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SHARIPOV R. A.

CLASSICAL ELECTRODYNAMICS AND THEORY OF RELATIVITY

the manual

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This book is a manual for the course of electrodynamics and theory of relativity. It is recommended primarily for students of mathematical departments. This defines its style: I use elements of vectorial and tensorial analysis, differential geometry, and theory of distributions in it.

In preparing Russian edition of this book I used computer typesetting on the base of $\mathcal{AMS}$-TeX package and I used cyrillic fonts of Lh-family distributed by CyrTUG association of Cyrillic TeX users. English edition is also typeset by $\mathcal{AMS}$-TeX.

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PREFACE.

Theory of relativity is a physical discipline which arose in the beginning of XX-th century. It has dramatically changed traditional notion about the structure of the Universe. Effects predicted by this theory becomes essential only when we describe processes at high velocities close to light velocity

$$c = 2.998 \cdot 10^5\,km/sec.$$

In XIX-th century there was the only theory dealing with such processes, this was theory of electromagnetism. Development of theory of electromagnetism in XIX-th century became a premise for arising theory of relativity.

In this book I follow historical sequence of events. In Chapter I electrostatics and magnetostatics are explained starting with first experiments on interaction of charges and currents. Chapter II is devoted to classical electrodynamics based on Maxwell equations.

In the beginning of Chapter III Lorentz transformations are derived as transformations keeping form of Maxwell equations. Physical interpretation of such transformation requires uniting space and time into one four-dimensional continuum (Minkowsky space) where there is no fixed direction for time axis. Upon introducing four-dimensional space-time in Chapter III classical electrodynamics is rederived in the form invariant with respect to Lorentz transformations.

In Chapter IV variational approach to describing electromagnetic field and other material fields in special relativity is considered. Use of curvilinear coordinates in Minkowsky space and appropriate differential-geometric methods prepares background for passing to general relativity.

In Chapter V Einstein's theory of gravitation (general relativity) is considered, this theory interprets gravitational field as curvature of space-time itself.

This book is addressed to Math. students. Therefore I paid much attention to logical consistence of given material. References to physical intuition are minimized: in those places, where I need additional assumptions which do not follow from previous material, detailed comment is given.

I hope that assiduous and interested reader with sufficient preliminary background could follow all mathematical calculations and, upon reading this book, would get pleasure of understanding how harmonic is the nature of things.

I am grateful to N. T. Ahtyamov, D. I. Borisov, Yu. P. Mashentseva, and A. I. Utarbaev for reading and correcting Russian version of book.

November, 1997;
November, 2003. R. A. Sharipov.

CHAPTER I

ELECTROSTATICS AND MAGNETOSTATICS

§ 1. Basic experimental facts and unit systems.

Quantitative description of any physical phenomenon requires measurements. In mechanics we have three basic quantities and three basic units of measure: for mass, for length, and for time.

Quantity	Unit in SI	Unit in SGS	Relation of units
mass	kg	g	$1\,kg = 10^3\,g$
length	m	cm	$1\,m = 10^2\,cm$
time	sec	sec	$1\,sec = 1\,sec$

Units of measure for other quantities are derived from the above basic units. Thus, for instance, for measure unit of force due to Newton's second law we get:

(1) $N = kg \cdot m \cdot sec^{-2}$ in SI,
(2) $dyn = g \cdot cm \cdot sec^{-2}$ in SGS.

Unit systems SI and SGS are two most popular unit systems in physics. Units for measuring mechanical quantities (velocity, acceleration, force, energy, power) in both systems are defined in quite similar way. Proportions relating units for these quantities can be derived from proportions for basic quantities (see table above). However, in choosing units for electric and magnetic quantities these systems differ essentially.

Choice of measure unit for electric charge in SGS is based on Coulomb law describing interaction of two charged point.

Coulomb law. Two charged points with charges of the same sign are repulsing, while points with charges of opposite signs are attracting with force proportional to quantities of their charges and inverse proportional to square of distance between them:

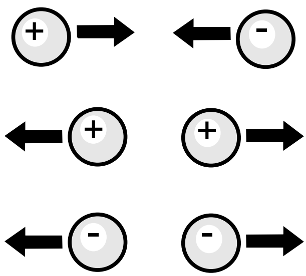

Fig. 1.1

$$F \sim \frac{Q_1\,Q_2}{r^2}. \tag{1.1}$$

Measure unit of charge in SGS is chosen such that coefficient in formula (1.1) is equal to unity. Hence we have the following relation:

$$\textit{unit of charge in SGS} = dyn^{1/2} \cdot cm = g^{1/2} \cdot cm^{3/2} \cdot sec^{-1}.$$

Coulomb law itself then is written in form of the equality

$$F = \frac{Q_1\,Q_2}{r^2}. \tag{1.2}$$

Force F defined by the relationship (1.2) is very strong. However, in everyday life it does not reveal itself. This is due to the *screening*. The numbers of positive and negative charges in nature are exactly balanced. Atoms and molecules, which constitute all observable matter around us, have the same amount of positive and negative charges. Therefore they are electrically neutral in whole. Force (1.2) reveals itself in form of chemical links only when atoms are pulled together.

Electric current arises as a result of motion of charged points. This occurs in metallic conductor, which usually have lengthy form (form of wire). Current in such conductor is determined by the *amount of charge passing through it within the unit of time.* Therefore for unit of current we have:

$$\begin{gathered}\textit{unit of current in SGS} = \textit{unit of charge in SGS} \cdot \textit{sec}^{-1} = \\ = g^{1/2} \cdot cm^{3/2} \cdot sec^{-2}.\end{gathered}$$

Let's consider straight conducting rod of the length l. Current in it leads to misbalance of charges in its ends. Charges of definite sign move to one end of the rod, while lack of these charges in the other end of the rod is detected as the charge of opposite sign. Then Coulomb force (1.2) arises that tends to recover balance of charges in electrically neutral rod. This means that in such rod current could not flow in constant direction during long time. Another situation we have with conductor of the form of ring or circuit. Here current does not break the balance of charges. Direct current can flow in it during unlimitedly long time. Circular conductor itself thereby remains electrically neutral and no Coulomb forces arise.

In spite of absence of Coulomb forces, in experiments the interaction of two circular conductors with currents was detected. This interaction has other nature, it is not due to electrical, but due to magnetic forces. The magnitude of magnetic forces depends essentially on the shape and mutual arrangement of circular conductors. In order to reveal quantitative characteristics for magnetic forces one should maximally simplify the geometry of conductors. For this purpose they are deformed so that each possesses straight rod-shaped part of sufficiently big length l. These rod-

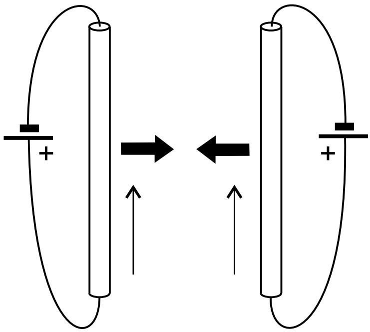

Fig. 1.2

shaped parts are arranged parallel to each other with the distance r between them. In the limit, when l is much larger than r, this configuration of conductors can be treated as a pair of infinitely long parallel conductors. In experiments it was found that such conductors do interact according to the following law.

Ampere law. Force of interaction of two infinite parallel conductors with currents per unit length of them is proportional ti the values of currents in them and inverse proportional to the distance between them:

$$\frac{F}{l} \sim \frac{I_1\,I_2}{r}. \tag{1.3}$$

Two co-directed currents attract each other, while opposite directed currents repulse each other.

The unit of current in SGS was already introduced above. Therefore coefficient of proportionality in formula (1.3) is unique quantity that should be determined in experiment. Here is the measure unit for this coefficient: $sec^2 \cdot cm^{-2}$. It coincides with inverse square of velocity. Therefore formula (1.3) in SGS is written as

$$\frac{F}{l} = \frac{2}{c^2}\frac{I_1\,I_2}{r}. \tag{1.4}$$

Constant c in (1.4) is a velocity constant. The value of this constant is determined experimentally:

$$c \approx 2.998 \cdot 10^{10}\ \ cm/sec. \tag{1.5}$$

As we shall see below, constant c in (1.5) coincides with velocity of light in vacuum. Numeric coefficient 2 in (1.4) is introduced intentionally for to provide such coincidence.

In SI measure unit of current $1\,A$ (one *ampere*) is a basic unit. It is determined such that formula (1.3) is written as

$$\frac{F}{l}=\frac{2\,\mu_0}{4\pi}\frac{I_1\,I_2}{r}. \tag{1.6}$$

Here $\pi=3.14\ldots$ is exact (though it is irrational) mathematical constant with no measure unit. Constant μ_0 is called *magnetic susceptibility* of vacuum. It has the measure unit:

$$\mu_0=4\pi\cdot 10^{-7}N\cdot\,A^{-2}. \tag{1.7}$$

But, in contrast to constant c in (1.5), it is exact constant. Its value should not be determined experimentally. One could choose it to be equal to unity, but the above value (1.7) for this constant was chosen by convention when SI system was established. Due to this value of constant (1.7) current of 1 *ampere* appears to be in that range of currents, that really appear in industrial and household devices. Coefficient 4π in denominator (1.6) is used in order to simplify some other formulas, which are more often used for engineering calculations in electric technology.

Being basic unit in SI, unit of current *ampere* is used for defining unit of charge of 1 *coulomb*: $1C=1A\,\cdot\,1sec.$ Then coefficient of proportionality in Coulomb law (1.1) appears to be not equal to unity. In SI Coulomb law is written as

$$F=\frac{1}{4\pi\epsilon_0}\frac{Q_1Q_2}{r^2}. \tag{1.8}$$

Constant ϵ_0 is called dielectric permittivity of vacuum. In contrast to constant μ_0 in (1.7) this is physical constant determined experimentally:

$$\epsilon_0\approx 8.85\cdot 10^{-12}\ C^2\,\cdot\,N^{-1}\,\cdot\,m^{-2}. \tag{1.9}$$

Constants (1.5), (1.7), and (1.9) are related to each other by the following equality:

$$c = \frac{1}{\sqrt{\epsilon_0\,\mu_0}} \approx 2.998 \cdot 10^8 \ m/sec. \tag{1.10}$$

From the above consideration we see that SGS and SI systems differ from each other not only in the scale of units, but in formulas for two fundamental laws: Coulomb law and Ampere law. SI better suits for engineering calculations. However, derivation of many formulas in this system appears more huge than in SGS. Therefore below in this book we use SGS system.

Comparing Coulomb law and Ampere law we see that electrical and magnetic forces reveal themselves in quite different way. However, they have common origin: they both are due to electric charges. Below we shall see that their relation is much more close. Therefore theories of electricity and magnetism are usually united into one theory of electromagnetic phenomena. Theory of electromagnetism is a theory with one measurable constant: this is light velocity c. Classical mechanics (without Newton's theory of gravitation) has no measurable constants. Newton's theory of gravitation has one constant:

$$\gamma \approx 6.67 \cdot 10^{-8} \ cm^3 \cdot g^{-1} \cdot sec^{-2}. \tag{1.11}$$

This theory is based on Newton's fourth law formulated as follows.

Universal law of gravitation. Two point masses attract each other with the force proportional to their masses and inverse proportional to the square of distance between them.

Universal law of gravitation is given by the same formula

$$F = \gamma \frac{M_1\,M_2}{r^2} \tag{1.12}$$

in both systems: in SGS and in SI.

According to modern notion of nature classical mechanics and Newton's theory of gravitation are approximate theories. Currently they are replaced by special theory of relativity and general theory of relativity. Historically they appeared as a result of development of the theory of electromagnetism. Below we keep this historical sequence in explaining all three theories.

Exercise 1.1. *On the base of above facts find quantitative relation of measure units for charge and current in SGS and SI.*

§ 2. Concept of near action.

Let's consider pair of charged bodies, which are initially fixed, and let's do the following mental experiment with them. When we start moving second body apart from first one, the distance r begins increasing and consequently force of Coulomb interaction (1.2) will decrease. In this situation we have natural question: how soon after second body starts moving second body will feel change of Coulomb force of interaction? There are two possible answers to this question:

(1) immediately;
(2) with some delay depending on the distance between bodies.

First answer is known as concept of *distant action.* Taking this concept we should take formula (1.2) as absolutely exact formula applicable for charges at rest and for moving charges as well.

Second answer is based on the concept of *near action.* According to this concept, each interaction (and electric interaction among others) can be transmitted immediately only to the point of space infinitesimally close to initial one. Transmission of any action to finite distance should be considered as a process of successive transmission from point to point. This process always leads to some finite velocity of transmission for any action. In the framework of the concept of near action Coulomb law (1.2) is treated as approximate law, which is exact only for the charges at rest that stayed at rest during sufficiently long time so that process of transmission of electric interaction has been terminated.

Theory of electromagnetism has measurable constant c (light velocity (1.5)), which is first pretender for the role of transmission velocity of electric and magnetic interactions. For this reason electromagnetic theory is much more favorable as compared to Newton's theory of gravitation.

The value of light velocity is a very large quantity. If we settle an experiment of measuring Coulomb force at the distances of the order of $r \approx 10\ cm$, for the time of transmission of interaction we would get times of the order of $t \approx 3 \cdot 10^{-10}\ sec.$ Experimental technique of XIX-th century was unable to detect such a short interval of time. Therefore the problem of choosing concept could not be solved experimentally. In XIX-th century it was subject for contests. The only argument against the concept of distant action that time, quite likely, was its straightness, its self-completeness, and hence its scarcity.

In present time concept of near action is commonly accepted. Now we have the opportunity for testing it experimentally in the scope of electromagnetic phenomena. Let's study this concept more attentively. According to the concept of near action, process of transmitting interaction to far distance exhibits an inertia. Starting at one point, where moving charge is placed, for some time this process exist in hidden form with no influence to both charges. In order to describe this stage of process we need to introduce new concept. This concept is a *field*.

Field is a material entity able to fill the whole space and able to act upon other material bodies transmitting mutual interaction of them.

The number of fields definitely known to scientists is not big. There are only four fundamental fields: *strong field, weak field,* *electromagnetic field*, and *gravitational field*. Strong and weak fields are very short distance fields, they reveal themselves only in atomic nuclei, in collisions and decay of elementary particles, and in stellar objects of extremely high density, which are called neutron stars. Strong and weak interactions and fields are not considered in this book.

There are various terms using the word field: *vector field, tensor field, spinor field, gauge field,* and others. These are mathematical terms reflecting some definite properties of real physical fields.

§ 3. Superposition principle.

Let's apply concept of near action to Coulomb law for two charged points. Coulomb force in the framework of this concept can be interpreted as follows: first charge produces electric field around itself, and this field acts upon other charge. Result of such action is detected as a force F applied to second charge. Force is vectorial quantity. Let's denote by $\mathbf{F}$ vector of force and take into account the direction of this vector determined by verbal statement of Coulomb law above. This yields

$$\mathbf{F} = Q_1\,Q_2\,\frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}. \tag{3.1}$$

Here $\mathbf{r}_1$ and $\mathbf{r}_2$ are radius-vectors of points, where charges Q_1 and Q_2 are placed. Let's consider vector $\mathbf{E}$ determined as the ratio $\mathbf{E} = \mathbf{F}/Q_2$. For this vector from formula (3.1) we derive

$$\mathbf{E} = Q_1\,\frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}. \tag{3.2}$$

Vector $\mathbf{E}$ depends upon the position of first charge and upon its value. It depends also on the position of second charge, but it doesn't depend on the value of second charge. One can take vector $\mathbf{E}$ for quantitative measure of electric field produced by first charge Q_1 at the point $\mathbf{r}_2$, where second charge is placed. Vector $\mathbf{E}$ can be determined by formula (3.2) or it can be measured experimentally. For this purpose one should place test charge q to the point $\mathbf{r}_2$ and one should measure Coulomb force $\mathbf{F}$ acting upon this test charge. Then vector $\mathbf{E}$ is determined by division of $\mathbf{F}$ by the value of test charge q:

$$\mathbf{E} = \mathbf{F}/q. \tag{3.3}$$

Now consider more complicated situation. Suppose that charges $Q_1, \ldots, Q_n$ are placed at the points $\mathbf{r}_1, \ldots, \mathbf{r}_n$. They produce electric field around them, and this field acts upon test charge q placed at the point $\mathbf{r}$. This action reveals as a force $\mathbf{F}$ applied to the charge q. Again we can define vector $\mathbf{E}$ of the form (3.3) and take it for the quantitative measure of electric field at the point $\mathbf{r}$. This vector is called *vector of intensity of electric field* or simply *vector of electric field* at that point.

Generally speaking, in this case one cannot be a priori sure that vector $\mathbf{E}$ does not depend on the quantity of test charge q. However, there is the following experimental fact.

Superposition principle. Electric field $\mathbf{E}$ at the point $\mathbf{r}$ produced by a system of point charges $Q_1, \ldots, Q_n$ is a vectorial sum of electric fields that would be produced at this point by each charge $Q_1, \ldots, Q_n$ separately.

Superposition principle combined with Coulomb law leads to the following formula for the intensity of electric field produced by a system of point charges at the point $\mathbf{r}$:

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{n} Q_i \, \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}. \tag{3.4}$$

Using superposition principle, one can pass from point charges to continuously distributed charges. Suppose that the number of point charges tends to infinity: $n \to \infty$. In such limit sum in formula (3.4) is replaced by integral over 3-dimensional space:

$$\mathbf{E}(\mathbf{r}) = \int \rho(\tilde{\mathbf{r}})\,\frac{\mathbf{r}-\tilde{\mathbf{r}}}{|\mathbf{r}-\tilde{\mathbf{r}}|^3}\,d^3\tilde{\mathbf{r}}. \tag{3.5}$$

Here $\rho(\tilde{\mathbf{r}})$ is spatial density of charge at the point $\tilde{\mathbf{r}}$. This value designates the amount of charge per unit volume.

In order to find force acting on test charge q we should invert formula (3.3). As a result we obtain

$$\mathbf{F} = q\,\mathbf{E}(\mathbf{r}). \tag{3.6}$$

Force acting on a charge q in electric field is equal to the product of the quantity of this charge by the vector of intensity of field at the point, where charge is placed. However, charge q also produces electric field. Does it experience the action of its own field? For point charges the answer to this question is negative. This fact should be treated as a supplement to principle of superposition. Total force acting on a system of distributed charges in electric field is determined by the following integral:

$$\mathbf{F} = \int \rho(\mathbf{r})\,\mathbf{E}(\mathbf{r})\,d^3\mathbf{r}. \tag{3.7}$$

Field $\mathbf{E}(\mathbf{r})$ in (3.7) is external field produced by external charges. Field of charges with density $\rho(\mathbf{r})$ is not included into $\mathbf{E}(\mathbf{r})$.

Concluding this section, note that formulas (3.4) and (3.5) hold only for charges at rest, which stayed at rest for sufficiently long time so that process of interaction transmitting reached the point of observation $\mathbf{r}$. Fields produced by such systems of charges are called *static fields*, while branch of theory of electromagnetism studying such fields is called *electrostatics*.

§ 4. Lorentz force and Biot-Savart-Laplace law.

