— Notes —

for an Introductory Course in

Theoretical Seismology

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Contents

1 Introduction to Continuum Mechanics **5**

1.1 The Continuum Postulate 6

1.2 The Goal of Continuum Mechanics (in Geophysics) . . . 7

1.3 Decomposing Newton's Second Law 11

2 Force, Traction, and Stress **15**

2.1 Whence Stress 16

2.2 There is a Stress Tensor 19

2.3 Invariants 25

3 Displacement, Deformation, and Strain **29**

3.1 Whence Strain 30

3.2 Some Properties of Strain 33

3.3 Compatibility 34

4 Constitutive Relations **35**

5 Mathematical Preliminaries **39**

5.1 Gauss Theorem (A Review) 40

5.2 Flux Across a Surface 42

5.3 Kinematics: Material and Spatial Descriptions 43

5.4 Material Derivatives 45

6 Conservation Relations **47**

6.1 Conservation of Mass 48

6.2 More Conservation of Mass 49

6.3 Reynold's Transport Theorem 50

6.4 Conservation of Linear Momentum 52

7 Equations of Motion for the Earth **55**

8 Adiabatic versus Isothermal **61**

9 Elastic Wave Equations **63**

9.1 Gravity *versus* Elasticity 64

10 Wave Equations **67**

10.1 The Elastic Wave Equation 68

10.2 Scalar Wave Equations 71

11 Traveling Waves **73**

11.1 Particular Forms 74

11.2 The Sommerfeld Radiation Condition 76

11.3 Reflection 77

11.4 Cartesian Harmonic Waves 80

11.5 Phase Velocity, Group Velocity, and Dispersion 81

12 Ray Theory 87

12.1 The Eikonal Equation 88

12.2 Rays 91

12.3 Fermat's Principle and Other Topics 93

13 Normal Modes 99

13.1 Particulars 100

13.2 Generalities and Rayleigh's Principle 110

Chapter 1

Introduction to Continuum Mechanics

Comfortable physical quantities such as *force*, *acceleration*, *etc.*, take on new forms when we describe the physics of a continuous medium. The simple equations of Newtonian dynamics acquire a complexity that is at the least annoying, and at its worst threatening, to the scholar. This complexity is intrinsic — no simple modification of the mathematics will banish it — and the best we can do is to try to keep it manageable.

One way to manage complexity is to keep a broad view of what the mathematics is really trying to do. This section is devoted to a qualitative forecast of where the several sections after this are going to take us, and why we have to go to those places.

1.1 The Continuum Postulate

The philosophical foundation of continuum mechanics is the *continuum postulate*. Some useful forms of the continuum postulate are:

- *The properties of matter can be described by piecewise continuous differentiable functions.*
- *A piece of matter can be subdivided an arbitrary number of times and still remain a recognizable piece of the same type of matter.*
- *It is alright to use differential calculus to study macroscopic properties.*

The continuum postulate is rigorously wrong but practically useful. Like Newtonian mechanics, it is often an excellent approximation.

There's never a continuum postulate around when you need one:

The fact that this puffy little section comes at the start of the notes does not mean that the Continuum Postulate preceded the development of continuum mechanics. Fundamental claims, such as the Continuum Postulate, are not made until a field is known to be successful and clean-up operations commence.

When your only tool is a hammer, everything looks like a nail: [1]

Suppose that we rejected the continuum postulate. The only other possibility I can think of is to model a chunk of matter as a (very, very) large number of interacting atoms. Is there any way of extracting useful information from such an approach which does not, one way or another, slide into continuus mathematics?

[1] Quotation stolen from someone at Bell Labs.

1.2 The Goal of Continuum Mechanics (in Geophysics)

We want to know how a particular material body (such as the Earth, a rubber ball, the state of California, *etc.*) responds to a given disturbance. Here are a few examples:

- Calculate the depression and uplift history of central Canada given the history of glacial loading during the most recent ice age.
- Find the rate at which the Chandler Wobble, one of the free motions of a rotating Earth, decays because of energy dissipated by deformation of the Earth.
- Compute the motion of the ground surface in Los Angeles after a large earthquake.

We approach these problems by seeking a way to apply Newton's Second Law, which we usually call "the conservation of linear momentum," to continua.

——— *Applying Newton's Second Law* ———

We can express the conservation of linear momentum as a vector equation which tells us that the rate of change of a body's momentum is equal to the net force applied to the body:

$$\mathbf{F} = d_t\,(M\mathbf{v}) \tag{1.1}$$

Here $\mathbf{F}$ is the total instantaneous force applied to some body,
M is the body's mass,
$\mathbf{V}$ is the instantaneous velocity of the body's center of mass,
and d_t is a convenient notation for time differentiation.

When we think about applying equation (1.1), we often think about applying it to rigid bodies. Rigid bodies are a nice tractable case because they always consist of the same set of atoms and those atoms bear a fixed relationship to one another.

Deformable bodies are more complicated because we have to be careful about specifying the particular volume or set of particles to which equation (1.1) applies, and how that domain may change with time. Sometimes we want to think about a fixed volume in space (which may be occupied by different sets of material particles at different times): we may want to predict the temperature measured at a thermometer on a fixed stand in the atmosphere as the winds blow around it. Sometimes we want to think about a particular set of material particles (which may occupy different volumes in space at different times): suppose we want to follow the chemical evolution of a puff of smoke as it is carried along by wind.

Later on we will give this topic substantial attention.

Newton's Second Law, equation (1.1), tells us how to compute the change in a body's linear momentum from knowledge of the applied force. If we know all the forces, all we would have to do is integrate over time once or twice and we would be finished.

In many problems of interest, we don't know all of the forces (at least not until the problem has been solved). What we have instead are one or more additional relations that tell us how to compute components of the total force that vary with the object's motion. This is particularly likely to be the case when we are studying the behavior of a system which has been disturbed away from a state of stable equilibrium. This point is both trivially obvious and extremely important. and I am going to belabor it.

Some examples are:

- the linear (phenomenological) relation between the extension of an ideal spring and the force it exerts,

$$f = -kx$$

- the relationship between the angle a simple pendulum makes with the local vertical and the component of gravitational force which acts to restore the pendulum to a vertical orientation,

$$\begin{aligned} f &= -mg\sin(\theta) \\ f &\approx -mg\theta \text{ as } \theta \to 0 \end{aligned}$$

These additional relations are often (but not always) phenomenological. They are often (but not always) linear. They *always* represent additional physics, physics which is not a simple consequence of the conservation of momentum. They usually (*always?*) represent forces which act to return the system to an equilibrium state.

One of the reasons I'm beating this point to death is that the principal point of the next few chapters (which deal with stress, strain, and consitutive relations) is to provide the "additional relations" needed for continuum mechanics. The path through all this is long (but not too hard) and it may help to remember the reason for taking it.

Another reason for all this harping is that the equations of motion for a continuous medium are relatively complicated. Try to remember that those complex equations are almost always just particular examples of these underlying forms: *Put this in a box:*

$$d_t\,(\text{momentum}) = \text{total force}$$

$$\text{total force} = \text{some function of}\,(\text{deviation from equilibrium})$$

Even though each of these elements may consist of several messy-looking expressions, the relationships among these elements are, at bottom, very simple.

Let's complete the problem of a point mass attached to a spring. We suppose that it is a particle of mass m, constrained to move only in the x direction, and bound to the origin ($x = 0$) by an ideal spring. Conservation of linear momentum demands

$$md_t^2 x = f \tag{1.2}$$

where f is the force exerted by the spring, Combining this with the earlier expression for the force exerted by an ideal spring as a function of its extension x,

$$md_t^2 x = -kx$$

This is a second-order ordinary differential equation for $x(t)$. The solution to this equation, and the object's motion, is just simple harmonic motion. *The general solution is*

$$x = A\sin(\omega t) + B\cos(\omega t)$$

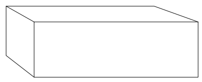
where $\omega = \sqrt{k/m}$ and A and B are determined from the object's initial conditions.

1.3 Decomposing Newton's Second Law

We want to characterize the rate-of-change of linear momentum (that is, $M\mathbf{A}$) and the totality of the applied forces (that is, $\mathbf{F}$) in a way which is both comprehensible and productive when applied to a continuum. Before we jump into the detailed mathematics that are the heart of this quest, we want to take a brief look at what is going to happen. This section is devoted to that look.

——— *Another Little Block of Stuff* ———

Consider a tiny piece of matter, say a little rectangular solid with infinitesimal sides dx, dy, and dz :



Obviously $M\mathbf{A}$ for this tiny block is just its total mass times its acceleration.

Of course, if different parts were to accelerate at appreciably different rates, we would have to make much more elaborate calculations. We avoid this difficulty by making the box smaller and smaller until the difference in acceleration between its parts is too small to worry about.

Now let's consider all of the ways in which the rest of the universe can act upon our little block. We claim that all such interactions fall into one of these two categories:

- body forces, and
- surface forces.

——— *Body Forces* ———

Body forces are forces exerted upon our little block by interaction between the body itself and some sort of field. Some of the common examples of body forces are

1. electromagnetic forces arising from the interaction between electrical charge in our little body and electromagnetic fields generated elsewhere in the universe.
2. gravitational forces due to interaction between our little body's mass (its *gravitational charge*) and the gravitational field due to all of the other masses in the universe.

Body forces are familiar to us from elementary physics. Their role is continuum mechanics is *mostly* straightforward.

Body forces are always proportional to the strength of some property of the material medium. (Gravitational forces are proprotional to the material's density.) The total body force acting on a small volume element is proportional to the element's volume — it is proportional to the amount *of some property contained in the volume. (The total gravitational force exerted on some little piece of matter is proportional to the total mass contained in the little piece.) Body forces vanish when the volume of the little mass goes to zero.*

——— *Surface Forces* ———

There will also be forces which act across the surface of our little block. These occur because our block is part of a continuum and, usually, is an integral glued-in part of a greater macroscopic body. In general these forces

- are different on each faces of the block, and
- are not necessarily perpendicular to the surface across which they act.

We will develop the notions of *traction* and *stress* to deal with these forces.

Although some numerical techniques actually solve continuum problems by adding up the effect of lots of little blocks acting on each other, we will have to develop some theoretical methods that are more attuned to human use.

Stress — Specifying Force in a Continuum

We are going to create a quantity named *stress* that characterizes all of the forces acting across all of the possible surfaces at each point in the continuum. Stress will be represented by a *second-order tensor*.

The stress in a continuum is always the result of efforts to deform the material away from its preferred shape (or *natural state*). It is easy to show that displacement alone is not sufficient to cause stress: intuition tells us that neither uniform translation nor rigid rotation should cause stress to appear, even though both of these actions produce distinct displacements.

Deformation

What we need is some way to characterize *distortion*, that is displacements that act to change a body's *shape*. The crucial difference between displacements which deform a body, and those such as uniform translation and rigid rotation which do not, is that the former (the "deformers") give rise to changes in the *distances* between different points in the medium.

It might help to imagine that a material body's configuration is specified by the geometry of the atomic bonds that hold the body together. Uniform translations and rigid rotations do nothing to disturb that preferred geometry. Squeezing the body, however, tends to shorten some bonds, and perhaps lengthen others, and the resistance of the bonds to this deformation produces stress.

——— *Deformation: Strain* ———

It will turn out that the quantity we want is a second-order tensor which we call *strain*. Strain, as we willl see later, is computed directly from displacement.

——— *Constitutive Relation* ———

The precise connection between stress and strain (which is always a function that converts one into the other) is called the *constitutive relation*. There is a thick forest, a veritable universe, of constitutive relations. Some are linear and some are non-linear; some are pointwise in time and some have memory; *etc.*

——— *Constitutive Relation: Linear Elasticity* ———

The only consitutive relation we will consider here is the *linearly elastic* one. In this rheology the stress at a moment in time and at a point in space depends linearly upon the strain at that same instant and place.

——— *Closure* ———

Continuum mechanics is built from a number of distinct pieces. For us, for now, those pieces are:

$$
\begin{array}{lcl}
\text{displacement} & \Longrightarrow & \text{strain} \\
\text{strain} & \Longrightarrow & \text{stress} \\
\text{stress} & \Longrightarrow & \text{force} \\
\text{force} & \Longrightarrow & \text{acceleration} \\
\text{acceleration} & \Longrightarrow & \text{displacement}
\end{array}
$$

Chapter 2

Force, Traction, and Stress

Usefully specifying the forces acting internally in a continuum is the single most challenging aspect of continuum mechanics. A demonstration due to Cauchy proves that all of the internal forces can be specified in terms of a single second-order symmetric tensor.

2.1 Whence Stress

In "normal" mechanics we can simply enumerate all of the forces acting on a given object. Computing the *total* force, then, is simply a matter of summing all of the individual forces.

The forces acting on a lunar orbiter are, to a good approximation, just the gravitational forces due to the Moon, the Earth, and the Sun acting on the orbiter's center of mass. Knowing those three force vectors amounts to knowing all of the "force" side of Newton's Second Law.

——— *Newton's Second Law Looks Funny* ———

Continuum mechanics is trickier. The total force acting on a little material blob is clearly the sum of all of the body forces acting on points in the blob *plus* the sum of all of the surface forces acting across the blob's surface.[1] Newton's Second Law, then, will look about like this:

$$D_t \int_{\mathcal{B}} \rho \mathbf{v} \, dv = \int_{\mathcal{B}} \mathbf{f}_{\mathrm{b}} \, dv + \int_{\partial\mathcal{B}} \mathbf{f}_{\mathrm{s}} \, ds \tag{2.1}$$

where $\mathcal{B}$ is the volume of our little blob,
$\partial\mathcal{B}$ is its surface,
$\mathbf{f}_{\mathrm{b}}$ is the total body force (in units of *force/volume*), and
$\mathbf{f}_{\mathrm{s}}$ is the total surface force (in units of *force/area*).

Equation (2.1) is not in a useful form. It tells us about the relationships among various volume and surface averages in a continuum but it does not, in general, tells us how to predict acceleration, given knowledge of the force state in a body.

[1] We assume that the blob is simply-connected, *etc*. This assumption costs us no generality and saves us from some grief. Not all gift horses are packed with Greeks.

Making Newton Work in a Continuum

The saving grace of equation (2.1) is, as you probably know, that it is true for all possible choices of blobs over which we integrate. In particular it is true for all very, very tiny blobs, and we can use that to turn the integral equation into a differential equation which is just what we want.

The way to achieve this is to recast the equation so that it consists purely of volume integrals. If we can reduce it to an equation of the form

$$\int_{\mathcal{B}} \text{AThing}\; dv = \int_{\mathcal{B}} \text{AnotherThing}\; dv$$

and which must be true for all choices of blob, $\mathcal{B}$, then we can let $\mathcal{B}$ shrink to a point and infer that

$$\text{AThing} = \text{AnotherThing}$$

everywhere in the continuum. Usually *AThing* and *AnotherThing* will involve various differential operators and the point-wise equation will be a differential equation.

Our manipulation took us from the integral form of a conservation relation (Newton's Second Law expresses the conservation of momentum) to a differential equation. This strategy is used more than once in continuum mechanics and it's worth pondering a bit.

Both the integral and differential relations express the same physics; in the present case that physics is: linear momentum is conserved*. Changing the* way *we express the physics is very helpful, but we should bear in mind that it does not add* new *physics. Problems which were incompletely or inconsistently specified before we waved our arms will remain so after the smoke has cleared.*

It is not in the least difficult to lose sight of the physics when we are hacking through the mathematics.

Two of the three terms in equation (2.1) are already in the form of volume integrals. The third term, however, is

$$\int_{\partial\mathcal{B}} \mathbf{f}_{\mathrm{S}}\; ds$$

which is a *surface* integral. That's not good because in general knowing that

$$\int_{\mathcal{B}} \text{AThing}\; dv = \int_{\partial\mathcal{B}} \text{ASurfaceThing}\; ds$$

does not, in the general case, allows us to infer any point-wise relationship between *AThing* and *ASurfaceThing*.

In order to get what we want, that is in order to recast our physics into differential form, we must convert this term into a volume integral. If the surface force distribution, $\mathbf{f}_\mathrm{S}$ is simply an unknown function of position, then there is nothing we can do about making the surface integral into a volume integral and these notes should stop here.

——— *Here is What Comes Next:* ———

There is, however, a way to achieve what we want and it centers on a quantity named "stress." The rest of this section is devoted to this topic, but here is the short form on how it works.

Using a simple, geometric proof, we will show that in any continuum there is a second-order symmetric tensor, $\mathbf{T}$, which we name "stress" and which has the property that the force acting across a surface with unit normal vector $\hat{\mathbf{n}}$ has the value

$$\mathbf{f}_\mathrm{S} = \mathbf{T} \cdot \hat{\mathbf{n}} \tag{2.2}$$

Using this expression in our earlier troublesome surface integral,

$$\begin{aligned} \int_{\partial\mathcal{B}} \mathbf{f}_\mathrm{S}\; ds &= \int_{\partial\mathcal{B}} \mathbf{T} \cdot \hat{\mathbf{n}}\; ds \\ &= \int_{\mathcal{B}} \nabla \cdot \mathbf{T}\; dv \end{aligned}$$

which is what we wanted.

I think that equation (2.2) is the single most powerful result in continuum mechanics. It is not at all obvious, at least to me, that we can expect a such a quantity to exist.

