e^{i\underline{k}\cdot(\underline{x}-\underline{y})}\, \frac{1}{(k^2-\omega^2/c_p^2)(k^2-\omega^2/c_s^2)}\,\underline{k}\underline{k}^T \qquad (15)$$

$$= \frac{1}{8\pi^3}\,\frac{c_s^2-c_p^2}{\rho c_s^2c_p^2}\left[-\frac{\partial^2}{\partial x_i\partial x_j}\int_{-\infty}^{\infty} d\underline{k}\,\frac{e^{i\underline{k}\cdot(\underline{x}-\underline{y})}}{(k^2-\omega^2/c_p^2)(k^2-\omega^2/c_s^2)}\right] .$$

Note that the term in brackets is a matrix, say $B = (b_{ij})$, with elements

$$b_{ij} = - \frac{\partial^2}{\partial x_i \partial x_j} \int_{-\infty}^{\infty} d\underline{k} \frac{e^{i\underline{k}\cdot(\underline{x}-\underline{y})}}{(k^2 - \omega^2/c_p^2)(k^2 - \omega^2/c_s^2)} .$$

Observe that this integral is like the integral for G_1, in (13), except for the additional poles at $k = \pm \omega/c_s$. In evaluating the integrals in (13) and (15) the usual procedure is as follows. Convert to spherical coordinates, $\underline{k} \rightarrow (r,\theta,\phi)$, with $r = |\underline{k}|$ and $\underline{x}-\underline{y}$ serving as the "north pole." The angular integrals are done routinely; then the r integral is done by the residue theorem. As in Aki and Richards [1], one assumes $\mathrm{Im}(\omega) \geq 0$ which dictates that one take the residue integration path with $\mathrm{Im}(r) \geq 0$. This results in the two poles at $k = + \omega/c_p$ and $k = + \omega/c_s$ contributing and gives

$$G_2(\underline{x},\omega;\underline{y}) = \frac{1}{4\pi\rho\omega^2} \left[\frac{\partial^2}{\partial x_i \partial x_j} \left[\frac{1}{|x-y|} (e^{i\omega|x-y|/c_s} - e^{i\omega|x-y|/c_p}) \right] \right] . \quad (16)$$

In some applications this form of $G = G_1 + G_2$ (i.e. in $(\underline{x}, \omega)$ domain) is desirable. If not, the inversion $(\omega \rightarrow t)$ proceeds as follows. Referring to (14) we have

$$G_1(\underline{x},t;\underline{y}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, G_1(\underline{x},\omega,\underline{y})$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \frac{1}{4\pi\mu|\underline{x}-\underline{y}|} \exp\left[\frac{i\omega}{c_s} |\underline{x}-\underline{y}| \right] I$$

$$= \frac{1}{4\pi\mu|\underline{x}-\underline{y}|} \delta(t-|\underline{x}-\underline{y}|/c_s)\, I .$$

Similarly inverting $G_2(\underline{x},\omega;\underline{y})$ in (16) leads to

$G_2(\underline{x},t;\underline{y})$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \; e^{-i\omega t}\left[\frac{1}{4\pi\rho\omega^2}\frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{|\underline{x}-\underline{y}|}\left(e^{i\omega|\underline{x}-\underline{y}|/c_s} - e^{i\omega|\underline{x}-\underline{y}|/c_p}\right)\right]$$

$$= \frac{1}{4\pi\rho}\frac{\partial^2}{\partial x_i \partial x_j}\left[\frac{1}{2\pi|\underline{x}-\underline{y}|}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega^2}\left[e^{i\omega|\underline{x}-\underline{y}|/c_s} - e^{i\omega|\underline{x}-\underline{y}|/c_p}\right]\right] .$$

Recognizing the ω integral as the second integral (in t) of the delta functions $\delta(t-|\underline{x}-\underline{y}|/c_s)$ and $\delta(t-|\underline{x}-\underline{y}|/c_p)$, we have

$$G_2(\underline{x},t;\underline{y}) = \frac{1}{4\pi\rho}\frac{\partial^2}{\partial x_i \partial x_j}\left[\frac{1}{|\underline{x}-\underline{y}|}\left[L(t - \frac{|\underline{x}-\underline{y}|}{c_p}) - L(t - \frac{|\underline{x}-\underline{y}|}{c_s})\right]\right] ,$$

where, if $H(t'-\tau)$ is the Heaviside function,

$$L(t-\tau) = \int_{-\infty}^{t} H(t'-\tau)\, dt' = \begin{cases} 0 , & t \le \tau \\ t-\tau , & t > \tau \end{cases} .$$

Note that if the two spatial derivatives were to be carried out on the L functions in G_2 one would obtain the delta function character similar to that in $G_1(\underline{x},t;\underline{y})$ above. Finally $G(\underline{x},t;\underline{y}) = G_1 + G_2$ produces the free space Green's function in $(\underline{x},t)$ space.

