D) KIRSCH's PROBLEM - THE CIRCULAR BORE HOLE / TUNNEL

D.1) The model.

It is assumed that the whole 3-dimensional space is represented by an elastic, homogeneous, isotropic medium, having the elastic constants denoted by E and ν, respectively λ and μ. A co-ordinate system having the third axis positive upward will be used. The initial state of stress is represented by the homogeneous tensor σ^0, corresponding to a planar state of deformation, i.e.

$$\sigma^0 = \begin{pmatrix} \sigma_{11}^0 & \sigma_{12}^0 & 0 \\ \sigma_{12}^0 & \sigma_{22}^0 & 0 \\ 0 & 0 & \nu\left(\sigma_{11}^0 + \sigma_{22}^0\right) \end{pmatrix}, \tag{d1}$$

where the components σ_{ij}^0 have constant values. The mass forces are ignored, hence the equilibrium equation

$$\operatorname{div} \sigma^0 = \vec{0} \tag{d2}$$

is identically satisfied.
Suppose that a circular, infinite bore hole / tunnel is performed along the third axis, its material being instantly removed. The origin of the co-ordinate system is placed at the centre of the cavity. On the wall of the bore hole is acting now the atmospheric pressure (or the pressure of the drilling mud), denoted by p_0. Consequently, a new (non-homogeneous) stress value is obtained and the circular shape of the bore hole is changing too. It follows to obtain the new stress, denoted by σ^f, and the new shape of the bore hole in the final equilibrium stage, where

$$\operatorname{div} \sigma^f = \vec{0} \ . \tag{d3}$$

It is also assumed that the deformation is an elastic one, i.e. the stress perturbation $\sigma = \sigma^f - \sigma^0$ is related to the strain tensor by

$$\sigma = \lambda \operatorname{tr} \varepsilon \, \mathbf{1} + 2\mu \varepsilon \ . \tag{d4}$$

The unknown components of the displacement vector are supposed to correspond to a planar deformation state, i.e.

$$u_1 = u_1(x_1, x_2) \quad , \quad u_2 = u_2(x_1, x_2) \quad , \quad u_3 = 0 . \tag{d5}$$

Because the symmetry of the problem, the cylindrical co-ordinate system (r, θ, z) will be used, having the unit vectors denoted by $\left(\vec{e}_r, \vec{e}_\theta, \vec{e}_z\right)$ (see Fig.D1).

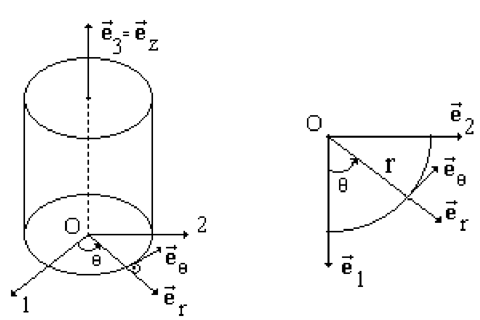

Fig.D1. The cylindrical co-ordinate system.

D.2) The planar sate of deformation in cylindrical co-ordinate system.

With respect to Fig.D1 it follows that

$$\begin{cases} \vec{e}_r = \cos\theta\, \vec{e}_1 + \sin\theta\, \vec{e}_2 , \\ \vec{e}_\theta = -\sin\theta\, \vec{e}_1 + \cos\theta\, \vec{e}_2 , \\ \vec{e}_z = \vec{e}_3 . \end{cases} \tag{d6}$$

Hence the matrix for passing from the Cartesian co-ordinates to cylindrical co-ordinates is

$$Q = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{d7}$$

It represents a rotation of angle equal to θ in a positive (counter clockwise) sense. From (d4) and (d5) it follows that the stress matrix in Cartesian co-ordinates is

$$[\sigma]^{crt} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \nu(\sigma_{11}+\sigma_{22}) \end{pmatrix} . \tag{d8}$$

Let the stress matrix in cylindrical co-ordinates be

$$[\sigma]^{cyl} = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{r\theta} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{rz} & \sigma_{\theta z} & \sigma_{zz} \end{pmatrix} . \tag{d9}$$

It follows that

$$[\sigma]^{cyl} = Q[\sigma]^{crt} Q^t = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \nu(\sigma_{11}+\sigma_{12}) \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{d10}$$

By performing the computations in (d10), it follows

$$\sigma_{rr} = \frac{\sigma_{11}+\sigma_{22}}{2} + \sigma_{12}\sin 2\theta + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta \;, \tag{d11}$$

$$\sigma_{\theta\theta} = \frac{\sigma_{11}+\sigma_{22}}{2} - \sigma_{12}\sin 2\theta - \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta \;, \tag{d12}$$

$$\sigma_{r\theta} = -\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta \;, \tag{d13}$$

$$\sigma_{rz} = \sigma_{\theta z} = 0 \quad , \quad \sigma_{zz} = \nu(\sigma_{11}+\sigma_{22}) = \nu(\sigma_{rr}+\sigma_{\theta\theta}) \;. \tag{d14}$$

D.3) The circle of MOHR.

Suppose the Cartesian co-ordinate system is selected in order its axes to be along the first two eigen vectors of the stress tensor. In that case, σ_{11} and σ_{22} are eigenvalues of the stress tensor and $\sigma_{12}=0$. From equations (d11)-(d13) it follows that

$$\left(\sigma_{rr}-\frac{\sigma_{11}+\sigma_{22}}{2}\right)^2+\sigma_{r\theta}^2=\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 , \qquad \text{(d15)}$$

and an identical relation obtained by replacing σ_{rr} with $\sigma_{\theta\theta}$. Eq.(d15) shows that σ_{rr} and $\sigma_{r\theta}$ are placed on a circle of radius equal to $\left|\sigma_{11}-\sigma_{22}\right|/2$. Suppose now that σ_{rr} (or $\sigma_{\theta\theta}$) (i.e. the radial stress component, usually denoted by σ) and $\sigma_{r\theta}$ (i.e. the tangential stress, usually denoted by τ) are obtained at various angles θ and the MOHR's circle represented by eq.(d15) is obtained. Its radius and its position of the centre allow one to obtain graphically the eigenvalues of the stress tensor. Further discussion will be presented in relation to the empirical failure criteria of materials.