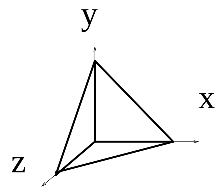
2.2 There is a Stress Tensor

A *tetrahedron* is a three-dimensional object with four plane sides. It does not have any other general regularity.

In this section we're going to examine the forces acting on a particular, slightly specialized tetrahedron and from that examination infer the existence of a stress tensor.

Cauchy's Tetrahedron

Consider a little tetrahedral piece of matter built from three sides that are mutually orthogonal and a fourth side that can be any plane which is not parallel to any of the first three sides. To be definite, let us suppose that this fourht plane intersects each of the coordinate axes at a positive intercept. Create a cartesian coordinate frame aligned so that the three mutually orthogonal faces have the *outward* unit normal vectors $-\hat{\mathbf{x}}$, $-\hat{\mathbf{y}}$, and $-\hat{\mathbf{z}}$. It should all look somewhat like this:



Geometric Relationships Among the Faces

Let $\hat{\mathbf{n}}$ be the outward unit normal for the remaining face. Since $\hat{\mathbf{n}}$ is a vector in 3-space, there must be three real numbers, a_1, a_2, and a_3, such that

$$\hat{\mathbf{n}} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}} \tag{2.3}$$

Since all of $\hat{\mathbf{n}}$, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ have unit length, then we must have

$$a_x^2 + a_y^2 + a_z^2 = 1$$

Let $\Delta S_x \ldots \Delta S_z$ be the areas of the three mutually orthogonal faces, and let ΔS_n be the area of the fourth face. It is easy to show that

$$\Delta S_x = |a_x| \Delta S_n \tag{2.4}$$

etc.

Well, maybe it's not absolutely obvious. If we form the dot product of equation (2.3) with, for example, $\hat{\mathbf{x}}$*, we end up with*

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{n}} = a_x$$

This says that a_x *is equal to the cosine of the angle between* $\hat{\mathbf{n}}$ *and* $\hat{\mathbf{x}}$*. The relationship between the areas of the two faces follows directly. (The three numbers* a_x*,* a_y*, and* a_z *are sometimes called the* direction cosines *of* $\hat{\mathbf{n}}$*.)*

Naming the Forces

Let $\hat{\mathbf{f}}_x, \ldots, \hat{\mathbf{f}}_z$ be the *net force* on the three mutually orthogonal faces, and let $\hat{\mathbf{f}}_n$ be the net force on the fourth face. Apart from body forces, these forces acting across the tetrahedron's outer surface are the sole means by which the tetrahedron interacts with the surrounding continuum. Recall that the force acting across a face is not necessarily normal to that face.

Traction

Traction is the force acting per unit area and it is given by

$$\mathbf{t}_x = \mathbf{f}_x / \Delta S_x \tag{2.5}$$

etc. Traction is more useful than force because if we shrink the size of the tetrahedron to zero, both the forces, $\mathbf{f}_x, \ldots$, and the surface areas, $\Delta S_x, \ldots$, will vanish but the tractions, $\mathbf{t}_x, \ldots$, will not. The $\mathbf{f}$'s reflect both the state of the continuum and the size of the little tetrahedron, while the $\mathbf{t}$'s reflect only the state of the continuum.

— *Newton's Second Law* —

Now let's apply Newton's second law,

$$d_t\,(M\mathbf{v}) = \mathbf{F}\ ,$$

to our tetrahedron. Let

ΔV	be the tetrahedron's volume,
ρ	its density,
$\mathbf{v}$	its velocity, and
$\mathbf{g}$	the total acceleration due to body forces.

We suppose that the little body is so small that its density, velocity, and body forces are sensibly constant across the body. Newton's law takes the form:

$$(\rho\Delta V)\partial_t\mathbf{v} = (\rho\Delta V)\mathbf{g} + \mathbf{f}_x + \mathbf{f}_y + \mathbf{f}_z + \mathbf{f}_n$$

This expression is quite straightforward. The left-hand side is the total mass of the little body times its acceleration, while the right-hand side is just the sum of all of the forces acting on it. Rewriting this with equations (2.4) and (2.5) gives

$$\left(\frac{\rho\Delta V}{\Delta S_n}\right)(\partial_t\mathbf{v} - \mathbf{g}) = a_x\mathbf{t}_x + a_y\mathbf{t}_y + a_z\mathbf{t}_z + \mathbf{t}_n.$$

Now consider the limit when we shrink the tetrahedron to a point.

— *Make the Tetrahedron Vanish...* —

To be specific, suppose that we do this by uniformly scrunching all of the tetrahedron's dimensions together. The tractions remain unaltered while the left-hand side vanishes because

$$\frac{\Delta V}{\Delta S_n} \propto \text{ the height, say, of the tetrahedron}$$

so

$$\frac{\Delta V}{\Delta S_n} \to 0 \text{ as height} \to 0.$$

——— ... *and Show the Tractions are not Independent* ———

Then

$$\mathbf{t}_n = -a_x\mathbf{t}_x - a_y\mathbf{t}_y - a_z\mathbf{t}_z$$

or

$$\mathbf{t}_n = -(\hat{\mathbf{n}}\cdot\hat{\mathbf{x}})\mathbf{t}_x - (\hat{\mathbf{n}}\cdot\hat{\mathbf{y}})\mathbf{t}_y - (\hat{\mathbf{n}}\cdot\hat{\mathbf{z}})\mathbf{t}_z$$

or

$$\mathbf{t}_n = -\hat{\mathbf{n}}\cdot\{\hat{\mathbf{x}}\mathbf{t}_x + \hat{\mathbf{y}}\mathbf{t}_y + \hat{\mathbf{z}}\mathbf{t}_z\}$$

This is a truly profound result. From Newton's Second Law of Motion and the scale dependence of the ratio of volume to surface area, it follows that if we know the tractions on three orthogonal surfaces at some point in a continuum, then we can compute the traction on any surface (at that same point) by knowing only the normal to that surface. Note that this result did not depend upon the little tetrahedral chunk being stationary; its instantaneous velocity $\mathbf{v}$ was arbitrary (but finite, of course).

——— *At Last: the Stress Tensor* ———

The quantity $\{-\hat{\mathbf{x}}\mathbf{t}_x - \ldots\}$ is a linear operator which takes a vector ($\hat{\mathbf{n}}$) and turns it into another vector ($\mathbf{t}$). Thus this quantity, which we denote by

$$\mathbf{T} = -\hat{\mathbf{x}}\mathbf{t}_x - \hat{\mathbf{y}}\mathbf{t}_y - \hat{\mathbf{z}}\mathbf{t}_z$$

is a second-order tensor.

This is a particular, and very handy, representation of a second-order tensor called a dyadic. *It is a sum of* dyads. *If you don't know what dyads and dyadics are, this is a good time to go find out. A good place to look is Chapter 1 of* **??**.

We call $\mathbf{T}$ the *stress tensor* (more properly the "Cauchy stress tensor"). (Note that we could eliminate all of the minus signs by selecting tractions for orthogonal surfaces having outward normals parallel to the coordinate vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$, rather than antiparallel.)

In any particular coordinate system we can always represent $\mathbf{T}$ as a collection of numbers T_{ij}, where i, j range over 1, 2, 3. ("T_{ij}" is usually used to refer specifically to a local *cartesian* or x, y, z coordinate system.) Then the plane with outward normal $\{n_i\}$ experiences the traction $\{t_i\}$ given by

$$t_i = T_{ij} n_j$$

or more elegantly

$$\mathbf{t} = \mathbf{T} \cdot \hat{\mathbf{n}}.$$

The Stress Tensor is Symmetric

Further, in the absence of local torque or couple-stress distributions (a virtually universal geophysical assumption), we can show that $\mathbf{T}$ is a symmetric tensor:

$$\mathbf{T} = \mathbf{T}^{\mathrm{T}} \quad \text{or} \quad T_{ij} = T_{ji}.$$

This can be done in several ways. The simplest is by a line of reasoning analogous to that used with the Cauchy tetrahedron. Consider a thin slice of material with a square cross-section and normal $\hat{\mathbf{z}}$.

A Figure

The torque about $\hat{\mathbf{z}}$ due to tractions on the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ faces is clearly[2]

$$\text{Torque} \propto \delta(T_{21} - T_{12}).$$

This, of course, must equal the rate of change of angular momentum. If we let $\delta \to 0$, the angular momentum term vanishes like δ^2 while the torque only vanishes like δ. Thus we are left with, in the limit,

$$T_{21} = T_{12}.$$

Similarly,

$$\begin{aligned} T_{13} &= T_{31} \\ T_{23} &= T_{32}. \end{aligned}$$

Thus there are only *six* independent elements in T_{ij}

$$\mathbf{T} = \left\{ \begin{array}{ccc} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{array} \right\}$$

Note that this result is dynamically true. We only require that the velocity be finite.

[2]In maneuvers like this we suppose stress to be essentially constant across our little test body. We can get away with this because (i) we can start with δ as small as we like and (ii) if stress is continuous there must be a non-zero delta which is small enough that the change in stress across the body is less than any desired amount. If you don't believe me go ask a mathematician.

2.3 Invariants

The values of the components T_{ij} of some stress tensor $\mathbf{T}$ depend, of course, on the particular cartesian coordinate system in which they are evaluated. There are also a number of coordinate-independent ways by which we can characterize $\mathbf{T}$ (or any other symmetric second-order tensor) and we will summarize some of those here.

Since $\mathbf{T}$ is real and symmetric (and therefor Hermitian) we can always find three orthogonal unit vectors $\hat{\alpha}^1$, $\hat{\alpha}^2$, and $\hat{\alpha}^3$ such that

$$\hat{\alpha}^i \cdot \hat{\alpha}^j = \delta_{ij}$$

and three real numbers, $\lambda_1, \lambda_2, \lambda_3$ such that

$$\mathbf{T} \cdot \hat{\alpha}^i = \lambda_i \hat{\alpha}^i \quad (i \text{ not summed}).$$

Thus the traction across a surface with normal $\hat{\alpha}^i$ is normal to the surface and of magnitude λ_i.

The λ_i are the *principal stresses.* A sometimes convenient representation of the stress tensor is

$$\mathbf{T} = \lambda_1 \hat{\alpha}^1 \hat{\alpha}^1 + \lambda_2 \hat{\alpha}^2 \hat{\alpha}^2 + \lambda_3 \hat{\alpha}^3 \hat{\alpha}^3.$$

The largest λ_i is called the *greatest principal stress*, and the smallest is called the *least principal stress.* It is a well-known result of linear algebra, and you can easily prove, that

$$\mathrm{tr}(\mathbf{T}) = \lambda_1 + \lambda_2 + \lambda_3$$

where $\mathrm{tr}(\cdot)$ means "trace of $\cdot$".

Don't let statements like "you can easily prove" intimidate you. If he's so damn smart, why isn't he rich?

Invariants are, usually, scalar functions of a tensor's components which are coordinate independent. For example,

$$\mathrm{tr}(\mathbf{T}) = T_{ii} \;\text{ is an invariant}$$

$$T_{13} \;\text{ is not.}$$

The stress tensor has three principal invariants you should be aware of. In order to crack them out, consider the eigenvalue equation for the principal stresses:

$$\mathbf{T} \cdot \hat{\alpha} = \lambda \hat{\alpha}.$$

As you (should) know, λ must be a solution to

$$\det\left(\mathbf{T} - \lambda \mathbf{T}\right) = 0$$

or, in some cartesian coordinate systems

$$\begin{vmatrix} \mathbf{T}_{11} - \lambda & \mathbf{T}_{12} & \mathbf{T}_{13} \\ \mathbf{T}_{12} & \mathbf{T}_{22} - \lambda & \mathbf{T}_{23} \\ \mathbf{T}_{13} & \mathbf{T}_{23} & \mathbf{T}_{33} - \lambda \end{vmatrix} = 0.$$

This leads to a cubic for λ of the form

$$\lambda^3 - \mathrm{I_T}\lambda^2 - \mathrm{II_T}\lambda - \mathrm{III_T} - 0$$

where

$$\begin{aligned} \mathrm{I_T} &= T_{11} + T_{22} + T_{33} = \mathrm{tr}(\mathbf{T}) \\ \mathrm{II_T} &= \frac{1}{2}(T_{ij}T_{ij} - T_{ii}T_{jj}) = \frac{1}{2}(\mathbf{T} : \mathbf{T} - \mathrm{tr}(\mathbf{T})^2) \\ \mathrm{III_T} &= \det\left(\mathbf{T}\right). \end{aligned}$$

The coefficients of the cubic uniquely determine its roots. The roots (λ_i) are frame independent and thus the coefficients must also be. Thus they are invariants. You can show

$$\begin{aligned} \mathrm{I_T} &= \lambda_1 + \lambda_2 + \lambda_3 \\ \mathrm{II_T} &= -\lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_3\lambda_1 \\ \mathrm{III_T} &= \lambda_1\lambda_2\lambda_3. \end{aligned}$$

another way to demonstrate invariance.

Define

$$p = -\frac{1}{3}\mathrm{tr}(\mathbf{T}).$$

The *spherical stress* is defined to be

$$\mathbf{T}_{\mathrm{SPH}} = -p\mathbf{T}$$

and the *stress deviator* is

$$\mathbf{T}_D = \mathbf{T} - \mathbf{T}_{\mathrm{SPH}}.$$

Clearly $\mathbf{T}_D$ has a vanishing trace. In hydrostatic equilibrium the deviatoric stresses vanish.

Chapter 3

Displacement, Deformation, and Strain

3.1 Whence Strain

Let $\mathcal{V}_0$ denote the volume occupied by some piece of matter in its *equilibrium* or *reference* state. Let $\mathbf{x}$ denote position in $\mathcal{V}_0$. Suppose that something happens to $\mathcal{V}_0$ so that the particle which *was* as $\mathbf{x}$ moves to a new position

$$\mathbf{r} = \mathbf{x} + \mathbf{s}.$$

We call $\mathbf{s}$ the "displacement" of the particle initially at $\mathbf{x}$. In general the displacement $\mathbf{s}$ may be different for different particles so we have

$$\mathbf{r}(\mathbf{x}) = \mathbf{x} + \mathbf{s}(\mathbf{x}).$$

Clearly we can regard $\mathbf{x}$ as being a "label" attached uniquely to each material particle, as well as being a vector in three-dimensional space. The vector-valued function $\mathbf{r}(\mathbf{x})$ maps $\mathcal{V}_0$ into a new volume $\mathcal{V}_1$ which we may regard as the "disturbed state" of the material.

Our goal now is to characterize those properties of $\mathbf{s}(\mathbf{x})$ which produce stress in the material. If $\mathbf{s}(\mathbf{x})$ is a rigid-body translation

$$\mathbf{s}(\mathbf{x}) = \mathbf{s}_0, \quad \mathbf{s}_0 \text{ a constant vector}$$

or an infinitesimal rigid-body rotation

$$\mathbf{s}(\mathbf{x}) = \Theta \times \mathbf{x}, \quad \Theta \text{ a constant vector}$$

intuition assures us that stresses will *not* be generated. What *does* generate stress is the change in separation between two material particles.

To be more specific let $\mathbf{x}$ and $\mathbf{x} + \Delta\mathbf{x}$ be two particles in $\mathcal{V}_0$. The squared distance between them is

$$ds_0^2 = |\mathbf{x} + \Delta\mathbf{x}|^2 = (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2.$$

The corresponding point in $\mathcal{V}_1$ are

$$\mathbf{r}(\mathbf{x}) = \mathbf{x} + \mathbf{s}(\mathbf{x})$$

and

$$\begin{aligned}\mathbf{r}(\mathbf{x}+\Delta\mathbf{x}) &= \mathbf{x}+\Delta\mathbf{x}+\mathbf{s}(\mathbf{x}+\Delta\mathbf{x}) \\ &= \mathbf{x}+\Delta\mathbf{x}+\mathbf{s}(\mathbf{x})+(\Delta\mathbf{x})\cdot\nabla\mathbf{s}(\mathbf{x}).\end{aligned}$$

So

$$\mathbf{r}(\mathbf{x}+\Delta\mathbf{x})-\mathbf{r}(\mathbf{x}) = \mathbf{x}+(\Delta\mathbf{x})\cdot\nabla\mathbf{s}(\mathbf{x})$$

and the squared distance between the *deformed* points is

$$\begin{aligned}ds_1^2 &= \sum_{i=1}^{3}(\Delta x_i+\Delta x_j\partial_j s_i(\mathbf{x}))^2 \\ &= \sum_{i=1}^{3}((\Delta x_i)^2+2\Delta x_j(\partial_j s_i)\Delta x_i+(\Delta x_j\partial_j s_i)^2).\end{aligned}$$

From here on we shall always suppose that $\mathbf{s}$ is sufficiently small that powers of $\mathbf{s}$ higher than the first are negligible. We are restricting our attention to the case of linearly small strain. Then

$$\begin{aligned}ds_1^2-ds_0^2 &= 2\Delta x_j(\partial_j s_i)\Delta x_i \\ &= 2\Delta x_j\left[\frac{1}{2}(\partial_i s_j+\partial_j s_i)\right]\Delta x_i\end{aligned}$$

or

$$ds_1^2-ds_0^2 = 2\Delta\mathbf{x}\cdot\mathbf{E}\cdot\Delta\mathbf{x} \quad E_{ij}=\frac{1}{2}(\partial_i s_j+\partial_j s_i).$$

$\mathbf{E}$ is the infinitesimal strain tensor. (There are a variety of strain tensors but in the limit of small displacements all reduce to $\mathbf{E}$.) $\mathbf{E}$ extracts from a displacement field $\mathbf{s}$ the relative internal deformation caused by $\mathbf{s}$.