THE GREEN'S FUNCTION FOR THE LAMB PROBLEM

The traction-free boundary conditions at the $x_3 = 0$ plane can be expressed

$$B\underline{u}(x_3=0) \begin{bmatrix} \mu\partial_3 & 0 & \mu\partial_1 \\ 0 & \mu\partial_3 & \mu\partial_2 \\ \lambda\partial_1 & \lambda\partial_2 & (\lambda+2\mu)\partial_3 \end{bmatrix} \underline{u}(x_1,x_2,x_3)\Big|_{x_3=0} = \underline{0} \ , \qquad (17)$$

where $\partial_j = \partial/\partial x_j$. We are now seeking solutions to equation (3) with $f = I$ for $x_3 \geq 0$ and satisfying (17). Because of the restriction (17), we cannot now fully transform ($\underline{x} \to \underline{k}$) the problem. Hence we begin by inverting ($k_3 \to x_3$) the matrix $G(\underline{k},\omega;\underline{y})$ given by (11) and (9); this will provide the free-space portion, F, of the Green's function we seek in $(\underline{k}',x_3,\omega)$ space. Then we will work on the portion, R, defined below, that will cause (17) to be satisfied; and our final Green's function will be

$H = F - R.$

We define the free-space portion in $(\underline{k}',x_3)$-space by

$$F(\underline{k}',x_3,\omega;\underline{y}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_3 \ e^{ik_3x_3}\, G(\underline{k},\omega;\underline{y})$$

(18)

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_3 \ e^{ik_3x_3}\, e^{-i\underline{k}\cdot\underline{y}}\,\frac{1}{\rho}\left[\frac{1}{(c_p^2\underline{k}^2-\omega^2)}\,\underline{p}\underline{p}^T + \frac{1}{(c_s^2\underline{k}^2-\omega^2)}\left[\underline{h}\underline{h}^T + \underline{v}\underline{v}^T\right]\right] .$$

Since the residue theorem is to be applied it is necessary to clearly identify the poles, relative to k_3, for fixed $\underline{k}' = (k_1,k_2)$. Consider the (reciprocal of) first term involving a pole above,

$$c_p^2k^2-\omega^2 = c_p^2(k^2-\frac{\omega^2}{c_p^2}) = c_p^2(k_3^2 + k'^2-\frac{\omega^2}{c_p^2})$$

$$= c_p^2(k_3^2- k_{3p}^2) \quad ,$$

where

$$k_{3p} \equiv \mathrm{sgn}(\omega)\sqrt{\frac{\omega^2}{c_p^2} - k'^2} \tag{19}$$

and similarly,

$$c_s^2k^2- \omega^2 = c_s^2(k_3^2 - k_{3s}^2) \quad ,$$

$$k_{3s} = \mathrm{sgn}(\omega)\sqrt{\frac{\omega^2}{c_s^2} - k'^2} \quad . \tag{20}$$

The sgn(ω) will be explained below. In this notation we have

$$F(\underline{k}',x_3\omega;\underline{y}) =$$

$$\frac{e^{-i\underline{k}'\cdot\underline{y}'}}{2\pi\rho}\int_{-\infty}^{\infty} dk_3 \left[\frac{1}{c_p^2(k_3^2 - k_{3p}^2)}\,\underline{p}\underline{p}^T + \frac{1}{c_s^2(k_3^2 - k_{3s}^2)}\left[\underline{h}\underline{h}^T + \underline{v}\underline{v}^T\right]\right] \quad . \tag{21}$$