Ampere law of interaction of parallel conductors with currents is an analog of Coulomb law for magnetic interactions. According to near action principle, force F arises as a result of action of magnetic field produced by a current in first conductor upon second conductor. However, parallel conductors cannot be treated as point objects: formula (1.4) holds only for $l \gg r$. In order to get quantitative measure of magnetic field at some point $\mathbf{r}$ let's consider current I_2 in (1.4) as a flow of charged particles of charge q each, and each moving along conductor with constant velocity v. If we denote by ν the number of such particles per unit length of conductor, then in the whole length l we would have $N = \nu\, l$ particles. Then during time interval t we would have $n = \nu\, v\, t$ particles passing through a fixed cross-section of the conductor. They carry charge amounting to $Q = q\,\nu\, v\, t$. Therefore for current I_2 in second conductor we get

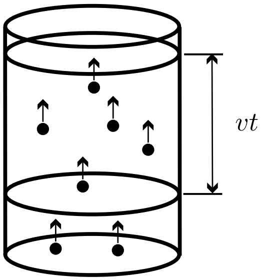

Fig. 4.1

$$I_2 = Q/t = q\,\nu\, v.$$

Upon calculating force acting on a segment of conductor of the length l by formula (1.4) we should divide it by the number of particles N contained in this segment. Then for the force per each particle we derive

$$F = \frac{2}{c^2}\frac{I_1\, I_2\, l}{r\, N} = \frac{2}{c^2}\frac{I_1\, q\, v}{r}. \tag{4.1}$$

Formula determines (4.1) qualitative dependence of F on q and on v: each charged particle moving in magnetic field experiences a force proportional to its charge q and to the magnitude of its velocity vector $v = |\mathbf{v}|$, i. e. we have proportionality

$$F \sim q\,v. \tag{4.2}$$

Force and velocity both are vectorial quantities. Simplest way to relate two vectors $\mathbf{F}$ and $\mathbf{v}$ is to use vector product of $\mathbf{v}$ with some third vectorial quantity $\mathbf{H}$:

$$\mathbf{F} = \frac{q}{c}\,[\mathbf{v},\,\mathbf{H}(\mathbf{r})]. \tag{4.3}$$

Here c is scalar constant equal to light velocity. Vectorial quantity $\mathbf{H}(\mathbf{r})$ is a quantitative measure of magnetic field at the point $\mathbf{r}$. It is called *intensity of magnetic field* at that point. Scalar factor $1/c$ in (4.3) is used for to make $\mathbf{H}$ to be measured by the same units as intensity of electric field $\mathbf{E}$ in (3.6). Force $\mathbf{F}$ acting on a point charge in magnetic field is called *Lorentz force*. Total Lorentz force acting on a charge in electromagnetic field is a sum of two components: electric component and magnetic component:

$$\mathbf{F} = q\,\mathbf{E} + \frac{q}{c}\,[\mathbf{v},\,\mathbf{H}]. \tag{4.4}$$

Formula (4.4) extends formula (3.6) for the case of general electromagnetic fields. It holds not only for static but for time-dependent (non-static) fields. Surely the above derivation of formula (4.4) is empiric. Actually, one should treat formula (4.4) as experimental fact that do not contradict to another experimental fact (1.4) within theory being developed.

Let's turn back to our conductors. Formula (4.3) can be interpreted in terms of currents. Each segment of unit length of a conductor with current I in magnetic field $\mathbf{H}$ experiences the force

$$\frac{\mathbf{F}}{l} = \frac{I}{c}\,[\boldsymbol{\tau},\,\mathbf{H}] \tag{4.5}$$

acting on it. Here $\boldsymbol{\tau}$ is unit vector tangent to conductor and directed along current in it. Total force acting on circular conductor with current I is determined by contour integral

$$\mathbf{F} = \oint \frac{I}{c}\,[\boldsymbol{\tau}(s),\,\mathbf{H}(\mathbf{r}(s))]\,ds, \tag{4.6}$$

where s is natural parameter on contour (length) and $\mathbf{r}(s)$ is vector-function determining shape of contour in parametric form.

Let's consider the case of two parallel conductors. Force $\mathbf{F}$ now can be calculated by formula (4.5) assuming that first conductor produces magnetic field $\mathbf{H}(\mathbf{r})$ that acts upon second conductor. Auxiliary experiment shows that vector $\mathbf{H}$ is perpendicular to the plane of these two parallel conductors. The magnitude of magnetic field $H = |\mathbf{H}|$ can be determined by formula (4.1):

$$H = \frac{2}{c}\frac{I_1}{r}. \tag{4.7}$$

Here r is the distance from observation point to the conductor producing field at that point.

Magnetic field produced by conductor with current satisfies superposition principle. In particular, field of infinite straight line conductor (4.7) is composed by fields produced by separate segments of this conductor. One cannot measure magnetic field of separate segment experimentally since one cannot keep constant current is such separate segment for sufficiently long time. But theoretically one can consider infinitesimally small segment of conductor with current of the length ds. And one can write formula for magnetic field produced by such segment of conductor:

$$d\mathbf{H}(\mathbf{r}) = \frac{1}{c}\frac{[I\,\boldsymbol{\tau},\,\mathbf{r}-\tilde{\mathbf{r}}]}{|\mathbf{r}-\tilde{\mathbf{r}}|^3}\,ds. \tag{4.8}$$

Here $\boldsymbol{\tau}$ is unit vector determining spatial orientation of infinitesimal conductor. It is always taken to be directed along current I. In practice, when calculating magnetic fields produced by circular conductors, formula (4.8) is taken in integral form:

$$\mathbf{H}(\mathbf{r}) = \oint \frac{1}{c} \frac{[I\,\boldsymbol{\tau}(s),\,\mathbf{r} - \tilde{\mathbf{r}}(s)]}{|\mathbf{r} - \tilde{\mathbf{r}}(s)|^3}\,ds. \tag{4.9}$$

Like in (4.6), here s is natural parameter on the contour and $\tilde{\mathbf{r}}(s)$ is vectorial function determining shape of this contour. Therefore $\boldsymbol{\tau}(s) = d\tilde{\mathbf{r}}(s)/ds$. The relationship (4.8) and its integral form (4.9) constitute Biot-Savart-Laplace law for circular conductors with current.

Biot-Savart-Laplace law in form (4.8) cannot be tested experimentally. However, in integral form (4.9) for each particular conductor it yields some particular expression for $\mathbf{H}(\mathbf{r})$. This expression then can be verified in experiment.

Exercise 4.1. *Using relationships* (4.6) *and* (4.9), *derive the law of interaction of parallel conductors with current in form* (1.4).

Exercise 4.2. *Find magnetic field of the conductor with current having the shape of circle of the radius a.*

§ 5. Current density and the law of charge conservation.

Conductors that we have considered above are kind of idealization. They are linear, we assume them having no thickness. Real conductor always has some thickness. This fact is ignored when we consider long conductors like wire. However, in some cases thickness of a conductor cannot be ignored. For example, if we consider current in electrolytic bath or if we study current in plasma in upper layers of atmosphere. Current in bulk conductors can be distributed non-uniformly within volume of conductor.

The concept of *current density* $\mathbf{j}$ is best one for describing such situation.

Current density is vectorial quantity depending on a point of conducting medium: $\mathbf{j} = \mathbf{j}(\mathbf{r})$. Vector of current density $\mathbf{j}(\mathbf{r})$ indicate the direction of charge transport at the point $\mathbf{r}$. Its magnitude $j = |\,\mathbf{j}\,|$ is determined by the amount of charge passing through unit area perpendicular to vector $\mathbf{j}$ per unit time. Let's mark mentally some restricted domain Ω within bulk conducting medium. Its boundary is smooth closed surface. Due to the above definition of current density the amount of charge flowing out from marked domain per unit time is determined by surface integral over the boundary of this domain, while charge enclosed within this domain is given by spatial integral:

$$Q = \int\limits_{\Omega} \rho \, d^3\mathbf{r}, \qquad\qquad J = \int\limits_{\partial\Omega} \langle\, \mathbf{j},\, \mathbf{n} \rangle \, dS. \tag{5.1}$$

Here $\mathbf{n}$ is unit vector of external normal to the surface $\partial\Omega$ restricting domain Ω.

Charge conservation law is one more fundamental experimental fact reflecting the nature of electromagnetism. In its classical form it states that charges cannot appear from nowhere and cannot disappear as well, they can only move from one point to another. Modern physics insert some correction to this statement: charges appear and can disappear in processes of creation and annihilation of pairs of elementary particles consisting of particle and corresponding antiparticle. However, even in such creation-annihilation processes total balance of charge is preserved since total charge of a pair consisting of particle and antiparticle is always equal to zero. When applied to integrals (5.1) charge conservation law yields: $\dot{Q} = -J$. This relationship means that decrease of charge enclosed within domain Ω is always due to charge leakage through the boundary and conversely increase of charge is due to incoming flow through the boundary of this domain. Let's write charge conservation law in the following form:

$$\frac{d}{dt}\left(\int\limits_{\Omega}\rho\,d^3\mathbf{r}\right)+\int\limits_{\partial\Omega}\langle\mathbf{j},\,\mathbf{n}\rangle\,dS=0. \tag{5.2}$$

Current density $\mathbf{j}$ is a vector depending on a point of conducting medium. Such objects in differential geometry are called *vector fields*. Electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ are other examples of vector fields. Surface integral J in (5.1) is called *flow of vector field* $\mathbf{j}$ through the surface $\partial\Omega$. For smooth vector field any surface integral like J can be transformed to spatial integral by means of Ostrogradsky-Gauss formula. When applied to (5.2), this yields

$$\int\limits_{\Omega}\left(\frac{\partial\rho}{\partial t}+\operatorname{div}\mathbf{j}\right)\,d^3\mathbf{r}=0. \tag{5.3}$$

Note that Ω in (5.3) is an arbitrary domain that we marked mentally within conducting medium. This means that the expression being integrated in (5.3) should be identically zero:

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\mathbf{j}=0. \tag{5.4}$$

The relationships (5.2) and (5.4) are integral and differential forms of charge conservation law respectively. The relationship (5.4) also is known as *continuity equation* for electric charge.

When applied to bulk conductors with distributed current $\mathbf{j}$ within them, formula (4.6) is rewritten as follows:

$$\mathbf{F}=\int\frac{1}{c}\,[\mathbf{j}(\mathbf{r}),\,\mathbf{H}(\mathbf{r})]\;d^3\mathbf{r}. \tag{5.5}$$

Biot-Savart-Laplace law for such conductors also is written in terms of spatial integral in the following form:

$$\mathbf{H}(\mathbf{r}) = \int \frac{1}{c}\,\frac{[\,\mathbf{j}(\tilde{\mathbf{r}}),\,\mathbf{r}-\tilde{\mathbf{r}}]}{|\mathbf{r}-\tilde{\mathbf{r}}|^3}\,d^3\tilde{\mathbf{r}}. \tag{5.6}$$

In order to derive formulas (5.5) and (5.6) from formulas (4.6) and (4.8) one should represent bulk conductor as a union of linear conductors, then use superposition principle and pass to the limit by the number of linear conductors $n \to \infty$.

§ 6. Electric dipole moment.

Let's consider some configuration of distributed charge with density $\rho(\mathbf{r})$ which is concentrated within some restricted domain Ω. Let R be maximal linear size of the domain Ω. Let's choose coordinates with origin within this domain Ω and let's choose observation point $\mathbf{r}$ which is far enough from the domain of charge concentration: $|\mathbf{r}| \gg R$. In order to find electric field $\mathbf{E}(\mathbf{r})$ produced by charges in Ω we use formula (3.5):

$$\mathbf{E}(\mathbf{r}) = \int\limits_{\Omega} \rho(\tilde{\mathbf{r}})\,\frac{\mathbf{r}-\tilde{\mathbf{r}}}{|\mathbf{r}-\tilde{\mathbf{r}}|^3}\,d^3\tilde{\mathbf{r}}. \tag{6.1}$$

Since domain Ω in (6.1) is restricted, we have inequality $|\tilde{\mathbf{r}}| \leq R$. Using this inequality along with $|\mathbf{r}| \gg R$, we can write Taylor expansion for the fraction in the expression under integration in (6.1). As a result we get power series in powers of ratio $\tilde{\mathbf{r}}/|\mathbf{r}|$:

$$\frac{\mathbf{r}-\tilde{\mathbf{r}}}{|\mathbf{r}-\tilde{\mathbf{r}}|^3} = \frac{\mathbf{r}}{|\mathbf{r}|^3} + \frac{1}{|\mathbf{r}|^2}\cdot\left(3\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\left\langle\frac{\mathbf{r}}{|\mathbf{r}|},\,\frac{\tilde{\mathbf{r}}}{|\mathbf{r}|}\right\rangle - \frac{\tilde{\mathbf{r}}}{|\mathbf{r}|}\right) + \ldots\,. \tag{6.2}$$

Substituting (6.2) into (6.1), we get the following expression for the vector of electric field $\mathbf{E}(\mathbf{r})$ produced by charges in Ω:

$$\mathbf{E}(\mathbf{r}) = Q\frac{\mathbf{r}}{|\mathbf{r}|^3} + \frac{3\,\langle\mathbf{r},\,\mathbf{D}\rangle\,\mathbf{r} - |\mathbf{r}|^2\,\mathbf{D}}{|\mathbf{r}|^5} + \ldots\,. \tag{6.3}$$

First summand in (6.3) is Coulomb field of point charge Q placed at the origin, where Q is total charge enclosed in the domain Ω. It is given by integral (5.1).

Second summand in (6.3) is known as field of point dipole placed at the origin. Vector $\mathbf{D}$ there is called *dipole moment.* For charges enclosed within domain Ω it is given by integral

$$\mathbf{D} = \int\limits_{\Omega} \rho(\tilde{\mathbf{r}})\,\tilde{\mathbf{r}}\,d^3\tilde{\mathbf{r}}. \tag{6.4}$$

For point charges dipole moment is determined by sum

$$\mathbf{D} = \sum_{i=1}^{n} Q_i\,\tilde{\mathbf{r}}_i. \tag{6.5}$$

For the system of charges concentrated near origin, which is electrically neutral in whole, the field of point dipole

$$\mathbf{E}(\mathbf{r}) = \frac{3\,\langle\mathbf{r},\,\mathbf{D}\rangle\,\mathbf{r} - |\mathbf{r}|^2\,\mathbf{D}}{|\mathbf{r}|^5} \tag{6.6}$$

is leading term in asymptotics for electrostatic field (3.4) or (3.5) as $\mathbf{r} \to \infty$. Note that for the system of charges with $Q = 0$ dipole moment $\mathbf{D}$ calculated by formulas (6.4) and (6.5) is invariant quantity. This quantity remains unchanged when we move all charges to the same distance at the same direction without changing their mutual orientation: $\tilde{\mathbf{r}} \to \tilde{\mathbf{r}} + \mathbf{r}_0$.

Exercise 6.1. *Concept of charge density is applicable to point charges as well. However, in this case* $\rho(\mathbf{r})$ *is not ordinary function. It is distribution. For example point charge* Q *placed at the point* $\mathbf{r} = 0$ *is represented by density* $\rho(\mathbf{r}) = Q\,\delta(\mathbf{r})$, *where* $\delta(\mathbf{r})$ *is Dirac's delta-function. Consider the density*

$$\rho(\mathbf{r}) = \bigl\langle \mathbf{D},\, \operatorname{grad}\delta(\mathbf{r})\bigr\rangle = \sum_{i=1}^{3} D^i\,\frac{\partial\delta(\mathbf{r})}{\partial r^i}. \tag{6.7}$$

Applying formula (5.1), *calculate total charge* Q *corresponding to this density* (6.7). *Using formula* (6.4) *calculate dipole moment for distributed charge* (6.7) *and find electrostatic field produced by this charge. Compare the expression obtained with* (6.6) *and explain why system of charges described by the above density* (6.7) *is called point dipole.*

Exercise 6.2. *Using formula* (3.7) *find the force acting on point dipole in external electric field* $\mathbf{E}(\mathbf{r})$.

§ 7. Magnetic moment.

Let's consider situation similar to that of previous section. Suppose some distributed system of currents is concentrated in some restricted domain near origin. Let R be maximal linear size of this domain Ω. Current density $\mathbf{j}(\mathbf{r})$ is smooth vector-function, which is nonzero only within Ω and which vanishes at the boundary $\partial\Omega$ and in outer space. Current density $\mathbf{j}(\mathbf{r})$ is assumed to be stationary, i. e. it doesn't depend on time, and it doesn't break charge balance, i. e. $\rho(\mathbf{r}) = 0$. Charge conservation law applied to this situation yields

$$\operatorname{div}\mathbf{j} = 0. \tag{7.1}$$

In order to calculate magnetic field $\mathbf{H}(\mathbf{r})$ we use Biot-Savart-Laplace law written in integral form (5.6):

$$\mathbf{H}(\mathbf{r}) = \int\limits_{\Omega} \frac{1}{c}\,\frac{[\mathbf{j}(\tilde{\mathbf{r}}),\,\mathbf{r}-\tilde{\mathbf{r}}]}{|\mathbf{r}-\tilde{\mathbf{r}}|^3}\,d^3\tilde{\mathbf{r}}. \tag{7.2}$$

Assuming that $|\mathbf{r}| \gg R$, we take Taylor expansion (6.2) and substitute it into (7.2). As a result we get

$$\begin{aligned}\mathbf{H}(\mathbf{r}) = &\int\limits_{\Omega} \frac{[\mathbf{j}(\tilde{\mathbf{r}}),\,\mathbf{r}]}{c\,|\mathbf{r}|^3}\,d^3\tilde{\mathbf{r}}\,+\\ &+\int\limits_{\Omega} \frac{3\,\langle\mathbf{r},\,\tilde{\mathbf{r}}\rangle\,[\mathbf{j}(\tilde{\mathbf{r}}),\,\mathbf{r}] - |\mathbf{r}|^2\,[\mathbf{j}(\tilde{\mathbf{r}}),\,\tilde{\mathbf{r}}]}{c\,|\mathbf{r}|^5}\,d^3\tilde{\mathbf{r}} + \ldots\,.\end{aligned} \tag{7.3}$$

Lemma 7.1. *First integral in* (7.3) *is identically equal to zero.*

Proof. Denote this integral by $\mathbf{H}_1(\mathbf{r})$. Let's choose some arbitrary constant vector $\mathbf{e}$ and consider scalar product

$$\langle\mathbf{H}_1,\,\mathbf{e}\rangle = \int\limits_{\Omega} \frac{\langle\mathbf{e},\,[\mathbf{j}(\tilde{\mathbf{r}}),\,\mathbf{r}]\rangle}{c\,|\mathbf{r}|^3}\,d^3\tilde{\mathbf{r}} = \int\limits_{\Omega} \frac{\langle\mathbf{j}(\tilde{\mathbf{r}}),\,[\mathbf{r},\,\mathbf{e}]\rangle}{c\,|\mathbf{r}|^3}\,d^3\tilde{\mathbf{r}}. \tag{7.4}$$

Then define vector $\mathbf{a}$ and function $f(\tilde{\mathbf{r}})$ as follows:

$$\mathbf{a} = \frac{[\mathbf{r},\,\mathbf{e}]}{c\,|\mathbf{r}|^3}, \qquad\qquad f(\tilde{\mathbf{r}}) = \langle\mathbf{a},\,\tilde{\mathbf{r}}\rangle.$$

Vector $\mathbf{a}$ does not depend on $\tilde{\mathbf{r}}$, therefore in calculating integral (7.4) we can take it for constant vector. For this vector we derive

$\mathbf{a} = \operatorname{grad} f$. Substituting this formula into the (7.4), we get

$$\begin{aligned}\big\langle \mathbf{H}_1,\, \mathbf{e}\big\rangle &= \int\limits_{\Omega} \big\langle\, \mathbf{j},\, \operatorname{grad} f\big\rangle\, d^3\tilde{\mathbf{r}} = \\ &= \int\limits_{\Omega} \operatorname{div}(f\,\mathbf{j})\, d^3\tilde{\mathbf{r}} - \int\limits_{\Omega} f\, \operatorname{div}\mathbf{j}\, d^3\tilde{\mathbf{r}}.\end{aligned} \tag{7.5}$$