In this derivation we replaced the second-order tensor $\partial_j s_i$ by a symmetric version $\frac{1}{2}(\partial_i s_j+\partial_j s_i)$. In general, any second-order tensor $\mathbf{Q}$ can be divided into a symmetric part and an anti-symmetric part:

$$\begin{aligned}\text{Symmetric}(\mathbf{Q}) &= \frac{1}{2}\left(\mathbf{Q}+\mathbf{Q}^{\mathrm{T}}\right) \\ \text{Antisymmetric}(\mathbf{Q}) &= \frac{1}{2}\left(\mathbf{Q}-\mathbf{Q}^{\mathrm{T}}\right).\end{aligned}$$

The first has six independent components and the latter has three; their sum is the original tensor $\mathbf{Q}$. As you can easily demonstrate, only the symmetric part of $\partial_j s_i$ appeared we managed to streamline the expression by eliminating those three of the nine degrees-of-freedom of $\partial_j s_i$ which do not contribute to stretching (namely the antisymmetric part of $\partial_j s_i$). This is important later because only the six remaining degrees-of-freedom could induce stress. We could always, as an alternative, retain terms which in the end cannot have any effect; doing so only muddies the waters.

In cartesian coordinates, if $\mathbf{s} = \{u, v, w\}$

$$\mathbf{E} = \left\{ \begin{array}{ccc} \partial_x u & \frac{1}{2}(\partial_x v + \partial_y u) & \frac{1}{2}(\partial_x w + \partial_z u) \\ \frac{1}{2}(\partial_x v + \partial_y u) & \partial_y v & \frac{1}{2}(\partial_y w + \partial_z v) \\ \frac{1}{2}(\partial_x w + \partial_z u) & \frac{1}{2}(\partial_y w + \partial_z v) & \partial_z w \end{array} \right\}$$

Another way of writing $\mathbf{E}$ is

$$\mathbf{E} = \frac{1}{2}(\nabla \mathbf{u} + \mathbf{u}\nabla).$$

Aside: The strain due to a pure infinitesimal rigid rotation:

$$\begin{aligned} \mathbf{s} &= \Theta \times \mathbf{x}, \quad \Theta \text{ a constant vector} \\ s_i &= \epsilon_{ijkl}\theta_x x_l \\ \partial_j s_i &= \epsilon_{ikj}\theta_x \delta_l j = \epsilon_{ikj}\theta_k \\ \partial_i s_j &= \epsilon_{jki}\theta_k = -\epsilon_{ikj}\theta_k \\ \Rightarrow \mathbf{E} &= 0 \end{aligned}$$

3.2 Some Properties of Strain

Volume Strain: (Consider a cube with sides Δx, Δy, Δz.)

$$\frac{\Delta v}{v} = \mathrm{tr}(\mathbf{E}) = E_{ij}$$

Strain Deviator:

$$\mathbf{E}^{\Delta} = \mathbf{E} - \frac{1}{3}\mathrm{tr}(\mathbf{E})\mathbf{I} \quad \mathrm{tr}(\mathbf{E}^{\Delta}) = 0$$

$$\mathbf{E}^{\Delta} = (\mathbf{E}) \text{ if } \mathrm{tr}(\mathbf{E}) = 0$$

3.3 Compatibility

A second-order symmetric tensor is said to be "compatible" if there exists a piecewise continuously differentiable displacement field having the given tensor as its infinitesimal strain tensor. Not all continuous symmetric second-order tensors are possible strain tensors. (This topic is not overly important in geophysics. I mention it mostly by way of background.)

In 1860, St. Venant pointed out that a twice-differentiable second-order symmetric tensor which satisfies the eighty-one second-order partial differential equations

$$\partial_k \partial_l E_{ij} + \partial_i \partial_j E_{kl} - \partial_j \partial_l E_{ik} - \partial_i \partial_k E_{jl} = 0$$

could be a strain tensor. These are called "St. Venant's compatibility equations." They express the condition that displacement inferred by integrating strain around a closed path be single-valued. Seventy-five of the eighty-one differential equations are redundant leaving only six essential ones. These equations are principally useful when one is solving problems using strain as the primary variable. In geophysics, we fortunately almost never do that.

Chapter 4

Constitutive Relations

Constitutive relations are the equations connecting stress and strain. A perusal of any of the references mentioned above reveals the large number and rich variety of useful constitutive equations. We are here concerned only with the simplest, most common, and most useful such relation.

What references mentioned above?

The linear connection between stress and strain was first suggested by Robert Hooke in 1676 in the anagram

C E I I I N O S S S T T U V

which he revealed in 1678 to be

UT TENSIO SIC VIS

Hooke's observation, made long before the ideas of stress and strain were formulated, was that restoring forces are proportional to extension.[1]

Cauchy generalized this to a linear relation between stress, $\mathbf{T}$, and strain, $\mathbf{E}$. The most general linear connection between two second-order tensors involves a fourth-order tensor and has the form:

$$T_{ij} = C_{ijkl}E_{kl}$$

[1]It did not receive a great deal of use being, in that form, a touch hard to apply in most instances.

or

$$\mathbf{T} = \mathbf{C} : \mathbf{E}$$

$\mathbf{C}$ is an elastic coefficient tensor, or for short an "elastic tensor." $\mathbf{C}$ is a property of the medium and is independent of either $\mathbf{T}$ or $\mathbf{E}$. In general the elements of $\mathbf{C}$ are functions of position. A proper specification of $\mathbf{C}$ requires specification of the thermodynamic conditions under which strain occurs. Among other possibilities strain could be adiabatic or isothermal. We will ignore this matter here; later we will briefly see why we are principally interested in the adiabatic case.

In general, a fourth-order tensor has

$$3 \times 3 \times 3 \times 3 = 81$$

components. $\mathbf{C}$, however, relates two symmetric tensors and therefore can be required to have the symmetries

$$C_{ijkl} = C_{jikl} = C_{ijlk}$$

which leaves only 36 independent components. Thermodynamic considerations lead to one more symmetry:

$$C_{ijkl} = C_{klij}$$

which leaves only 21 independent ocmponents. This is the most general form of the adiabatic elastic tensor for a linearly elastic solid.[2]

We shall impose one further assumption on the medium and its elastic tensor: we shall suppose them to be isotropic. Suppose that $\mathbf{T}$ is the stress tensor due to some strain tensor $\mathbf{E}$

$$\mathbf{T} = \mathbf{C} : \mathbf{E}.$$

If the medium is isotropic, and if we perform an orthogonal transformation on $\mathbf{E}$, then the stress tensor associated with the transformed strain

[2] As usual exploiting symmetries is technically optional; we are free to carry as much excess baggage as we like. We choose to work with the minimum number of necessary coefficients.

tensor is simply the transformed version of the stress tensor associated with the original strain tensor, $\mathbf{E}$. This condition boils down to the requirement the $\mathbf{C}$ is invariant under any rigid rotation—the numerical components of $\mathbf{C}$ are exactly the same in any right-handed cartesian coordinate system.

Some examples of isotropic tensors of various orders are: The most

order	**example**
0 (a scalar)	all scalars
1 (a vector)	only the zero vector is isotropic
2	only scalar multiples of $\mathbf{I}$ (or δ_{ij})
3	only scalar multiples of ϵ_{ijk}

general isotropic fourth-order tensor has the cartesian components

$$C_{ijkl} = \lambda\delta_{ij}\delta kl + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \nu(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

Further, when this tensor connects two second-order tensors at least one of which is symmetric, $\nu \equiv 0$. So, the most general isotropic elastic tensor is

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl}\delta_{il}\delta_{jk})$$

The coefficients λ and μ may be functions of position. The equivalent scalar machine is

Is the following C supposed to be a tensor?

$$C(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) = \lambda(\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{d}) + \mu((\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}))$$

The associated stress-strain relation is

$$T_{ij} = (\lambda E_{kk})\delta_{ij} + 2\mu E_{ij}$$

or

$$\mathbf{T} = \lambda \mathrm{tr}(\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}$$

where λ and μ are called *Lamé parameters*; μ is sometimes called the *shear modulus.*

There are many versions of the elastic constants. Most of these are not widely used in seismology. One which is sometimes used in geophysics is the *bulk modulus*. Define $p = -\frac{1}{3}T_{kk}$, to be the *mean pressure*. Then

$$
\begin{aligned}
p &= -\frac{1}{3}(3\lambda\nabla\cdot\mathbf{s} + 2\mu\nabla\cdot\mathbf{s}) \\
&= -(\lambda + \frac{2}{3}\mu)\nabla\cdot\mathbf{s} \\
&= -\kappa\nabla\cdot\mathbf{s} \quad \kappa \equiv \textit{bulk modulus}
\end{aligned}
$$

We will mostly use the stress-displacement relations

$$T_{ij} = (\lambda\partial_k s_k)\delta_{ij} + \mu(\partial_i s_j + \partial_j s_i)$$

or

$$\mathbf{T} = (\lambda\nabla\cdot\mathbf{s})\mathbf{I} + \mu(\nabla\mathbf{s} + \mathbf{s}\nabla).$$

Chapter 5

Mathematical Preliminaries

Before we actually grind out the equations of motion of an elastic continuum, we shall first go over a few of the mathematical methods for handling deformable media.

The methods of vector calculus enable us to manipulate objects which are defined with respect to a fixed spatial coordinate system. The laws of physics, on the other hand, apply to collections of matter. One of the difficulties of continuum mechanics if that a fixed point in space does not generally coincide with a fixed material point. In this section, and the next, we shall examine ways to reconcile these.

5.1 Gauss Theorem (A Review)

$$
\begin{aligned}
\mathcal{V} &= \text{some fixed volume in 3 space} \\
\mathbf{x} &= (x_1, x_2, x_3) = (x, y, z) = \text{ position in 3-space} \\
\partial\mathcal{V} &= \text{surface of } \mathcal{V} \\
\hat{\mathbf{n}} &= (n_1, n_2, n_3) = \text{outward normal on } \partial\mathcal{V}\text{, the surface of } \mathcal{V}.
\end{aligned}
$$

A Figure

If f is some scalar field over $\mathcal{V}$ with continuous first derivatives,

$$\int_{\partial\mathcal{V}} f\hat{\mathbf{n}}d\sigma = \int_{\mathcal{V}} \nabla f dv$$

or

$$\int_{\partial\mathcal{V}} f n_i dv = \int_{\mathcal{V}} \partial_i f dv$$

This applies to any scalar function f. In particular it applies when f is some component of an n-th order tensor. With a little fiddling we can crack out a variety of forms:

$$\int_{\partial\mathcal{V}} \mathbf{f} \cdot \hat{\mathbf{n}} d\sigma = \int_{\mathcal{V}} \nabla \cdot \mathbf{f} d\sigma$$

or

$$\int_{\partial\mathcal{V}} f_i n_i d\sigma = \int_{\mathcal{V}} \partial_i f_i d\sigma$$

and

$$\int_{\partial\mathcal{V}} \mathbf{F} \cdot \hat{\mathbf{n}} d\sigma = \int_{\mathcal{V}} \nabla \cdot \mathbf{F} dv$$

or

$$\int_{\partial\mathcal{V}} F_{ij} n_j d\sigma = \int_{\mathcal{V}} \partial_j F_{ij} dv$$

et cetera.

5.2 Flux Across a Surface

f = local density per unit mass of some material property:

Examples

Mass	f	$=$	1	
Momentum	f	$=$	$\mathbf{v}$	where $\mathbf{v}$ = particle velocity
Kinetic Energy	f	$=$	$\frac{1}{2}v^2$	

The flux, or transport per unit time, across a fixed surface S with outward normal $\hat{\mathbf{n}}$ is

$$\text{flux} = \int_S \rho f \mathbf{v} \cdot \hat{\mathbf{n}} d\sigma$$

We can define flux for any quantity but it will only be useful (and sensible) if f is some property that is advected with material elements.

5.3 Kinematics: Material and Spatial Descriptions

As before

$$\begin{aligned}
\mathcal{V}_0 &= \text{initial volume (or state)} \\
\mathbf{x} &= \text{position in } \mathcal{V}_0 \\
\mathcal{V}_1(t) &= \text{instantaneous volume (or state)} \\
\mathbf{r}(\mathbf{s}, t) &= \mathbf{x} + \mathbf{s}(\mathbf{x}, t) \text{ instantaneous position of particle } \mathbf{x}.
\end{aligned}$$

A Figure

Consider the following two ways of describing some material property, say, $\mathcal{V}$ the instantaneous particle velocity

$$\begin{aligned}
\text{Spatial} \quad \mathbf{v}_E(\mathbf{r}, t) &= \text{velocity of particle at } \mathbf{r} \text{ at time } t. \\
\text{Material} \quad \mathbf{v}_L(\mathbf{x}, t) &= \text{velocity of particle} \\
& \ \mathbf{x} \text{ at time } t \text{ (currently at } \mathbf{x} + \mathbf{s}(\mathbf{x}, t))
\end{aligned}$$

Clearly, $\mathbf{v}_E(\mathbf{x} + \mathbf{s}(\mathbf{x}, t), t) = \mathbf{v}_L(\mathbf{x}, t)$.

A *spatial description* of the instantaneous state of a continuum is a specification of the state in terms of the relevant material properties as functions of position (in space) and time. Spatial descriptions (often called *eulerian* descriptions) are common in fluid mechanics where experimental sensors measure, for example, the temperature at a fixed point in space as a function of time.

A *material description* of the instantaneous state of a continuum is a specification of the relevant material properties as a function of initial position (or particle label) and time. A material description is useful for keeping track of particular material elements. The material position of any surface, for example, does not change when the continuum is deformed. This is often called a *Langrangian* even though it is apparently due to Euler.

When we admit only infinitesimal strain, as is the case here, there are only a few practical differences between the two descriptions and we can move rather easily back and forth between them. It is intellectually very useful, and sometimes very important, to keep careful track of which one we are using.

5.4 Material Derivatives

Suppose we have a spatial description of the instantaneous particle velocity field $\mathbf{v}_E(\mathbf{r}, t)$ and we wish to compute the instantaneous particle acceleration, $\mathbf{a}(\mathbf{r}, t)$ also as a spatial description. First note that $\mathbf{a}(\mathbf{r}, t) \neq (\partial_t \mathbf{v}_E(\mathbf{r}, t))_r$. For a fixed $\mathbf{r}$, $\mathbf{v}_E(\mathbf{r}, t)$ at different instants in time is the instantaneous velocity of different particles. What we need to compute is the rate-of-change of velocity for a fixed particle. That is,

check the v's in the following, in orig, some were cap.

$$
\begin{array}{lrcl}
\text{if} & \mathbf{r} & = & \mathbf{x} + \mathbf{s}(\mathbf{x}, t) \\
\text{we want} & \mathbf{a}(\mathbf{r}, t) & = & (\partial_t \mathbf{v}_L(\mathbf{x}, t))_{\mathbf{x}} \\
\text{so,} & \mathbf{a}(\mathbf{r}, t) & = & \partial_t \mathbf{v}_E(\mathbf{x} + \mathbf{s}(\mathbf{x}, t), t) \\
\text{or} & \mathbf{a}(\mathbf{r}, t) & = & \partial_t \mathbf{v}_E(\mathbf{r}, t) + \frac{\partial r_i}{\partial_t} \cdot \frac{\partial}{\partial r_i} \\
\text{since} & \mathbf{v}_E(\mathbf{r}, t) & = & \partial_t \mathbf{r} \\
\text{we have} & \mathbf{a} & = & \partial_t \mathbf{v}_E + \mathbf{v}_E \cdot \nabla \mathbf{v}_E \quad \text{in spatial coordinates.}
\end{array}
$$

The operator

$$d_t \equiv \partial_t + \mathbf{v}_E \cdot \nabla$$

is the material derivative expressed in spatial coordinates. For any function $f(\mathbf{r}, t)$ (scalar or tensor)

$$d_t f \equiv \partial_t f + \mathbf{v}_E \cdot \nabla f$$

is the time rate-of-change of f at the particle instantaneously located at $(\mathbf{r}, t)$. One more time: $d_t f(\mathbf{r}, t)$ is the time rate-of-change of f as perceived by an observer attached to the particle passing through $\mathbf{r}$ at the instant t; $\partial_t f(\mathbf{r}, t)$ is the time rate-of-change of f perceived by an observed fixed at the point $\mathbf{r}$ in space.

Note, however, that iv $\mathbf{v}_E$ is infinitesimal and *if* f is of the same order of smallness,[1] that

$$d_t f = \partial_t f + \text{ higher-order terms.}$$

[1] Which is note always the case in coninuum mechanics.

Chapter 6

Conservation Relations

The lasw of physics come to us as conservation relations. They tell us how certain quantities are conserved or, more precisely, they tell us exactly how certain quantities are not conserved and by how much.

6.1 Conservation of Mass

We start with conservation of mass. We do not admit the direct destruction of matter and the famous creation of energy thereby. Therefor, mass is exactly conserved.