It remains to select the proper integration paths for the application of the residue theorem. There are four poles, $k_3 = \pm k_{3p}$, $\pm k_{3s}$. (Since $\underline{p} = \underline{k}/k$ and etc. for $\underline{h}$ and $\underline{v}$, there is an apparent pole for $k^2 = k'^2+k_3^2 = 0$. A little algebra shows that there is no pole there). Since we are seeking a free-space solution due to a point source (at $\underline{x} = \underline{y}$) we want our solution $F(\underline{k}', x_3, \omega; \underline{y})$ to be "outgoing" or $x_3 \rightarrow \pm \infty$. (By "outgoing" we mean the following. Should our F be inverse transformed ($\omega \rightarrow t$) one would have a composition of wave forms like $\exp[-i\omega t \pm k_{3p}(x_3 - y_3)]$; and similarly for k_{3s}. For all such forms to be outgoing, for $t > 0$, it is necessary that $\mathrm{sgn}(\pm k_{3p}(x_3 - y_3)) = \mathrm{sgn}(\omega)$ under all conditions). With the specification $\mathrm{sgn}(k_{3p}) = \mathrm{sgn}(k_{3s}) = \mathrm{sgn}(\omega)$ in (18) and (19), it remains to include in our integration path the poles in (21) by the rule:

$$\begin{aligned} &\text{for } x_3 > y_3 \,, \quad \text{include poles } k_3 = k_{3p},\ k_{3s} \ ; \\ &\text{for } x_3 < y_3 \,, \quad \text{include poles } k_3 = -k_{3p},\ -k_{3s} \ . \end{aligned} \tag{22}$$

With the integration paths determined, the residue theorem is applied to (21) producing

$$F(\underline{k}', x_3, \omega, \underline{y}) = \\ i \, \frac{e^{-i\underline{k}' \cdot \underline{y}'}}{2\rho} \left[\frac{e^{ik_{3p}|x_3 - y_3|}}{c_p^2 k_{3p}^2} \underline{\tilde{p}}\underline{\tilde{p}}^T + \frac{e^{ik_{3s}|x_3 - y_3|}}{c_s^2 k_{3s}^2} \left[\underline{\tilde{h}}\underline{\tilde{h}}^T + \underline{\tilde{v}}\underline{\tilde{v}}^T \right] \right] \tag{23}$$

where $\underline{\tilde{p}}$, $\underline{\tilde{h}}$, $\underline{\tilde{v}}$ are the p, $\underline{h}$, $\underline{v}$ vectors with k_3 evaluated at the proper pole. That is,

$$\underline{\tilde{p}} = \frac{1}{k_p} (k_1, k_2, k_{3p} \mathrm{sgn}(x_3 - y_3))^T$$

$$\underline{\tilde{h}} = \underline{h} = \frac{1}{k'} (k_2, -k_1, 0)^T$$

$$\underline{\tilde{v}} = \frac{1}{k_s k'} (k_1 k_{3s} \mathrm{sgn}(x_3 - y_3) \ , \ k_2 k_{3s} \mathrm{sgn}(x_3 - y_3) \ , \ -|k'|^2)^T$$

where $k_s = |(k_1, k_2, k_{3s})|$ and $k_p = |(k_1, k_2, k_{3p})|$. Note that since

$k_{3p} \neq k_{3s}$, the $\tilde{\underline{p}}$ and $\tilde{\underline{v}}$ vectors are not orthogonal. However, $\tilde{\underline{h}}$ remains orthogonal to $\tilde{\underline{p}}$ and $\tilde{\underline{v}}$. We will also use the notation

$$\underline{p}^{\pm} = \frac{1}{k_p} (k_1, k_2, \pm k_{3p})^T$$

$$\underline{v}^{\pm} = \frac{1}{k_s k'} (\pm k_1 k_{3s} , \pm k_2 k_{3s} , - |k'|^2)^T .$$

The wave vectors with the + are associated with the upward moving waves ($x_3 > y_3$) and those with the - with downward moving waves ($x_3 < y_3$).

Next we apply the boundary operator B to F; however, we do this in ($\underline{k}', x_3$) - space. Hence define the transform ($\underline{x}' \rightarrow \underline{k}'$) of B in (17) by

$$B(\underline{k}', x_3) = \begin{bmatrix} \mu\partial_3 & 0 & i\mu k_1 \\ 0 & \mu\partial_3 & i\mu k_2 \\ i\lambda k_1 & i\lambda k_2 & (\lambda+2\mu)\partial_3 \end{bmatrix} , \qquad (24)$$

i.e. the operator in (17) with $\partial_1 \rightarrow ik_1$, $\partial_2 \rightarrow ik_2$.