Last integral in (7.5) is equal to zero due to (7.1). Previous integral is transformed to surface integral by means of Ostrogradsky-Gauss formula. It is also equal to zero since $\mathbf{j}(\tilde{\mathbf{r}})$ vanishes at the boundary of domain Ω. Therefore

$$\big\langle \mathbf{H}_1,\, \mathbf{e}\big\rangle = \int\limits_{\partial\Omega} f\,\big\langle\, \mathbf{j},\, \mathbf{n}\big\rangle\, dS = 0. \tag{7.6}$$

Now vanishing of vector $\mathbf{H}_1(\mathbf{r})$ follows from formula (7.6) since $\mathbf{e}$ is arbitrary constant vector. Lemma 7.1 is proved. □

Let's transform second integral in (7.3). First of all we denote it by $\mathbf{H}_2(\mathbf{r})$. Then, taking an arbitrary constant vector $\mathbf{e}$, we form scalar product $\big\langle \mathbf{H}_2,\, \mathbf{e}\big\rangle$. This scalar product can be brought to

$$\big\langle \mathbf{H}_2,\, \mathbf{e}\big\rangle = \frac{1}{c\,|\mathbf{r}|^5} \int\limits_{\Omega} \big\langle\, \mathbf{j}(\tilde{\mathbf{r}}),\, \mathbf{b}(\tilde{\mathbf{r}})\big\rangle\, d^3\tilde{\mathbf{r}}, \tag{7.7}$$

where $\mathbf{b}(\tilde{\mathbf{r}}) = 3\big\langle \mathbf{r},\, \tilde{\mathbf{r}}\big\rangle\,[\mathbf{r},\, \mathbf{e}] - |\mathbf{r}|^2\,[\tilde{\mathbf{r}},\, \mathbf{e}]$. If one adds gradient of arbitrary function $f(\tilde{\mathbf{r}})$ to $\mathbf{b}(\tilde{\mathbf{r}})$, this wouldn't change integral in (7.7). Formulas (7.5) and (7.6) form an example of such invariance. Let's specify function $f(\tilde{\mathbf{r}})$, choosing it as follows:

$$f(\tilde{\mathbf{r}}) = -\frac{3}{2}\big\langle \mathbf{r},\, \tilde{\mathbf{r}}\big\rangle\,\big\langle \tilde{\mathbf{r}},\, [\mathbf{r}, \mathbf{e}]\big\rangle. \tag{7.8}$$

For gradient of function (7.8) by direct calculations we find

$$\begin{aligned}\operatorname{grad} f(\tilde{\mathbf{r}}) &= -\frac{3}{2}\big\langle\tilde{\mathbf{r}},\, [\mathbf{r}, \mathbf{e}]\big\rangle\,\mathbf{r} - \frac{3}{2}\big\langle\mathbf{r},\, \tilde{\mathbf{r}}\big\rangle\,[\mathbf{r}, \mathbf{e}] =\\ &= -3\,\big\langle\mathbf{r},\, \tilde{\mathbf{r}}\big\rangle\,[\mathbf{r}, \mathbf{e}] - \frac{3}{2}\,\big(\mathbf{r}\,\big\langle\tilde{\mathbf{r}},\, [\mathbf{r}, \mathbf{e}]\big\rangle - [\mathbf{r}, \mathbf{e}]\,\big\langle\mathbf{r},\, \tilde{\mathbf{r}}\big\rangle\big)\,.\end{aligned}$$

Now let's use well-known identity $[\mathbf{a},\, [\mathbf{b},\, \mathbf{c}]] = \mathbf{b}\,\big\langle\mathbf{a},\, \mathbf{c}\big\rangle - \mathbf{c}\,\big\langle\mathbf{a},\, \mathbf{b}\big\rangle$. Assuming that $\mathbf{a} = \tilde{\mathbf{r}}$, $\mathbf{b} = \mathbf{r}$, and $\mathbf{c} = [\mathbf{r}, \mathbf{e}]$, we transform the above expression for $\operatorname{grad} f$ to the following form:

$$\operatorname{grad} f(\tilde{\mathbf{r}}) = -3\,\big\langle\mathbf{r},\, \tilde{\mathbf{r}}\big\rangle\,[\mathbf{r}, \mathbf{e}] - \frac{3}{2}\,[\tilde{\mathbf{r}},\, [\mathbf{r},\, [\mathbf{r},\, \mathbf{e}]]]. \tag{7.9}$$

Right hand side of (7.9) contains triple vectorial product. In order to transform it we use the identity $[\mathbf{a},\, [\mathbf{b},\, \mathbf{c}]] = \mathbf{b}\,\big\langle\mathbf{a},\, \mathbf{c}\big\rangle - \mathbf{c}\,\big\langle\mathbf{a},\, \mathbf{b}\big\rangle$ again, now assuming that $\mathbf{a} = \mathbf{r}$, $\mathbf{b} = \mathbf{r}$, and $\mathbf{c} = \mathbf{e}$:

$$\operatorname{grad} f(\tilde{\mathbf{r}}) = -3\,\big\langle\mathbf{r},\, \tilde{\mathbf{r}}\big\rangle\,[\mathbf{r}, \mathbf{e}] - \frac{3}{2}\,\big\langle\mathbf{r}, \mathbf{e}\big\rangle\,[\tilde{\mathbf{r}},\, \mathbf{r}] + \frac{3}{2}\,|\mathbf{r}|^2\,[\tilde{\mathbf{r}},\, \mathbf{e}].$$

Let's add this expression for $\operatorname{grad} f$ to vector $\mathbf{b}(\tilde{\mathbf{r}})$. Here is resulting new expression for this vector:

$$\mathbf{b}(\tilde{\mathbf{r}}) = -\frac{3}{2}\,\big\langle\mathbf{r}, \mathbf{e}\big\rangle\,[\tilde{\mathbf{r}},\, \mathbf{r}] + \frac{1}{2}\,|\mathbf{r}|^2\,[\tilde{\mathbf{r}},\, \mathbf{e}]. \tag{7.10}$$

Let's substitute (7.10) into formula (7.7). This yields

$$\big\langle\mathbf{H}_2,\, \mathbf{e}\big\rangle = \int\limits_{\Omega} \frac{-3\big\langle\mathbf{r}, \mathbf{e}\big\rangle\,\big\langle\mathbf{r},\, [\mathbf{j}(\tilde{\mathbf{r}}),\, \tilde{\mathbf{r}}]\big\rangle + |\mathbf{r}|^2\,\big\langle\mathbf{e},\, [\mathbf{j}(\tilde{\mathbf{r}}),\, \tilde{\mathbf{r}}]\big\rangle}{2\,c\,|\mathbf{r}|^5}\,d^3\tilde{\mathbf{r}}.$$

Note that quantities $\mathbf{j}(\tilde{\mathbf{r}})$ and $\tilde{\mathbf{r}}$ enter into this formula in form of vector product $[\mathbf{j}(\tilde{\mathbf{r}}),\,\tilde{\mathbf{r}}]$ only. Denote by $\mathbf{M}$ the following integral:

$$\mathbf{M}=\int\limits_{\Omega}\frac{[\tilde{\mathbf{r}},\,\mathbf{j}(\tilde{\mathbf{r}})]}{2\,c}\,d^3\tilde{\mathbf{r}}. \tag{7.11}$$

Vector $\mathbf{M}$ given by integral (7.11) is called *magnetic moment* for currents with density $\mathbf{j}(\tilde{\mathbf{r}})$. In terms of $\mathbf{M}$ the above relationship for scalar product $\langle\mathbf{H}_2,\,\mathbf{e}\rangle$ is written as follows:

$$\langle\mathbf{H}_2,\,\mathbf{e}\rangle=\frac{3\,\langle\mathbf{r},\mathbf{e}\rangle\,\langle\mathbf{r},\,\mathbf{M}\rangle-|\mathbf{r}|^2\,\langle\mathbf{e},\,\mathbf{M}\rangle}{|\mathbf{r}|^5}. \tag{7.12}$$

If we remember that $\mathbf{e}$ in formula (7.12) is an arbitrary constant vector, then from (7.3) and lemma 7.1 we can conclude that the field of point magnetic dipole

$$\mathbf{H}(\mathbf{r})=\frac{3\,\langle\mathbf{r},\,\mathbf{M}\rangle\,\mathbf{r}-|\mathbf{r}|^2\,\mathbf{M}}{|\mathbf{r}|^5} \tag{7.13}$$

is leading term in asymptotical expansion of static magnetic field (4.9) and (5.6) as $\mathbf{r}\to\infty$.

Like electric dipole moment $\mathbf{D}$ of the system with zero total charge $Q=0$, magnetic moment $\mathbf{M}$ is invariant with respect to displacements $\mathbf{r}\to\mathbf{r}+\mathbf{r}_0$ that don't change configuration of currents within system. Indeed, under such displacement integral (7.11) is incremented by

$$\triangle\mathbf{M}=\int\limits_{\Omega}\frac{[\mathbf{r}_0,\,\mathbf{j}(\tilde{\mathbf{r}})]}{2\,c}\,d^3\tilde{\mathbf{r}}=0. \tag{7.14}$$

Integral in formula (7.14) is equal to zero by the same reasons as in proof of lemma 7.1.

Exercise 7.1. *Consider localized system of currents* $\mathbf{j}(\mathbf{r})$ *with current density given by the following distribution:*

$$\mathbf{j}(\mathbf{r}) = -c\,[\mathbf{M},\,\operatorname{grad}\delta(\mathbf{r})]. \tag{7.15}$$

Verify the relationship (7.1) *for the system of currents* (7.15) *and find its magnetic moment* $\mathbf{M}$. *Applying formula* (5.6), *calculate magnetic field of this system of currents and explain why this system of currents is called* point magnetic dipole.

Exercise 7.2. *Using formula* (5.5), *find the force acting upon point magnetic dipole in external magnetic field* $\mathbf{H}(\mathbf{r})$.

Exercise 7.3. *By means of the following formula for the torque*

$$\mathcal{M} = \int \frac{1}{c}\,[\mathbf{r},\,[\mathbf{j}(\mathbf{r}),\,\mathbf{H}]]\,d^3\mathbf{r}$$

find torque $\mathcal{M}$ *acting upon point magnetic dipole* (7.15) *in homogeneous magnetic field* $\mathbf{H} = \text{const}$.

§ 8. Integral equations for static electromagnetic field.

Remember that we introduced the concept of flow of vector field through a surface in considering charge conservation law (see integral J in (5.1)). Now we consider flows of vector fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$, i. e. for electric field and magnetic field:

$$\mathcal{E} = \int\limits_S \langle \mathbf{E},\,\mathbf{n}\rangle\,dS, \qquad\qquad \mathcal{H} = \int\limits_S \langle \mathbf{H},\,\mathbf{n}\rangle\,dS. \tag{8.1}$$

Let S be closed surface enveloping some domain Ω, i. e. $S = \partial\Omega$. Electrostatic field $\mathbf{E}$ is determined by formula (3.5). Let's substitute (3.5) into first integral (8.1) and then let's change order of integration in resulting double integral:

$$\mathcal{E} = \int \rho(\tilde{\mathbf{r}}) \int\limits_{\partial\Omega} \frac{\bigl\langle \mathbf{r} - \tilde{\mathbf{r}},\, \mathbf{n}(\mathbf{r}) \bigr\rangle}{|\mathbf{r} - \tilde{\mathbf{r}}|^3}\, dS\, d^3\tilde{\mathbf{r}}. \tag{8.2}$$

Inner surface integral in (8.2) is an integral of explicit function. This integral can be calculated explicitly:

$$\int\limits_{\partial\Omega} \frac{\bigl\langle \mathbf{r} - \tilde{\mathbf{r}},\, \mathbf{n}(\mathbf{r}) \bigr\rangle}{|\mathbf{r} - \tilde{\mathbf{r}}|^3}\, dS = \begin{cases} 0, \text{ if } \tilde{\mathbf{r}} \notin \overline{\Omega}, \\ 4\pi, \text{ if } \tilde{\mathbf{r}} \in \Omega. \end{cases} \tag{8.3}$$

Here by $\overline{\Omega} = \Omega \cup \partial\Omega$ we denote closure of the domain Ω.

In order to prove the relationship (8.3) let's consider vector field $\mathbf{m}(\mathbf{r})$ given by the following formula:

$$\mathbf{m}(\mathbf{r}) = \frac{\mathbf{r} - \tilde{\mathbf{r}}}{|\mathbf{r} - \tilde{\mathbf{r}}|^3}. \tag{8.4}$$

Vector field $\mathbf{m}(\mathbf{r})$ is smooth everywhere except for one special point $\mathbf{r} = \tilde{\mathbf{r}}$. In all regular points of this vector field by direct calculations we find $\operatorname{div}\mathbf{m} = 0$. If $\tilde{\mathbf{r}} \notin \overline{\Omega}$ special point of the field $\mathbf{m}$ is out of the domain Ω. Therefore in this case we can apply Ostrogradsky-Gauss formula to (8.3):

$$\int\limits_{\partial\Omega} \langle \mathbf{m},\, \mathbf{n} \rangle\, dS = \int\limits_{\Omega} \operatorname{div}\mathbf{m}\, d^3\mathbf{r} = 0.$$

This proves first case in formula (8.3). In order to prove second case, when $\tilde{\mathbf{r}} \in \Omega$, we use tactical maneuver. Let's consider spherical ϵ-neighborhood $O = O_\epsilon$ of special point $\mathbf{r} = \tilde{\mathbf{r}}$. For sufficiently small ϵ this neighborhood $O = O_\epsilon$ is completely enclosed into the domain Ω. Then from zero divergency condition $\operatorname{div}\mathbf{m} = 0$ for the field given by formula (8.4) we derive

$$\int\limits_{\partial\Omega} \langle \mathbf{m},\, \mathbf{n}\rangle\, dS = \int\limits_{\partial O} \langle \mathbf{m},\, \mathbf{n}\rangle\, dS = 4\pi. \tag{8.5}$$

The value of last integral over sphere ∂O in (8.5) is found by direct calculation, which is not difficult. Thus, formula (8.3) is proved. Substituting (8.3) into (8.2) we get the following relationship:

$$\int\limits_{\partial\Omega} \langle \mathbf{E},\, \mathbf{n}\rangle\, dS = 4\pi \int\limits_{\Omega} \rho(\mathbf{r})\, d^3\mathbf{r}. \tag{8.6}$$

This relationship (8.6) can be formulated as a theorem.

Theorem (on the flow of electric field). *Flow of electric field through the boundary of restricted domain is equal to total charge enclosed within this domain multiplied by* 4π.

Now let's consider flow of magnetic field $\mathcal{H}$ in (8.1). Static magnetic field is determined by formula (5.6). Let's substitute $\mathbf{H}(\mathbf{r})$ given by (5.6) into second integral (8.1), then change the order of integration in resulting double integral:

$$\mathcal{H} = \int \int\limits_{\partial\Omega} \frac{1}{c}\, \frac{\langle [\mathbf{j}(\tilde{\mathbf{r}}),\, \mathbf{r} - \tilde{\mathbf{r}}],\, \mathbf{n}(\mathbf{r})\rangle}{|\mathbf{r} - \tilde{\mathbf{r}}|^3}\, dS\, d^3\tilde{\mathbf{r}}. \tag{8.7}$$

It's clear that in calculating inner integral over the surface $\partial\Omega$ vector $\mathbf{j}$ can be taken for constant. Now consider the field

$$\mathbf{m}(\mathbf{r}) = \frac{[\mathbf{j},\, \mathbf{r} - \tilde{\mathbf{r}}]}{c\, |\mathbf{r} - \tilde{\mathbf{r}}|^3}. \tag{8.8}$$

Like (8.4), this vector field (8.8) has only one singular point $\mathbf{r} = \tilde{\mathbf{r}}$. Divergency of this field is equal to zero, this fact can be verified by direct calculations. As appears in this case, singular point makes no effect to the value of surface integral in (8.7). Instead of (8.3) in this case we have the following formula:

$$\int\limits_{\partial\Omega} \frac{1}{c} \frac{\langle[\mathbf{j},\,\mathbf{r}-\tilde{\mathbf{r}}],\,\mathbf{n}(\mathbf{r})\rangle}{|\mathbf{r}-\tilde{\mathbf{r}}|^3}\,dS = 0. \tag{8.9}$$

For $\tilde{\mathbf{r}} \notin \overline{\Omega}$ the relationship (8.9) follows from $\operatorname{div}\mathbf{m} = 0$ by applying Ostrogradsky-Gauss formula. For $\tilde{\mathbf{r}} \in \Omega$ we have the relationship similar to the above relationship (8.5):

$$\int\limits_{\partial\Omega} \langle\mathbf{m},\,\mathbf{n}\rangle\,dS = \int\limits_{\partial O} \langle\mathbf{m},\,\mathbf{n}\rangle\,dS = 0. \tag{8.10}$$

However, the value of surface integral over sphere ∂O in this case is equal to zero since vector $\mathbf{m}(\mathbf{r})$ is perpendicular to normal vector $\mathbf{n}$ at all points of sphere ∂O. As a result of substituting (8.9) into (8.7) we get the relationship

$$\int\limits_{\partial\Omega} \langle\mathbf{H},\,\mathbf{n}\rangle\,dS = 0, \tag{8.11}$$

which is formulated as the following theorem.

Theorem (on the flow of magnetic field). *Total flow of magnetic field through the boundary of any restricted domain is equal to zero.*

Let $\mathbf{r}(s)$ be vectorial parametric equation of some closed spatial curve Γ being the rim for some open surface S, i.e. $\Gamma = \partial S$. Open surface S means that S and Γ have empty intersection. By $\overline{S}$ we denote the closure of the surface S. Then $\overline{S} = S \cup \Gamma$. Taking s for natural parameter on Γ, we define *circulation* for electric and magnetic fields in form of the following contour integrals:

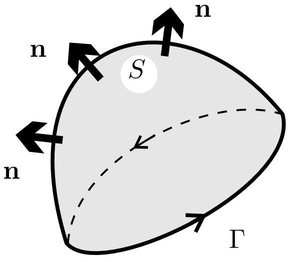

Fig. 8.1

$$\begin{aligned}\mathfrak{e} &= \oint\limits_{\Gamma} \bigl\langle \mathbf{E},\, \boldsymbol{\tau} \bigr\rangle\, ds,\\ \mathfrak{h} &= \oint\limits_{\Gamma} \bigl\langle \mathbf{H},\, \boldsymbol{\tau} \bigr\rangle\, ds.\end{aligned} \tag{8.12}$$

Substituting (3.5) into (8.12) and changing the order of integration in resulting double integral, we get the following equality for circulation of electric field:

$$\mathfrak{e} = \int \rho(\tilde{\mathbf{r}}) \oint\limits_{\Gamma} \frac{\bigl\langle \mathbf{r}(s) - \tilde{\mathbf{r}},\, \boldsymbol{\tau}(s) \bigr\rangle}{|\mathbf{r}(s) - \tilde{\mathbf{r}}|^3}\, ds\, d^3\tilde{\mathbf{r}}. \tag{8.13}$$

Due to (8.13) we need to consider vector field (8.4) again. For $\tilde{\mathbf{r}} \notin \Gamma$, taking into account $\Gamma = \partial S$ and applying Stokes formula, we can transform contour integral in (8.13) to surface integral:

$$\oint\limits_{\Gamma} \frac{\bigl\langle \mathbf{r}(s) - \tilde{\mathbf{r}},\, \boldsymbol{\tau}(s) \bigr\rangle}{|\mathbf{r}(s) - \tilde{\mathbf{r}}|^3}\, ds = \int\limits_{S} \bigl\langle \operatorname{rot} \mathbf{m},\, \mathbf{n} \bigr\rangle\, dS = 0. \tag{8.14}$$

Values of integral (8.14) at those points $\tilde{\mathbf{r}} \in \Gamma$ are of no matter since when substituting (8.14) into integral (8.13) such points constitute a set of zero measure.