D.4) AIRY's potential in cylindrical co-ordinates. The bi-harmonic equation.

Consider the representation of the stress components with the AIRY's potential in Cartesian co-ordinates, i.e.

$$\sigma_{11}=A_{,22} \quad ,\sigma_{22}=A_{,11} \quad ,\sigma_{12}=-A_{,12} \quad , \qquad \text{(d16)}$$

where the AIRY's potential verifies the bi-harmonic equation

$$\Delta^*\Delta^* A \;=0 \quad . \qquad \text{(d17)}$$

In the beginning, the derivatives in eq.(d16) will be evaluated by using the polar co-ordinates

$$\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{cases} \quad ,\begin{cases} r=\sqrt{x_1^2+x_2^2} \\ \theta = a\tan(x_2/x_1) \end{cases} \qquad \text{(d18)}$$

Than, a representation of the stress components in cylindrical co-ordinates with the help of the AIRY's potential will be obtained from (d11)-(d13).
But

$$\frac{\partial r}{\partial x_1}=\frac{x_1}{r}=\cos\theta \quad ,\frac{\partial r}{\partial x_2}=\frac{x_2}{r}=\sin\theta \quad ,\frac{\partial\theta}{\partial x_1}=-\frac{x_2}{r^2}=-\frac{\sin\theta}{r} \quad ,\frac{\partial\theta}{\partial x_2}=\frac{x_1}{r^2}=\frac{\cos\theta}{r} \qquad \text{(d19)}$$

It follows

$$A_{,1}=\frac{\partial A}{\partial x_1}=\frac{\partial A}{\partial r}\frac{\partial r}{\partial x_1}+\frac{\partial A}{\partial\theta}\frac{\partial\theta}{\partial x_1}=\cos\theta\, A_{,r}-\frac{\sin\theta}{r}A_{,\theta} \qquad \text{(d20)}$$

In the same way,

$$A_{,2}=\sin\theta\, A_{,r}+\frac{\cos\theta}{r}A_{,\theta} \quad . \qquad \text{(d21)}$$

Also,

$$\begin{aligned} A_{,12} &= \left(A_{,1}\right)_{,2}=\left(\cos\theta\, A_{,r}-\frac{\sin\theta}{r}A_{,\theta}\right)_{,2} \\ &= \sin\theta\left(\cos\theta\, A_{,r}-\frac{\sin\theta}{r}A_{,\theta}\right)_{,r}+\frac{\cos\theta}{r}\left(\cos\theta\, A_{,r}-\frac{\sin\theta}{r}A_{,\theta}\right)_{,\theta} \\ &= \left(A_{,rr}-\frac{A_{,r}}{r}-\frac{A_{,\theta\theta}}{r^2}\right)\frac{\sin 2\theta}{2}+\left(\frac{A_{,r\theta}}{r}-\frac{A_{,\theta}}{r^2}\right)\cos 2\theta \end{aligned} \qquad \text{(d22)}$$

In the same way

$$A_{,11}=\cos^2\theta\, A_{,rr}+\sin^2\theta\left(\frac{A_{,r}}{r}+\frac{A_{,\theta\theta}}{r^2}\right)+2\sin\theta\cos\theta\left(-\frac{A_{,r\theta}}{r}+\frac{A_{,\theta}}{r^2}\right) \;, \qquad \text{(d23)}$$

$$A_{,22} = \sin^2\theta\, A_{,rr} + \cos^2\theta\left(\frac{A_{,r}}{r} + \frac{A_{,\theta\theta}}{r^2}\right) - 2\sin\theta\cos\theta\left(-\frac{A_{,r\theta}}{r} + \frac{A_{,\theta}}{r^2}\right) \quad , \quad \text{(d24)}$$

Hence

$$\sigma_{11} + \sigma_{22} = A_{,22} + A_{,11} = A_{,rr} + \frac{A_{,r}}{r} + \frac{A_{,\theta\theta}}{r^2} \quad , \quad \text{(d25)}$$

$$\sigma_{11} - \sigma_{22} = A_{,22} - A_{,11} = \left(-A_{,rr} + \frac{A_{,r}}{r} + \frac{A_{,\theta\theta}}{r^2}\right)\cos 2\theta - 2\sin 2\theta\left(-\frac{A_{,r\theta}}{r} + \frac{A_{,\theta}}{r^2}\right) \quad , \quad \text{(d26)}$$

From eqs.(d11)-(d13) it follows

$$\sigma_{rr} = \frac{A_{,r}}{r} + \frac{A_{,\theta\theta}}{r^2} \quad , \quad \sigma_{\theta\theta} = A_{,rr} \quad , \quad \sigma_{r\theta} = -\frac{A_{,r\theta}}{r} + \frac{A_{,\theta}}{r^2} \quad . \quad \text{(d27)}$$

D.5) The divergence of a tensor in cylindrical co-ordinates.