We use a spatial description. Let

$\mathcal{V}$	be some fixed spatial volume,
$\rho(\mathbf{r}, t)$	be a spatial description of the density field

then

$$M(t) = \int_{\mathcal{V}} \rho(\mathbf{r}, t) dv = \text{ total mass in } \mathcal{V} \text{ at time } t$$

and

$$\partial_t M(t) = \int_{\mathcal{V}} \partial_t \rho(\mathbf{r}, t).dv$$

The only way to change $M(t)$ is by transport across the surface of $\mathcal{V}$:

$$\partial_t M(t) = -\int_{\partial\mathcal{V}} \rho \mathbf{v}(\mathbf{r}, t) \cdot \hat{\mathbf{n}}(\mathbf{r}) d\sigma.$$

So

$$\int_{\mathcal{V}} \partial_t \rho dv = -\int_{\partial\mathcal{V}} \rho \mathbf{v} \cdot \hat{\mathbf{n}} d\sigma.$$

Applying Gauss' theorem

$$\int_{\mathcal{V}} \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) dv = 0.$$

Therefore

$$\partial_t \rho(\mathbf{r}, t) + \nabla \cdot (\rho(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t)) = 0$$

or

$$\partial_t \rho + \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho = 0$$

or

$$d_t \rho + \rho \nabla \cdot \mathbf{v} = 0.$$

The last form says that changes in the density of a material packet must be a result of volume changes $(\nabla \cdot \mathbf{v})$ of the packet.

6.2 More Conservation of Mass

We briefly sketch here an alternate manner of arriving at the same result. If $\mathcal{V}_0$ and $\mathcal{V}_1(t)$ are as defined earlier and $\rho_0(\mathbf{x})$ is the equilibrium density (over $\mathcal{V}_0$) then

$$M = \int_{\mathcal{V}_0} \rho_0(\mathbf{x}) df = \int_{\mathcal{V}_1(t)} \rho(\mathbf{r}, t) dv' = \text{ a constant}$$

The prime (dv') reminds us that we're integrating over $\mathcal{V}_1$. Transform the second integral

$$\int_{\mathcal{V}_1(t)} \rho(cmvectorr, t) dv' = \int_{\mathcal{V}_0} \rho(\mathbf{r}(\mathbf{x}, t), t)|J| dv$$

where

$$J = \det \begin{vmatrix} \frac{\partial s_1}{\partial x_1} & \cdots & \cdots & \cdots & \frac{\partial s_1}{\partial x_3} \\ \frac{\partial s_3}{\partial x_1} & \cdots & \cdots & \cdots & \frac{\partial s_3}{\partial x_3} \end{vmatrix}$$

check the number of cdots in the following

is the *Jacobian* of the transformation $\mathbf{x} \to \mathbf{r}$. Then

$$\int_{\mathcal{V}_0} \{\rho_0(\mathbf{x}) - |J|\rho(\mathbf{r}(\mathbf{x}, t), t)\} dv = 0$$

so

$$\rho_0(\mathbf{x}) = |J|\rho(\mathbf{r}, t).$$

(Remember $J = J(\mathbf{x}, t)$.) Can we show this is equivalent to the earlier, more sensible looking version? Let's settle for the easy case of small displacements. First,

$$J(\mathbf{x}, t) = J(\mathbf{r}, t) = 1 + \nabla \cdot \mathbf{s}$$

so

$$\begin{aligned} \rho_0(\mathbf{x}) &= (1 + \nabla \cdot \mathbf{s})\rho(\mathbf{r}, t) \text{ which makes sense.} \\ \partial_t \rho_0(\mathbf{x}) = 0 &= d_t\rho(\mathbf{r}, t)(1 + \nabla \cdot \mathbf{s}) + \rho(\mathbf{r}, t)\nabla \cdot \mathbf{v} \\ d_t\rho(\mathbf{r}, t)(1 + \nabla \cdot \mathbf{s} &= d_t\rho(\mathbf{r}, t) + \text{ higher order terms} \\ \Rightarrow d_t\rho + \rho\nabla \cdot \mathcal{V} &= 0. \end{aligned}$$

6.3 Reynold's Transport Theorem

(This isn't really a conservation law but we needed to derive the conservation of mass first.)

We want to define the *material derivative of a volume intergral* so that we can compute the time-rate-of-change of the total amount of some quantity carried by a given mass. We want to do this using spatial coordinates. This is very much like our effort to compute the acceleration of a particle given a spatial description of the velocity field.

Let $\mathcal{V}$ be a fixed spatial volume and let $A(cmvectorr, t)$ be some quantity, given per unit mass, that we wish to study. Then

$$Q(t) = \int_{\mathcal{V}} A\rho dv = \text{ amount of } A \text{ in } \mathcal{V} \text{ at } t.$$

Clearly $\partial_t Q =$ rate of change of amount of A in $\mathcal{V}$, the fixed volume.

$$\partial_t Q(t) = \int_{\mathcal{V}} \partial_t(A\rho) dv.$$

On the other hand $d_t Q(t) =$ rate of change of the amount of A in the matter which at time t exactly fills $\mathcal{V}$ (by definition).

If the particle velocity $\mathbf{v}$ vanishes in and around $\mathcal{V}$,

$$d_t Q = \partial_t Q \text{ when } \mathbf{v} = 0.$$

In the more general case we must have

$$\partial_t Q = d_t Q - \int_{\partial\mathcal{V}} (A\rho\mathbf{v}) \cdot \hat{\mathbf{n}} d\sigma$$

where the surface integral is the rate at which A is carried out of $\mathcal{V}$ by mass transport. In words, the rate of change of the amount of A in $\mathcal{V}$ is due to changes in the particle population inhabiting $\mathcal{V}$ (the surface integral) as well as "real" changes in the amount of A carried by the particles in $\mathcal{V}(d_t Q)$. Then

$$\begin{aligned}
\partial_t Q &= d_t Q - \int_{\mathcal{V}} \nabla \cdot (A\rho\mathbf{v}) dv \\
d_t \int_{\mathcal{V}} A\rho dv &= \int_{\mathcal{V}} \{\rho\partial_t A + A\partial_t\rho + A\nabla\cdot(\rho\mathbf{v}) + \rho\mathbf{v}\cdot\nabla A\} dv \\
d_t \int_{\mathcal{V}} A\rho dv &= \int_{\mathcal{V}} \rho(\partial_t A + \mathbf{v}\cdot\nabla A) dv \\
\text{since } \partial_t\rho + \nabla\cdot(\rho\mathbf{v}) &= 0.
\end{aligned}$$

Then

$$d_t \int_{\mathcal{V}} A\rho dv = \int_{\mathcal{V}} \rho d_t A dv$$

which is a very handy result. The material derivative of an integral over a fixed spatial volume is the integral of the material derivative. Now we can derive the equations of motion.

6.4 Conservation of Linear Momentum

The conservation of linear momentum, in the everyday form $\mathbf{F} = \mathbf{MA}$ is just Newton's Second Law. We will now derive the form of this principle appropriate to continuum mechanics. We will do this in spatial coordinates. (The material approach is somewhat more complicated although much easier to generalize.)

Consider a fixed volume, $\mathcal{V}$, and sum up all of the forces acting on the mass in $\mathcal{V}$ at some instant, t.

is the T in the following a tensor as i have it?

$$\sum \textbf{force} \ = \int_{\mathcal{V}} \rho \mathbf{g} dv + \int_{\partial \mathcal{V}} \mathbf{T} \cdot \hat{\mathbf{n}} dv$$

where $\mathbf{g}$ is the sum of all of the body forces. The net linear momentum at time t of the mass in $\mathcal{V}$ is

$$\textbf{momentum} \ = \int_{\mathcal{V}} \rho \mathbf{v} dv$$

$\mathbf{F} = \mathbf{MA}$, then, says

$$d_t\{momentum\} = \{force\}$$

or

$$d_t \int_{\mathcal{V}} \rho \mathbf{v} dv = \int_{\mathcal{V}} \rho \mathbf{g} dv + \int_{\partial \mathcal{V}} \mathbf{T} \cdot \hat{\mathbf{n}} dv.$$

Applying Gauss' theorem and the Reynold's transport theorem leads to

$$\int_{\mathcal{V}} \{\rho \partial_t \mathbf{v} - \rho \mathbf{g} - \nabla \cdot \mathbf{t}\} dv = 0$$

for any fixed $\mathcal{V}$. Therrefore

$$\rho d_t \mathbf{v} = \rho \mathbf{g} + \nabla \cdot \mathbf{T}.$$

This relation, which is sometimes called "Cauchy's equation of motion," is an exact form of Newton's second law for a continuum. It is true for displacements of any size and stresses of any origin. The unusual term

$\nabla \cdot \mathbf{T}$ appears because only the change in stress across a mass element contributes to the net force acting onthat element.

When the medium is quiescent, $\mathbf{v} = 0$ and

$$\nabla \cdot \mathbf{T} = -\rho \mathbf{g}.$$

This is sometimes called the "equation of mechanical equilibrium."

Chapter 7

Equations of Motion for the Earth

Make sure you go over the p's versus the ρ's in this section. I think I got them the same as the original, but you never know.

We will now derive the equations governing the infinitesimal, elastic-gravitational deformation of an Earth model which is spherically symmetric and non-rotating, has an isotropic and perfectly elastic incremental constitutive relation, and has an initial stress field which is perfectly hydrostatic. This is kind of a lot at one time, and some of these complications are not important in many seismological applications. It is better in the long run, however, to do the most complicated case initially and then simplify it later.

We characterize the Earth's equilibrium (non-moving) state by its equilibrium stress field, $\mathbf{T}_0$, its equilibrium density field, ρ_0, and its equilibrium gravitational potential field, ϕ_0. These are related by

$$\nabla \cdot \mathbf{T}_0(\mathbf{x}) = \rho_0(\mathbf{x}) \nabla \phi_0(\mathbf{x})$$

and by Poisson's equation

$$\nabla^2 \phi_0(\mathbf{x}) = 4\pi G \rho_0(\mathbf{x})$$

plus some boundary conditions. We also suppose that $\mathbf{T}_0$ is purely hydrostatic. Then

$$\mathbf{T}_0 = -p_0 \mathbf{I} \quad p_0 = \text{ equilibrium pressure field.}$$

This is reasonable if the Earth has no significant strength. The equilibrium equations are then

$$\begin{aligned} \nabla p_0 &= -\rho_0 \nabla \phi_0 \\ \nabla^2 \phi_0 &= 4\pi G \rho_0 \end{aligned}$$

Suppose, now, that the Earth is disturbed slightly from its equilibrium configuration. We shall have to be a little careful with our coordinates. We express particle motion in the usual form

$$\mathbf{r}(\mathbf{x}, t) = \mathbf{x} + \mathbf{s}(\mathbf{x}, t)$$

where $\mathbf{x}$ can be regarded as the particle's equilibrium position.

We express the time-dependent stress, density, and gravitational potential at a fixed spatial point $\mathbf{r}$ as

$$\begin{aligned} \phi(\mathbf{r}, t) &= \phi_0(\mathbf{r}) + \phi_1(\mathbf{r}, t) \\ \rho(\mathbf{r}, t) &= \rho_0(\mathbf{r}) + \rho_1(\mathbf{r}, t) \\ \mathbf{T}(\mathbf{r}, t) &= \mathbf{T}_0(\mathbf{r}) + \mathbf{T}_1(\mathbf{r}, t) \\ &= -p_0(\mathbf{r})\mathbf{I} + \mathbf{T}_1(\mathbf{r}, t) \end{aligned}$$

These equations define the "incremental" equantities ϕ_1, ρ_1, and $\mathbf{T}_1$. Note that the incremental quantities are defined as differences from the equilibrium values at the observing point; they are not, in general, the changes experienced by a material particle. We also need the particle acceleration

$$d_t \mathbf{v}(\mathbf{r}, t) = \partial_t^2 \mathbf{s}(\mathbf{x}, t) = \partial_t^2 \mathbf{s}(\mathbf{x}, t) \text{ to first order in } \mathbf{s}$$

The equations of motion are, then to first order

$$\rho_0 \partial_t^2 \mathbf{s} = -\rho_0 \nabla \phi_0 = \rho_0 \nabla \phi_1 - \rho_1 \nabla \phi_0 - \nabla p_0 + \nabla \cdot \mathbf{T}_1$$

using the fact that the incremental quantities are all of order $\mathbf{s}$ and kiiping only terms through the first order. If we subtract the equilibrium condition we are left with

$$\rho_0 \partial_t^2 \mathbf{s} = -\rho_0 \nabla \phi_1 - \rho_1 \nabla \phi_0 + \nabla \cdot \mathbf{T}_1$$

Each term in this equation is evaluated at the spatial point $\mathbf{r}$, but the equation governs the acceleration of the material packet which may be at $\mathbf{r}$ only for the single instant of time, t.

We also must have

$$\partial_t(\rho_0+\rho_1)+(\rho_0+\rho_1)\nabla\cdot(\partial_t\mathbf{s})+(\partial_t\mathbf{s})\cdot\nabla(\rho_0+\rho_1)=0$$

so to first order

$$\begin{aligned}\partial_t\rho_1+\rho_0\nabla\cdot(\partial_t\mathbf{s})+(\partial_t\mathbf{s})\cdot\nabla\rho_0 &= 0\\ \partial_t\{\rho_1+\nabla\cdot(\rho_0\mathbf{s})\} &= 0\\ \rho_1 &= -\nabla\cdot(\rho_0\mathbf{s})\\ \text{since } \rho_1=0 \quad \text{if} &\quad \mathbf{s}=0.\end{aligned}$$

Further,

$$\begin{aligned}\nabla^2(\phi_0+\phi_1) &= 4\pi G(\rho_0+\rho_1)\\ \nabla^2\phi_1=4\pi G\rho_1 &= -4\pi G\nabla\cdot(\rho_0\mathbf{s})\end{aligned}$$

The last bit we have to sort out is how to compute $\mathbf{T}_1$, the incremental stress at the point $\mathbf{r}$. This is more complicated than it might appear because $\mathbf{T}_1$ is *not* simply the product of the local infinitesimal strain tensor and the local elastic tensor. The reason we have this problem is the presence of an initial stress field, $\mathbf{T}_0$ which is not infinitesimal but is of order zero in the displacement.

Suppose, for expamle, we keep our attention on the point $\mathbf{r}$ while we rigidly translate the Earth a small amount. After the translation, we will be looking at the same spatial point $\mathbf{r}$ but at a different piece of matter. This new piece will in general have a different initial stress field than our old piece. If we translated the Earth an amount $\mathbf{d}$ we would perceive an incremental stress of

$$\mathbf{T}_1(\mathbf{r},t)=\mathbf{T}_0(\mathbf{r}-\mathbf{d})-\mathbf{T}_0(\mathbf{r})\neq 0 \text{ in general.}$$

However, a rigid translation produces no strain, as we well know, and thus cannot cause any elastic stress. Thus $\mathbf{T}_1$, above, has nothing whatever to do with elasticity or strain but is solely a result of the initial stress field.

To see how to handle this problem, consider the stress tensor at a material particle, which we express as

$$-p_0(\mathbf{x})\mathbf{I} + \mathbf{T}_m \text{ at } \mathbf{r} = \mathbf{x} + \mathbf{s}(\mathbf{x}, t)$$

We have expressed the total stress at the particle $\mathbf{x}$, which is now located at $\mathbf{r}$, as the initial stress *at the particle* plus an increment $\mathbf{T}_m$. (Note that we can use either $\mathbf{T}_m(\mathbf{x})$ or $\mathbf{T}_m(\mathbf{r})$ since $\mathbf{T}_m$, as we will see, is first-order in $\mathbf{s}$. We cannot use $p_0(\mathbf{r})$, however, if we want $\mathbf{T}_m$ to mean what we claim it to.) We now claim

$$\mathbf{T}_m = \lambda(\nabla \cdot \mathbf{s})\mathbf{I} + \mu(\nabla\mathbf{s} + \mathbf{s}\nabla)$$

because the only way to alter the stress at a particle is by straining the continuum. Remember that the stress tensor describes the forces between parts of the continuum across intervening surfaces. These forces, in turn, are exactly a result of the material being distorted out of its preferred state. The expression for $\mathbf{T}_m$, in fact, defines what we mean by an "incremental constitutive relation."

If $-p_0(\mathbf{x})\mathbf{I} + \mathbf{T}_m$ is the stress tensor at a particle, then the stress tensor at a pint in space is simply

$$(\mathbf{I} - \mathbf{s} \cdot \nabla)(-p_0\mathbf{I} + \mathbf{T}_m) = \mathbf{T}_0 + \mathbf{T}_1$$

which is just a Taylor series. To first order, we must have

$$\mathbf{T}_1 = \mathbf{T}_m + (\mathbf{s} \cdot \nabla p_0)\mathbf{I}$$

Now, the full set of linearized equations is

$$\begin{aligned}
\rho_0 \partial_t^2 \mathbf{s} &= -\rho_0 \nabla \phi_1 - \rho_1 \nabla \phi_0 + \nabla \cdot \mathbf{T}_m + \nabla \cdot (\mathbf{s} \cdot \nabla p_0) \\
\rho_1 &= -\nabla \cdot (\rho_0 \mathbf{s}) \\
\nabla^2 \phi_1 &= 4\pi G \rho_1 \\
\mathbf{T}_m &= \lambda(\nabla \cdot \mathbf{s})\mathbf{I} + \mu(\nabla\mathbf{s} + \mathbf{s}\nabla) \\
\nabla p_0 &= -\rho_0 \nabla \phi_0
\end{aligned}$$

Remember that we have assumed that the Earth

- was not rotating
- had a hydrostatic prestress field
- was self-gravitating
- was spherically symmetric
- had an isotropic, perfectly elastic incremental constitutive relation

Remember, too, that in formulating these equations we encountered a situation in which the distinction between spatial and material concepts was crucial. It is often said that in linear (infinitesimal) elasticity ther is no difference between Eulerian (spatial) and Lagrangian (material) descriptions. That is wrong. There is no difference in problems which do not have zeroeth order initial stress (or other) fields.