Before applying B to F, we will decide the general location of the source, i.e. $\underline{y}$. We will assume that $y_3 > 0$ (source above the $x_3 = 0$ plane). In the common $y_3 = 0$ situation, our point of view will be that $y_3 > 0$ first and then one lets $y_3 \rightarrow 0$. Hence as we apply boundary operator B to F, at $x_3 = 0$, we use the expression of F in (23) with $x_3 < y_3$. This results in $\tilde{\underline{p}} = \underline{p}^-$, etc. and produces

$$BF(x_3{=}0) = B(\underline{k}',x_3)\ F(\underline{k}',\underline{x}_3,\omega;\underline{y})\Big|_{x_3=0}$$

$$= B(\underline{k}',x_3) \left[\frac{ie^{-i\underline{k}'\cdot\underline{y}'}}{2\rho} \left[\frac{e^{-ik_{3p}(x_3-y_3)}}{c_p^2 k_{3p}^2} \underline{p}^-\underline{p}^{-T} \right.\right.$$

$$\left.\left. + \frac{e^{-ik_{3s}(x_3-y_3)}}{c_s^2 k_{3s}^2} (\underline{h}\underline{h}^T + \underline{v}^-\underline{v}^{-T}) \right] \right]_{x_3=0} \tag{25}$$

$$= \frac{ie^{-\underline{k}'\cdot\underline{y}'}}{2\rho} \left[\frac{e^{ik_{3p}y_3}}{c_p^2 k_{3p}^2} B(k',-k_{3p})\ \underline{p}^-\underline{p}^{-T} \right.$$

$$\left. + \frac{e^{ik_{3s}y_3}}{c_s^2 k_{3s}^2} B(k',-k_{3s})\ (\underline{h}\underline{h}^T + \underline{v}^-\underline{v}^{-T}) \right]$$

where $B(\underline{k},-k_{3p})$ is $B(\underline{k}',x_3)$ with the ∂_3 replaced by the appropriate value of ik_3; i.e. $-ik_{3p}$ in the first case in (25); similarly for $B(\underline{k}',-k_{3s})$.

Since F accommodates the forcing term ($\delta(\underline{x}-\underline{y})\delta(t)I$ in $(\underline{x},t)$-space) and has $BF(x_3{=}0)$ given by (25), it remains to find matrix R satisfying the homogeneous equation with boundary conditions $BR(x_3{=}0) = BF(x_3{=}0)$. Moreover, as discussed above for F, we want R to be "outgoing" as $x_3 \to +\infty$.

The homogeneous vector equation in $(\underline{k}',x_3)$ space appears

$$(L(\underline{k}',x_3) - \rho\omega^2 I)\ \underline{u} = \underline{0} \tag{26}$$

where $L(\underline{k}',\underline{x}_3)$ is $L(\underline{k})$ with ik_3 replaced by ∂_3. Equation (26) represents a sixth order ordinary differential equation in that it consists of three second order equations in x_3. Its general solution consists of linear combinations of six vector solutions; however, all outgoing solutions (as

$x_3 \to \infty$) consists of combinations of the following three vector solutions

$$\phi_1(x_3;\underline{k}') = e^{ik_{3p}x_3}\,\underline{p}^+$$

$$\phi_2(x_3;\underline{k}') = e^{ik_{3s}x_3}\,\underline{h} \tag{27}$$

$$\phi_3(x_3;k') = e^{ik_{3s}x_3}\,\underline{v}^+ \quad .$$

That these ϕ_i solve (26) follows from earlier work on F; however, it is easy to verify this directly. For example, putting $\underline{u} = \phi_1$ in (26) we have

$$(L(\underline{k}',x_3)-\rho\omega^2 I)\, e^{ik_{3p}x_3}\,\underline{p}^+$$

$$= e^{ik_{3p}x_3}\,(L(\underline{k}',k_{3p})\,\underline{p}^+-\rho\omega^2\underline{p}^+)$$

$$= e^{ik_{3p}x_3}\,((\lambda+2\mu)k^2\underline{p}^+-\rho\omega^2\underline{p}^+)$$

$$= e^{ik_{3p}x_3}\rho\,(c_p^2k^2-\omega^2)\,\underline{p}^+ = \underline{0} \quad ,$$