In the case of the cylindrical co-ordinates, the square of the elementary arc is equal to

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 = dr^2 + r^2 d\theta^2 + dz^2 \quad \text{(d28)}$$

Hence the differential parameters of Lamé are

$$h^1 = 1 \quad , \quad h^2 = r \quad , \quad h^3 = 1 \quad , \quad \text{(d29)}$$

the orthogonal curvilinear co-ordinates are equal to

$$c^1 = 1 \quad , \quad c^2 = \theta \quad , \quad c^3 = z \quad , \quad \text{(d30)}$$

and the unit vectors are

$$\vec{n}^1 = \vec{e}_r \quad , \quad \vec{n}^2 = \vec{e}_\theta \quad , \quad \vec{n}^3 = \vec{e}_z \quad . \quad \text{(d31)}$$

It follows that

$$\frac{\partial h^q}{\partial c^\beta} = \delta^{q2}\delta^{\beta 1} \quad . \quad \text{(d32)}$$

Substituting the above results in the formula (a2.77) (Ivan 1996) it follows the next formula for the divergence of a tensor in cylindrical co-ordinates

$$\operatorname{div}T = \left(\frac{\partial T_{rr}}{\partial r} + \frac{\partial T_{r\theta}}{r\partial\theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r}\right)\vec{e}_r + \left(\frac{\partial T_{r\theta}}{\partial r} + \frac{\partial T_{\theta\theta}}{r\partial\theta} + \frac{\partial T_{\theta z}}{\partial z} + 2\frac{T_{r\theta}}{r}\right)\vec{e}_\theta + \left(\frac{\partial T_{rz}}{\partial r} + \frac{\partial T_{\theta z}}{r\partial\theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{rz}}{r}\right)\vec{e}_z \quad \text{(d33)}$$

D.6) The gradient of a vector and the strain tensor in cylindrical co-ordinates.

Substituting the above results in the formula (a2.68) (Ivan 1996) it follows the next formula for the gradient of a vector in cylindrical co-ordinates

$$\mathrm{grad}\ \vec{u} = \frac{\partial u_r}{\partial r}\vec{e}_r \otimes \vec{e}_r + \frac{\partial u_r}{r\partial\theta}\vec{e}_r \otimes \vec{e}_\theta + \frac{\partial u_r}{\partial z}\vec{e}_r \otimes \vec{e}_z$$
$$+\frac{\partial u_\theta}{\partial r}\vec{e}_\theta \otimes \vec{e}_r + \frac{\partial u_\theta}{r\partial\theta}\vec{e}_\theta \otimes \vec{e}_\theta + \frac{\partial u_\theta}{\partial z}\vec{e}_\theta \otimes \vec{e}_z \quad . \qquad \text{(d34)}$$
$$+\frac{\partial u_z}{\partial r}\vec{e}_z \otimes \vec{e}_r + \frac{\partial u_z}{r\partial\theta}\vec{e}_z \otimes \vec{e}_\theta + \frac{\partial u_z}{\partial z}\vec{e}_z \otimes \vec{e}_z - \frac{u_\theta}{r}\vec{e}_r \otimes \vec{e}_\theta + \frac{u_r}{r}\vec{e}_\theta \otimes \vec{e}_\theta$$

The components of the strain tensor $\varepsilon = \frac{1}{2}\left[\mathrm{grad}\ \vec{u} + \left(\mathrm{grad}\ \vec{u}\right)^t\right]$ are

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \ , \ \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u_r}{r\partial\theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) \ , \ \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$
$$\varepsilon_{\theta\theta} = \frac{\partial u_\theta}{r\partial\theta} + \frac{u_r}{r} \ , \ \varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{\partial u_z}{r\partial\theta}\right) \ , \ \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \quad . \qquad \text{(d35)}$$

For the particular displacement field represented by (d5), the components of the vector $\vec{u}$ are

$$u_r = u_r(r,\theta) \ , \ u_\theta = u_\theta(r,\theta) \ , \ u_z = 0 \quad . \qquad \text{(d36)}$$

Hence

$$\varepsilon_{rz} = \varepsilon_{\theta z} = \varepsilon_{zz} = 0 \quad . \qquad \text{(d37)}$$

It is the case of a planar state of deformation, i.e.

$$\varepsilon = \begin{pmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & 0 \\ \varepsilon_{r\theta} & \varepsilon_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \qquad \text{(d38)}$$

and all the strain elements are functions of r and θ.
Consequently, the stress is

$$\sigma = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & 0 \\ \sigma_{r\theta} & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right) \end{pmatrix} \ , \qquad \text{(d39)}$$

all the components of the stress being too functions of r and θ, according to HOOKE's reversed law. It follows from (d33) the next equations of equilibrium are obtained in the absence of mass forces:

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{r\theta}}{r\partial\theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \qquad \text{(d40)}$$
$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{\partial\sigma_{\theta\theta}}{r\partial\theta} + 2\frac{\sigma_{r\theta}}{r} = 0 \qquad \text{(d41)}$$

D.7) The bi-harmonic equation in cylindrical co-ordinates.

By using (a2.69) and (a2.43) (Ivan 1996), it follows the LAPLACE operator in cylindrical co-ordinates is

$$\Delta f = \mathrm{div}(\mathrm{grad}\, f) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \quad . \tag{d42}$$

The AIRY's potential is also a function of r and θ. Hence the AIRY's potential is the solution of the bi-harmonic equation

$$\Delta^* \Delta^* \mathcal{A} = 0 \quad , \tag{d43}$$

where the LAPLACE operator in polar co-ordinates is

$$\Delta^* = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \quad . \tag{d44}$$

It should be noted that the singular point $r = 0$ is avoided in (d44) because $r \geq R > 0$, where R is the radius of the bore hole.
It follows to solve (d43) by using (d44) in order to derive the AIRY's potential. The stress components will be obtained from (d27), imposing the boundary conditions on the wall of the bore hole. The components of the strain will be derived by using the HOOKE's reversed law. The displacement vector will be obtained from the definition of strain elements, allowing one to find the final shape of the deformed bore hole wall.
Consider the FOURIER expansion of the AIRY's potential, having the coefficients equal to functions of r

$$\mathcal{A}(r,\theta) = \mathcal{A}_0(r) + \sum_{n=1}^{\infty}\left[\mathcal{A}_n(r)\cos n\theta + \mathcal{B}_n(r)\sin n\theta\right] \tag{d45}$$