Chapter 8

Adiabatic versus Isothermal

This short section is denoted to clearing up a minor question raised earlier: why are the adiabatic elastic constants used in seismology?

Consider a compressional wave propagating across the page and examine the stess field at some instant in time.

A Figure

Let f be the frequency of this wave train and $\tau = 1/f$ its period. The distance ehat can travel during one cycle is proportional to the square root of time,

$$L_\theta \propto \tau^{\frac{1}{2}} = \frac{1}{\sqrt{f}}$$

while the distance the wave itself travels is proportional to time

$$L_w \propto \tau = \frac{1}{f}$$

Therefore

$$\frac{L_\theta}{L_w} \propto f^{\frac{1}{2}}.$$

At *low frequencies*, $L_\theta \ll L_w$. In this case, heat does not propagate appreciably and thus the process is essentiall *adiabatic*. At *high frequencies*, $L_\theta \gg L_w$. In this case, heat propagates "faster" than the wave motion and the process is essentiall *isothermal*.

The dividing line between the two processes occurs at frequencies of the order of 10^{10} Hz or more. Seismology, on the other hand, is concerned with waves having frequencies less than 10^2 Hz. A thorough analysis of this point is given by R. N. Thurston in Volume 1 of the series *Physical Acousitics*, edited by Warren P. Mason.

Don't forget to add this to the bibl.

Chapter 9

Elastic Wave Equations

9.1 Gravity *versus* Elasticity

Our most recent version of the equations of motion,

$$\begin{aligned} \rho_0 \partial_t^2 \mathbf{s} &= -\rho_1 \nabla \phi_0 - \rho_0 \nabla \phi_1 + \nabla(\mathbf{s} \cdot p_0) + \nabla \cdot \mathbf{T}_m \\ &+ \quad \text{four more relations} \end{aligned}$$

is a little overwhelming and we might hope that not every geophysical application requires solving the full-blown system.

There is a very important class of problems requireing the simultaneous solution of the entire system, namely the elastic-gravitational normal mode problem. On the other hand, as we will crudely demonstrate here, at the higher portion of the range of frequencies of seismological interest, a lorge portion of the equation of motion can be neglected.

To see this, we shall have to make premature use of the notion of a propagating seismic wave. Suppose that some sort of wave with angular frequency ω and wave-vector $\mathbf{k}$ is propagating through the Earth. Then, being deliberately vague,

$$\mathbf{s} \text{ goes like } e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \text{ more or less.}$$

Let us consider how the various terms in the momentum equation scale with ω and $k = |\mathbf{k}|$. We neglect the spatial variation of ρ, λ, and μ and use S to denote the scale size of displacement:

$$\begin{aligned} \rho_0 \partial_t^2 \mathbf{s} &\approx \omega^2 \rho_0 S \\ \rho_1 &\approx \rho_0 k S \\ \rho_1 \nabla \phi_0 &\approx \rho_0 k g S \text{ where } g = \text{gravity} \\ \nabla(\mathbf{s} \cdot \nabla p_0) &\approx \rho_0 k g S \\ \nabla \cdot \mathbf{T} &\approx k^2 \lambda \text{ or } k^2 \mu \end{aligned}$$

(We have skipped over ϕ_1, for simplicity. Generally speaking, ϕ_1 is less significant than ϕ_0.)

The right-hand side of the momentum equation is the sum of various elastic *or* gravitational terms. Elastic terms scale like $k^2\lambda$; gravitational terms scale like $\rho_0 kg$. Thus the ratio of gravitation to elastic forces goes like

$$\begin{aligned}\frac{\text{gravity}}{\text{elasticity}} &\propto \frac{\rho k g}{k^2\lambda} = \frac{\rho g}{k\lambda} \\ \frac{\rho}{\lambda} &\approx c^2 \text{ the elastic velocity} \\ \frac{\rho g}{k\lambda} &\approx \frac{g}{c^2 k}.\end{aligned}$$

(I confess to reaching ahead somewhat.) We see that $k \to \infty$ results in an elastically-dominated system while $k \to 0$ results in a gravitationally-dominated system. The two influences are crudely equal when

$$\frac{g}{c^2 k} \approx 1$$

or

$$k \approx \frac{g}{c^2} \approx \frac{10^3}{10^12} = 10^{-9} \text{ in the Earth.}$$

Let us further suppose that we are essentially in the elastic domain. Then $k \approx \omega/c$. This is obviously somewhat contradictory but it does give us a useful idea of where gravity becomes important

$$\begin{aligned}\frac{\omega}{c} &\approx 10^{-9} \\ \omega &\approx 10^{-3} \\ t &= \frac{2\pi}{\omega} \approx 6000 \text{ seconds.}\end{aligned}$$

This asserts that at periods short compared to 6×10^3 seconds, the equations of motion are dominated by elasticity. In the high-frequency limit, then, we can make do with

$$\rho_0 \partial_t^2 \mathbf{s} = \nabla \cdot \mathbf{T}_m$$

$$\mathbf{T}_m = \lambda(\nabla \cdot \mathbf{s})\mathbf{I} + \mu(\nabla \mathbf{s} + \mathbf{s}\nabla)$$

These are the equations of motion used in conventional short-period seismology (10^{-2} sec to 10 sec). At longer periods the influence of gravity becomes non-negligible and we must find ways to study the complete equations of motion. (A much better way to determine the point at which gravity becomes significant is to solve both versions of the equations of motion and compare the solutions.)

Chapter 10

Wave Equations

10.1 The Elastic Wave Equation

We will first boil down the equations of motion (for high frequencies and therefore without gravity) into a form called "the elastic wave equation." Then we will briefly review why equations of the form are called wave equations.

For sufficiently high frequencies we have

$$\rho_0 \partial_t^2 \mathbf{s} = \nabla \cdot \mathbf{T}_m$$

and

$$\mathbf{T}_m = \lambda(\nabla \cdot \mathbf{s})\mathbf{I} + \mu(\nabla \mathbf{s} + \mathbf{s}\nabla).$$

Then

$$\begin{aligned} \rho_0 \partial_t^2 \mathbf{s} &= \nabla(\lambda(\nabla \cdot \mathbf{s})) + \nabla\mu \cdot [\mu(\nabla \mathbf{s} + \mathbf{s}\nabla)] \\ \rho_0 \partial_t^2 \mathbf{s} &= (\nabla \cdot \mathbf{s})\nabla\lambda + \nabla\mu \cdot (\nabla \mathbf{s} + \mathbf{s}\nabla) \\ &+ \lambda\nabla(\nabla \cdot \mathbf{s}) + \mu\nabla \cdot (\nabla \mathbf{s} + \mathbf{s}\nabla) \end{aligned}$$

The first two terms result from spatial gradients of the elastic constants. We are going to drop them for two reasons:

1. As we could show by scale analysis, they are relatively unimportant as long as the wavelength of the solution is short compared to the scale length of changes in the elastic properties. For any continuous distribution of properties there will be some scale frequency *above which* we may ignore these terms.

2. These terms can cause an incredible amount of grief.

After surgery we have

$$\rho_0 \partial_t^2 \mathbf{s} = \lambda\nabla(\nabla \cdot \mathbf{s}) + \mu\nabla \cdot (\nabla \mathbf{s} + \mathbf{s}\nabla)$$

where λ and μ are "sufficiently constant." Since

$$\begin{aligned}
\nabla \cdot (\nabla \mathbf{s} + \mathbf{s}\nabla) &= 2\nabla(\nabla \cdot \mathbf{s}) - \nabla \times \nabla \times \mathbf{s} \\
\rho_0 \partial_t^2 \mathbf{s} &= (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{s}) - \mu\nabla \times \nabla \times \mathbf{s}
\end{aligned}$$

This is sometimes called the "elastic wave equation," although we know it to be only a high-frequency pretender.

Now express $\mathbf{s}$ as

$$\mathbf{s} = \nabla\psi + (\mathbf{r}\chi) + \nabla \times (\nabla \times \mathbf{r}\sigma)$$

where ψ, χ, and σ are scalar fields. The above expression is a representation for $\mathbf{s}$ by which we mean that we believe that for any $\mathbf{s}$ of interest there exist scalar fields ψ, χ, σ fulfilling the above. Further we believe that recasting our problem in terms of these scalars will do some good.

You can verify, if you wish, that with this representation the elastic wave equation becomes

$$\begin{aligned}
\nabla(\rho_0 \partial_t^2 \psi - (\lambda + 2\mu)\nabla^2\psi) &+ \nabla \times \mathbf{r}(\rho_0 \partial_t^2 \chi - \mu\nabla^2\chi) \\
&+ \nabla \times \nabla \times t(\rho_0 \partial_t^2 \sigma - \mu\nabla^2\sigma) = 0.
\end{aligned}$$

If

$$\begin{aligned}
\rho_0 \partial_t^2 \psi &= (\lambda + 2\mu)\nabla^2\psi \\
\rho_0 \partial_t^2 \chi &= \mu\nabla^2\chi \text{ and} \\
\rho_0 \partial_t^2 \sigma &= \mu\nabla^2\sigma
\end{aligned}$$

then $\mathbf{s}$ as gotten from these scalars will be a solution of the elastic wave equation. However, as a little ghought shows, we have not shown that every $\mathbf{s}$ satisfying the elastic wave equatoin can be found from scalar fields satisfying these three relationsships. Fortunately, however, is is so and we can replace "if" by "if and only if." (See Steinberg, *Arch. for Rat. Mech. and Anal.*, 1960, vol. 6.) *Do reference.*

If terms involving the gradients of λ and μ appear, all of this, alas goes down the drain and we have a much more intractable system. Qualitatively the gradient terms couple shear and compressional motions together. When this coupling is weak we can regard those terms as small sources, as is done in scattering theory. If the coupling is strong enought the notions of distinct shear and compressional waves break down.

The three scalar equations are called "scalar wave equations." This form of equation is discussed later. Observe that three perfectly valid classes of solutions can be gotten by setting two of the scalars to zero and requiring the third to be a non-zero solution of its respective wave equatoin. Let $\{\mathbf{s}_\psi\}$, $\{\mathbf{s}_\chi\}$, $\{\mathbf{s}_\sigma\}$ denote, in the obvious way, these three sets of solutions. Because everything is linear, any general solution to the elastic wave equation can be expressed as a linear combination of members of these three sets. Note that every $\mathbf{s}_\psi$ satisfies $\nabla \times \mathbf{s}_\psi = 0$ and every $\mathbf{s}_\chi$, $\mathbf{s}_\sigma$ satisfies $\nabla \cdot \mathbf{s}_\chi = \nabla \cdot \mathbf{s}_\sigma = 0$. Also χ and σ satisfy exactly the same scalar wave equation.

These solutions have certain characteristic properties which we will simply describe, and not prove, here. $\mathbf{s}_\psi$ is called a "P wave" or "compressional wave." It is a propagating dilatation with phase velocity

$$V_P = \alpha = \sqrt{\lambda + 2\mu/\rho}.$$

$\mathbf{s}_\chi$ and $\mathbf{s}_\sigma$ are called "S waves" or "shear waves." Particle motions for these waves are at right angles to the direction of propagation. These waves have a phase velocity

$$V_S = \beta = \sqrt{\mu/\rho} < V_P$$

There are two classes of shear waves simply because there are two possible orthogonal polarizations. Shear waves have no associated dilatation ($\nabla \cdot \mathbf{s} = 0$).

10.2 Scalar Wave Equations

We will now consider why equations of the form

$$\partial_t^2 \psi = c^2 \nabla^2 \psi$$

are called "scalar wave equations." c, which we suppose to be nearly constant, is called the "wave speed."

Let $\hat{\mathbf{n}}$ be a fixed direction in space and let $f(x)$ be any twice differentiable function of one variable. If

$$\psi(\mathbf{r}, t) = f(\hat{\mathbf{n}} \cdot \mathbf{r} - ct)$$

then

$$\begin{aligned} \nabla\psi &= \hat{\mathbf{n}} f' \\ \nabla \cdot (\nabla\psi) &= \hat{\mathbf{n}} \cdot \hat{\mathbf{n}} f'' = f'' \end{aligned}$$

and

$$\begin{aligned} \partial_t \psi &= -cf' \\ \partial_t^2 \psi &= c^2 f'' \end{aligned}$$

The wave equation becomes

$$c^2 f'' = c^2 f''$$

which is true. Thus any ψ of the above form satisfies the wave equation. Such a ψ is constant on the planes defined by

$$\hat{\mathbf{n}} \cdot \mathbf{r} - ct = A, \text{ a constant}$$

Also, the plane associated with a fixed A, and therefore a particular value of ψ moves in the direction $\hat{\mathbf{n}}$ with speed c (or to be pedantic, velocity $c\hat{\mathbf{n}}$). Thus the pattern of ψ in space is propagating with the velocity $c\hat{\mathbf{n}}$, which is what we expect of a "wave."

We generally think in terms of simple harmonic time dependence, and use ω to denote angular frequency. If we define the wave number

$$k = \frac{\omega}{c}$$

and the wave-number vector

$$\mathbf{k} = k\hat{\mathbf{n}} = \frac{\omega}{c}\hat{\mathbf{n}}$$

then

$$\hat{\mathbf{n}} \cdot \mathbf{r} - ct = \frac{\omega}{c}(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

a more conventional form.

Chapter 11

Traveling Waves

11.1 Particular Forms

We begin with the scalar wave equation:

$$\partial_t^2\psi = c^2\nabla^2\psi$$

where we assume c is constant. We have already seen that if $f(x)$ is any bounded, twice-differentiable function of a single variable, then

$$\psi(\mathbf{r},t) = f(\hat{\mathbf{n}}\cdot\mathbf{r} - ct)$$

satisfies the wave equation for any direction $\hat{\mathbf{n}}$.

There are other simple wave solutions. In spherical coordinates, r, θ, ϕ, if we assume ψ is a function only of r and t (*i.e.*, spherical symmetry), then

$$\nabla^2\psi = \partial_r^2\psi + \frac{2}{r}\partial_r\psi.$$

(Remember we are assuming that the *wave* is spherically symmetric about the origin of the coordinate system. We might imagine a purely symmetric, explosion-like source for such a wave.) The substitution

$$\psi(r,t) = \frac{1}{r}\eta(r,t)$$

produces

$$\nabla^2\psi = \frac{1}{r}\partial_r^2\eta$$

and the wave equation reduces to the one-dimensional wave equation

$$\partial_r^2\eta = \frac{1}{c^2}\delta_t^2\eta.$$

So,

$$\eta = f(r-ct) + g(r+ct)$$

and thus

$$\psi = \frac{1}{r}\{f(r-ct) + g(r+ct)\}.$$

This solution represents two spherically symmetric wave—one traveling outward and one traveling inward. The leading factor of $\frac{1}{r}$ tells us how the amplitudes of the waves diminish as they progress.

We can see that $\frac{1}{r}$ is a physically reasonable rate of decay. Consider just the outward traveling wave:

$$\psi^{\text{out}} = \frac{1}{r} f(r - ct).$$

ψ is virtually always proportional to a quantity such as displacement, strain, stress, electromagnetic field strength, *etc.*, and thus ψ^2 is usually a measure of energy density. The energy in the wavefront crossing a sphere of radius r at time t is then

$$4\pi r^2 (\psi^2) = 4\pi f^2 (r - ct)$$

the size of which is independent of radius (other than as a "phase" argument). Thus energy is conserved by the $\frac{1}{r}$ dependence.

11.2 The Sommerfeld Radiation Condition

The otehr spherically symmetric solution,

$$\psi^{\text{in}} = \frac{1}{r} g(R + ct)$$

is a wave traveling inward toward the origin. If the problem we were studying involved a source at the origin and if the medium was homogeneous and infinite, we might discard the inward-bound solution on the grounds that energy must flow outward from the source to infinity and not the other way around. This requirement is formally called the "Somerfeld radiation condition" and is ferquently applied in wave propagation problems.

The Sommerfeld condition applies equally in non-spherically symmetric problems; in those cases the separation into incoming and outgoing radiation may be a little more complicated. It is important to remember that the radiation condition only applies to radiation which is truly coming in from infinity; we cannot apply it inside of a region where incoming energy might legitimately appear as a result of distant scattering (unless, of course, we wish to neglect that scattering).

11.3 Reflection

The imposition of boundary conditions at various surfaces in the medium generally gives rise to additional reflected wave fields. (In elasticity, boundaries also act, in general, to convert one type of wave into another upon reflection.) Reflected energy plays an essential role in many important wave propagation phenomena.

We will here examine only a simple case. Suppose we have a semi-infinite medium ending at $x = 0$.

A Figure

At the boundary surface, $x = 0$, we impose the boundary condition $\psi = 0$. Finally let

$$\psi^+ = f(x + ct)$$

be an incoming wave (that is, coming from the right). What is the solution to the wave equation which satisfies the boundary conditions?