Vanishing of integral (8.14) for $\tilde{\mathbf{r}} \notin \Gamma$ follows from $\operatorname{rot} \mathbf{m} = 0$, this equality can be verified by direct calculations. Singular point $\mathbf{r} = \tilde{\mathbf{r}}$ of vector field (8.4) is unessential since surface S, for which

Γ is a boundary, can be deformed so that $\tilde{\mathbf{r}} \notin S$. The result of substituting (8.14) into (8.13) can be written as an equation:

$$\oint\limits_{\partial S} \langle \mathbf{E}, \boldsymbol{\tau} \rangle \, ds = 0. \tag{8.15}$$

Theorem (on the circulation of electric field). *Total circulation of static electric field along the boundary of any restricted open surface is equal to zero.*

Formula like (8.15) is available for magnetic field as well. Here is this formula that determines circulation of magnetic field:

$$\oint\limits_{\partial S} \langle \mathbf{H}, \boldsymbol{\tau} \rangle \, ds = \frac{4\,\pi}{c} \int\limits_{S} \langle \mathbf{j}, \mathbf{n} \rangle \, dS. \tag{8.16}$$

Corresponding theorem is stated as follows.

Theorem (on the circulation of magnetic field). *Circulation of static magnetic field along boundary of restricted open surface is equal to total electric current penetrating this surface multiplied by fraction $4\,\pi/c$.*

Integral over the surface S now is in right hand side of formula (8.16) explicitly. Therefore surface spanned over the contour Γ now is fixed. We cannot deform this surface as we did above in proving theorem on circulation of electric field. This leads to some technical complication of the proof. Let's consider ε-blow-up of surface S. This is domain $\Omega(\varepsilon)$ being union of all ε-balls surrounding all point $\mathbf{r} \in S$. This domain encloses surface S and contour $\Gamma = \partial S$. If $\varepsilon \to 0$, domain $\Omega(\varepsilon)$ contracts to S.

Denote by $D(\varepsilon) = \mathbb{R}^3 \setminus \Omega(\varepsilon)$ exterior of the domain $\Omega(\varepsilon)$ and then consider the following modification of formula (5.6) that expresses Biot-Savart-Laplace law for magnetic field:

$$\mathbf{H}(\mathbf{r}) = \lim_{\varepsilon\to 0}\int\limits_{D(\varepsilon)} \frac{1}{c}\,\frac{[\mathbf{j}(\tilde{\mathbf{r}}),\,\mathbf{r}-\tilde{\mathbf{r}}]}{|\mathbf{r}-\tilde{\mathbf{r}}|^3}\,d^3\tilde{\mathbf{r}}. \tag{8.17}$$

Let's substitute (8.17) into integral (8.12) and change the order of integration in resulting double integral. As a result we get

$$\mathfrak{h} = \lim_{\varepsilon\to 0}\int\limits_{D(\varepsilon)}\oint\limits_{\Gamma} \frac{1}{c}\,\frac{\bigl\langle[\mathbf{j}(\tilde{\mathbf{r}}),\,\mathbf{r}(s)-\tilde{\mathbf{r}}],\,\boldsymbol{\tau}(s)\bigr\rangle}{|\mathbf{r}(s)-\tilde{\mathbf{r}}|^3}\,ds\,d^3\tilde{\mathbf{r}}. \tag{8.18}$$

In inner integral in (8.18) we see vector field (8.8). Unlike vector filed (8.4), rotor of this field is nonzero:

$$\operatorname{rot}\mathbf{m} = \frac{3\,\bigl\langle\mathbf{r}-\tilde{\mathbf{r}},\,\mathbf{j}\bigr\rangle\,(\mathbf{r}-\tilde{\mathbf{r}}) - |\mathbf{r}-\tilde{\mathbf{r}}|^2\,\mathbf{j}}{c\,|\mathbf{r}-\tilde{\mathbf{r}}|^5}. \tag{8.19}$$

Using Stokes formula and taking into account (8.19), we can transform contour integral (8.18) to surface integral:

$$\oint\limits_{\Gamma}\frac{1}{c}\,\frac{\bigl\langle[\mathbf{j}(\tilde{\mathbf{r}}),\,\mathbf{r}(s)-\tilde{\mathbf{r}}],\,\boldsymbol{\tau}(s)\bigr\rangle}{|\mathbf{r}(s)-\tilde{\mathbf{r}}|^3}\,ds =$$

$$= \int\limits_{S}\frac{3\,\bigl\langle\mathbf{r}-\tilde{\mathbf{r}},\,\mathbf{j}(\tilde{\mathbf{r}})\bigr\rangle\,\bigl\langle\mathbf{r}-\tilde{\mathbf{r}},\,\mathbf{n}(\mathbf{r})\bigr\rangle - |\mathbf{r}-\tilde{\mathbf{r}}|^2\,\bigl\langle\mathbf{j}(\tilde{\mathbf{r}}),\,\mathbf{n}(\mathbf{r})\bigr\rangle}{c\,|\mathbf{r}-\tilde{\mathbf{r}}|^5}\,dS.$$

Denote by $\widetilde{\mathbf{m}}(\tilde{\mathbf{r}})$ vector field of the following form:

$$\widetilde{\mathbf{m}}(\tilde{\mathbf{r}}) = \frac{3\,\bigl\langle\tilde{\mathbf{r}}-\mathbf{r},\,\mathbf{n}(\mathbf{r})\bigr\rangle\,(\tilde{\mathbf{r}}-\mathbf{r}) - |\tilde{\mathbf{r}}-\mathbf{r}|^2\,\mathbf{n}(\mathbf{r})}{c\,|\tilde{\mathbf{r}}-\mathbf{r}|^5}.$$

In terms of the field $\widetilde{\mathbf{m}}(\tilde{\mathbf{r}})$ formula for $\mathfrak{h}$ is written as

$$\mathfrak{h} = \lim_{\varepsilon\to 0} \int\limits_{D(\varepsilon)} \int\limits_{S} \bigl\langle \widetilde{\mathbf{m}}(\tilde{\mathbf{r}}),\, \mathbf{j}(\tilde{\mathbf{r}}) \bigr\rangle \, dS \, d^3\tilde{\mathbf{r}}.$$

Vector field $\widetilde{\mathbf{m}}(\tilde{\mathbf{r}})$ in this formula has cubic singularity $|\tilde{\mathbf{r}} - \mathbf{r}|^{-3}$ at the point $\tilde{\mathbf{r}} = \mathbf{r}$. Such singularity is not integrable in $\mathbb{R}^3$ (if we integrate with respect to $d^3\tilde{\mathbf{r}}$). This is why we use auxiliary domain $D(\varepsilon)$ and limit as $\varepsilon \to 0$.

Let's change the order of integration in resulting double integral for circulation $\mathfrak{h}$. This leads to formula

$$\int\limits_{S} \int\limits_{D(\varepsilon)} \bigl\langle \widetilde{\mathbf{m}}(\tilde{\mathbf{r}}),\, \mathbf{j}(\tilde{\mathbf{r}}) \bigr\rangle \, d^3\tilde{\mathbf{r}} \, dS = \int\limits_{S} \int\limits_{D(\varepsilon)} \bigl\langle \operatorname{grad} f(\tilde{\mathbf{r}}),\, \mathbf{j}(\tilde{\mathbf{r}}) \bigr\rangle \, d^3\tilde{\mathbf{r}} \, dS,$$

since vector field $\widetilde{\mathbf{m}}(\tilde{\mathbf{r}})$ apparently is gradient of the function $f(\tilde{\mathbf{r}})$:

$$f(\tilde{\mathbf{r}}) = -\frac{\bigl\langle \tilde{\mathbf{r}} - \mathbf{r},\, \mathbf{n}(\mathbf{r}) \bigr\rangle}{c\, |\tilde{\mathbf{r}} - \mathbf{r}|^3}. \tag{8.20}$$

Function $f(\tilde{\mathbf{r}})$ vanishes as $\tilde{\mathbf{r}} \to \infty$. Assume that current density also vanishes as $\tilde{\mathbf{r}} \to \infty$. Then due to the same considerations as in proof of lemma 7.1 and due to formula (7.1) spatial integral in the above formula can be transformed to surface integral:

$$\mathfrak{h} = \lim_{\varepsilon\to 0} \int\limits_{S} \int\limits_{\partial D(\varepsilon)} f(\tilde{\mathbf{r}}) \bigl\langle \mathbf{j}(\tilde{\mathbf{r}}),\, \tilde{\mathbf{n}}(\tilde{\mathbf{r}}) \bigr\rangle \, d\widetilde{S} \, dS. \tag{8.21}$$

Let's change the order of integration in (8.21) then take into account common boundary $\partial D(\varepsilon) = \partial \Omega(\varepsilon)$. Outer normal to the surface $\partial D(\varepsilon)$ coincides with inner normal to $\partial\Omega(\varepsilon)$. This coincidence and explicit form of function (8.20) lead to the following expression for circulation of magnetic field $\mathfrak{h}$:

$$\mathfrak{h} = \lim_{\varepsilon\to 0} \int\limits_{\partial\Omega(\varepsilon)} \frac{\bigl\langle \mathbf{j}(\tilde{\mathbf{r}}),\, \tilde{\mathbf{n}}(\tilde{\mathbf{r}})\bigr\rangle}{c} \int\limits_{S} \frac{\bigl\langle \tilde{\mathbf{r}} - \mathbf{r},\, \mathbf{n}(\mathbf{r})\bigr\rangle}{|\tilde{\mathbf{r}} - \mathbf{r}|^3}\, dS\, d\widetilde{S}. \tag{8.22}$$

Let's denote by $V(\tilde{\mathbf{r}})$ inner integral in formula (8.22):

$$V(\tilde{\mathbf{r}}) = \int\limits_{S} \frac{\bigl\langle \tilde{\mathbf{r}} - \mathbf{r},\, \mathbf{n}(\mathbf{r})\bigr\rangle}{|\tilde{\mathbf{r}} - \mathbf{r}|^3}\, dS. \tag{8.23}$$

Integral (8.23) is well-known in mathematical physics. It is called *potential of double layer*. There is the following lemma, proof of which can be found in [1].

Lemma 8.1. *Double layer potential* (8.23) *is restricted function in* $\mathbb{R}^3 \setminus \overline{S}$. *At each inner point* $\tilde{\mathbf{r}} \in S$ *there are side limits*

$$V_{\pm}(\tilde{\mathbf{r}}) = \lim_{\mathbf{r}\to\pm S} V(\mathbf{r}),$$

inner limit $V_{-}(\tilde{\mathbf{r}})$ *as* $\mathbf{r}$ *tends to* $\tilde{\mathbf{r}} \in S$ *from inside along normal vector* $\mathbf{n}$, *and outer limit* $V_{+}(\tilde{\mathbf{r}})$ *as* $\mathbf{r}$ *tends to* $\tilde{\mathbf{r}} \in S$ *from outside against the direction of normal vector* $\mathbf{n}$. *Thereby* $V_{+} - V_{-} = 4\pi$ *for all points* $\tilde{\mathbf{r}} \in S$.

In order to calculate limit in formula (8.22) we need to study the geometry of ε-blow-up of the surface S. On Fig. 8.2 below we see cross-section of the domain $\Omega(\varepsilon)$ obtained from the surface S shown on Fig. 8.1. For sufficiently small ε boundary of the domain $\Omega(\varepsilon)$ is composed of three parts:

$$\partial\Omega(\varepsilon) = S_0 \cup S_{+} \cup S_{-}. \tag{8.24}$$

Surface S_0 is a part of ε-blow-up of the contour Γ Area of this surface S_0 satisfies the relationship

$$S_0 \sim \varepsilon\pi L \quad \text{as} \quad \varepsilon \to 0, \tag{8.25}$$

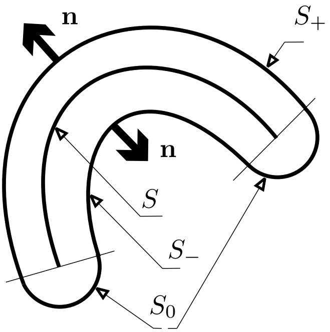

Fig. 8.2

where L is length of contour Γ. Surfaces S_+ and S_- are obtained as a result of normal shift of surface S to the distance ε along normal vector $\mathbf{n}$, and to the same distance against normal vector $\mathbf{n}$.

Substituting (8.24) into (8.22) we break surface integral over $\partial\Omega(\varepsilon)$ into three parts. Since double layer potential and function $|\,\mathbf{j}(\tilde{\mathbf{r}})|$ are restricted, we get the relationship

$$\lim_{\varepsilon\to 0}\int\limits_{S_0} V(\tilde{\mathbf{r}})\,\frac{\bigl\langle\,\mathbf{j}(\tilde{\mathbf{r}}),\,\tilde{\mathbf{n}}(\tilde{\mathbf{r}})\bigr\rangle}{c}\,d\widetilde{S} = 0. \tag{8.26}$$

For other two summand we also can calculate limits as $\varepsilon \to 0$:

$$\int\limits_{S_\pm} V(\tilde{\mathbf{r}})\,\frac{\bigl\langle\,\mathbf{j}(\tilde{\mathbf{r}}),\,\tilde{\mathbf{n}}(\tilde{\mathbf{r}})\bigr\rangle}{c}\,d\widetilde{S} \longrightarrow \pm\int\limits_{S} V_\pm(\mathbf{r})\,\frac{\bigl\langle\,\mathbf{j}(\mathbf{r}),\,\mathbf{n}(\mathbf{r})\bigr\rangle}{c}\,dS. \tag{8.27}$$

We shall not load reader with the proof of formulas (8.24), (8.25) and (8.27), which are sufficiently obvious. Summarizing (8.26) and (8.27) and taking into account lemma 8.1, we obtain

$$\mathfrak{h} = \frac{4\pi}{c}\int\limits_{S}\bigl\langle\,\mathbf{j}(\mathbf{r}),\,\mathbf{n}(\mathbf{r})\bigr\rangle\,dS. \tag{8.28}$$

This relationship (8.28) completes derivation of formula (8.16) and proof of theorem on circulation of magnetic field in whole.

Exercise 8.1. *Verify the relationship* $\operatorname{div} \mathbf{m} = 0$ *for vector fields* (8.4) *and* (8.8).

Exercise 8.2. *Verify the relationship* (8.19) *for vector field given by formula* (8.8).

Exercise 8.3. *Calculate* $\operatorname{grad} f$ *for the function* (8.20).

§ 9. Differential equations for static electromagnetic field.

In § 8 we have derived four integral equations for electric and magnetic fields. They are used to be grouped into two pairs. Equations in first pair have zero right hand sides:

$$\int\limits_{\partial\Omega} \langle \mathbf{H}, \mathbf{n}\rangle\, dS = 0, \qquad \oint\limits_{\partial S} \langle \mathbf{E}, \boldsymbol{\tau}\rangle\, ds = 0. \tag{9.1}$$

Right hand sides of equations in second pair are non-zero. They are determined by charges and currents:

$$\begin{aligned} &\int\limits_{\partial\Omega} \langle \mathbf{E}, \mathbf{n}\rangle\, dS = 4\pi \int\limits_{\Omega} \rho\, d^3\mathbf{r}, \\ &\oint\limits_{\partial S} \langle \mathbf{H}, \boldsymbol{\tau}\rangle\, ds = \frac{4\,\pi}{c} \int\limits_{S} \langle \mathbf{j}, \mathbf{n}\rangle\, dS. \end{aligned} \tag{9.2}$$

Applying Ostrogradsky-Gauss formula and Stokes formula, one can transform surface integrals to spatial ones, and contour integrals to surface integrals. Then, since Ω is arbitrary domain and

S is arbitrary open surface, integral equations (9.1) and (9.2) can be transformed to differential equations:

$$\operatorname{div} \mathbf{H} = 0, \qquad \operatorname{rot} \mathbf{E} = 0, \tag{9.3}$$

$$\operatorname{div} \mathbf{E} = 4\pi\rho, \qquad \operatorname{rot} \mathbf{H} = \frac{4\pi}{c}\,\mathbf{j}. \tag{9.4}$$

When considering differential equations (9.3) and (9.4), we should add conditions for charges and currents being stationary:

$$\frac{\partial\rho}{\partial t} = 0, \qquad \frac{\partial\mathbf{j}}{\partial t} = 0. \tag{9.5}$$

The relationship (7.1) then is a consequence of (9.5) and charge conservation law.

Differential equations (9.3) and (9.4) form complete system of differential equations for describing stationary electromagnetic fields. When solving them functions $\rho(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$ are assumed to be known. If they are not known, one should have some additional equations relating ρ and $\mathbf{j}$ with $\mathbf{E}$ and $\mathbf{H}$. These additional equations describe properties of medium (for instance, continuous conducting medium is described by the equation $\mathbf{j} = \sigma\,\mathbf{E}$, where σ is conductivity of medium).

CHAPTER II

CLASSICAL ELECTRODYNAMICS

§ 1. Maxwell equations.

Differential equations (9.3) and (9.4), which we have derived in the end of Chapter I, describe fields generated by stationary charges and currents. They are absolutely unsuitable if we are going to describe the process of haw electromagnetic interaction is transmitted in space. Note that the notion of field was introduced within framework of the concept of near action for describing the object that transmit interaction of charges and currents. For static fields this property is revealed in a very restrictive form, i. e. we use fields only to divide interaction of charges and currents into two processes: creation of a field by charges and currents is first process, action of this field upon other currents and charges is second process. Dynamic properties of the field itself appears beyond our consideration.

More exact equations describing process of transmitting electromagnetic interaction in its time evolution were suggested by Maxwell. They are the following ones:

$$\text{div}\,\mathbf{H} = 0, \qquad \text{rot}\,\mathbf{E} = -\frac{1}{c}\,\frac{\partial \mathbf{H}}{\partial t}, \tag{1.1}$$

$$\text{div}\,\mathbf{E} = 4\pi\rho, \qquad \text{rot}\,\mathbf{H} = \frac{4\pi}{c}\,\mathbf{j} + \frac{1}{c}\,\frac{\partial \mathbf{E}}{\partial t}. \tag{1.2}$$

It is easy to see that equations (1.1) and (1.2) are generalizations for the (9.3) and (9.4) from Chapter I. They are obtained from latter ones by modifying right hand sides. Like equations (9.3) and (9.4) in Chapter I, Maxwell equations (1.1) and (1.2) can be written in form of integral equations:

$$\begin{aligned} &\int\limits_{\partial\Omega} \langle \mathbf{H},\, \mathbf{n}\rangle\, dS = 0,\\ &\oint\limits_{\partial S} \langle \mathbf{E},\, \boldsymbol{\tau}\rangle\, ds = -\frac{1}{c}\frac{d}{dt}\int\limits_{S}\langle\, \mathbf{H},\, \mathbf{n}\rangle\, dS, \end{aligned} \tag{1.3}$$

$$\begin{aligned} &\int\limits_{\partial\Omega} \langle \mathbf{E},\, \mathbf{n}\rangle\, dS = 4\pi\int\limits_{\Omega} \rho\, d^3\mathbf{r},\\ &\oint\limits_{\partial S} \langle \mathbf{H},\, \boldsymbol{\tau}\rangle\, ds = \frac{4\,\pi}{c}\int\limits_{S}\langle\, \mathbf{j},\, \mathbf{n}\rangle\, dS + \frac{1}{c}\frac{d}{dt}\int\limits_{S}\langle\, \mathbf{E},\, \mathbf{n}\rangle\, dS. \end{aligned} \tag{1.4}$$

Consider contour integral in second equation (1.3). Similar contour integral is present in second equation (1.4). However, unlike circulation of magnetic field, circulation of electric field

$$\mathfrak{e} = \oint\limits_{\partial S} \langle \mathbf{E},\, \boldsymbol{\tau}\rangle\, ds \tag{1.5}$$

possess its own physical interpretation. If imaginary contour $\Gamma = \partial S$ in space is replaced by real circular conductor, then electric field with nonzero circulation induces electric current in conductor. The quantity $\mathfrak{e}$ from (1.5) in this case is called *electromotive force* of the field $\mathbf{E}$ in contour. Electromotive force $\mathfrak{e} \neq 0$ in contour produce the same effect as linking electric cell with voltage $\mathfrak{e}$ into this contour. Experimentally it reveals as follow: alternating magnetic field produces electric field with nonzero circulation, this induces electric current in circular conductor. This phenomenon is known as *electromagnetic induction*. It was first discovered by Faraday. Faraday gave qualitative description of this phenomenon in form of the following induction law.