It follows

$$\frac{\partial \mathcal{A}}{\partial r} = \mathcal{A}_0' + \sum_{n=1}^{\infty}\left(\mathcal{A}_n'\cos n\theta + \mathcal{B}_n'\sin n\theta\right) \quad , \tag{d46}$$

$$\frac{\partial^2 \mathcal{A}}{\partial r^2} = \mathcal{A}_0'' + \sum_{n=1}^{\infty}\left(\mathcal{A}_n''\cos n\theta + \mathcal{B}_n''\sin n\theta\right) \quad , \tag{d47}$$

$$\frac{\partial^2 \mathcal{A}}{\partial \theta^2} = -\sum_{n=1}^{\infty} n^2\left(\mathcal{A}_n\cos n\theta + \mathcal{B}_n\sin n\theta\right) \quad . \tag{d48}$$

Hence

$$\Delta \mathcal{A} = \Delta \mathcal{A}_0 + \sum_{n=1}^{\infty}\left[\left(\mathcal{A}'' + \frac{1}{r}\mathcal{A}' - \frac{n^2}{r^2}\mathcal{A}\right)\cos n\theta + \left(\mathcal{B}'' + \frac{1}{r}\mathcal{B}' - \frac{n^2}{r^2}\mathcal{B}\right)\sin n\theta\right] \tag{d49}$$

and

$$\begin{aligned}\Delta\Delta \mathcal{A} = \Delta\Delta \mathcal{A}_0 + \sum_{n=1}^{\infty}\Bigg[&\left(\mathcal{A}'''' + \frac{2}{r}\mathcal{A}''' - \frac{2n^2+1}{r^2}\mathcal{A}'' + \frac{2n^2+1}{r^3}\mathcal{A}' + \frac{n^4-4n^2}{r^4}\mathcal{A}\right)\cos n\theta \\ &+ \left(\mathcal{B}'''' + \frac{2}{r}\mathcal{B}''' - \frac{2n^2+1}{r^2}\mathcal{B}'' + \frac{2n^2+1}{r^3}\mathcal{B}' + \frac{n^4-4n^2}{r^4}\mathcal{B}\right)\sin n\theta\Bigg] \quad .\end{aligned} \tag{d50}$$

By using (d50), the bi-harmonic (d43) is verified if

$$\Delta\Delta \mathcal{A}_0 = 0 \quad , \tag{d51}$$

and if the functions $\mathcal{A}$, $\mathcal{B}$ are the solutions of the next differential equation

$$\Phi'''' + \frac{2}{r}\Phi''' - \frac{2n^2+1}{r^2}\Phi'' + \frac{2n^2+1}{r^3}\Phi' + \frac{n^4-4n^2}{r^4}\Phi = 0 \quad . \tag{d52}$$

Because $\mathcal{A}_0$ is a function of r only, eq.(d51) is

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathcal{A}_0}{dr}\right)\right]\right\} = 0 \quad . \tag{d53}$$

Hence

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_0}{dr}\right)\right]=a_0 \quad , \quad \frac{d}{dr}\left(r\frac{dA_0}{dr}\right)=a_0 r\ln r+b_0 r \quad . \tag{d54}$$

But

$$\int r\ln r dr=\frac{r^2}{2}\left(\ln r-\frac{1}{2}\right)+\text{const.} \tag{d55}$$

Hence

$$r\frac{dA_0}{dr}=\frac{a_0}{2}r^2\left(\ln r-\frac{1}{2}\right)+\frac{b_0}{2}r^2+c_0 \tag{d56}$$

Finally, denoting again the constants, it follows

$$A_0(r)=a_0 r^2\ln r+b_0 r^2+c_0\ln r+d_0 \quad . \tag{d57}$$

In order to solve eq.(52), a solution of the form

$$\Phi=r^m \tag{d58}$$

is considered. Substituting (d58) in (d52), it follows that the exponent m is the solution of the algebraic equation

$$m(m-1)(m-2)(m-3)+2m(m-1)(m-2)-(2n^2+1)m(m-1)+(2n^2+1)m+n^4-4n^2=0 \text{ ,(d59)}$$

having the roots

$$m_1=-n \quad , \quad m_2=-n+2 \quad , \quad m_3=n \quad , \quad m_4=n+2 \quad . \tag{d60}$$

Hence the AIRY's potential is

$$A(r,\theta)=a_0 r^2\ln r+b_0 r^2+c_0\ln r+d_0+\sum_{n=1}^{\infty}\left(a_n r^{n+2}+b_n r^n+c_n r^{-n+2}+d_n r^{-n}\right)\cos n\theta$$
$$+\sum_{n=1}^{\infty}\left(\alpha_n r^{n+2}+\beta_n r^n+\gamma_n r^{-n+2}+\delta_n r^{-n}\right)\sin n\theta \text{ ,(d61)}$$

where the unknown coefficients $a_i, b_i, c_i, d_i, \alpha_j, \beta_j, \gamma_j, \delta_j, \quad i=\overline{0,1,2,\dots} \quad , \quad j=\overline{1,2,\dots}$ follows to be obtained.

D.8) The stress elements. Conditions at infinity for the stress elements.

By using (d61) and (d27), it follows

$$\sigma_{\theta\theta}=A_{,rr}=a_0(2\ln r+3)+2b_0-\frac{c_0}{r^2}$$
$$+\sum_{n=1}^{\infty}\left(a_n(n+2)(n+1)r^n+b_n n(n-1)r^{n-2}+c_n(n-2)(n-1)r^{-n}+d_n n(n+1)r^{-n-2}\right)\cos n\theta \quad \text{(d62)}$$
$$+\sum_{n=1}^{\infty}\left(\alpha_n(n+2)(n+1)r^n+\beta_n n(n-1)r^{n-2}+\gamma_n(n-2)(n-1)r^{-n}+\delta_n n(n+1)r^{-n-2}\right)\sin n\theta$$

At great distances from the cylindrical cavity, the elastic perturbation has to vanish, i.e.

$$\lim_{r\to\infty}\sigma_{\theta\theta}=0 \quad . \tag{d63}$$

It follows

$$a_0=0 \quad , \quad 2b_0+2b_2+2\beta_2=0 \quad , \quad a_n=\alpha_n=0 \quad , n=\overline{1,2,\dots}$$
$$b_n=\beta_n=0 \quad , n=\overline{3,4,\dots} \tag{d64}$$