The general form of the solution is

$$\psi(x, t) = f(x + ct) + g(x - ct)$$

reflection. Where f is the known incoming wave and g is the (unknown) outgoing. The boundary condition has the form

$$\psi(0, t) = 0$$

or

$$\begin{aligned} f(ct) + g(-ct) &= 0 \\ g(-ct) &= -f(ct). \end{aligned}$$

So

$$\psi(x,t) = f(x + ct) - f(ct - x)$$

The first term is just energy traveling to the left; the second is energy traveling to the right. Notice the reflected wave has the same form as the incident wave but is of opposite sign; obviously that is because the boundary condition we have applied requires the two waves to exactly cancel at $x = 0$. The argument of $f(x)$ is also reversed in sign in the reflected wave; this is because the spatial shape of the reflected wave is the reverse of that of the incident wave.

We can see the behavior of the waves a little more easily by studying the path of a particular point of the wave in the xt plane. Suppose $f(x)$ has the shape

A Figure

and x_0 is the peak of the wave. Then we plot the position, in space, of this peak as a function of time for both incoming and outgoing waves.

A Figure

The region $x < 0$ is not really a part of our problem, but it does no harm to imagine it filled with continuum. Obviously we get the same results whether we have a single wave-plus-reflection in our half-infinite medium, or a wholly infinite medium and a pair of carefully symmetric waves going in opposite directions.

11.4 Cartesian Harmonic Waves

The most useful simple solution is the simple-harmonic plane wave:

$$\psi(\mathbf{r}, t) = e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

where $\mathbf{k}$ is any fixed vector satisfying $\mathbf{k}\cdot\mathbf{k} = \omega^2/c^2$ (that is, having a specified length). For fixed ω, there is a family of these solutions, each member corresponding to one point on the surface of the sphere $\mathbf{k}\cdot\mathbf{k} = \omega^2/c^2$.

The set of harmonic plane waves for all frequencies, ω, is complete. Thus any traveling wave can be guilt up from a sum of plane waves, although this is not all that often useful. This point is discussed at some length in

do reference.

Ewing, M., Jardetzky, W.S., and Press, F., *Elastic Waves in Layered Media*, McGraw Hill, New York, 380 pages, 1957.

This book precedes the widespread use of computers in seismology and its applications are somewhat outdated. The principles, however, remain the same.

11.5 Phase Velocity, Group Velocity, and Dispersion

The next topic is somewhat outside of the subject of simple wave solutions to the scalar wave equation. It is, however, one that we must deal with and now is as good a point as any.

Suppose that we have a system governed bya wave equation which differs from the object of our current fascination in a rather strange way. We suppose that this system still supports plane harmonic waves of the form

$$\psi = e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}$$

where $\mathbf{k}$ is any vector but now we have

$$\omega = f(\mathbf{k})$$

where $f(\mathbf{k})$ is some as yet unspecified function. In our previous case,

$$\omega = c|\mathbf{k}|.$$

We want to study the properties of wave motion in this system. The dependence of ω on $\mathbf{k}$ is called "dispersion" for reasons that will emerge.

This unusual behavior is not at all unusual and is more than just a fanciful assumption. Many types of wave propagation are noticeable dispersive including almost everything of interest to seismologists except, perhaps, body waves.

——— *Model A* ———

We shall first construct a relatively uncomplicated model of the propagation of wave energy to see if we can infer some of the phenomena associated with dispersoin. Consider the simultaneous propagation in the $+\hat{\mathbf{x}}$ direction of two plane waves. We have

$$\psi(x,t) = e^{i(\omega_1 t - k_1 x)} + e^{i(\omega_2 t - k_2 x)}$$

where

$$\omega_1 = f(k_1) \quad \omega_2 = f(k_2)$$

(For simplicity we have assumed ω depends only upon $|\mathbf{k}|$). Now take

$$\begin{array}{rcl} k_1 & = & k - \epsilon \\ k_2 & = & k + \epsilon \end{array} \quad \text{where } \epsilon \ll k.$$

Then

$$\begin{array}{rcl} \omega_1 & = & \omega - \epsilon U \\ \omega_2 & = & \omega + \epsilon U \end{array} \quad \text{where} \left\{ \begin{array}{rcl} \omega & = & f(k) \\ U & = & \partial_k f. \end{array} \right.$$

So

$$\begin{array}{rcl} \psi & = & e^{i(\omega t - kx)} \left(e^{i\epsilon(-Ut+x)} + e^{i\epsilon(Ut-x)} \right) \\ \psi & = & 2e^{i(\omega t - kx)} \cos\{\epsilon(x - Ut)\} \end{array}$$

The first factor in ψ is just the mean plane wave wiggling off to infinity. Because of interference between the two components, however, the amplitude of the wave is multiplied by a second factor which variew much more slowly in space (since $\epsilon \ll k$). Schematically,

A Figure

which is a modulated version of the mean plane wave.

The zero-crossings of the composite wave travel with the speed

$$C = \frac{\omega}{k}$$

whether the wave is modulated or not. This quantity is called the "phase velocity." The modulation peaks and troughs, however, travel with the speed

$$U = \frac{\partial \omega}{\partial k}$$

which is called the "group velocity." The modulation envelope controls the local energy density ($\psi\psi^\dagger$) of the wave; hence we associate group velocity with the speed of energy transport. Notice that both C and U depend only upon $\omega(k)$ and not upon ϵ, the wavenumber separation of the two interfering waves. The wavenumber separation, ϵ, does not control the modulation wavelength.

Observe that when $\omega = c/k$ then

$$C = U = c$$

In this case both phase and group velocity are equal to the velocity appearing in the wave equation. This condition is true for elastic waves propagating in an unbounded homogeneous medium.

This example serves more as a demonstration than a proof. The result we derived indicates a sinusoidal modulation envelope which itself extends infinitely far in both directions, and we might fairly wonder if our interpretation of the results is unfounded or at least suspect.

Such suspicion, while commendable, would be in error. The trouble with this demonstration lies in our use of only a finite number of interfering waves; with a finite set (two in this case) it is impossible to construct a spatially limited interference product. On the other hand, we have not assembled in this course all of the mathematical apparatus necessary for a more convincing treatment.

For those familiar with the ideas of three-dimensional Fourier transforms, we now present Model B. If the following is too involved for you, take heart in the fact that the answers are virtually the save as given above. Model A is all that you really need.

——— *Model B* ———

Construct a spatially bounded wave using a wavenumber integral over K, the 3-space of wavenumber vectors:

$$\psi(\mathbf{r},t) = \int_K \phi(\mathbf{k}) e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} dv_k$$

where $\omega = f(\mathbf{k})$. All we are doing is adding up a continuum of plane waves; since each of these satisfies the wave equation (by hypothesis), so does the sum.

In particular, we take $\phi(\mathbf{k})$ to be a three-dimensional gaussian distribution centered about $\mathbf{k}_0$,

$$\begin{aligned} \phi(\mathbf{k}) &= (2\pi a^2)^{\frac{3}{4}} e^{-a^2(\mathbf{k}-\mathbf{k}_0)^2/4} \\ &= N e^{-a^2(\mathbf{k}-\mathbf{k}_0)^2/4}, \quad \text{for short.} \end{aligned}$$

We have selected a gaussian distribution because it is band-limited in a practical sense, and because its inverse transform is spatially limited (the inverse is also gaussian) in the same sense. So,

$$\psi(\mathbf{r},t) = N \int_K e^{-a^2(\mathbf{k}-\mathbf{k}_0)^2/4} e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} dv_k.$$

Since $\phi(\mathbf{k})$ is effectively constrained to a region around $\mathbf{k}_0$, the size of which we control by selecting a, let us linearize around $\mathbf{k}_0$:

$$\begin{aligned} \mathbf{k} &= \mathbf{k}_0 + \delta \\ \omega(\mathbf{k}) &= \omega_0 + \delta\cdot\mathbf{U} \end{aligned} \qquad \text{where} \left\{ \begin{aligned} \omega_0 &= \omega(\mathbf{k}_0) \\ \mathbf{U} &= \nabla_{\mathbf{k}}\omega(\mathbf{k}) \end{aligned} \right.$$

$$\begin{aligned} \psi(\mathbf{r},t) &= N \int_K \exp\left(-a^2\delta\cdot\delta/4 - i\mathbf{k}_0\cdot\mathbf{r} - i\delta\cdot\mathbf{r} + i\omega_0 t + i\delta\cdot\mathbf{U}t\right) dv_\delta \\ \psi(\mathbf{r},t) &= N e^{i(\omega_0 t - \mathbf{k}_0\cdot\mathbf{r})} \int_K \exp\left(-a^2\delta\cdot\delta/4 - i\delta\cdot\mathbf{r} + i\delta\cdot\mathbf{U}t\right) dv_\delta \\ \psi(\mathbf{r},t) &= N e^{i(\omega_0 t - \mathbf{k}_0\cdot\mathbf{r})} \int_K \exp\left(-a^2\delta\cdot\delta/4 + i\delta\cdot\mathbf{P}\right) dv_\delta \end{aligned}$$

$$\text{where } \mathbf{P} = \mathbf{U}t - \mathbf{r}$$

Call the integral $H(\mathbf{P})$. Convert it to spherical polar coordinates (δ,θ,ϕ) by taking $\mathbf{P}$ as the pole and let $\alpha = \cos\theta$.

$$H(\mathbf{P}) = 2\pi \int_0^\infty \delta^2 d\delta \int_{-1}^{+1} e^{-a^2\delta^2/4 + i\delta P\alpha} d\alpha$$

$$P = |\mathbf{P}|.$$

Since

$$\begin{aligned}
\int_{-1}^{+1} e^{i\delta P\alpha} d\alpha &= \frac{2}{\delta P}\sin(\delta P) \\
H(\mathbf{P}) &= \frac{4\pi}{P}\int_0^{\infty} \delta \sin(\delta P) e^{-a^2\delta^2/4} d\delta \\
H(\mathbf{P}) &= \frac{4\pi}{P}\int_{-\infty}^{\infty} \frac{\delta}{2i} e^{i\delta P} e^{-a^2\delta^2/4} d\delta.
\end{aligned}$$

Substitute

$$\begin{aligned}
q &= \frac{\delta a}{2} + \frac{iP}{a} \\
\delta &= \frac{2}{a}\left(q - \frac{iP}{a}\right)
\end{aligned}$$

to get

$$\begin{aligned}
H(\mathbf{P}) &= \frac{4\pi}{P}\frac{2}{a} e^{-P^2/a^2} \int_{-\infty}^{\infty} \left(q - \frac{iP}{a}\right) e^{-q^2} dq \\
H(\mathbf{P}) &= \frac{-i8\pi}{a^2}\sqrt{\pi} e^{-P^2/a^2}
\end{aligned}$$

since

$$\int_{-\infty}^{\infty} q e^{-q^2} dq = 0$$

So,

$$\psi(\mathbf{r}, t) = \frac{-8\pi i N}{a^2}\sqrt{\pi} e^{i(\omega_0 t - \mathbf{k}_0\cdot\mathbf{r})} e^{-(\mathbf{U}t - \mathbf{r})^2/a^2}$$

This is the three-dimensional version of our earlier result. Note that $\mathbf{U} = \nabla_k f$ is a vector quantity and need not be parallel to $\mathbf{k}_0$. We could have a wave group like this:

A Figure

Chapter 12

Ray Theory

12.1 The Eikonal Equation

We will concentrate on the canonical scalar wave equation:

$$\partial_t^2 \psi = c^2 \nabla^2 \psi$$

where ψ is some potential. We Fourier-transform from time, t, to angular frequency, ω, so that

$$\partial_t \rightarrow i\omega$$

and we have

$$-\omega^2 \psi = c^2 \nabla^2 \psi.$$

Plane-wave solutions for (transformed) ψ are of the general form

$$\begin{aligned} \psi &= P_0 e^{i\Phi} \\ \Phi &= \mathbf{k} \cdot \mathbf{r} \\ |k| &= \left| \frac{\omega}{c} \right| \end{aligned}$$

for the case where c, the wave speed, is everywhere constant.

Now suppose that c varies with position but slowly enough that we can use the wave equatoin

$$-\omega^2 \psi = c^2(\mathbf{r}) \nabla^2 \psi.$$

Basically such a supposition amounts to neglecting the coupling terms (involving things like ∇c or $\nabla \lambda$) which most exact wave equations possess. In general, as frequency increases we expect this approximation to become increasingly valid. Define the index of refraction, γ, with respect to some (arbitrary) reference speed c_0 by

$$\gamma(\mathbf{r}) = \frac{c_0}{c(\mathbf{r})}.$$

Now we will investigate the *possibility* that ψ may have (possibly approximate) solutions of the form

$$\psi = P(\mathbf{r}) e^{i\Phi(\mathbf{r})}$$

where $P(\mathbf{r})$ is a real, "slowly-varying" wave amplitude and $\Phi(\mathbf{r})$ is a real generalized phase. (Remember that we are in the frequency domain; we regard ω as fixed, of course.)

We plug our assumed form of ψ into the wave equation and see what comes out:

$$\begin{aligned}\nabla\psi &= (\nabla P)e^{i\Phi} + (i\nabla\Phi)(Pe^{i\Phi})\\ \nabla^2\psi &= \nabla\cdot\nabla\psi\\ &= (\nabla^2 P_0)e^{i\Phi} + 2i(\nabla P\cdot\nabla\Phi)e^{i\Phi}\\ &+ i\nabla^2\Phi(Pe^{i\Phi}) - (\nabla\Phi\cdot\nabla\Phi)(Pe^{i\Phi})\end{aligned}$$

so,

$$\begin{aligned}-\omega^2 Pe^{i\Phi} &= c^2(\nabla^2 P_0 - (\nabla\Phi\cdot\nabla\Phi)P)e^{i\Phi}\\ &+ ic^2(2\nabla P\cdot\nabla\Phi + P\nabla^2\Phi)e^{i\Phi}.\end{aligned}$$

If P, Φ satisfy this equation, then ψ as specified above satisfies the wave equatoin (to be more precise, the approximate wave equation when c is a slowly-varying function of position). Extract the in-phase part (the real part of the various coefficients):

$$-\omega^2 P = c^2\nabla^2 P = c^2 P\nabla\Phi\cdot\nabla\Phi$$

or

$$(\nabla\Phi)^2 = \frac{\omega^2}{c^2} - \frac{\nabla^2 P}{P}.$$

We now suppose that

$$(\nabla\Phi)^2 \gg \frac{\nabla P}{P}$$

to such an extent that the second term is negligible. When c is a constant in space the P is also a constant and $\nabla^2 P$ is exactly zero while $(\nabla\Phi)^2$ is exactly equal to ω^2/c^2. When c is no longer constant but still reasonably smooth, neglecting $\nabla^2 P/P$ is, of course, an approximation. We justify this by claiming that at sufficiently high frequencies the wavenumber of the solution is much, much greater than the rate at which wave amplitude, or energy, varies in space. There are two important features of this approximation:

1. The approximation in general improves as frequency increases. In the limit of unbounded frequency it becomes arbitrarily good. Because of this feature, we sometimes call this approach "asymptotic ray theory."

2. At any finite frequency there are always cases in which ray theory fails. We must alsays be careful in exploiting this technique.

The equation

$$(\nabla\Phi)^2 = \frac{\omega^2}{c^2}$$

is called the "eikonal" equation, for the greek word $\epsilon\iota\kappa\tilde{\omega}\nu$ meaning "image." The solution, Φ, is sometimes called the "eikonal." The solutions, Φ, to this non-linear partial differential equation are the phase survaces associated with high-frequency wave propagation through a medium with, at worst, a slowly-varying wave speed.

Note that $\nabla\Phi$ is essentially the local wave-number. To see this, we neglect possible variation in P and expand

$$\begin{aligned}\psi(\mathbf{r}+\delta) &= P\exp(i\{\Phi(\mathbf{r}+\delta)\}) \\ &= P\exp(i\{\Phi(\mathbf{r})+\delta\cdot\nabla\Phi(\mathbf{r})\}).\end{aligned}$$

Basically, the eikonal equation is telling us how the local solution wavenumber is related to the local (and only the local) material properties.