Faraday's law of electromagnetic induction. *Electromotive force of induction in circular conductor is proportional to the rate of changing of magnetic flow embraced by this conductor.*

Faraday's induction law was a hint for Maxwell when choosing right hand side in second equation (1.1). As for similar term in right hand side of second equation (1.2), Maxwell had written it by analogy. Experiments and further development of technology proved correctness of Maxwell equations.

Note that charge conservation law in form of relationship (5.4) from Chapter I is a consequence of Maxwell equations. One should calculate divergency of both sides of second equation (1.2):

$$\operatorname{div}\operatorname{rot}\mathbf{H} = \frac{4\pi}{c}\operatorname{div}\mathbf{j} + \frac{1}{c}\,\frac{\partial \operatorname{div}\mathbf{E}}{\partial t},$$

then one should apply the identity $\operatorname{div}\operatorname{rot}\mathbf{H} = 0$. When combined with the first equation (1.2) this yields exactly the relationship (5.4) from Chapter I.

Equations (1.1) and (1.2) form complete system for describing arbitrary electromagnetic fields. In solving them functions $\rho(\mathbf{r}, t)$ and $\mathbf{j}(\mathbf{r}, t)$ should be given, or they should be determined from medium equations. Then each problem of electrodynamics mathematically reduces to some boundary-value problem or mixed initial-value/boundary-value problem for Maxwell equations optionally completed by medium equations. In this section we consider only some very special ones among such problems. Our main goal is to derive important mathematical consequences from Maxwell equations and to interpret their physical nature.

§ 2. Density of energy and energy flow for electromagnetic field.

Suppose that in bulk conductor we have a current with density $\mathbf{j}$, and suppose that this current is produced by the flow of charged particles with charge q. If ν is the number of such particles per unit volume and if $\mathbf{v}$ is their velocity, then $\mathbf{j} = q\,\nu\,\mathbf{v}$. Recall that current density is a charge passing through unit area per unit time (see § 5 in Chapter I).

In electromagnetic field each particle experiences Lorentz force determined by formula (4.4) from Chapter I. Work of this force per unit time is equal to $\bigl\langle \mathbf{F},\,\mathbf{v}\bigr\rangle = q\,\bigl\langle \mathbf{E},\,\mathbf{v}\bigr\rangle$. Total work produced by electromagnetic field per unit volume is obtained if one multiplies this quantity by ν, then $w = q\,\nu\,\bigl\langle \mathbf{E},\,\mathbf{v}\bigr\rangle = \bigl\langle \mathbf{E},\,\mathbf{j}\bigr\rangle$. This work increases kinetic energy of particles (particles are accelerated by field). Otherwise this work is used for to compensate forces of viscous friction that resist motion of particles. In either case total power spent by electromagnetic field within domain Ω is determined by the following integral:

$$W = \int\limits_{\Omega} \bigl\langle \mathbf{E},\,\mathbf{j}\bigr\rangle\,d^3\mathbf{r}. \tag{2.1}$$

Let's transform integral (2.1). Let's express current density $\mathbf{j}$ through $\mathbf{E}$ and $\mathbf{H}$ using second equation (1.2) for this purpose:

$$\mathbf{j} = \frac{c}{4\,\pi}\operatorname{rot}\mathbf{H} - \frac{1}{4\,\pi}\frac{\partial\mathbf{E}}{\partial t}. \tag{2.2}$$

Substituting this expression (2.2) into formula (2.1), we get

$$W = \frac{c}{4\,\pi}\int\limits_{\Omega}\bigl\langle\mathbf{E},\,\operatorname{rot}\mathbf{H}\bigr\rangle\,d^3\mathbf{r} - \frac{1}{8\,\pi}\int\limits_{\Omega}\frac{\partial}{\partial t}\bigl\langle\mathbf{E},\,\mathbf{E}\bigr\rangle\,d^3\mathbf{r}. \tag{2.3}$$

In order to implement further transformations in formula (2.3) we use well-known identity $\operatorname{div}[\mathbf{a},\,\mathbf{b}] = \bigl\langle\mathbf{b},\,\operatorname{rot}\mathbf{a}\bigr\rangle - \bigl\langle\mathbf{a},\,\operatorname{rot}\mathbf{b}\bigr\rangle$. Assuming $\mathbf{a} = \mathbf{H}$ and $\mathbf{b} = \mathbf{E}$, for W we get

$$W = \frac{c}{4\,\pi}\int\limits_{\Omega}\operatorname{div}[\mathbf{H},\,\mathbf{E}]\,d^3\mathbf{r} + \frac{c}{4\,\pi}\int\limits_{\Omega}\bigl\langle\mathbf{H},\,\operatorname{rot}\mathbf{E}\bigr\rangle\,d^3\mathbf{r} - \frac{d}{dt}\int\limits_{\Omega}\frac{|\mathbf{E}|^2}{8\,\pi}\,d^3\mathbf{r}.$$

First integral in this expression can be transformed by means of Ostrogradsky-Gauss formula, while for transforming $\operatorname{rot}\mathbf{E}$ one can use Maxwell equations (1.1):

$$W + \int\limits_{\partial\Omega}\frac{c}{4\,\pi}\bigl\langle[\mathbf{E},\,\mathbf{H}],\,\mathbf{n}\bigr\rangle\,dS + \frac{d}{dt}\int\limits_{\Omega}\frac{|\mathbf{E}|^2 + |\mathbf{H}|^2}{8\,\pi}\,d^3\mathbf{r} = 0. \tag{2.4}$$

Let's denote by $\mathbf{S}$ and ε vectorial field and scalar field of the form

$$\mathbf{S} = \frac{c}{4\,\pi}\,[\mathbf{E},\mathbf{H}], \qquad\qquad \varepsilon = \frac{|\mathbf{E}|^2 + |\mathbf{H}|^2}{8\,\pi}. \tag{2.5}$$

The quantity ε in (2.5) is called *density of energy* of electromagnetic field. Vector $\mathbf{S}$ is known as *density of energy flow.* It also called *Umov-Pointing vector.* Under such interpretation of quantities (2.5) the relationship (2.4) can be treated as the equation of energy balance. First summand in (2.4) is called dissipation power, this is the amount of energy dissipated per unit time at the expense of transmitting it to moving charges. Second summand is the amount of energy that flows from within domain Ω to outer space per unit time. These two forms of energy losses lead to diminishing the energy stored by electromagnetic field itself within domain Ω (see third summand in (2.4)).

Energy balance equation (2.4) can be rewritten in differential form, analogous to formula (5.4) from Chapter I:

$$\frac{\partial\varepsilon}{\partial t} + \operatorname{div}\mathbf{S} + w = 0. \tag{2.6}$$

Here $w = \langle\mathbf{E},\,\mathbf{j}\rangle$ is a density of energy dissipation. Note that in some cases w and integral (2.1) in whole can be negative. In such a case we have energy pumping into electromagnetic field. This energy then flows to outer space through boundary of the domain Ω. This is the process of radiation of electromagnetic waves from the domain Ω. It is realized in antennas (aerials) of radio and TV transmitters. If we eliminate or restrict substantially the energy leakage from the domain Ω to outer space, then we would have the device like microwave oven, where electromagnetic field is used for transmitting energy from radiator to beefsteak.

Electromagnetic field can store and transmit not only the energy, but the momentum as well. In order to derive momentum balance equations let's consider again the current with density $\mathbf{j}$ due to the particles with charge q which move with velocity $\mathbf{v}$. Let ν be concentration of these particles, i. e. number of particles per unit volume. Then $\mathbf{j} = q\,\nu\,\mathbf{v}$ and $\rho = q\,\nu$. Total force acting on all particles within domain Ω is given by integral

$$\mathbf{F} = \int\limits_{\Omega} \rho\,\mathbf{E}\,d^3\mathbf{r} + \int\limits_{\Omega} \frac{1}{c}\,[\mathbf{j},\,\mathbf{H}]\,d^3\mathbf{r}. \tag{2.7}$$

In order to derive formula (2.7) one should multiply Lorentz force acting on each separate particle by the number of particles per unit volume ν and then integrate over the domain Ω.

Force $\mathbf{F}$ determines the amount of momentum transmitted from electromagnetic field to particles enclosed within domain Ω. Integral (2.7) is vectorial quantity. For further transformations of this integral let's choose some constant unit vector $\mathbf{e}$ and consider scalar product of this vector $\mathbf{e}$ and vector $\mathbf{F}$:

$$\langle \mathbf{F},\, \mathbf{e}\rangle = \int\limits_{\Omega} \rho\,\langle\, \mathbf{E},\, \mathbf{e}\rangle\, d^3\mathbf{r} + \int \frac{1}{c}\,\langle \mathbf{e},\, [\mathbf{j},\, \mathbf{H}]\rangle\, d^3\mathbf{r}. \tag{2.8}$$

Substituting (2.2) into (2.8), we get

$$\begin{aligned}\langle \mathbf{F},\, \mathbf{e}\rangle = \int\limits_{\Omega} \rho\,\langle \mathbf{E},\, \mathbf{e}\rangle\, d^3\mathbf{r} + \frac{1}{4\,\pi}\int\limits_{\Omega}\langle \mathbf{e},\, [\operatorname{rot}\mathbf{H},\, \mathbf{H}]\rangle\, d^3\mathbf{r}\, -\\ -\,\frac{1}{4\,\pi\,c}\int\limits_{\Omega}\langle \mathbf{e},\, [\partial\mathbf{E}/\partial t,\, \mathbf{H}]\rangle\, d^3\mathbf{r}.\end{aligned} \tag{2.9}$$

Recalling well-known property of mixed product, we do cyclic transposition of multiplicands in second integral (2.9). Moreover, we use obvious identity $[\partial\mathbf{E}/\partial t,\, \mathbf{H}] = \partial\,[\mathbf{E},\, \mathbf{H}]\,/\partial t - [\mathbf{E},\, \partial\mathbf{H}/\partial t]$. This yields the following expression for $\langle \mathbf{F},\, \mathbf{e}\rangle$:

$$\begin{aligned}\langle \mathbf{F},\, \mathbf{e}\rangle = \int\limits_{\Omega} \rho\,\langle\, \mathbf{E},\, \mathbf{e}\rangle\, d^3\mathbf{r} + \frac{1}{4\,\pi}\int\limits_{\Omega}\langle \operatorname{rot}\mathbf{H},\, [\mathbf{H},\, \mathbf{e}]\rangle\, d^3\mathbf{r}\, -\\ -\,\frac{1}{4\,\pi\,c}\,\frac{d}{dt}\int\limits_{\Omega}\langle \mathbf{e},\, [\mathbf{E},\, \mathbf{H}]\rangle\, d^3\mathbf{r} + \frac{1}{4\,\pi\,c}\int\limits_{\Omega}\langle \mathbf{e},\, [\mathbf{E},\, \partial\mathbf{H}/\partial t]\rangle\, d^3\mathbf{r}.\end{aligned}$$

Now we apply second equation of the system (1.1) written as

$\partial\mathbf{H}/\partial t = -c\,\operatorname{rot}\mathbf{E}$. Then we get formula

$$\begin{gathered}\bigl\langle\mathbf{F},\,\mathbf{e}\bigr\rangle+\frac{d}{dt}\int\limits_{\Omega}\frac{\bigl\langle\mathbf{e},\,[\mathbf{E},\,\mathbf{H}]\bigr\rangle}{4\,\pi\,c}\,d^3\mathbf{r}=\int\limits_{\Omega}\rho\,\bigl\langle\mathbf{E},\,\mathbf{e}\bigr\rangle\,d^3\mathbf{r}\,+\\+\int\limits_{\Omega}\frac{\bigl\langle\operatorname{rot}\,\mathbf{H},\,[\mathbf{H},\,\mathbf{e}]\bigr\rangle+\bigl\langle\operatorname{rot}\,\mathbf{E},\,[\mathbf{E},\,\mathbf{e}]\bigr\rangle}{4\,\pi}\,d^3\mathbf{r}.\end{gathered}\tag{2.10}$$

In order to transform last two integrals in (2.10) we use the following three identities, two of which we already used earlier:

$$\begin{aligned}&[\mathbf{a},\,[\mathbf{b},\,\mathbf{c}]]=\mathbf{b}\,\bigl\langle\mathbf{a},\,\mathbf{c}\bigr\rangle-\mathbf{c}\,\bigl\langle\mathbf{a},\,\mathbf{b}\bigr\rangle,\\&\operatorname{div}\,[\mathbf{a},\,\mathbf{b}]=\bigl\langle\mathbf{b},\,\operatorname{rot}\,\mathbf{a}\bigr\rangle-\bigl\langle\mathbf{a},\,\operatorname{rot}\,\mathbf{b}\bigr\rangle,\\&\operatorname{rot}\,[\mathbf{a},\,\mathbf{b}]=\mathbf{a}\,\operatorname{div}\,\mathbf{b}-\mathbf{b}\,\operatorname{div}\mathbf{a}-\{\mathbf{a},\,\mathbf{b}\}.\end{aligned}\tag{2.11}$$

Here by curly brackets we denote commutator of two vector fields $\mathbf{a}$ and $\mathbf{b}$ (see [2]). Traditionally square brackets are used for commutator, but here by square brackets we denote vector product of two vectors. From second identity (2.11) we derive

$$\bigl\langle\operatorname{rot}\mathbf{H},\,[\mathbf{H},\,\mathbf{e}]\bigr\rangle=\operatorname{div}\,[\mathbf{H},\,[\mathbf{H},\,\mathbf{e}]]+\bigl\langle\mathbf{H},\,\operatorname{rot}\,[\mathbf{H},\,\mathbf{e}]\bigr\rangle.$$

In order to transform $\operatorname{rot}[\mathbf{H},\,\mathbf{e}]$ we use third identity (2.11): $\operatorname{rot}[\mathbf{H},\,\mathbf{e}]=-\mathbf{e}\,\operatorname{div}\mathbf{H}-\{\mathbf{H},\,\mathbf{e}\}$. Then

$$\begin{gathered}\bigl\langle\mathbf{H},\,\operatorname{rot}\,[\mathbf{H},\,\mathbf{e}]\bigr\rangle=-\bigl\langle\mathbf{H},\,\mathbf{e}\bigr\rangle\,\operatorname{div}\mathbf{H}+\sum_{i=1}^{3}H_i\sum_{j=1}^{3}e^j\,\frac{\partial H^i}{\partial r^j}=\\=-\bigl\langle\mathbf{H},\,\mathbf{e}\bigr\rangle\,\operatorname{div}\mathbf{H}+\frac{1}{2}\,\bigl\langle\mathbf{e},\,\operatorname{grad}|\mathbf{H}|^2\bigr\rangle.\end{gathered}$$

Let's combine two above relationships and apply first identity (2.11) for to transform double vectorial product $[\mathbf{H},\,[\mathbf{H},\,\mathbf{e}]]$ in first of them. As a result we obtain

$$\begin{aligned}\bigl\langle \operatorname{rot}\mathbf{H},\,[\mathbf{H},\,\mathbf{e}]\bigr\rangle = \operatorname{div}\bigl(\mathbf{H}\,\langle \mathbf{H},\,\mathbf{e}\rangle\bigr) - \operatorname{div}\bigl(\mathbf{e}\,|\mathbf{H}|^2\bigr) -\\ - \bigl\langle \mathbf{H},\,\mathbf{e}\bigr\rangle\,\operatorname{div}\mathbf{H} + \frac{1}{2}\bigl\langle \mathbf{e},\,\operatorname{grad}|\mathbf{H}|^2\bigr\rangle.\end{aligned}$$

But $\operatorname{div}\bigl(\mathbf{e}\,|\mathbf{H}|^2\bigr) = \bigl\langle \mathbf{e},\,\operatorname{grad}|\mathbf{H}|^2\bigr\rangle$. Hence as a final result we get

$$\begin{aligned}\bigl\langle \operatorname{rot}\mathbf{H},\,[\mathbf{H},\,\mathbf{e}]\bigr\rangle = -\bigl\langle \mathbf{H},\,\mathbf{e}\bigr\rangle\,\operatorname{div}\mathbf{H} +\\ + \operatorname{div}\left(\mathbf{H}\,\langle \mathbf{H},\,\mathbf{e}\rangle - \frac{1}{2}\,\mathbf{e}\,|\mathbf{H}|^2\right).\end{aligned}\tag{2.12}$$

Quite similar identity can be derived for electric field $\mathbf{E}$:

$$\begin{aligned}\bigl\langle \operatorname{rot}\mathbf{E},\,[\mathbf{E},\,\mathbf{e}]\bigr\rangle = -\bigl\langle \mathbf{E},\,\mathbf{e}\bigr\rangle\,\operatorname{div}\mathbf{E} +\\ + \operatorname{div}\left(\mathbf{E}\,\langle \mathbf{E},\,\mathbf{e}\rangle - \frac{1}{2}\,\mathbf{e}\,|\mathbf{E}|^2\right).\end{aligned}\tag{2.13}$$

The only difference is that due to Maxwell equations $\operatorname{div}\mathbf{H} = 0$, while divergency of electric field $\mathbf{E}$ is nonzero: $\operatorname{div}\mathbf{E} = 4\pi\rho$.