Hence the AIRY's potential is

$$A(r,\theta) = c_0 \ln r + d_0 + (b_1 + \beta_1) r + \sum_{n=1}^{\infty} (c_n \cos n\theta + \gamma_n \sin n\theta) r^{-n+2} + \sum_{n=1}^{\infty} (d_n \cos n\theta + \delta_n \sin n\theta) r^{-n} \tag{d65}$$

and

$$\sigma_{\theta\theta} = -\frac{c_0}{r^2} + \sum_{n=1}^{\infty} (n-2)(n-1)(c_n \cos n\theta + \gamma_n \sin n\theta) r^{-n} + \sum_{n=1}^{\infty} n(n+1)(d_n \cos n\theta + \delta_n \sin n\theta) r^{-n-2} \quad . \tag{d66}$$

From (d65), it follows that

$$A_{,r} = \frac{c_0}{r} + b_1 + \beta_1 + \sum_{n=1}^{\infty} (-n+2)(c_n \cos n\theta + \gamma_n \sin n\theta) r^{-n+1} + \sum_{n=1}^{\infty} (-n)(d_n \cos n\theta + \delta_n \sin n\theta) r^{-n-1} \quad , \tag{d67}$$

$$A_{,\theta} = \sum_{n=1}^{\infty} n(-c_n \sin n\theta + \gamma_n \cos n\theta) r^{-n+2} + \sum_{n=1}^{\infty} n(-d_n \sin n\theta + \delta_n \cos n\theta) r^{-n} \quad , \tag{d68}$$

$$A_{,\theta\theta} = -\sum_{n=1}^{\infty} n^2 (c_n \cos n\theta + \gamma_n \sin n\theta) r^{-n+2} - \sum_{n=1}^{\infty} n^2 (d_n \cos n\theta + \delta_n \sin n\theta) r^{-n} \quad . \tag{d69}$$

Hence

$$\sigma_{rr} = \frac{A_{,r}}{r} + \frac{A_{,\theta\theta}}{r^2} = \frac{c_0}{r^2} + \frac{b_1 + \beta_1}{r} - \sum_{n=1}^{\infty} \left(n^2 + n - 2\right)(c_n \cos n\theta + \gamma_n \sin n\theta) r^{-n} - \sum_{n=1}^{\infty} \left(n^2 + n\right)(d_n \cos n\theta + \delta_n \sin n\theta) r^{-n-2} \tag{d70}$$

From (d70), it follows that

$$\lim_{r \to \infty} \sigma_{rr} = 0 \quad . \tag{d71}$$

Also,

$$A_{,r\theta} = \sum_{n=1}^{\infty} n(n-2)(c_n \sin n\theta - \gamma_n \cos n\theta) r^{-n+1} + \sum_{n=1}^{\infty} n^2 (d_n \sin n\theta - \delta_n \cos n\theta) r^{-n-1} \quad . \tag{d72}$$

Hence

$$\sigma_{r\theta} = -\frac{A_{,r\theta}}{r} + \frac{A_{,\theta}}{r^2} = -\sum_{n=1}^{\infty} (n^2 - n)(c_n \sin n\theta - \gamma_n \cos n\theta) r^{-n} - \sum_{n=1}^{\infty} \left(n^2 + n\right)(d_n \sin n\theta - \delta_n \cos n\theta) r^{-n-2} \tag{d73}$$

From (d73), it follows that

$$\lim_{r \to \infty} \sigma_{r\theta} = 0 \quad . \tag{d74}$$

D.9) Strain and displacement vector. Conditions at infinity.

From (d39) and HOOKE's reversed law it follows that

$$\boldsymbol{\varepsilon} = \frac{1}{E}\left[(1+\nu)\boldsymbol{\sigma} - \nu \operatorname{tr}\boldsymbol{\sigma}\,\mathbf{1}\right] = \frac{1+\nu}{E}\left[\boldsymbol{\sigma} - \nu\left(\sigma_{rr}+\sigma_{\theta\theta}\right)\mathbf{1}\right] \tag{d75}$$

i.e.

$$\varepsilon_{rr} = \frac{1-\nu^2}{E}\left[\sigma_{rr} - \frac{\nu}{1-\nu}\sigma_{\theta\theta}\right] = \frac{\partial u_r}{\partial r} \quad , \tag{d76}$$

Hence

$$\begin{aligned}\frac{E}{1-\nu^2}\frac{\partial u_r}{\partial r} = {} & \frac{c_0}{1-\nu}\frac{1}{r^2} + \frac{b_1+\beta_1}{r} - \sum_{n=2}^{\infty}(n-1)\left[n+2+\frac{\nu}{1-\nu}(n-2)\right]\left(c_n\cos n\theta + \gamma_n \sin n\theta\right)r^{-n} \\ & - \frac{1}{1-\nu}\sum_{n=1}^{\infty} n(n+1)\left(d_n \cos n\theta + \delta_n \sin n\theta\right)r^{-n-2}\end{aligned} \tag{d77}$$

Integrating (d77) it follows

$$\begin{aligned}\frac{E}{1-\nu^2}u_r = {} & -\frac{c_0}{1-\nu}\frac{1}{r} + \left(b_1+\beta_1\right)\ln r + \sum_{n=2}^{\infty}\left[n+2+\frac{\nu}{1-\nu}(n-2)\right]\left(c_n\cos n\theta + \gamma_n \sin n\theta\right)r^{-n+1} \\ & + \frac{1}{1-\nu}\sum_{n=1}^{\infty} n\left(d_n \cos n\theta + \delta_n \sin n\theta\right)r^{-n-1} + \varphi(\theta)\end{aligned} \tag{d78}$$