12.2 Rays

Consider the surface associated with some constant value, $\Phi = \Phi_0$, of the eikonal:

A Figure

The normal, $\hat{\mathbf{n}}$, is obviously given by

$$\hat{\mathbf{n}} = \frac{\nabla\Phi}{|\nabla\Phi|} = \frac{\nabla\Phi}{k_0\gamma} \text{ where } k_0 = \frac{\omega}{c_0}.$$

We define a trajectory by the locus of points, $\mathbf{p}(s)$, where s is arclength, such that

$$d_s\mathbf{p} = \hat{\mathbf{n}}(\mathbf{p}).$$

We call this trajectory a "ray." We will see later why we attach so much significance to the ray. Right now, we wish to find a way to construct the ray without having to directly solve the eikonal equation. We do this as follows:

$$\begin{aligned} d_s(\mathbf{p}) = \hat{\mathbf{n}} &= \frac{\nabla\Phi}{k_0\gamma} \\ \gamma d_s\mathbf{p} &= \frac{1}{k_0}\nabla\Phi \\ d_s(\gamma d_s\mathbf{p}) &= \frac{1}{k_0}d_s(\nabla\Phi). \end{aligned}$$

Since

$$\begin{aligned} d_s &= \hat{\mathbf{n}} \cdot \nabla \\ d_s(\gamma d_s \mathbf{p}) &= \frac{1}{k_0} \hat{\mathbf{n}} \cdot \nabla(\nabla\Phi) \\ &= \frac{1}{\gamma k_0^2} \nabla\Phi \cdot \nabla(\nabla\Phi) \\ &= \frac{1}{\gamma k_0^2} \frac{\nabla(\nabla\Phi \cdot \nabla\Phi)}{2} \quad \text{(why?)} \\ &= \frac{1}{2\gamma k_0^2} \nabla(\gamma^2 k_0^2) \end{aligned}$$

so

$$d_s(\gamma d_s \mathbf{p}) = \nabla\gamma$$

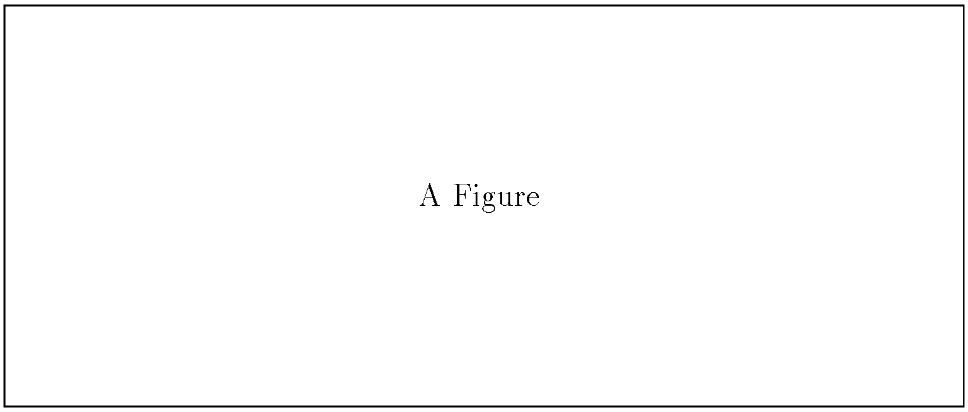
or

$$d_s(\gamma \hat{\mathbf{n}}) = \nabla\gamma.$$

This equation describes the precise manner in which a ray, as we have defined it, twists and turns in response to the spatial variation of the wave speed, c. Imagine starting at some initial point, $\mathbf{p}(0)$, and in some initial direction, $\hat{\mathbf{n}}(0)$. Then in the presence of some particular $\gamma(\mathbf{r})$, this equation tells us how to take little local steps along the unique ray specified by $\mathbf{p}(0)$, $\hat{\mathbf{n}}(0)$. Now we shall explore why this particular path is significant.

12.3 Fermat's Principle and Other Topics

We begin with Huygen's Principle which asserts that as a wave disturbance spreads through a medium, we may, at each instant, regard the points disturbed by the ray at that moment as a new set of radiating sources. To take a very simple example, consider a pulse from some source which at time t has spread outward to the surface ∂V:



Huygen's Principle, as we shall cast it, says that at any later time the pulse may be found by computing the signal from a set of sources appropriately distributed over the disturbed area contained by ∂V. In this form, Huygen's Principle is simply a statement of the obvious fact that we can regard the calculatoin for times later than some moment, t, as being the solution of an initial-value problem starting at t having initial displacements and velocities equal to those they actually had at t.

Huygen's Principle suggests a rather incoherent, jumbled progress of wave energy outward from the source. It implies that wave energy traverses every possible path available to it. We, on the other hand, hope to find some more orderly scheme of wave propagation.

Harmony among these divergent notions comes as follows: Consider a medium in which the wave speed strictly increases with depth (but "slowly" of course) and in which we have a source and a receiver, both at some common depth.

A Figure

We have drawn in three trial paths by which radiated energy might reach the receiver; two of them are somewhat fanciful. Any continuous piecewise differentiable curve is obviously a possible path. Because of the multitude of unrelated paths, we might expect, and it is so, that the signals arriving from the vast majority of them will interfere destructively to product no net observable effect at the receiver.

Now contemplate the path with the distinguishing property that it is the path from source to receiver having the shortest time-of-flight (travel-time). In our particular case this path will not be a straight line from source to receiver but, because the wave speed increases downward, the minimum-time path will "dip" down somewhat into the higher velocity region. The path is sketched (qualitatively) as the lowermost path on the figure.

Let Γ symbolically denote this path and let η be a small path-variation which vanishes at either end. (We are playing a little fast and loose with algebra here, but it will work out all right.) Symbolically, the set of paths

$$\Gamma + \alpha\eta$$

for small α, is a group of paths from source to receiver near to, and containing, the minimum-time path Γ. Since Γ is a *minimum*-time path, none of the nearby paths can be any faster than Γ and most must be slower. For any particular η, travel-time as a function of α must look like

A Figure

Because the bottom of this curve is flat, as is always the case for a minimum, there must be a cluster of paths all having the same travel-time and which will therefore interfere constructively to produce a signal at the receiver. One of the most important properties of a minimum-time path, then, is that it delivers a significant amount of coherent energy at the far end.

Notice that this argument only really required that the minimum-time path be a *local* minimum. Between any pair of points in a given medium, there may be a number of minimum-time (in the local sense) paths and each should contribute an "arrival."

Fermat's Principle asserts that a *ray* is a miminum-time path through the medium. When we have shown that this is so, we shall have found out why ray theory is useful.

We begin by proving Lagrange's integral invariant. Since

$$\gamma\hat{\mathbf{n}} = \frac{1}{k_0}\nabla\Phi$$

where k_0 is constant, we must have

$$\nabla \times (\gamma \hat{\mathbf{n}}) = 0$$

is that a d-"ell"

and therefore

$$\int_{\text{closed path}} (\gamma \hat{\mathbf{n}}) \cdot d\mathbf{l} = 0$$

if $\nabla \Phi$ is continuous and where the integral is around any closed path. This result is called "Lagrange's integral invariant."

Now let Γ_1 be the ray from source A to receiver B, let Γ_2 be a nearby ray for the same eikonal which does not necessarily strike either A or B, and let C be some path from A to B which is close to, but in general different from, the ray Γ_1. Pictorially,

A Figure

Also shown are two equal generalized phase surfaces and five labeled intersections that we shall need below.

in the original, the subscript of the integral was abcd, *i changed it to* abca *to match the following equation.*

Lagrange's integral invariant for the path $a - b - c$ gives:

$$\int_{abca} (\gamma \hat{\mathbf{n}}) \cdot d\mathbf{l} = 0$$

$$a \to b \to c \to a$$

Taking $a - b - c$ infinitesimally small,

$$(\gamma \hat{\mathbf{n}}) \cdot (d\mathbf{l})_{ab} + (\gamma \hat{\mathbf{n}}) \cdot (d\mathbf{l})_{bc} - (\gamma \hat{\mathbf{n}}) \cdot (d\mathbf{l})_{ac} = 0.$$

Since

$$(\gamma\hat{\mathbf{n}})\cdot(d\mathbf{l})_{bc}=0$$

we can always arrange that

i changed this formula. check the orig to see if i'm right.

$$\begin{aligned}(\gamma\hat{\mathbf{n}})\cdot(d\mathbf{l})_{ab}=(\gamma\hat{\mathbf{n}})\cdot(d\mathbf{l})_{ac} &= \gamma|d\mathbf{l}|_{ac}=\gamma(dl)_{ac}\\ \gamma(dl)_{ab}\geq(\gamma\hat{\mathbf{n}})\cdot(d\mathbf{l})_{ab} &= \gamma(dl)_{ac}=\gamma(dl)_{ef}.\end{aligned}$$

So

$$\begin{aligned}\gamma(dl)_{ab} &\geq \gamma(dl)_{ef}\\ \Rightarrow\int_C\gamma dl &\geq \int_{\Gamma_1}\gamma dl.\end{aligned}$$

Since each of these integrals is equal to C_0 times the time-of-flight (or travel-time) along that path,

$$T(C)\geq T(\Gamma_1).$$

Therefore the ray-path which we have defined in terms of the eikonal is a (local) minimum-time path.

Also note that although the quality strictly holds only when both paths are the same, the amount of inequality when C is not the same as Γ_1 depends upon the amount by which

$$(\gamma\hat{\mathbf{n}})\cdot(d\mathbf{l})_{ab}\text{ differs from }\gamma|d\mathbf{l}|_{ab}.$$

As you can easily show, then $\hat{\mathbf{n}}$ and $d\mathbf{l}$ differ in directions by only a small angle β (that is, when C differs only slightly in direction from the local ray), then

$$(\gamma\hat{\mathbf{n}})\cdot(d\mathbf{l})_{ab}=(1-\beta^2-\cdots)\gamma|d\mathbf{l}|_{ab}$$

so the inequality is only affected to second order in β. That is why the set of paths close to a ray path all have the same travel-time to first order in the deviation from the ray path.

Ray theory is the basis for geometrical optics and much of the discussion above was purloined from

Do references.

Born, M., and E. Wolf, *Principles of Optics*, Pergamon Press, Oxford, 808 pages, 1975.

A seismologically oriented discussion, less elegant than that in the above, is given in

Officer, C.B., *Introduction to Theoretical Geophysics*, Springer-Verlag, Berlin, 385 pages, 1974.

Chapter 13

Normal Modes

In this section we will investigate some of the properties of an important class of exact solutions to the wave equation, called "normal modes." The properties of normal modes which we shall develop here apply to a much broader class of problems than just the scalar wave equation. In order to derive and describe these properties more generally, we will also develop in this section some additional mathematical tools.

13.1 Particulars

Eigenfunctions, Eigenvalues, and Eigenfrequencies

We start with the scalar wave equation over a square two-dimensional domain with the boundary condition $\psi = 0$.

A Figure

Look for solutions of the form

$$\psi(\mathbf{r}, t) = \sigma(\mathbf{r})e^{i\omega t}$$

so

$$c^2\nabla^2\sigma(\mathbf{r}) = -\omega^2\sigma(\mathbf{r}), \text{ where } \sigma = 0 \text{ on } \partial V.$$

We can solve this by separation of variables. The answer is

$$\sigma(\mathbf{r}) = \sin(\alpha x)\sin(\beta y)$$

where α, β must be integers to satisfy the boundary conditions. Also we must have

$$\omega^2 = c^2(\alpha^2 + \beta^2).$$

This equation means that solutions only exist for certain values of frequency, ω.

Define

$$\begin{aligned}\sigma_{\alpha\beta}(\mathbf{r}) &= \sin(\alpha x)\sin(\beta y)\\ \omega^2_{\alpha\beta} &= c^2(\alpha^2+\beta^2).\end{aligned}$$

The functions $\sigma_{\alpha\beta}$ are called "eigenfunctions" of this problem, and the frequencies $\pm\omega_{\alpha\beta}$ are called "eigenfrequencies." The set of solutions

$$\psi_{\alpha\beta}(\mathbf{r},t) = \sigma_{\alpha\beta}(\mathbf{r})e^{\pm i\omega_{\alpha\beta}t}$$

i guessed at this equation. i couldn't read the orig, pg73.

are called the "normal modes" or "eigensolutions."

The normal modes are *complete*. This means that every solution to the wave equation, for this domain and these boundary conditions, can be expressed as

$$\psi(\mathbf{r},t) = \sum_{\alpha,\beta} \sigma_{\alpha\beta}(\mathbf{r})A_{\alpha\beta}e^{+i\omega_{\alpha\beta}t} + B_{\alpha\beta}e^{-i\omega_{\alpha\beta}t}.$$

i guessed at this equation. i couldn't read the orig, pg73.

One way to see this is to note that over V, the $\sigma_{\alpha\beta}$ are a two-dimensional Fourier sine series which is complete. We can expand any solution ψ, at some instant, in terms of these functions. Values of ψ at later times must agree with the above formula since

1. ψ satisfies the wave equation,
2. each term satisfies the wave equaton, and
3. the problem we have posed in unique.

Note that the eigenfunction σ_{00} vanishes everywhere. Also note that

$$-\sigma_{-\alpha,\beta} = -\sigma_{\alpha,-\beta} = \sigma_{-\alpha,-\beta} = \sigma_{\alpha\beta}$$

so that, without loss of generality, we may restrict α, β to the set of non-negative integers, $1, 2, 3, 4, \ldots$.

——— *Orthogonality and Orthogonalization* ———

These solutions are called normal modes because

$$\int_v \sigma_{\gamma\eta}(\mathbf{r})\sigma_{\delta\partial}(\mathbf{r})dv = \delta_{\alpha\gamma}\delta_{\beta\eta}\frac{\pi^2}{4}$$

so any two distinct solutions are orthogonal, or "normal," in the above sense.

We an get this in a different way. Suppose

$$\nabla^2\phi = \lambda^2\phi$$

and

$$\nabla^2\psi = \omega^2\psi$$

are both solutions to the problem (where time goes as $e^{\pm i\lambda t}$, $e^{\pm i\omega t}$, respectively). Now consider the quantity

$$\begin{aligned}
\int_v \psi\nabla^2\phi dv &= \int_v \nabla\cdot(\psi\nabla\phi) - \nabla\psi\cdot\nabla\phi dv \\
&= \int_{\partial v} \hat{\mathbf{n}}\cdot(\psi\nabla\phi)da - \int_v \nabla\psi\cdot\nabla\phi dv \\
&= \int_v -\nabla\psi\cdot\nabla\phi dv = \int_v \phi\nabla^2\psi - \nabla\cdot(\phi\nabla\psi)dv \\
\int_v \psi\nabla^2\phi dv &= \int_v \phi\nabla^2\psi dv.
\end{aligned}$$

So,

$$(\lambda^2 - \omega^2)\int_v \phi\psi dv = 0.$$

This equation says that either λ^2 and ω^2 must be equal or ϕ and ψ must be orthogonal. This proves our result so long as λ^2 and ω^2 are not equal. When they are equal we do a little extra work and we *make* ϕ and ψ orthogonal. Suppose $\phi_1, \phi_2, \ldots, \phi_N$ are eigenfunctions all having the same eigenvalue ω^2. $\phi_1, \ldots, \phi_N$ are, of course, all different. Now define a new set of functions

$$\begin{aligned}
\psi_1 &= \phi_1 \\
\psi_2 &= \phi_2 - \frac{(\psi_1, \phi_2)}{(\psi_1, \psi_1)}\psi_1 \\
\psi_3 &= \phi_3 - \frac{(\psi_1, \phi_3)}{(\psi_1, \psi_1)}\psi_1 - \frac{(\psi_2, \phi_3)}{(\psi_2, \psi_2)}\psi_2
\end{aligned}$$

in previous eqns, you have ψ first, then ϕ. now you switch them.

et cetera where

$$(\phi, \psi) = \int_v \phi\psi dv$$

is just a number and is called the "inner-product" of ϕ and ψ. (It is the generalization of the dot product to functions.) The new sequence $\psi_1, \ldots, \psi_N$ has two important properties which you should verify:

1. Every ψ_i is a linear combination of solutions to the transformed wave equation with eigenvalue ω^2 and is, therefore, itself a solution with eighenvalue ω^2.

2. Every ψ_i is orthogonal to all other ψ_j.

The circumstance of having more than one linearly-independent eigenfunction associated with a single eigenvalue is called "degeneracy." Degeneracy occurs quite often in practice but the process of defining a new eigenfunctions set (which, by the way, is called *Gram-Schmidt orthogonalization*) is seldom necessary. As in our case here, it is almost always true that the set $\phi_1, \ldots, \phi_N$ one finds is already orthogonal.

Modes and Waves

The normal modes

$$\psi_{\alpha\beta}(\mathbf{r}, t) = \sigma_{\alpha\beta}(\mathbf{r}) e^{\pm i\omega_{\alpha\beta} t}$$

are standing waves in the sense that although ψ at a fixed point varies sinusoidally in time, lines of constant phase do not move about with time. The motion of lines of constant hase is precisely the characteristic property of traveling waves.

Traveling waves and standing waves are both solutions to the wave equation and we would like to look for a connection between them. In general,

$$\psi_{\alpha\beta}(\mathbf{r}, t) = \sin(\alpha x)\sin(\beta y) e^{\pm i\omega_{\alpha\beta} t},$$

However,

$$\sin(\alpha x) = \frac{1}{2i}(e^{i\alpha x} - e^{-i\alpha x})$$
$$\sin(\beta y) = \frac{1}{2i}(e^{i\beta y} - e^{-i\beta y})$$

i've modified the following eqn. pls chk orig, pg 76.

so

$$\begin{aligned}\psi_{\alpha\beta} &= -\frac{1}{4}(e^{i\alpha x} - e^{-i\alpha x})(e^{i\beta y} - e^{-i\beta y})e^{\pm\omega_{\alpha\beta}t} \\ &= \frac{1}{4}[e^{i(\alpha x - \beta y)} - e^{-i(\alpha x - \beta y)} + \ldots]e^{\pm i\omega_{\alpha\beta}t}.\end{aligned}$$

Thus for each of $\pm\omega_{\alpha\beta}$, we can express ψ as the sum of four traveling waves with wave vectors as shown:

A Figure

Each of these alone will satisfy the wave equation, but the sum of the four is necessary to satisfy the boundary conditions.

Consider the wave

$$e^{i(\alpha x + \beta y - \omega_{\alpha\beta}t)}.$$

Since

$$\begin{aligned} k &= \sqrt{\alpha^2 + \beta^2} \\ \omega &= c\sqrt{\alpha^2 + \beta^2} \end{aligned}$$

the phase velocity of the wave is

$$C = \frac{\omega}{k} = c.$$

The group velocity of the wave is *is U a tensor?*

$$\mathbf{U} = \nabla_{\mathbf{k}}\omega = \frac{\alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{y}}}{\sqrt{\alpha^2+\beta^2}}c$$

which has magnitude c and is parallel to $\alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{y}}$, the wave-number vector.