Now, if we take into account (2.12) and (2.13), formula (2.10) can be transformed to the following one:

$$\begin{aligned}\bigl\langle \mathbf{F},\,\mathbf{e}\bigr\rangle - \int\limits_{\partial\Omega} \frac{\langle \mathbf{E},\,\mathbf{e}\rangle\langle \mathbf{n},\,\mathbf{E}\rangle + \langle \mathbf{H},\,\mathbf{e}\rangle\langle \mathbf{n},\,\mathbf{H}\rangle}{4\,\pi}\,dS +\\ + \int\limits_{\partial\Omega} \frac{(|\mathbf{E}|^2 + |\mathbf{H}|^2)\,\langle \mathbf{e},\,\mathbf{n}\rangle}{8\,\pi}\,dS + \frac{d}{dt}\int\limits_{\Omega} \frac{\langle \mathbf{e},\,[\mathbf{E},\,\mathbf{H}]\rangle}{4\,\pi\,c}\,d^3\mathbf{r} = 0.\end{aligned}$$

Denote by σ linear operator such that the result of applying this operator to some arbitrary vector $\mathbf{e}$ is given by formula

$$\sigma\,\mathbf{e} = -\frac{\mathbf{E}\,\bigl\langle\mathbf{E},\,\mathbf{e}\bigr\rangle + \mathbf{H}\,\bigl\langle\mathbf{H},\,\mathbf{e}\bigr\rangle}{4\,\pi} + \frac{|\mathbf{E}|^2 + |\mathbf{H}|^2}{8\,\pi}\,\mathbf{e}. \tag{2.14}$$

Formula (2.14) defines tensorial field $\boldsymbol{\sigma}$ of type $(1, 1)$ with the following components:

$$\sigma^i_j = \frac{|\mathbf{E}|^2 + |\mathbf{H}|^2}{8\,\pi}\,\delta^i_j - \frac{E^i\,E_j + H^i\,H_j}{4\,\pi}. \tag{2.15}$$

Tensor $\boldsymbol{\sigma}$ with components (2.15) is called tensor of the *density of momentum flow*. It is also known as *Maxwell tensor*. Now let's define vector of *momentum density* $\mathbf{p}$ by formula

$$\mathbf{p} = \frac{[\mathbf{E},\,\mathbf{H}]}{4\,\pi\,c}. \tag{2.16}$$

In terms of the notations (2.15) and (2.16) the above relationship for $\bigl\langle\mathbf{F},\,\mathbf{e}\bigr\rangle$ is rewritten as follows:

$$\bigl\langle\mathbf{F},\,\mathbf{e}\bigr\rangle + \int\limits_{\partial\Omega}\bigl\langle\sigma\,\mathbf{e},\,\mathbf{n}\bigr\rangle\,dS + \frac{d}{dt}\int\limits_{\Omega}\bigl\langle\mathbf{p},\,\mathbf{e}\bigr\rangle\,d^3\mathbf{r} = 0. \tag{2.17}$$

Operator of the density of momentum flow $\boldsymbol{\sigma}$ is symmetric, i. e. $\bigl\langle\sigma\,\mathbf{e},\,\mathbf{n}\bigr\rangle = \bigl\langle\mathbf{e},\,\sigma\,\mathbf{n}\bigr\rangle$. Due to this property and because $\mathbf{e}$ is arbitrary vector we can rewrite (2.17) in vectorial form:

$$\mathbf{F} + \int\limits_{\partial\Omega}\sigma\,\mathbf{n}\,dS + \frac{d}{dt}\int\limits_{\Omega}\mathbf{p}\,d^3\mathbf{r} = 0. \tag{2.18}$$

This equation (2.18) is the equation of momentum balance for electromagnetic field. Force $\mathbf{F}$, given by formula (2.7) determines loss of momentum stored in electromagnetic field due to transmitting it to moving particles. Second term in (2.18) determines loss of momentum due to its flow through the boundary of the domain Ω. These two losses lead to diminishing the momentum stored by electromagnetic field within domain Ω (see third summand in (2.18)).

The relationship (2.18) can be rewritten in differential form. For this purpose we should define vectorial divergency for tensorial field $\boldsymbol{\sigma}$ of the type $(1, 1)$. Let

$$\boldsymbol{\mu} = \operatorname{div} \sigma, \quad \text{where } \mu_j = \sum_{i=1}^{3} \frac{\partial \sigma^i_j}{\partial r^i}. \tag{2.19}$$

Then differential form of (2.18) is written as

$$\frac{\partial \mathbf{p}}{\partial t} + \operatorname{div} \sigma + \mathbf{f} = 0, \tag{2.20}$$

where $\mathbf{f} = \rho\,\mathbf{E} + [\mathbf{j},\,\mathbf{H}]/c$ is a density of Lorentz force, while vectorial divergency is determined according to (2.19).

Thus, electromagnetic field is capable to accumulate within itself the energy and momentum:

$$\mathcal{E} = \int\limits_{\Omega} \frac{|\mathbf{E}|^2 + |\mathbf{H}|^2}{8\,\pi}\, d^3\mathbf{r}, \qquad \mathbf{P} = \int\limits_{\Omega} \frac{[\mathbf{E},\,\mathbf{H}]}{4\,\pi\,c}\, d^3\mathbf{r}. \tag{2.21}$$

It is also capable to transmit energy and momentum to material bodies. This confirms once more our assertion that electromagnetic field itself is a material entity. It is not pure mathematical abstraction convenient for describing interaction of charges and currents, but real physical object.

Exercise 2.1. *Verify that relationships* (2.11) *hold. Check on the derivation of* (2.12) *and* (2.13).

§ 3. Vectorial and scalar potentials of electromagnetic field.

In section 2 we have found that electromagnetic field possess energy and momentum (2.21). This is very important consequence of Maxwell equations (1.1) and (1.2). However we have not studied Maxwell equations themselves. This is system of four equations, two of them are scalar equations, other two are vectorial equations. So they are equivalent to eight scalar equations. However we have only six undetermined functions in them: three components of vector $\mathbf{E}$ and three components of vector $\mathbf{H}$. So observe somewhat like excessiveness in Maxwell equations.

One of the most popular ways for solving systems of algebraic equations is to express some variable through other ones by solving one of the equations in a system (usually most simple equation) and then substituting the expression obtained into other equations. Thus we exclude one variable and diminish the number of equations in a system also by one. Sometimes this trick is applicable to differential equations as well. Let's consider Maxwell equation $\operatorname{div} \mathbf{H} = 0$. Vector field with zero divergency is called *vortex field*. For vortex fields the following theorem holds (see proof in book [3]).

Theorem on vortex field. *Each vortex field is a rotor of some other vector field.*

Let's write the statement of this theorem as applied to magnetic field. It is given by the following relationship:

$$\mathbf{H} = \operatorname{rot} \mathbf{A}. \tag{3.1}$$

Vector field $\mathbf{A}$, whose existence is granted by the above theorem, is called *vector-potential* of electromagnetic field.

Let's substitute vector $\mathbf{H}$ as given by (3.1) into second Maxwell equation (1.1). This yields the equality

$$\operatorname{rot}\mathbf{E}+\frac{1}{c}\frac{\partial}{\partial t}\operatorname{rot}\mathbf{A}=\operatorname{rot}\left(\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right)=0. \tag{3.2}$$

Vector field with zero rotor is called *potential field.* It is vector field $\mathbf{E}+(\partial\mathbf{A}/\partial t)/c$ in formula (3.2) which is obviously potential field. Potential fields are described by the following theorem (see proof in book [3]).

Theorem on potential field. *Each potential field is a gradient of some scalar field.*

Applying this theorem to vector field (3.2), we get the relationship determining *scalar potential* of electromagnetic field φ:

$$\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}=-\operatorname{grad}\varphi. \tag{3.3}$$

Combining (3.1) and (3.3), we can express electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ through newly introduced fields $\mathbf{A}$ and φ:

$$\begin{aligned}&\mathbf{E}=-\operatorname{grad}\varphi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t},\\&\mathbf{H}=\operatorname{rot}\mathbf{A}.\end{aligned} \tag{3.4}$$

Upon substituting (3.4) into first pair of Maxwell equations (1.1) we find them to be identically fulfilled. As for second pair of Maxwell equations, substituting (3.4) into these equations, we get

$$\begin{aligned}&-\triangle\varphi-\frac{1}{c}\frac{\partial}{\partial t}\operatorname{div}\mathbf{A}=4\,\pi\,\rho,\\&\operatorname{grad}\operatorname{div}\mathbf{A}-\triangle\mathbf{A}+\frac{1}{c}\frac{\partial}{\partial t}\operatorname{grad}\varphi+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=\frac{4\,\pi\,\mathbf{j}}{c}.\end{aligned} \tag{3.5}$$

In deriving (3.5) we used relationships

$$\begin{aligned}&\operatorname{div}\operatorname{grad}\varphi=\triangle\varphi,\\&\operatorname{rot}\operatorname{rot}\mathbf{A}=\operatorname{grad}\operatorname{div}\mathbf{A}-\triangle\mathbf{A}.\end{aligned}\tag{3.6}$$

Second order differential operator $\triangle$ is called *Laplace operator.* In rectangular Cartesian coordinates it is defined by formula

$$\triangle=\sum_{i=1}^{3}\left(\frac{\partial}{\partial r^i}\right)^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.\tag{3.7}$$

In order to simplify the equations (3.5) we rearrange terms in them. As a result we get

$$\begin{aligned}&\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}-\triangle\varphi=4\,\pi\,\rho+\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\varphi}{\partial t}+\operatorname{div}\mathbf{A}\right),\\&\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-\triangle\mathbf{A}=\frac{4\,\pi\,\mathbf{j}}{c}-\operatorname{grad}\left(\frac{1}{c}\frac{\partial\varphi}{\partial t}+\operatorname{div}\mathbf{A}\right).\end{aligned}\tag{3.8}$$

Differential equations (3.8) are Maxwell equations written in terms of $\mathbf{A}$ and φ. This is system of two equations one of which is scalar equation, while another is vectorial equation. As we can see, number of equations now is equal to the number of undetermined functions in them.

§ 4. Gauge transformations and Lorentzian gauge.

Vectorial and scalar potentials $\mathbf{A}$ and φ were introduced in § 3 as a replacement for electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$. However, fields $\mathbf{A}$ and φ are not physical fields. Physical fields $\mathbf{E}$ and $\mathbf{H}$ are expressed through $\mathbf{A}$ and φ according to formulas (3.4), but backward correspondence is not unique, i. e. fields $\mathbf{A}$

and φ are not uniquely determined by physical fields $\mathbf{E}$ and $\mathbf{H}$. Indeed, let's consider transformation

$$\begin{aligned} \tilde{\mathbf{A}} &= \mathbf{A} + \operatorname{grad}\psi, \\ \tilde{\varphi} &= \varphi - \frac{1}{c}\frac{\partial\psi}{\partial t}, \end{aligned} \tag{4.1}$$

where $\psi(\mathbf{r}, t)$ is an arbitrary function. Substituting (4.1) into formula (3.4), we immediately get

$$\tilde{\mathbf{E}} = \mathbf{E}, \qquad\qquad \tilde{\mathbf{H}} = \mathbf{H}.$$

This means that physical fields $\mathbf{E}$, $\mathbf{H}$ determined by fields $\tilde{\mathbf{A}}$, $\tilde{\varphi}$ and by fields $\mathbf{A}$, φ do coincide. Transformation (4.1) that do not change physical fields $\mathbf{E}$ and $\mathbf{H}$ is called *gauge transformation.*

We use gauge transformations (4.1) for further simplification of Maxwell equations (3.8). Let's consider the quantity enclosed in brackets in right hand sides of the equations (3.8):

$$\frac{1}{c}\frac{\partial\varphi}{\partial t} + \operatorname{div}\mathbf{A} = \frac{1}{c}\frac{\partial\tilde{\varphi}}{\partial t} + \operatorname{div}\tilde{\mathbf{A}} + \left(\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \triangle\psi\right). \tag{4.2}$$

Denote by $\square$ the following differential operator:

$$\square = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle. \tag{4.3}$$

Operator (4.3) is called *d'Alambert operator* or *wave operator.* Differential equation $\square u = v$ is called *d'Alambert equation.*

Using gauge freedom provided by gauge transformation (4.1), we can fulfill the following condition:

$$\frac{1}{c}\frac{\partial\varphi}{\partial t} + \operatorname{div}\mathbf{A} = 0. \tag{4.4}$$

For this purpose we should choose ψ solving d'Alambert equation

$$\Box\psi = -\left(\frac{1}{c}\frac{\partial\tilde{\varphi}}{\partial t} + \operatorname{div}\tilde{\mathbf{A}}\right).$$

It is known that d'Alambert equation is solvable under rather weak restrictions for its right hand side (see book [1]). Hence practically always we can fulfill the condition (4.4). This condition is called *Lorentzian gauge.*

If Lorentzian gauge condition (4.4) is fulfilled, then Maxwell equations (3.8) simplify substantially:

$$\Box\varphi = 4\,\pi\,\rho, \qquad \Box\mathbf{A} = \frac{4\,\pi\,\mathbf{j}}{c}. \tag{4.5}$$

They look like pair of independent d'Alambert equations. However, one shouldn't think that variables $\mathbf{A}$ and φ are completely separated. Lorentzian gauge condition (4.4) itself is an additional equation requiring concordant choice of solutions for d'Alambert equations (4.5).

D'Alambert operator (4.3) is a scalar operator, in (4.5) it acts upon each component of vector $\mathbf{A}$ separately. Therefore operator $\Box$ commutates with rotor operator and with time derivative as well. Therefore on the base of (3.4) we derive

$$\Box\mathbf{E} = -4\pi\ \operatorname{grad}\rho - \frac{4\pi}{c^2}\frac{\partial j}{\partial t}, \qquad \Box\mathbf{H} = \frac{4\pi}{c}\operatorname{rot}\mathbf{j}. \tag{4.6}$$

These equations (4.6) have no entries of potentials $\mathbf{A}$ and φ. They are written in terms of real physical fields $\mathbf{E}$ and $\mathbf{H}$, and are consequences of Maxwell equations (1.1) and (1.2). However, backward Maxwell equations do not follow from (4.6).

§ 5. Electromagnetic waves.

In previous Chapter we considered static electromagnetic fields. Such fields are uniquely determined by static configuration of charges and currents (see formulas (3.5) and (5.6) in Chapter I). They cannot exist in the absence of charges and currents. However, as we shall see just below, Maxwell equations have nonzero solutions even in the case of identically zero currents and charges in the space. Let's study one of such solutions. We choose some right-oriented rectangular Cartesian system of coordinates and take some constant vector $\mathbf{k}$ directed along x-axis (see Fig. 5.1). Then we choose another constant vector $\mathbf{A}_0$ directed along y-axis and consider the following two functions:

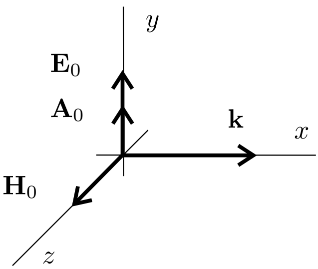

Fig. 5.1

$$\mathbf{A} = \mathbf{A}_0 \sin(k\,x - \omega\,t), \qquad \varphi = 0. \tag{5.1}$$

Here $k = |\mathbf{k}|$. Suppose $\rho = 0$ and $\mathbf{j} = 0$. Then, substituting (5.1) into (4.4) and into Maxwell equations (4.5), we get

$$k^2 = |\mathbf{k}|^2 = \frac{\omega}{c}. \tag{5.2}$$

It is not difficult to satisfy this condition (5.2). If it is fulfilled, then corresponding potentials (5.1) describe plane electromagnetic wave, ω being its *frequency* and $\mathbf{k}$ being its *wave-vector*, which determines the direction of propagation of that plane wave. Rewriting (5.1) in a little bit different form

$$\mathbf{A} = \mathbf{A}_0 \sin(k(x - c\,t)), \tag{5.3}$$

we see that the velocity of propagating of plane electromagnetic wave is equal to constant c (see (1.5) in Chapter I).

Now let's substitute (5.1) into (3.4) and calculate electric and magnetic fields in electromagnetic wave:

$$\begin{aligned} &\mathbf{E} = \mathbf{E}_0 \cos(k\,x - \omega\,t), && \mathbf{E}_0 = |\mathbf{k}|\,\mathbf{A}_0,\\ &\mathbf{H} = \mathbf{H}_0 \cos(k\,x - \omega\,t), && \mathbf{H}_0 = [\mathbf{k}, \mathbf{A}_0],\\ &|\mathbf{E}_0| = |\mathbf{H}_0| = |\mathbf{k}|\,|\mathbf{A}_0|. \end{aligned} \tag{5.4}$$

Vectors $\mathbf{k}$, $\mathbf{E}_0$, and $\mathbf{H}_0$ are perpendicular to each other, they form right triple. Wave (5.4) with such vectors is called *plane linear polarized electromagnetic wave.* Vector $\mathbf{E}_0$ is taken for *polarization vector* of this wave. Wave

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 \cos(k\,x - \omega\,t) + \mathbf{H}_0 \sin(k\,x - \omega\,t),\\ \mathbf{H} &= \mathbf{H}_0 \cos(k\,x - \omega\,t) - \mathbf{E}_0 \sin(k\,x - \omega\,t) \end{aligned}$$

is called *circular polarized* wave. It is superposition of two linear polarized waves. Natural light is also electromagnetic wave. It has no fixed polarization, however it is not circular polarized as well. Natural light is a superposition of numerous plane linear polarized waves with chaotically distributed polarization vectors.

§ 6. Emission of electromagnetic waves.

Plane wave (5.4) is an endless wave filling the whole space. It is certainly kind of idealization. Real electromagnetic waves fill only some restricted part of the space. Moreover, they are not eternal in time: there are sources (radiators) and absorbers of electromagnetic fields. Formula (5.4) is an approximate description of real electromagnetic field in that part of space which is far apart from radiators and absorbers.

In this section we consider process of generation and radiation of electromagnetic waves. Usually radiator is a system of charges and currents, which is not static. We describe it by means of functions $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$. Let's consider Maxwell equations transformed to the form (4.5). These are non-homogeneous differential equations. Their solutions are not unique: to each solution already found one can add arbitrary solution of corresponding homogeneous equations. However, if we assume $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ to be fast decreasing as $\mathbf{r}\to\infty$ and apply similar condition to $\varphi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$, then we restrict substantially the freedom in choosing solutions of the equations (4.5). In order to find one of such solutions we need fundamental solution of d'Alambert operator. This is distribution of the form:

$$u(\mathbf{r},t)=\frac{c}{2\pi}\,\theta(t)\,\delta(c^2t^2-|\mathbf{r}|^2), \tag{6.1}$$

where θ and δ are Heaviside theta-function and Dirac delta-function respectively. Function (6.1) satisfies d'Alambert equation with distribution in right hand side:

$$\Box u=\delta(t)\delta(\mathbf{r}).$$

In physics such objects are called *Green functions.* Knowing fundamental solution (6.1) of d'Alambert operator, now we can write solution for the equations (4.5) in form of contractions:

$$\varphi=4\pi\,u*\rho, \qquad\qquad \mathbf{A}=\frac{4\pi}{c}\,u*\mathbf{j}. \tag{6.2}$$

Here $*$ denotes contraction of two distributions, see [1]. Due to the properties of this operation from charge conservation law (see formula (5.4) in Chapter I) we derive Lorentzian gauge condition (4.4) for scalar and vectorial potentials (6.2). For smooth and sufficiently fast decreasing functions $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ potentials (6.2) are reduced to the following two integrals:

$$\varphi(\mathbf{r},t)=\int\frac{\rho(\tilde{\mathbf{r}},t-\tau)}{|\mathbf{r}-\tilde{\mathbf{r}}|}\,d^3\tilde{\mathbf{r}},$$
$$\mathbf{A}(\mathbf{r},t)=\int\frac{\mathbf{j}(\tilde{\mathbf{r}},t-\tau)}{c\,|\mathbf{r}-\tilde{\mathbf{r}}|}\,d^3\tilde{\mathbf{r}}. \tag{6.3}$$

Here the quantity $\tau=\tau(\mathbf{r},\tilde{\mathbf{r}})$ is called *time delay.* It is determined by the ratio $\tau=|\mathbf{r}-\tilde{\mathbf{r}}|/c$. Potentials (6.3) are called *retarded potentials.*

Retarded potentials have transparent physical interpretation. Scalar potential φ at the point $\mathbf{r}$ at time instant t is a superposition of contributions from charges at various points of the space, the contribution from the point $\tilde{\mathbf{r}}$ being determined not by charge density at present time instant t, but at previous time instant $t-\tau$. Time delay τ is exactly equal to the time required for the signal spreading with light velocity c from the source point $\tilde{\mathbf{r}}$ to get to the observation point $\mathbf{r}$. Similar time delay is present in formula for vector potential $\mathbf{A}$.

Note that fundamental solution of d'Alambert equation is not unique. For example there is a solution obtained from (6.1) by changing τ for $-\tau$. Such solution corresponds to *advanced potentials.* However, in physics advanced potentials have no meaning, since they would break causality principle.