From

$$\lim_{r\to\infty} u_r = 0 \quad , \tag{d79}$$

it follows that

$$b_1 + \beta_1 = 0 \quad , \quad \varphi(\theta) \equiv 0 \quad . \tag{d80}$$

Hence

$$\begin{aligned}\frac{E}{1-\nu^2}u_r = {} & -\frac{c_0}{1-\nu}\frac{1}{r} + \sum_{n=2}^{\infty}\left[n+2+\frac{\nu}{1-\nu}(n-2)\right]\left(c_n\cos n\theta + \gamma_n \sin n\theta\right)r^{-n+1} \\ & + \frac{1}{1-\nu}\sum_{n=1}^{\infty} n\left(d_n \cos n\theta + \delta_n \sin n\theta\right)r^{-n-1}\end{aligned} \quad . \tag{d81}$$

Finally

$$\begin{aligned}u_r = \frac{1+\nu}{E}\Bigg[& -\frac{c_0}{r} + \sum_{n=2}^{\infty}(n+2-4\nu)\left(c_n\cos n\theta + \gamma_n \sin n\theta\right)r^{-n+1} \\ & + \sum_{n=1}^{\infty} n\left(d_n \cos n\theta + \delta_n \sin n\theta\right)r^{-n-1}\Bigg]\end{aligned} \quad . \tag{d82}$$

In the same way,

$$\varepsilon_{\theta\theta} = \frac{1-\nu^2}{E}\left[\sigma_{\theta\theta} - \frac{\nu}{1-\nu}\sigma_{rr}\right] = \frac{\partial u_\theta}{r\partial\theta} + \frac{u_r}{r} \quad . \tag{d83}$$

It follows

$$\frac{E}{1-\nu^2}\frac{\partial u_\theta}{\partial\theta} = -\frac{E}{1-\nu^2}u_r + r\sigma_{\theta\theta} - \frac{\nu}{1-\nu}r\sigma_{rr} \quad . \tag{d84}$$

Substituting (d82), (d66) and (d70) into (d84) and integrating with respect to θ, it follows after some computations that

$$u_\theta = \frac{1+\nu}{E}\left[(1-\nu)\psi(r) + \sum_{n=2}^{\infty}(n-4+4\nu)(c_n \sin n\theta - \gamma_n \cos n\theta)r^{-n+1} + \sum_{n=1}^{\infty} n(d_n \sin n\theta - \delta_n \cos n\theta)r^{-n-1}\right] , \qquad \text{(d85)}$$

Form the condition

$$\lim_{r\to\infty} u_\theta = 0 , \qquad \text{(d86)}$$

it follows the unknown function $\psi = \psi(r)$ is subject to the condition

$$\lim_{r\to\infty} \psi(r) = 0 . \qquad \text{(d87)}$$

But

$$\varepsilon_{r\theta} = \frac{1+\nu}{E}\sigma_{r\theta} = \frac{1}{2}\left(\frac{\partial u_r}{r\partial\theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) , \qquad \text{(d88)}$$

or

$$2r\sigma_{r\theta} = \frac{E}{1+\nu}\left(\frac{\partial u_r}{\partial\theta} - u_\theta + r\frac{\partial u_\theta}{\partial r}\right) . \qquad \text{(d89)}$$

Substituting (d73), (d82) and (d85) into (d89), it follows after some computations that

$$r\frac{d\psi}{dr} = \psi , \qquad \text{(d90)}$$

i.e. $\psi = Cr$. From (d87) it follows that $\psi(r) \equiv 0$ and, finally,

$$u_\theta = \frac{1+\nu}{E}\left[\sum_{n=2}^{\infty}(n-4+4\nu)(c_n \sin n\theta - \gamma_n \cos n\theta)r^{-n+1} + \sum_{n=1}^{\infty} n(d_n \sin n\theta - \delta_n \cos n\theta)r^{-n-1}\right] \qquad \text{(d91)}$$

D.10) Boundary conditions for the stress elements on the wall of the circular cavity.

Using the previous results, the final expressions of the plane elements of the stress are equal to

$$\sigma_{rr} = \frac{c_0}{r^2} - \sum_{n=1}^{\infty}\left(n^2+n-2\right)(c_n \cos n\theta + \gamma_n \sin n\theta)r^{-n} - \sum_{n=1}^{\infty}\left(n^2+n\right)(d_n \cos n\theta + \delta_n \sin n\theta)r^{-n-2} \qquad \text{(d92)}$$

$$\sigma_{\theta\theta} = -\frac{c_0}{r^2} + \sum_{n=1}^{\infty}(n-2)(n-1)(c_n \cos n\theta + \gamma_n \sin n\theta)r^{-n} + \sum_{n=1}^{\infty} n(n+1)(d_n \cos n\theta + \delta_n \sin n\theta)r^{-n-2} \qquad \text{(d93)}$$

and

$$\sigma_{r\theta} = -\sum_{n=1}^{\infty}(n^2-n)(c_n \sin n\theta - \gamma_n \cos n\theta)r^{-n} - \sum_{n=1}^{\infty}\left(n^2+n\right)(d_n \sin n\theta - \delta_n \cos n\theta)r^{-n-2} \qquad \text{(d94)}$$

The cavity wall has the outer normal (with respect to the rock domain) equal to $-\vec{e}_r$ and radius equal to $r = R$. In the case of a bore hole, let Δp be the difference between the mud pressure and the pressure of the fluid contained by the porous rock (usually, because the atmospheric pressure is negligible, it follows in the case of a tunnel that $\Delta p = 0$). It follows the final stress σ^f satisfies the next boundary condition

$$\sigma^f\left(-\vec{e}_r\right) = \Delta p\, \vec{e}_r \quad , \quad \text{for} \quad r = R \tag{d92}$$

Hence

$$\begin{cases} \sigma_{rr}\big|_{r=R} = -\sigma^0_{rr} - \Delta p \\ \sigma_{r\theta}\big|_{r=R} = -\sigma^0_{r\theta} \end{cases} . \tag{d93}$$