Initial Value Problems

Now we will use the normal modes of our square domain to actually solve a problem. Suppose that at time $t = 0$ we specify

$$\begin{aligned}\psi(\mathbf{r}, 0) &= 0 \text{ in } \mathbf{U}\\ \partial_t\psi(\mathbf{r}, 0) &= \delta\left(x - \frac{\pi}{2}\right)\delta\left(y - \frac{\pi}{2}\right).\end{aligned}$$

Think of ψ as, say, the vertical displacement of a membrane. Then the initial conditions are that the membrane possesses no initial displacement but does have a velocity spike at $x = y = \pi/2$.

The general solution is

$$\psi(\mathbf{r}, t) = \sum_{\alpha,\beta}\sigma_{\alpha\beta}(\mathbf{r})A_{\alpha\beta}e^{i\omega_{\alpha\beta}t} + B_{\alpha\beta}e^{-i\omega_{\alpha\beta}t}.$$

Also *should there be parenthesis?*

$$\partial_t\psi(\mathbf{r}, 0) = \sum_{\alpha,\beta}\sigma_{\alpha\beta}(\mathbf{r})A_{\alpha\beta} + B_{\alpha\beta} = 0.$$

Our task is to find the set of constants $A_{\alpha\beta}$, $B_{\alpha\beta}$ which cause ψ and $\partial_t\psi$ to satisfy the initial conditions.

First,

$$\psi(\mathbf{r}, 0) = \sum_{\alpha,\beta}\sigma_{\alpha\beta}(\mathbf{r})(A_{\alpha\beta} + B_{\alpha\beta}) = 0.$$

Multiplying by $\sigma_{\gamma\delta}$ and integrating, we arrive at

$$A_{\alpha\beta} + B_{\alpha\beta} = 0 \quad \forall \alpha, \beta.$$

The second initial condition is

$$\partial_t \psi(\mathbf{r}, 0) = \sum_{\alpha,\beta} \sigma_{\alpha\beta}(i\omega_{\alpha\beta})(A_{\alpha\beta} - B_{\alpha\beta}) = \delta\left(x - \frac{\pi}{2}\right)\delta\left(y - \frac{\pi}{2}\right).$$

Now,

$$\left(\sigma_{\gamma\delta}, \delta\left(x - \frac{\pi}{2}\right)\delta\left(y - \frac{\pi}{2}\right)\right) = \sin\left(\gamma\frac{\pi}{2}\right)\sin\left(\delta\frac{\pi}{2}\right)$$

what in the world is this $(-)$ stuff?

which vanishes for *either* γ or δ *even*. When *both* γ and δ are *odd*,

$$\left(\sigma_{\gamma\delta}, \delta\left(x - \frac{\pi}{2}\right)\delta\left(y - \frac{\pi}{2}\right)\right) = (-)^{\frac{\gamma-1}{2}}(-)^{\frac{\delta-1}{2}}.$$

So we get

$$A_{\alpha\beta} - B_{\alpha\beta} = \begin{cases} (-)^{\frac{\alpha-1}{2}}(-)^{\frac{\beta-1}{2}}\frac{-i}{\omega_{\alpha\beta}}\frac{4}{\pi^2} & \forall \alpha, \beta \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Since $-B_{\alpha\beta} = A_{\alpha\beta}$, from the first condition,

$$\begin{aligned} A_{\alpha\beta} - B_{\alpha\beta} &= 2A_{\alpha\beta} \\ A_{\alpha\beta} &= \begin{cases} \frac{-2i}{\pi^2\omega^2\alpha\beta}(-)^{\frac{\alpha-1}{2}}(-)^{\frac{\beta-1}{2}} & \alpha\beta \text{ both odd} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

check this superscripts with previous formula.

The *exact* solution to our problem is

$$\psi(\mathbf{r}, t) = \sum_{\alpha\beta} \sigma_{\alpha\beta}(\mathbf{r})\frac{4}{\pi^2\omega_{\alpha\beta}}(-)^{\alpha-\frac{1}{2}}(-)^{\beta-\frac{1}{2}}\sin(\omega_{\alpha\beta}t)$$

The solution, $\psi(\mathbf{r}, t)$, is expressed as a sum of standing waves, $\sigma_{\alpha\beta}(\mathbf{r})$. Nevertheless, $\psi(\mathbf{r}, t)$ (which is the sum) *looks* like a traveling disturbance. We cannot see this property of $\psi(\mathbf{r}, t)$ by looking at the standing wave sum, but from our study of traveling waves we know that it must be so.

Further, at points away from the center of the square, the sum must exactly vanish at all times before the arrival of the first wave from the source, although as before we cannot determine this property by inspection of the formula. Note that each normal mode starts oscillating merrily along everywhere at time $t = 0$. The requirement that nothing happen before the first wave arrives (which we call "casuality") applies to the sum but not to each normal mode.

Forced Motions

Normal modes are often called "free vibrations" or "free oscillations" because they are solutions to the equations of motion in the absence of external forces. They can, however, be used to study *forced* motions of the system.

Suppose we apply a prescribed force, $f(\mathbf{r}, t)$, to the system; the equation of motion is

$$\partial_t^2 \psi = c^2 \nabla^2 \psi + f$$

plus, of course, some initial conditions. Express ψ as

$$\psi(\mathbf{r}, t) = \sum_{\alpha\beta} h_{\alpha\beta}(t) \sigma_{\alpha\beta}(\mathbf{r})$$

where we are now using the normal mode eigenfunctions as a basis to represent ψ; the coefficients $h_{\alpha\beta}(t)$ are yet-undetermined functions of time. The equations of motion become

$$\sum_{\alpha\beta} (\partial_t^2 h_{\alpha\beta}) \sigma_{\alpha\beta} = \sum_{\alpha\beta} c^2 \nabla^2 (h_{\alpha\beta} \sigma_{\alpha\beta}) + f$$

or

$$\sum_{\alpha\beta} (\partial_t^2 h_{\alpha\beta}) \sigma_{\alpha\beta} = \sum_{\alpha\beta} (-\omega_{\alpha\beta}^2 h_{\alpha\beta} \sigma_{\alpha\beta}) + f.$$

Form the inner product with a particular $\sigma_{\gamma\delta}$ and use the orthogonality of the eigenfunction to arrive at

$$d_t^2 h_{\alpha\beta} = -\omega_{\alpha\beta}^2 h_{\alpha\beta} + \frac{4}{\pi^2} (\sigma_{\alpha\beta}, f).$$

Now

$$g_{\alpha\beta}(t) = (\sigma_{\alpha\beta}, f)$$

is some known (or computable) function of time. We now have an infinite number of ordinary regular differential equations

$$d_t^2 h_{\alpha\beta} = -\omega_{\alpha\beta}^2 h_{\alpha\beta} + \frac{4}{\pi^2} g_{\alpha\beta}(t)$$

the solutions of which enable us to construct the response of the system to a prescribed force distribution, $(\mathbf{r}, t)$.

A complete specification of the problem requires specification of $\psi(\mathbf{r}, 0)$ and $\partial_t \psi(\mathbf{r}, t)$. This in turn leads to initial conditions on $h_{\alpha\beta}$.

Let's do a specific problem. Suppose

$$f(\mathbf{r}, t) = \sin(x)\sin(y)\sin(\lambda t)$$

and

$$\begin{aligned} \psi(\mathbf{r}, 0) &= 0 \\ \partial_t \psi(\mathbf{r}, 0) &= 0. \end{aligned}$$

Obviously

$$g_{\alpha\beta} = \begin{cases} \left(\frac{\pi^2}{4}\right)\sin(\lambda t) & \text{when } (\alpha, \beta) = (1, 1) \\ 0 & \text{all other } (\alpha, \beta). \end{cases}$$

Also

$$h_{\alpha\beta}(0) = d_t h_{\alpha\beta}(0) = 0 \quad \forall \alpha, \beta.$$

Therefore, except for $h_{11}(t)$, all $h_{\alpha\beta} = 0$ for all time. Now we must solve

$$d_t^2 h_{11}(t) = -\omega_{11}^2 h_{11}(t) + \sin(\lambda t)$$

or

$$d_t^2 h = -\omega^2 h + \sin(\lambda t) \text{ where } h(0) = d_t h(0) = 0.$$

The solution to this equation for the case $\lambda \neq \omega$ which satisfies the initial conditions is

$$h(t) = \frac{1}{\omega^2 - \lambda^2}\left[\sin(\lambda t) - \frac{\lambda}{\omega}\sin(\omega t)\right].$$

$h(t)$ has two parts. One portion vibrates with frequency ω and represents a free mode which was excited at $t = 0$ when the force was applied. If the physical system possesses any damping whatever, this portion will ultimately decay. The second portion varies with angular frequency λ and represents the forced response of the system. In the absence of the free motion,

$$h(t) = \frac{\sin(\lambda t)}{\omega^2 - \lambda^2}.$$

For λ^2 near ω^2, h may become very large.

13.2 Generalities and Rayleigh's Principle

Let us generalize our problem by expressing

$$c^2\nabla^2\sigma = -\omega^2\sigma$$

as

$$L\sigma = \lambda\sigma.$$

Here L stands for $c^2\nabla^2$ and is called an "operator." An operator is something which takes a function, does something to it, and gives us back another function. If this gets confusing, you can always think about $c^2\nabla^2$. When we speak of L, we always understand that it includes a set of boundary conditions, so L is a kind of shorthand for the whole problem.

We suppose the L is *linear*, so

$$L(\sigma_1 + \sigma_2) = L\sigma_1 + L\sigma_2$$

and *Hermitian*, a big word meaning that

$$\int_U \phi L\sigma dv = \int_U \sigma L\phi dv$$

or in more compact form

$$(\phi, L\sigma) = (\sigma, L\phi)$$

where ϕ and σ are any functions obeying the boundary conditions (and having enough derivates, *etc.*). We have already shown that $c^2\nabla^2$ is Hermitian.

The eigenvalues, λ_i, and eigenfunctions, σ_i, of a Hermitian operator, L, satisfy

$$L\sigma_i = \lambda_i\sigma_i$$

and we remember that we must also apply boundary conditions as well. This is just a fancy way of specifying a normal mode problem without getting too specific about it. There will usually be an infinite sequence of eigensolutions $\{\lambda_i, \sigma_i\}$.

The eigenvalues of a Hermitian operator are *real*, even if the operator or the eigenfunctions are complex. (We have not here developed the complex notation needed to express that result so we will not prove it here. But it is true.)

Also, eigenfunctions belonging to different eigenvalues are *orthogonal*. This follows because

$$\begin{aligned}(\sigma_i, L\sigma_j) - (\sigma_j, L\sigma_i) &= 0 \quad \text{(Hermitian)} \\ (\lambda_j - \lambda_i)(\sigma_j, \sigma_i) &= 0\end{aligned}$$

jst as before. Also as before, we can apply the Gram-Schmidt process to orthogonalize a set of eigenfunctions sharing a common eigenvalue.

Let us look at the initial-value problem in this light. This problem typically has the form

$$\begin{aligned}\partial_t^2 \psi &= L\psi \quad \text{in } V \\ \psi(\mathbf{r}, 0) &= f(\mathbf{r}) \\ \partial_t \psi(\mathbf{r}, 0) &= g(\mathbf{r})\end{aligned}$$

and L is a wholly spatial operator (that is, no time derivatives). We express ψ as an *eigenfunction expansion*

$$\psi = \sum_i h_i(t) \sigma_i(\mathbf{r}).$$

We also suppose the σ_i normalized so $(\sigma_i, \sigma_j) = \delta_{ij}$

$$\sum_i (\partial_t^2 h_i) \sigma_i = \sum h_i L \sigma_i$$

giving

$$\partial_t^2 h_i = \lambda_i h_i$$

after we have exercised orthonormality. The initial conditions are

$$\begin{aligned} h_i(0) &= (\sigma_i, f) \\ \partial_t h_i(0) &= (\sigma_i, g) \end{aligned}$$

which is enough information to compute $h_i(t)$. From the differential equation, we can see that $h_i(t)$ will be periodic with angular frequency

$$\omega_i = \sqrt{-\lambda_i}.$$

The forced problem proceeds in about the same way from the differential equation

$$\partial_t^2 \psi = L\psi + f(\mathbf{r}, t).$$

You might try working this through.

do reference.

Other general properties of eigenfunctions and eigenvalues of Hermitian operators are discussed in very straightforward terms in Chapter 9 of

> Mathews, J., and Walker, R.L., *Mathematical Methods of Physics*, W.A. Benjamin, Inc., New Your, 1965, 475 pages.

This book is a good, fairly compact source of the most frequent tricks of the trade.

Our principal motive for introducing this formulation was to develop a way to derive an important result of perturbation theory known to geophysicists as "Rayleigh's Principle." This result is a method for estimating the perturbation in a particular eigenvalue of a Hermitian operator resulting from a small perturbatoin in the operator—which amounts to a small perturbation in the problem. The significance of this result is due to the fact that some of the most useful seismological data are measurements of eigenvalues—namely, the frequencies of the Earth's free elastic-gravitational vibrations.

Suppose

$$L\sigma_i = \lambda_i \sigma_i$$

is an eigenvalue problem we can solve. We want to *estimate* the eigenvalues of a second problem

$$L^{\nabla}\sigma_i^{\nabla} = \lambda_i^{\nabla}\sigma_i^{\nabla}$$

which is "close" to our first problem in the sense that

$$L^{\nabla} = L + R$$

I've used ∇'s as the superscript as you had them. A $'$ might be better though.

where R is a "much smaller operator" than L. We must also assume that the eigenfunctions and eigenvalues of L^{∇} are close to those of L. (This will not always be the case for certain perturbations R no matter how small. Such instances are called "singular perturbations" and we will forbid them here.)

Let us concentrate on a specific eigenvalue, λ_i. We assumed

$$\lambda_i^{\nabla} = \lambda_i + \delta_i \quad \text{where } |\delta_i| \ll |\lambda_i|.$$

Since the set $\{\sigma_i\}$ is complete, we can express σ_i^{∇} as

$$\sigma_i^{\nabla} = \sigma_i + \sum_{k\neq i}\alpha_k\sigma_k \quad \text{where } |\alpha_k| \ll 1.$$

We want

$$(\zeta + R)(\sigma_i + \sum_{k\neq i}\alpha_k\sigma_k) = (\lambda_i + \delta_i)(\sigma_i + \sum_{k\neq i}\alpha_k\sigma_k).$$

Since R, α_k, and δ_i are first order, we have (through first order):

$$L\sigma_i + R\sigma_i + \sum_{k\neq i}\alpha_k L\sigma_k = \lambda_i\sigma_i + \delta_i\sigma_i + \lambda_i\sum_{k\neq i}\alpha_k\sigma_k.$$

But $L\sigma_i = \lambda_i\sigma_i$. Froming the inner product of σ_i with the remainder, and since

$$(\sigma_i, \sigma_j) = \delta_{ij}$$

and

$$(\sigma_i, L\sigma_j) = \lambda_i\delta_{ij}$$

we are left with

$$(\sigma_i, R\sigma_i) = \delta_i(\sigma_i, \sigma_j)$$

or

$$\delta_i = \frac{(\sigma_i, R\sigma_i)}{(\sigma_i, \sigma_i)}.$$

This is Rayleigh's Principle.

Let us try a short example to give this result some substance. Suppose that in our square (from the first part) we perturb the quantity c^2 slightly.

reference the figure number of the square.

$$c^{\nabla 2} = c^2 + \mu^2$$

where $|\mu^2| \ll c^2$ but is otherwise a general function of position. Then

$$L^{\nabla} = (c^2 + \mu^2)\nabla^2$$

and

$$R = \mu^2 \nabla^2.$$

The eigenvalues of the box are $\lambda_{\alpha\beta} = -\omega^2_{\alpha\beta}$. If

$$\begin{aligned} \omega^{\nabla}_{\alpha\beta} &= \omega_{\alpha\beta} + \nu_{\alpha\beta} \\ \lambda^{\nabla}_{\alpha\beta} &= -(\omega^{\nabla}_{\alpha\beta})^2 = -\omega^2_{\alpha\beta} - 2\omega_{\alpha\beta}\nu_{\alpha\beta}. \end{aligned}$$

So,

$$\Delta_{\alpha\beta} = -2\omega_{\alpha\beta}\nu_{\alpha\beta}$$

relates the perturbation in frequency to the perturbation in the eigenvalue. The latter is given by

$$\Delta_{\alpha\beta} = \frac{(\sigma_{\alpha\beta}, \mu^2\nabla^2\sigma_{\alpha\beta}}{(\sigma_{\alpha\beta}, \sigma_{\alpha\beta})}.$$

So,

$$-2\omega_{\alpha\beta}\nu_{\alpha\beta} = \frac{-4\omega^2_{\alpha\beta}}{\pi^2 c^2}(\sigma_{\alpha\beta}\sigma_{\alpha\beta}, \mu^2)$$

or

i put $\nu_{\alpha\beta}$ instead of $\nu^{\alpha\beta}$ as in orig.

$$\nu_{\alpha\beta} = \frac{2\omega_{\alpha\beta}}{c^2\pi^2}(\sigma_{\alpha\beta}, \mu^2\sigma_{\alpha\beta}).$$