Let's consider system of charges located in some small domain Ω surrounding the origin. Let R be maximal linear size of this domain Ω. Using formulas (6.3), we calculate we calculate electromagnetic field of the system of charges at the point $\mathbf{r}$ which is far distant from the domain Ω, i. e. $|\tilde{\mathbf{r}}|\leq R\ll|\mathbf{r}|$. Due to these inequalities the ratio $\tilde{\mathbf{r}}/|\mathbf{r}|$ is small vectorial quantity. Therefore we have the following asymptotic expansions for $|\mathbf{r}-\tilde{\mathbf{r}}|$ and $t-\tau$:

$$
\begin{aligned}
|\mathbf{r}-\tilde{\mathbf{r}}| &= |\mathbf{r}| - \frac{\langle \mathbf{r},\, \tilde{\mathbf{r}}\rangle}{|\mathbf{r}|} + \ldots\,, \\
t-\tau &= t - \frac{|\mathbf{r}|}{c} + \frac{\langle \mathbf{r},\, \tilde{\mathbf{r}}\rangle}{|\mathbf{r}|\, c} + \ldots\,.
\end{aligned} \tag{6.4}
$$

The ratio $|\mathbf{r}|/c$ in (6.4) determines the time required for electromagnetic signal to get from the domain Ω to the observation point $|\mathbf{r}|$. Posterior terms in the series for $t-\tau$ are estimated by small quantity R/c. This is the time of propagation of electromagnetic signal within domain Ω.

Denote $t' = t - |\mathbf{r}|/c$ and let $t-\tau = t' + \theta$. For the quantity θ we have the estimate $|\theta| \leq R/c$. Then let's consider the following Taylor expansions for ρ and $\mathbf{j}$:

$$
\begin{aligned}
\rho(\tilde{\mathbf{r}}, t-\tau) &= \rho(\tilde{\mathbf{r}}, t') + \frac{\partial \rho(\tilde{\mathbf{r}}, t')}{\partial t}\,\theta + \ldots\,, \\
\mathbf{j}(\tilde{\mathbf{r}}, t-\tau) &= \mathbf{j}(\tilde{\mathbf{r}}, t') + \frac{\partial \mathbf{j}(\tilde{\mathbf{r}}, t')}{\partial t}\,\theta + \ldots\,.
\end{aligned} \tag{6.5}
$$

The condition $R \ll |\mathbf{r}|$ is not sufficient for the expansions (6.5) to be consistent. Use of expansions (6.5) for approximating $\rho(\tilde{\mathbf{r}}, t-\tau)$ and $\mathbf{j}(\tilde{\mathbf{r}}, t-\tau)$ is possible only under some additional assumptions concerning these functions. Denote by T some specific time for which functions ρ and $\mathbf{j}$ within domain Ω change substantially. In case when one can specify such time T, the following quantities are of the same order, i. e. equally large or equally small:

$$
\begin{aligned}
\rho &\approx T\,\frac{\partial \rho}{\partial t} \approx \ldots \approx T^n\,\frac{\partial^n \rho}{\partial t^n}, \\
\mathbf{j} &\approx T\,\frac{\partial \mathbf{j}}{\partial t} \approx \ldots \approx T^n\,\frac{\partial^n \mathbf{j}}{\partial t^n}.
\end{aligned} \tag{6.6}
$$

Now (6.5) can be rewritten as follows:

$$\begin{aligned} &\rho(\tilde{\mathbf{r}}, t-\tau) = \rho(\tilde{\mathbf{r}}, t') + T\,\frac{\partial\rho(\tilde{\mathbf{r}}, t')}{\partial t}\,\frac{\theta}{T} + \ldots\,,\\ &\mathbf{j}(\tilde{\mathbf{r}}, t-\tau) = \mathbf{j}(\tilde{\mathbf{r}}, t') + T\,\frac{\partial\mathbf{j}(\tilde{\mathbf{r}}, t')}{\partial t}\,\frac{\theta}{T} + \ldots\,. \end{aligned} \tag{6.7}$$

Correctness of use of expansions (6.7) and (6.5) is provided by additional condition $R/c \ll T$. This yields $\theta/T \ll 1$.

The condition $R/c \ll T$ has simple meaning: the quantity $\omega = 2\pi/T$ is a frequency of radiated electromagnetic waves, while $\lambda = 2\pi c/\omega = cT$ is a wavelength. Hence condition $R/c \ll T$ means that wavelength is mach greater than the size of radiator.

Suppose that both conditions $R \ll cT$ and $R \ll |\mathbf{r}|$ are fulfilled. Let's calculate retarded vector potential $\mathbf{A}$ in (6.3) keeping only first term in the expansion (6.5):

$$\mathbf{A} = \int\limits_{\Omega} \frac{\mathbf{j}(\tilde{\mathbf{r}}, t')}{|\mathbf{r}|\,c}\,d^3\tilde{\mathbf{r}} + \ldots\,. \tag{6.8}$$

In order to transform integral in (6.8) let's choose some arbitrary constant vector $\mathbf{e}$ and consider scalar product $\langle\mathbf{A},\,\mathbf{e}\rangle$. Having defined vector $\mathbf{a}$ and function $f(\tilde{\mathbf{r}})$ by the relationships

$$\mathbf{a} = \frac{\mathbf{e}}{c|\mathbf{r}|} = \operatorname{grad} f, \qquad\qquad f(\tilde{\mathbf{r}}) = \langle\mathbf{a},\,\tilde{\mathbf{r}}\rangle,$$

we make calculations analogous to that of (7.5) in Chapter I:

$$\begin{aligned} &\int\limits_{\Omega} \langle\,\mathbf{j}, \operatorname{grad} f\,\rangle\,d^3\tilde{\mathbf{r}} = \int\limits_{\Omega} \operatorname{div}(f\,\mathbf{j})\,d^3\tilde{\mathbf{r}} - \int\limits_{\Omega} f\,\operatorname{div}\mathbf{j}\,d^3\tilde{\mathbf{r}} =\\ &= \int\limits_{\partial\Omega} f\langle\,\mathbf{j},\,\mathbf{n}\rangle\,dS + \int\limits_{\Omega} f\,\frac{\partial\rho}{\partial t}\,d^3\tilde{\mathbf{r}} = \int\limits_{\Omega} \frac{\partial\rho(\tilde{\mathbf{r}}, t')}{\partial t}\,\frac{\langle\mathbf{e},\,\tilde{\mathbf{r}}\rangle}{|\mathbf{r}|\,c}\,d^3\tilde{\mathbf{r}}. \end{aligned} \tag{6.9}$$

Since $\mathbf{e}$ is arbitrary vector, for vector potential $\mathbf{A}$ from (6.9) we derive the following formula:

$$\mathbf{A} = \int\limits_{\Omega} \frac{\partial\rho(\tilde{\mathbf{r}}, t')}{\partial t}\,\frac{\tilde{\mathbf{r}}}{|\mathbf{r}|\,c}\,d^3\tilde{\mathbf{r}} + \ldots = \frac{\dot{\mathbf{D}}}{|\mathbf{r}|\,c} + \ldots\,. \tag{6.10}$$

Here $\dot{\mathbf{D}} = \dot{\mathbf{D}}(t')$ is time derivative of dipole moment $\mathbf{D}$ of the system of charges at time instant t'.

In a similar way, keeping only initial terms in the expansions (6.4) and (6.5), for scalar potential φ in (6.3) we find

$$\varphi = \int\limits_{\Omega} \frac{\rho(\tilde{\mathbf{r}}, t')}{|\mathbf{r}|}\,d^3\tilde{\mathbf{r}} + \ldots = \frac{Q}{|\mathbf{r}|} + \ldots\,, \tag{6.11}$$

where Q is total charge enclosed within domain Ω. This charge does not depend on time since domain Ω is isolated and we have no electric current in outer space.

Let's compare the expressions under integration in (6.10) and (6.11) taking into account (6.6). This comparison yields

$$|\mathbf{A}| \approx \frac{R}{cT}\,\varphi.$$

The estimate $R/(cT) \ll 1$ following from $R \ll cT$ means that vectorial potential is calculated with higher accuracy than scalar potential. Hence in calculating φ one should take into account higher order terms in expansions (6.4) and (6.5). Then

$$\begin{aligned}\varphi = \frac{Q}{|\mathbf{r}|} &+ \int\limits_{\Omega} \frac{\partial\rho(\tilde{\mathbf{r}}, t')}{\partial t}\,\frac{\bigl\langle\mathbf{r},\,\tilde{\mathbf{r}}\bigr\rangle}{|\mathbf{r}|^2\,c}\,d^3\tilde{\mathbf{r}} + \\ &+ \int\limits_{\Omega} \frac{\rho(\tilde{\mathbf{r}}, t')}{|\mathbf{r}|}\,\frac{\bigl\langle\mathbf{r},\,\tilde{\mathbf{r}}\bigr\rangle}{|\mathbf{r}|^2}\,d^3\tilde{\mathbf{r}} + \ldots\,.\end{aligned} \tag{6.12}$$

Calculating integrals in formula (6.12), we transform it to

$$\varphi = \frac{Q}{|\mathbf{r}|} + \frac{\langle \dot{\mathbf{D}},\, \mathbf{r} \rangle}{|\mathbf{r}|^2\, c} + \frac{\langle \mathbf{D},\, \mathbf{r} \rangle}{|\mathbf{r}|^3} + \ldots\,. \tag{6.13}$$

Potentials (6.10) and (6.13) are retarded potentials of the system of charges in *dipole approximation.* Dependence of ρ and $\mathbf{j}$ on time variable t lead to the dependence of $\mathbf{D}$ on t' in them. Let's consider asymptotics of of these potentials as $\mathbf{r} \to \infty$. Thereby we can omit last term in (6.13). Then

$$\varphi = \frac{Q}{|\mathbf{r}|} + \frac{\langle \dot{\mathbf{D}},\, \mathbf{r} \rangle}{|\mathbf{r}|^2\, c} + \ldots\,, \qquad \mathbf{A} = \frac{\dot{\mathbf{D}}}{|\mathbf{r}|\, c} + \ldots\,. \tag{6.14}$$

Now on the base of formulas (3.4) and (6.14) we find

asymptotics of electric and magnetic fields at far distance from the system of charges. In calculating $\operatorname{rot} \mathbf{A}$ and $\operatorname{grad} \varphi$ we take into account that $t' = t - |\mathbf{r}|/c$ in argument of $\dot{\mathbf{D}}(t')$ is a quantity depending on $\mathbf{r}$. This dependence determines leading terms in asymptotics of $\mathbf{E}$ and $\mathbf{H}$:

$$\mathbf{E} = \frac{[\mathbf{r},\, [\mathbf{r}, \ddot{\mathbf{D}}]]}{|\mathbf{r}|^3\, c^2} + \ldots\,, \qquad \mathbf{H} = -\frac{[\mathbf{r}, \ddot{\mathbf{D}}]}{|\mathbf{r}|^2\, c^2} + \ldots\,. \tag{6.15}$$

Vectors $\mathbf{E}$ and $\mathbf{H}$ (more precisely, leading terms in their asymptotics) are perpendicular to each other and both are perpendicular to vector $\mathbf{r}$. This situation is similar to that of plane wave. However, in present case we deal with spherical wave being radiated from the origin. The magnitude of fields $|\mathbf{E}| \simeq |\mathbf{H}|$ decreases as $1/|\mathbf{r}|$, which is slower than for static Coulomb field. Using formula (2.5), one can find the density of energy flow for waves (6.15):

$$\mathbf{S} = \frac{|[\mathbf{r},\, \ddot{\mathbf{D}}]|^2}{4\pi\, |\mathbf{r}|^5\, c^3}\, \mathbf{r} + \ldots\,. \tag{6.16}$$

For modulus of vector $\mathbf{S}$ we have $|\mathbf{S}| \sim 1/|\mathbf{r}|^2$. This means that total flow of energy through the sphere of arbitrarily large sphere is nonzero. So we have real radiation of electromagnetic energy. The amount of radiated energy is determined by second time derivative of dipole moment. Therefore this case is called dipole approximation in theory of radiation.

Exercise 6.1. *Applying formula* (6.16), *find angular distribution of the intensity for dipole radiation. Also find total intensity of dipole radiation.*

Exercise 6.2. *Particle with charge q is moving along circular path of radius R with constant velocity $v = \omega R$ for infinitely long time (ω is angular velocity). Calculate retarding potentials and find angular distribution for intensity of electromagnetic radiation of this particle. Also find total intensity of such* cyclotronic *radiation.*

Exercise 6.3. *Assume that charge density ρ is zero, while current density $\mathbf{j}$ is given by the following distribution:*

$$\mathbf{j}(\mathbf{r}, t) = -c\,[\mathbf{M}(t),\, \operatorname{grad} \delta(\mathbf{r})] \tag{6.17}$$

(compare with (7.16) *in Chapter I). Find retarding potentials* (6.2) *for* (6.17). *Also find angular distribution and total intensity for* magnetic-dipole *radiation induced by current* (6.17).

CHAPTER III

SPECIAL THEORY OF RELATIVITY

§ 1. Galileo transformations.

Classical electrodynamics based on Maxwell equations historically was first field theory. It explained all electromagnetic phenomena and predicted the existence of electromagnetic waves. Later on electromagnetic waves were detected experimentally and nowadays they have broad scope of applications in our everyday life. However, along with successful development of this theory, some difficulties there appeared. It was found that classical electrodynamics contradicts to *relativity principle.* This principle in its classical form suggested by Galileo and Newton states that two Cartesian inertial coordinate systems moving with constant velocity with respect to each other are equivalent. All physical phenomena in these two systems happen identically and are described by the same laws.

Let's consider two such Cartesian inertial coordinate systems $(\mathbf{r}, t)$ and $(\tilde{\mathbf{r}}, \tilde{t})$. Suppose that second system moves with velocity $\mathbf{u}$ relative to first one so that coordinate axes in motion remain parallel to their initial positions. The relation of radius-vectors of points then can be written in form of the following transformations known as Galileo transformations:

$$t = \tilde{t}, \qquad\qquad \mathbf{r} = \tilde{\mathbf{r}} + \mathbf{u}\tilde{t}. \tag{1.1}$$

First relationship (1.1) means that watches in two systems are synchronized and tick synchronously. Let $\tilde{\mathbf{r}}(\tilde{t})$ be trajectory of some material point in coordinate system $(\tilde{\mathbf{r}}, \tilde{t})$. In first coordinate system this trajectory is given by vector $\mathbf{r}(t) = \tilde{\mathbf{r}}(\tilde{t}) + \mathbf{u}\tilde{t}$. Differentiating this relationship, due to $\tilde{t} = t$ in (1.1) we get

$$\frac{\partial \mathbf{r}}{\partial t} = \frac{\partial \tilde{\mathbf{r}}}{\partial \tilde{t}} + \mathbf{u}, \qquad\qquad \mathbf{v} = \tilde{\mathbf{v}} + \mathbf{u}. \tag{1.2}$$

Last relationship in (1.2) is known as *classical law of velocity addition*. Differentiating (1.2) once more, we find the relation for accelerations of material point in these two coordinate systems:

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} = \frac{\partial^2 \tilde{\mathbf{r}}}{\partial \tilde{t}^2}, \qquad\qquad \mathbf{a} = \tilde{\mathbf{a}}. \tag{1.3}$$

According to Newton's second law, acceleration of material point is determined by force $\mathbf{F}$ acting on it and by its mass $m\,\mathbf{a} = \mathbf{F}$. From (1.3) due to relativity principle we conclude that force $\mathbf{F}$ is invariant quantity. It doesn't depend on the choice of inertial coordinate system. This fact is represented by the relationship

$$\mathbf{F}(\tilde{\mathbf{r}} + \mathbf{u}\tilde{t}, \tilde{\mathbf{v}} + \mathbf{u}) = \tilde{\mathbf{F}}(\tilde{\mathbf{r}}, \tilde{\mathbf{v}}). \tag{1.4}$$

Now let's consider charged particle with charge q being at rest in coordinate system $(\tilde{\mathbf{r}}, \tilde{t})$. In this coordinate system it produces Coulomb electrostatic field. In coordinate system $(\mathbf{r}, t)$ this particle is moving. Hence it should produce electric field and magnetic field as well. This indicate that vectors $\mathbf{E}$ and $\mathbf{H}$ are not invariant under Galileo transformations (1.1). Even if in one coordinate system we have pure electric field, in second system we should expect the presence of both electric and magnetic fields.

Therefore transformation rules for $\mathbf{E}$ and $\mathbf{H}$ analogous to (1.4) for $\mathbf{F}$ should be written in the following form:

$$\begin{aligned} \mathbf{E}(\tilde{\mathbf{r}}+\mathbf{u}\tilde{t},\tilde{t}) &= \alpha(\tilde{\mathbf{E}}(\tilde{\mathbf{r}},\tilde{t}),\tilde{\mathbf{H}}(\tilde{\mathbf{r}},\tilde{t}),\mathbf{u}),\\ \mathbf{H}(\tilde{\mathbf{r}}+\mathbf{u}\tilde{t},\tilde{t}) &= \beta(\tilde{\mathbf{E}}(\tilde{\mathbf{r}},\tilde{t}),\tilde{\mathbf{H}}(\tilde{\mathbf{r}},\tilde{t}),\mathbf{u}). \end{aligned} \tag{1.5}$$

Due to superposition principle, which is fulfilled in both coordinate systems, functions α and β are linear and homogeneous with respect to $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$. Therefore (1.5) is rewritten as

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= \alpha_1\,\tilde{\mathbf{E}}(\tilde{\mathbf{r}},\tilde{t})+\alpha_2\,\tilde{\mathbf{H}}(\tilde{\mathbf{r}},\tilde{t}),\\ \mathbf{H}(\mathbf{r},t) &= \beta_1\,\tilde{\mathbf{E}}(\tilde{\mathbf{r}},\tilde{t})+\beta_2\,\tilde{\mathbf{H}}(\tilde{\mathbf{r}},\tilde{t}), \end{aligned} \tag{1.6}$$

where α_1, α_2, β_1, β_2 are some linear operators which depend on $\mathbf{u}$ only. Vectors $\mathbf{E}$ and $\mathbf{H}$ determine the action of electromagnetic field upon charges in form of Lorentz force (see formula (4.4) in Chapter I). Substituting (1.6) into that formula and taking into account (1.2) and (1.4), we get

$$\begin{aligned} &q\alpha_1\,\tilde{\mathbf{E}}+q\alpha_2\,\tilde{\mathbf{H}}+\frac{q}{c}\,[\tilde{\mathbf{v}}+\mathbf{u},\,\beta_1\,\tilde{\mathbf{E}}]+\\ &+\frac{q}{c}\,[\tilde{\mathbf{v}}+\mathbf{u},\,\beta_2\,\tilde{\mathbf{H}}]=q\tilde{\mathbf{E}}+\frac{q}{c}\,[\tilde{\mathbf{v}},\,\tilde{\mathbf{H}}]. \end{aligned} \tag{1.7}$$

The relationship (1.7) is an identity with three arbitrary parameters: $\tilde{\mathbf{v}}$, $\tilde{\mathbf{E}}$, $\tilde{\mathbf{H}}$. Therefore we can equate separately terms bilinear with respect to $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{E}}$. This yields $[\tilde{\mathbf{v}},\,\beta_1\,\tilde{\mathbf{E}}]=0$, hence $\beta_1=0$. Now let's equate terms bilinear with respect to $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{H}}$. This yields $[\tilde{\mathbf{v}},\,\beta_2\,\tilde{\mathbf{H}}]=[\tilde{\mathbf{v}},\,\tilde{\mathbf{H}}]$. Hence $\beta_2=1$. And finally we should equate terms linear with respect to $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{E}}$. This yields the following formulas for operators α_1 and α_2:

$$\alpha_2\,\tilde{\mathbf{H}}=-\frac{1}{c}\,[\mathbf{u},\,\tilde{\mathbf{H}}], \qquad\qquad \alpha_1=1.$$

Now, if we substitute the above expressions for operators α_1, α_2, β_1, β_2 into formula (1.6), we get the relationships

$$\mathbf{E} = \tilde{\mathbf{E}} - \frac{1}{c}\,[\mathbf{u},\,\tilde{\mathbf{H}}], \qquad\qquad \mathbf{H} = \tilde{\mathbf{H}}. \tag{1.8}$$

The relationships (1.8) should complete Galileo transformations (1.1) in electrodynamics. However, as we shall see just below, they cannot do this mission in non-contradictory form. For this purpose, let's transform Maxwell equations written as (1.1) and (1.2) in Chapter II to coordinate system $(\tilde{\mathbf{r}}, \tilde{t})$. For partial derivatives due to transformations (1.1) we have

$$\frac{\partial}{\partial r^i} = \frac{\partial}{\partial \tilde{r}^i}, \qquad\qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}} - \sum_{k=1}^{3} u^k \frac{\partial}{\partial \tilde{r}^k}. \tag{1.9}$$

Now, combining (1.8) and (1.9), we derive

$$\begin{aligned}
&\operatorname{div}\mathbf{H} = \operatorname{div}\tilde{\mathbf{H}},\\
&\operatorname{div}\mathbf{E} = \operatorname{div}\tilde{\mathbf{E}} + \frac{1}{c}\langle\mathbf{u},\,\operatorname{rot}\tilde{\mathbf{H}}\rangle,\\
&\operatorname{rot}\mathbf{H} = \operatorname{rot}\tilde{\mathbf{H}}\\
&\operatorname{rot}\mathbf{E} = \operatorname{rot}\tilde{\mathbf{E}} + \frac{1}{c}\{\mathbf{u},\,\tilde{\mathbf{H}}\} - \frac{1}{c}\,\mathbf{u}\,\operatorname{div}\tilde{\mathbf{H}},\\
&\frac{\partial\mathbf{H}}{\partial t} = \frac{\partial\tilde{\mathbf{H}}}{\partial\tilde{t}} - \{\mathbf{u},\,\tilde{\mathbf{H}}\},\\
&\frac{\partial\mathbf{E}}{\partial t} = \frac{\partial\tilde{\mathbf{E}}}{\partial\tilde{t}} - \{\mathbf{u},\,\tilde{\mathbf{E}}\} + \frac{1}{c}\,[\mathbf{u},\,\{\mathbf{u},\,\tilde{\mathbf{H}}\}] - \frac{1}{c}\,[\mathbf{u}, \partial\tilde{\mathbf{H}}/\partial\tilde{t}\,].
\end{aligned}$$

Here by curly brackets we denote commutator of vector fields (see [2]). Thereby vector $\mathbf{u}$ is treated as constant vector field.