With no loss of generality, it can be assumed that the stress at infinity is along its main axes, i.e. $\sigma^0_{12} = 0$. Hence

$$\begin{cases} \dfrac{c_0}{R^2} - \sum\limits_{n=2}^{\infty}\left(n^2+n-2\right)\left(c_n \cos n\theta + \gamma_n \sin n\theta\right)R^{-n} - \sum\limits_{n=1}^{\infty}\left(n^2+n\right)\left(d_n \cos n\theta + \delta_n \sin n\theta\right)R^{-n-2} \\ \qquad = -\dfrac{\sigma^0_{11}+\sigma^0_{22}}{2} - \dfrac{\sigma^0_{11}-\sigma^0_{22}}{2}\cos 2\theta - \Delta p \\ \sum\limits_{n=2}^{\infty}\left(n^2-n\right)\left(-c_n \sin n\theta + \gamma_n \cos n\theta\right)R^{-n} + \sum\limits_{n=1}^{\infty}\left(n^2+n\right)\left(-d_n \sin n\theta + \delta_n \cos n\theta\right)R^{-n-2} \\ \qquad = \dfrac{\sigma^0_{11}-\sigma^0_{22}}{2}\sin 2\theta \end{cases} \tag{d94}$$

Hence

$$\begin{cases} \dfrac{c_0}{R^2} = -\dfrac{\sigma^0_{11}+\sigma^0_{22}}{2} - \Delta p \\ d_1 = \delta_1 = 0 \end{cases} , \tag{d95}$$

and

$$\sum_{n=2}^{\infty}\left\{\left[\left(n^2+n-2\right)\frac{c_n}{R^n} + \left(n^2+n\right)\frac{d_n}{R^{n+2}}\right]\cos n\theta + \left[\left(n^2+n-2\right)\frac{\gamma_n}{R^n} + \left(n^2+n\right)\frac{\delta_n}{R^{n+2}}\right]\sin n\theta\right\} = \frac{\sigma^0_{11}-\sigma^0_{22}}{2}\cos 2\theta \tag{d96}$$

$$\sum_{n=2}^{\infty}\left\{-\left[\left(n^2-n\right)\frac{c_n}{R^n} + \left(n^2+n\right)\frac{d_n}{R^{n+2}}\right]\sin n\theta + \left[\left(n^2-n\right)\frac{\gamma_n}{R^n} + \left(n^2+n\right)\frac{\delta_n}{R^{n+2}}\right]\cos n\theta\right\} = \frac{\sigma^0_{11}-\sigma^0_{22}}{2}\sin 2\theta . \tag{d97}$$

It follows that

$$\begin{cases} \dfrac{c_0}{R^2} = -\dfrac{\sigma_{11}^0 + \sigma_{22}^0}{2} - \Delta p \\ 4\dfrac{c_2}{R^2} + 6\dfrac{d_2}{R^4} = \dfrac{\sigma_{11}^0 - \sigma_{22}^0}{2} \\ -2\dfrac{c_2}{R^2} - 6\dfrac{d_2}{R^4} = \dfrac{\sigma_{11}^0 - \sigma_{22}^0}{2} \end{cases} . \tag{d98}$$

and

$$\begin{cases} (n^2 + n - 2)\dfrac{c_n}{R^n} + (n^2 + n)\dfrac{d_n}{R^{n+2}} = 0 \quad , \quad n = \overline{3,4,\ldots} \\ (n^2 + n - 2)\dfrac{\gamma_n}{R^n} + (n^2 + n)\dfrac{\delta_n}{R^{n+2}} = 0 \quad , \quad n = \overline{2,3,\ldots} \\ (n^2 - n)\dfrac{c_n}{R^n} + (n^2 + n)\dfrac{d_n}{R^{n+2}} = 0 \quad , \quad n = \overline{3,4,\ldots} \\ (n^2 - n)\dfrac{\gamma_n}{R^n} + (n^2 + n)\dfrac{\delta_n}{R^{n+2}} = 0 \quad , \quad n = \overline{2,3,\ldots} \end{cases} \tag{d99}$$

Hence

$$\begin{cases} c_0 = -\left(\dfrac{\sigma_{11}^0 + \sigma_{22}^0}{2} + \Delta p\right) R^2 \\ c_2 = \dfrac{\sigma_{11}^0 - \sigma_{22}^0}{2} R^2 \quad , \quad c_n = 0 \quad , n = \overline{3,4,\ldots} \\ d_2 = -\left(\sigma_{11}^0 - \sigma_{22}^0\right)\dfrac{R^4}{4} \quad , \quad d_n = 0 \quad , n = \overline{3,4,\ldots} \\ \gamma_n = 0 \quad , \delta_n = 0 \quad , n = \overline{2,3,\ldots} \end{cases} . \tag{d100}$$

By using (d11-d13) for $\sigma_{12}^0 = 0$, the expressions of the final stress are equal to

$$\sigma_{rr}^f = \frac{\sigma_{11}^0 + \sigma_{22}^0}{2} + \frac{\sigma_{11}^0 - \sigma_{22}^0}{2}\cos 2\theta - \left(\frac{\sigma_{11}^0 + \sigma_{22}^0}{2} + \Delta p\right)\frac{R^2}{r^2} - 2\left(\sigma_{11}^0 - \sigma_{22}^0\right)\frac{R^2}{r^2}\cos 2\theta + \frac{3}{2}\left(\sigma_{11}^0 - \sigma_{22}^0\right)\frac{R^4}{r^4}\cos 2\theta \tag{d101}$$

and

$$\sigma_{\theta\theta}^f = \frac{\sigma_{11}^0 + \sigma_{22}^0}{2} - \frac{\sigma_{11}^0 - \sigma_{22}^0}{2}\cos 2\theta + \left(\frac{\sigma_{11}^0 + \sigma_{22}^0}{2} + \Delta p\right)\frac{R^2}{r^2} - \frac{3}{2}\left(\sigma_{11}^0 - \sigma_{22}^0\right)\frac{R^4}{r^4}\cos 2\theta \tag{d102}$$

$$\sigma_{r\theta}^f = -\frac{\sigma_{11}^0 - \sigma_{22}^0}{2}\sin 2\theta - \left(\sigma_{11}^0 - \sigma_{22}^0\right)\frac{R^2}{r^2}\sin 2\theta + \frac{3}{2}\left(\sigma_{11}^0 - \sigma_{22}^0\right)\frac{R^4}{r^4}\sin 2\theta \tag{d103}$$

In real cases, the value of the stress at infinity are positive ones for compression.