where we used equations (7) and (20); and in the last two lines $k^2 = k'^2+k_{3p}^2$. The key step in this verification is the first one, in which $L(\underline{k}',x_3)\exp(ik_{3p}) = \exp(ik_{3p})L(\underline{k}',k_{3p})$. This is a routine calculation. In the same manner ϕ_2 and ϕ_3 also solve (26). The other three solutions are the "incoming" (for $x_3 > 0$) solutions obtained by using $\underline{p}^-$, $\underline{v}^-$, $-k_{3p}$ and $-k_{3s}$ in (26). Since these are of no interest to us, we can express each column of $R(\underline{k}',x_3;\underline{y})$ as a linear combination of ϕ_1, ϕ_2 and ϕ_3. Thus, in matrix form we have

$$R(\underline{k}',x_3,\omega;\underline{y}) = \Phi(x_3;\underline{k}')\ C$$
$$= [\underline{\phi}_1,\underline{\phi}_2,\underline{\phi}_3]\ C \quad , \tag{28}$$

where $C = C(\underline{k}',\omega;\underline{y})$ is a "constant" matrix (relative to x_3). Our remaining task is to find the nine elements of C so that $BR(x_3{=}0) = BF(x_3{=}0)$. Of course one wants to avoid solving a system of nine equations for the c_{ij}, so we proceed with some care. We return briefy to $BF(x_3{=}0)$ in (25) and rewrite

$$BF(x_3{=}0) = \alpha_p B'(-k_{3p})\underline{p}^-\underline{p}^{-T} + \alpha_s B'(-k_{3s}) \left[\underline{h}\underline{h}^T + \underline{v}^-\underline{v}^{-T} \right] \quad , \tag{29}$$

where

$$B'(k_3) \equiv B(\underline{k}',k_3) \quad ,$$

$$\alpha_p \equiv \frac{ie^{-i\underline{k}'\cdot\underline{y}'} e^{ik_{3p}y_3}}{c_p^2 k_{3p}^2} \quad ,$$
$$\alpha_s \equiv \frac{ie^{-i\underline{k}'\cdot\underline{y}'} e^{ik_{3s}y_3}}{c_s^2 k_{3s}^2} \quad . \tag{30}$$

Referring to (27) and (28) we have

$$BR(x_3{=}0) = \left[B'(k_{3p})\underline{p}^+,\ B'(k_{3s})\underline{h},\ B'(k_{3s})\underline{v}^+ \right] C \quad . \tag{31}$$

We now anticipate equating $BF(x_3{=}0)$, in (29), to $BR(x_3{=}0)$ and work on matrix C. Write

$$C = \begin{bmatrix} \underline{c}_1^T \\ \underline{c}_2^T \\ \underline{c}_3^T \end{bmatrix} = \begin{bmatrix} \alpha_p r_{pp}\, \underline{p}^{-T} + \alpha_s (r_{hp}\, \underline{h} + r_{vp}\, \underline{v}^{-})^T \\ \alpha_p r_{ph}\, \underline{p}^{-T} + \alpha_s (r_{hh}\, \underline{h} + r_{vh}\, \underline{v}^{-})^T \\ \alpha_p r_{pv}\, \underline{p}^{-T} + \alpha_s (r_{hv}\, \underline{h} + r_{vv}\, \underline{v}^{-})^T \end{bmatrix} . \tag{32}$$

This is possible since $\underline{p}^{-}$, $\underline{h}$, $\underline{v}^{-}$ are independent vectors. (The α_p, α_s were inserted as a convenience, i.e. to cancel shortly with those in (29) above.) Note that with this form of C we obtain, using (27) and (28),

$$R(\underline{k}',x_3,\omega;\underline{y}) = \Phi(\underline{k}',x_3;\underline{y})\ C = \begin{bmatrix} e^{ik_{3p}x_3}\underline{p}^{+}, & e^{ik_{3s}x_3}\underline{h}, & e^{ik_{3s}x_3}\underline{v}^{+} \end{bmatrix} C \tag{33}$$

$$= e^{ik_{3p}x_3}\ \underline{p}^{+}\left[\ \alpha_p r_{pp}\ \underline{p}^{-} + \alpha_s (r_{hp}\ \underline{h} + r_{vp}\ \underline{v}^{-})\ \right]^T$$

$$+\ e^{ik_{3s}x_3}\ \underline{h}\left[\ \alpha_p r_{ph}\ \underline{p}^{-} + \alpha_s (r_{hh}\ \underline{h} + r_{vh}\ \underline{v}^{-})\ \right]^T$$

$$+\ e^{ik_{3s}x_3}\ \underline{v}^{+}\left[\ \alpha_p r_{pv}\ \underline{p}^{-} + \alpha_s (r_{hv}\ \underline{h} + r_{vv}\ \underline{v}^{-})\ \right]^T ,$$

where α_p and α_s are defined in (30) above. This follows from the algebra fact 4. above.