D.11) The final shape of the wall.

Consider again the case when the direction of the horizontal axes of the co-ordinate system is along the corresponding eigen vectors of the initial stress σ^0. In that case, $\sigma^0_{12} = 0$. Using the above results, it follows the displacement vector for the points initially placed on the wall of the circular cavity is

$$\begin{cases} u_r(r=R,\theta) = \frac{1+\nu}{E} R\left[\frac{\sigma^0_{11}+\sigma^0_{22}}{2} + \Delta p + (3-4\nu)\frac{\sigma^0_{11}-\sigma^0_{22}}{2}\cos 2\theta\right] \\ u_\theta(r=R,\theta) = -\frac{1+\nu}{E} R(3-4\nu)\frac{\sigma^0_{11}-\sigma^0_{22}}{2}\sin 2\theta \end{cases} . \quad \text{(d104)}$$

Consider an arbitrary point on the wall of the bore hole. In the initial state, it has the polar co-ordinates $(r = R, \theta)$. Its position vector with respect to the centre of the circle is

$$\vec{X} = R\left(\cos\theta\,\vec{e}_1 + \sin\theta\,\vec{e}_2\right) \quad \text{(d105)}$$

Using (d6) the position vector in the final stage is

$$\begin{aligned} \vec{\chi} &= \vec{X} + u_r(R,\theta)\vec{e}_r + u_\theta(R,\theta)\vec{e}_\theta = R\left(\cos\theta\,\vec{e}_1 + \sin\theta\,\vec{e}_2\right) \\ &+ \frac{1+\nu}{E} R\left[\frac{\sigma^0_{11}+\sigma^0_{22}}{2} + \Delta p + (3-4\nu)\frac{\sigma^0_{11}-\sigma^0_{22}}{2}\cos 2\theta\right]\left(\cos\theta\,\vec{e}_1 + \sin\theta\,\vec{e}_2\right) . \\ &- \frac{1+\nu}{E} R(3-4\nu)\frac{\sigma^0_{11}-\sigma^0_{22}}{2}\sin 2\theta\left(-\sin\theta\,\vec{e}_1 + \cos\theta\,\vec{e}_2\right) \end{aligned} \quad \text{(d106)}$$

Hence

$$\begin{cases} x_1 = R\cos\theta + \frac{1+\nu}{E} R\left[\frac{\sigma^0_{11}+\sigma^0_{22}}{2} + \Delta p + (3-4\nu)\frac{\sigma^0_{11}-\sigma^0_{22}}{2}\cos 2\theta\right]\cos\theta \\ \quad + \frac{1+\nu}{E} R(3-4\nu)\frac{\sigma^0_{11}-\sigma^0_{22}}{2}\sin 2\theta\sin\theta \\ x_2 = R\sin\theta + \frac{1+\nu}{E} R\left[\frac{\sigma^0_{11}+\sigma^0_{22}}{2} + \Delta p + (3-4\nu)\frac{\sigma^0_{11}-\sigma^0_{22}}{2}\cos 2\theta\right]\sin\theta \\ \quad - \frac{1+\nu}{E} R(3-4\nu)\frac{\sigma^0_{11}-\sigma^0_{22}}{2}\sin 2\theta\cos\theta \end{cases} . \quad \text{(d107)}$$

After elementary computations, it follows

$$\begin{cases} x_1 = \left[1 + \frac{1+\nu}{E}\left(\frac{\sigma_{11}^0 + \sigma_{22}^0}{2} + \Delta p + (3-4\nu)\frac{\sigma_{11}^0 - \sigma_{22}^0}{2}\right)\right] X_1 \\ x_2 = \left[1 + \frac{1+\nu}{E}\left(\frac{\sigma_{11}^0 + \sigma_{22}^0}{2} + \Delta p - (3-4\nu)\frac{\sigma_{11}^0 - \sigma_{22}^0}{2}\right)\right] X_2 \end{cases} . \tag{d108}$$

Taking into account that $X_1^2 + X_2^2 = R^2$, it follows the final shape of the cavity is an ellipse of equation

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \quad , \tag{d109}$$

where the semi-axes of the ellipse are equal to

$$\begin{cases} a = R\left[1 + \frac{1+\nu}{E}\left(\frac{\sigma_{11}^0 + \sigma_{22}^0}{2} + \Delta p + (3-4\nu)\frac{\sigma_{11}^0 - \sigma_{22}^0}{2}\right)\right] \\ b = R\left[1 + \frac{1+\nu}{E}\left(\frac{\sigma_{11}^0 + \sigma_{22}^0}{2} + \Delta p - (3-4\nu)\frac{\sigma_{11}^0 - \sigma_{22}^0}{2}\right)\right] \end{cases} . \tag{d110}$$

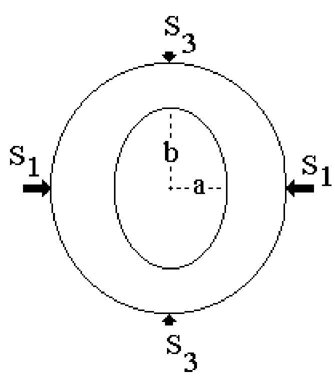

Fig.D2. The shape of the borehole (tunnel) in the initial state (the circle) and in the final state (the ellipse), corresponding to a compressive stress.

In real cases, the initial stress σ^0 is usually a compressive one, i.e. (see Fig.D2)

$$\sigma_{11}^0 = -S_1 \quad , \quad \sigma_{22}^0 = -S_3 \tag{d111}$$

where the maximum compressive stress S_1 and the minimum compressive stress S_3 have positive values $0 \le S_3 \le S_1$. It follows here that the major semi-axis a corresponds to the minimum stress and $a \le b$.