It remains to calculate the nine reflection coefficents. The computational advantage to the form in (33) is that it expedites the evaluation of the unknown r's. We now refer to $BR(x_3=0)$ in (31) where C is defined in (32) and equate the result to $BF(x_3=0)$ in (29). In particular, equating coefficients of $\underline{p}^{-T}$, $\underline{h}^T$, $\underline{v}^{-T}$ leads to these three 3 by 3 linear systems:

$$B'(-k_{3p})\underline{p}^- = r_{pp}B'(k_{3p})\underline{p}^+ + B'(k_{3s})\left[r_{ph}\,\underline{h} + r_{pv}\,\underline{v}^+\right] \quad (34a)$$

$$B'(-k_{3s})\underline{h} = r_{hp}B'(k_{3p})\underline{p}^+ + B'(k_{3s})\left[r_{hh}\,\underline{h} + r_{hv}\,\underline{v}^+\right] \quad (34b)$$

$$B'(-k_{3s})\underline{v}^- = r_{vp}B'(k_{3p})\underline{p}^+ + B'(k_{3s})\left[r_{vh}\,\underline{h} + r_{vv}\,\underline{v}^+\right] . \quad (34c)$$

Note in particular that the three systems (34) can each be solved independently for the unknowns (e.g. for r_{pp}, r_{ph}, r_{pv} in (34a)). Also note that the three systems have the same 3 by 3 matrix of coefficients,

$$A = \left[B'(k_{3p})\underline{p}^+ \ , \ B'(k_{3s})\underline{h} \ , \ B'(k_{3s})\underline{v}^+ \right] . \quad (35)$$

This fact can of course be used in solving for the r's by, e.g., inverting A and multiplying the left sides of (34) by A^{-1}. The terms in A come from evaluating the matrices $B'(\cdot) = B(\underline{k}',\cdot)$ and vectors $\underline{p}^+$ and $\underline{v}^+$ at the values of k_{3p} and k_{3s} given by (19) and (20). Similarly the quantities in the left sides of (34) are evaluated at $-k_{3p}$ and $-k_{3s}$.

When the three systems in (34) are solved we obtain

$$\begin{aligned}
r_{pp} &= -D'/D , & r_{ph} &= 0 , & r_{pv} &= -4\frac{k_{3p}k'k_s}{k_p}(k_{3s}^2-k'^2)/D, \\
r_{hp} &= 0 , & r_{hh} &= -1 , & r_{hv} &= 0 , \\
r_{vp} &= \frac{4k'k_{3s}k_p}{k_s}(k_{3s}^2-k'^2)/D , & r_{vh} &= 0 , & r_{vv} &= -D'/D ,
\end{aligned} \quad (36)$$

where

$$D = 4k_{3p}k_{3s}k'^2 + (k_{3s}^2-k'^2)^2 ,$$

$$D' = 4k_{3p}k_{3s}k'^2 - (k_{3s}^2-k'^2)^2 .$$

Notice in (36) the various aspects of decoupling that takes place in the reflection matrix R. For example, the middle three equations in (36) state the well known fact that an incidence h (i.e. SH) wave gives rise (in this setting) only to a reflected h wave. And similarly, incident p and v waves only produce p and v components.

In summary, in $(\underline{k}', x_3, \omega)$-space, the Green's function for the Lamb problem can be expressed

$$H(\underline{k}', x_3, \omega; \underline{y}) = F(\underline{k}', x_3, \omega; \underline{y}) - R(\underline{k}', x_3, \omega; \underline{y}) \tag{37}$$

where F is given by (23) and R by (33). In some applications this form of H is adequate, while on other occasions it is preferred to have H in $(\underline{x}, t)$space. Unfortunately, a complete closed form inversion back to $(\underline{x}, t)$ has not been accomplished. Perhaps the most successful attempt has been the ingenious methods due to Cagniard and deHoop [5], in which the inversion $(\underline{k}' \to \underline{x}')$ can be reduced to a single (finite) integral representation for $H(\underline{x}, t; \underline{y})$. Illustrations of this technique can be found, e.g., in Aki and Richards [1].

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