When substituting the above expressions into Maxwell equations we consider the case of zero charges and currents: $\rho = 0$, $\mathbf{j} = 0$. This yields the following equations:

$$
\begin{aligned}
&\operatorname{div}\tilde{\mathbf{H}} = 0,\\
&\operatorname{div}\tilde{\mathbf{E}} = -\frac{1}{c}\bigl\langle\mathbf{u},\,\operatorname{rot}\tilde{\mathbf{H}}\bigr\rangle,\\
&\operatorname{rot}\tilde{\mathbf{H}} = \frac{1}{c}\,\frac{\partial\tilde{\mathbf{E}}}{\partial\tilde{t}} - \frac{1}{c}\{\mathbf{u},\,\tilde{\mathbf{E}}\} +\\
&\qquad + \frac{1}{c^2}\,[\mathbf{u},\,\{\mathbf{u},\,\tilde{\mathbf{H}}\}] - \frac{1}{c^2}\,[\mathbf{u},\,\partial\tilde{\mathbf{H}}/\partial\tilde{t}\,],\\
&\operatorname{rot}\tilde{\mathbf{E}} = -\frac{1}{c}\,\frac{\partial\tilde{\mathbf{H}}}{\partial\tilde{t}}.
\end{aligned}
$$

Only two of the above four equations coincide with original Maxwell equations. Other two equations contain the entries of vector $\mathbf{u}$ that cannot be eliminated.

This circumstance that we have found is very important. In the end of XIX-th century it made a dilemma for physicists. The way how this dilemma was resolved had determined in most further development of physics in XX-th century. Indeed, one had to make the following crucial choice:

(1) to admit that Maxwell equations are not invariant with respect to Galileo transformations, hence they require the existence of some marked inertial coordinate system where they have standard form given in the very beginning of Chapter II;

(2) or to assume that formulas (1.1) are not correct, hence relativity principle claiming equivalence of all inertial coordinate systems is realized in some different way.

First Choice had lead to *ether theory*. According to this theory, marked inertial coordinate system is bound to some hypothetical matter, which was called *ether*. This matter has no mass, no color, and no smell. It fills the whole space and does not reveal itself otherwise, but as a carrier of electromagnetic interaction. Specified properties of ether look quite unusual, this makes ether theory too artificial (not natural). As a compromise this theory was admitted for a while, but later was refuted by experiments of Michaelson and Morley, who tried to measure the Earth velocity relative to ether (ether wind).

Second choice is more crucial. Indeed, refusing formulas (1.1), we refuse classical mechanics of Newton in whole. Nevertheless the development of science went through this second choice.

§ 2. Lorentz transformations.

Having refused formulas (1.1), one should replace them by something else. This was done by Lorentz. Following Lorentz, now we replace Galileo transformations (1.1) by general linear transformations relating $(\mathbf{r}, t)$ and $(\tilde{\mathbf{r}}, \tilde{t})$:

$$c\,t = S^0_0\,c\,\tilde{t} + \sum_{k=1}^{3} S^0_k\,\tilde{r}^k, \qquad r^i = S^i_0\,c\,\tilde{t} + \sum_{k=1}^{3} S^i_k\,\tilde{r}^k. \tag{2.1}$$

In (2.1) we introduced c as a factor for time variables t and $\tilde{t}$ in order to equalize measure units. Upon introducing this factor all components of matrix S appear to be purely numeric quantities that do not require measure units at all. It is convenient to denote $c\,t$ by r^0 and treat this quantity as additional (fourth) component of radius-vector:

$$r^0 = ct. \tag{2.2}$$

Then two relationships (2.1) can be united into one relationship:

$$r^i = \sum_{k=0}^{3} S^i_k\,\tilde{r}^k. \tag{2.3}$$

In order to have invertible transformation (2.3) one should assume that $\det S \neq 0$. Let $T = S^{-1}$. Then inverse transformation for (2.3) is written as follows:

$$\tilde{r}^i = \sum_{k=0}^{3} T^i_k\,r^k. \tag{2.4}$$

By their structure transformation (2.3) and (2.4) coincide with transformations of coordinates of four-dimensional vector under the change of base. Soon we shall see that such interpretation appears to be very fruitful.

Now the problem of deriving Lorentz transformations can be formulated as problem of finding components of matrix S in (2.3). The only condition we should satisfy thereby is the invariance of Maxwell equations with respect to transformations (2.3) upon completing them with transformations for ρ, $\mathbf{j}$, $\mathbf{E}$ and $\mathbf{H}$.

For the beginning let's consider the case with no currents and charges, i. e. the case $\rho = 0$, $\mathbf{j} = 0$. Instead of Maxwell equations let's study their differential consequences written in form of the equations (4.6) in Chapter II:

$$\Box\mathbf{E} = 0, \qquad\qquad \Box\mathbf{H} = 0. \tag{2.5}$$

Invariance of (2.5) under the transformations (2.3) and (2.4) is necessary (but possibly not sufficient) condition for invariance of Maxwell equations from which the equations (2.5) were derived.

Further we need the following formula for d'Alambert operator used in the above equations (2.5):

$$\square = \sum_{i=0}^{3} \sum_{j=0}^{3} g^{ij} \frac{\partial}{\partial r^i} \frac{\partial}{\partial r^j}. \tag{2.6}$$

Here by g^{ij} we denote components of matrix

$$g^{ij} = g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.7}$$

It is easy to see that inverse matrix g_{ij} for (2.7) has the same components, i. e. $g_{ij} = g^{ij}$.

From (2.3) and (2.4) we derive the following transformation rules for first order differential operators:

$$\frac{\partial}{\partial r^i} = \sum_{k=0}^{3} T_i^k \frac{\partial}{\partial \tilde{r}^k}, \qquad \frac{\partial}{\partial \tilde{r}^i} = \sum_{k=0}^{3} S_i^k \frac{\partial}{\partial r^k}. \tag{2.8}$$

Substituting (2.8) into formula (2.6), we get

$$\square = \sum_{p=0}^{3} \sum_{q=0}^{3} \tilde{g}^{pq} \frac{\partial}{\partial \tilde{r}^p} \frac{\partial}{\partial \tilde{r}^q},$$

where matrices g^{ij} and $\tilde{g}^{pq}$ are related by formula

$$\tilde{g}^{pq} = \sum_{i=0}^{3} \sum_{j=0}^{3} T_i^p \, T_j^q \, g^{ij}. \tag{2.9}$$

In terms of inverse matrices g_{pq} and $\tilde{g}_{pq}$ this relationship (2.9) can be rewritten as follows:

$$g_{ij}=\sum_{p=0}^{3}\sum_{q=0}^{3}T^p_i\,T^q_j\,\tilde{g}_{pq}. \tag{2.10}$$

Theorem 2.1. *For any choice of operator coefficients* α_1, α_2, β_1, *and* β_2 *in formulas* (1.6) *the invariance of the form of equations* (2.5) *under the transformations* (2.3) *and* (2.4) *is equivalent to proportionality of matrices* g *and* $\tilde{g}$, *i. e.*

$$\tilde{g}^{ij}=\lambda\,g^{ij}. \tag{2.11}$$

Numeric factor λ in formula (2.11) is usually chosen to be equal to unity: $\lambda=1$. In this case from (2.10) and (2.11) we derive

$$g_{ij}=\sum_{p=0}^{3}\sum_{q=0}^{3}T^p_i\,T^q_j\,g_{pq}. \tag{2.12}$$

In matrix form this relationship (2.12) looks like

$$T^t\,g\,T=g. \tag{2.13}$$

Here g is a matrix of the form (2.7), while by T^t in (2.13) we denote transposed matrix T.

Definition 2.1. Matrix T satisfying the relationship (2.13) is called *Lorentzian matrix.*

It is easy to check up that the set of Lorentzian matrices form a group. This group is usually denoted by $\mathrm{O}(1,3)$. It is called *matrix Lorentz group.*

From the relationship (2.13) for Lorentzian matrix we derive the equality $(\det T)^2 = 1$. Hence $\det T = \pm 1$. Lorentzian matrices with unit determinant form the group $\mathrm{SO}(1,3)$, it is called *special matrix Lorentz group.*

If $i = j = 0$, from (2.12) we obtain the following formula relating components of Lorentzian matrix T:

$$(T^0_0)^2 - (T^1_0)^2 - (T^2_0)^2 - (T^3_0)^2 = 1. \tag{2.14}$$

Inequality $|T^0_0| \geqslant 1$ is immediate consequence of the relationship (2.14). Hence $T^0_0 \geqslant 1$ or $T^0_0 \leqslant -1$. Lorentzian matrix with $T^0_0 \geqslant 1$ is called *orthochronous.* The set of orthochronous Lorentzian matrices form *orthochronous matrix Lorentz group* $\mathrm{O}^+(1,3)$. Intersection $\mathrm{SO}^+(1,3) = \mathrm{SO}(1,3) \cap \mathrm{O}^+(1,3)$ is called *special orthochronous matrix Lorentz group.*

Exercise 2.1. *Prove theorem 2.1 under the assumption that transformation* (1.6) *given by operator coefficients* α_1, α_2, β_1, *and* β_2 *is invertible.*

§ 3. Minkowsky space.

In previous section we have found that each Lorentzian matrix from group $\mathrm{O}(1,3)$ determines some transformation (2.1) preserving the form the equations (2.5). In deriving this fact we introduced notations (2.2) and united space and time into one four-dimensional "space-time". Let's denote it by M. Four-dimensional space M is basic object in special theory of relativity. Its points are called *events.* The space of events is equipped with quadratic form g with signature $(1,3)$. This quadratic form is called *Minkowsky metric.* Thereby inertial coordinate systems are interpreted as Cartesian coordinates for which Minkowsky metric has canonical form (2.7).

Equivalence principle. All physical laws in any two inertial coordinate systems are written in the same form.

Let's choose some inertial coordinate system. This choice determines separation of event space M into geometric space V (space of points) and time axis T:

$$M = T \oplus V. \tag{3.1}$$

Matrix of Minkowsky metric in chosen coordinate system has canonic form (2.7). Therefore geometric space V is orthogonal to time axis T with respect to Minkowsky metric g. Restriction of this metric to V is negative quadratic form. Changing its sign, we get positive quadratic form. This is standard Euclidean scalar product in V.

Now let's consider another inertial coordinate system. Like (3.1), it determines second expansion of M into space and time:

$$M = \tilde{T} \oplus \tilde{V}. \tag{3.2}$$

In general time axes T and $\tilde{T}$ in expansions (3.1) and (3.2) do not coincide. Indeed, bases of these two coordinate systems are related to each other by formula

$$\tilde{\mathbf{e}}_i = \sum_{j=0}^{3} S_i^j \, \mathbf{e}_j, \tag{3.3}$$

where S is Lorentzian matrix from (2.3). For base vector $\tilde{\mathbf{e}}_0$ directed along time axis $\tilde{T}$ from (3.3) we derive

$$\tilde{\mathbf{e}}_0 = S_0^0 \, \mathbf{e}_0 + S_0^1 \, \mathbf{e}_1 + S_0^2 \, \mathbf{e}_2 + S_0^3 \, \mathbf{e}_3. \tag{3.4}$$

In general components S_0^1, S_0^2, and S_0^3 in Lorentz matrix S are nonzero. Therefore vectors $\tilde{\mathbf{e}}_0$ and $\mathbf{e}_0$ are non-collinear. Hence $T \neq \tilde{T}$.

Non-coincidence of time axes $T \neq \tilde{T}$ for two inertial coordinate systems leads to non-coincidence of geometric spaces: $V \neq \tilde{V}$. This fact lead to quite radical conclusion when we interpret it physically: observers in two such inertial systems observe *two different three-dimensional geometric spaces* and have *two different time ticks*. However, in our everyday life this difference is very small and never reveals.

Let's calculate how big is the difference in the rate of time ticks for two inertial coordinate systems. From (2.4) we get

$$\tilde{t} = T^0_0\, t + \sum_{k=1}^{3} \frac{T^0_k}{c}\, r^k. \tag{3.5}$$

Let $t \to +\infty$. If Lorentzian matrix T is orthochronous, then $T^0_0 > 0$ and $\tilde{t} \to +\infty$. If matrix T is not orthochronous, then $t \to +\infty$ we get $\tilde{t} \to -\infty$. Transformations (2.4) with non-orthochronous matrices T invert the direction of time exchanging the future and the past. It would be very intriguing to have such a feature in theory. However, presently in constructing theory of relativity one uses more realistic approach. So we shall assume that two physically real inertial coordinate systems can be related only by orthochronous Lorentz matrices from $O^+(1,3)$.

Restriction of the set of admissible Lorentz matrices from $O(1,3)$ to $O^+(1,3)$ is due to the presence of additional structure in the space of events. It is called *polarization*. Let's choose some physical inertial coordinate system. Minkowsky metric in such system is given by matrix of canonical form (2.7). Let's calculate scalar square of four-dimensional vector $\mathbf{x}$ in Minkowsky metric:

$$g(\mathbf{x}, \mathbf{x}) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2. \tag{3.6}$$

By value of their scalar square $g(\mathbf{x}, \mathbf{x})$ in Minkowsky metric g

vectors of Minkowsky space M are subdivided into three parts:

(1) *tome-like vectors*, for which $g(\mathbf{x}, \mathbf{x})$ is positive;
(2) *light vectors*, for which $g(\mathbf{x}, \mathbf{x}) = 0$;
(3) *space-like vectors*, for which $g(\mathbf{x}, \mathbf{x})$ is negative.

Coordinates of light vectors satisfy the following equation:

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = 0. \tag{3.7}$$

It is easy to see that (3.7) is the equation of cone in four-dimensional space (see classification of quadrics in [4]). This cone (3.7) is called *light cone*.

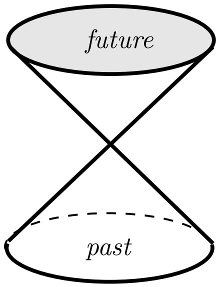

Fig. 3.1

Time-like vectors fill interior of light cone, while space-like vectors fill outer space outside this cone. Interior of light cone is a union of two parts: time-like vectors with $x^0 > 0$ are directed to the future, others with $x_0 < 0$ are directed to the past. Vector directed to the future can be continuously transformed to any other vector directed to the future. However, it cannot be continuously transformed to a vector directed to the past without making it space-like vector or zero vector at least once during transformation. This means that the set of time-like vectors is disjoint union of two connected components.

Definition 3.1. Geometric structure in Minkowsky space M that marks one of two connected components in the set of time-like vectors is called *polarization*. It is used to say that marked component *points to the future*.

Let $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be *orthonormal base* in Minkowsky metric*. In the space M with polarization one can consider only those such bases for which unit vector of time axis $\mathbf{e}_0$ is directed to the future. Then transition from one of such bases to another would be given by orthochronous matrix from group $\mathrm{O}^+(1,3)$.

Definition 3.2. Four-dimensional affine space M equipped with metric g of signature $(1,3)$ and equipped with orientation** and polarization is called *Minkowsky space.*

According to special theory of relativity Minkowsky space, which is equipped with orientation and polarization, is proper mathematical model for the space of real physical events. Now we can give strict mathematical definition of inertial coordinate system.

Definition 3.3. Orthonormal right inertial coordinate system is orthonormal right coordinate system in Minkowsky space with time base vector directed to the future.

It is easy to verify that any two inertial coordinate systems as defined above are related to each other by Lorentz transformation with matrix S from orthochronous Lorentz group $SO^+(1,3)$. Let's choose one of such coordinate systems and consider related expansion (3.1). It is clear that $\mathbf{e}_0 \in T$, while linear span of spatial vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ defines subspace V. Taking orthonormal base $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ for the standard of right bases in V, we equip this three-dimensional space with orientation. This is concordance with the fact that geometric space that we observe in our everyday life possesses orientation distinguishing left and right.

---

* i. e. base for which Minkowsky metric has the form (2.7).

** remember that orientation is geometric structure distinguishing left and right bases (see [4]).

Exercise 3.1. *By analogy with definition 3.3 formulate the definition of skew-angular inertial coordinate system.*

§ 4. Kinematics of relative motion.

Galileo transformations are used in mechanics for describing physical processes as they are seen by two observers representing two inertial coordinate systems. Lorentz transformations, which we have derived from the condition of invariance of electrodynamical equations (2.5), are designed for the same purpose. However, this is not immediately clear when looking at formulas (2.3) and (2.4). Therefore we shall bring these formulas to the form more convenient for studying their physical nature.

Let's fix two inertial coordinate systems related by Lorentz transformation (2.1). First one is related with orthonormal base $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ in Minkowsky space and with the expansion (3.1). Second is related with the base $\tilde{\mathbf{e}}_0$, $\tilde{\mathbf{e}}_1$, $\tilde{\mathbf{e}}_2$, $\tilde{\mathbf{e}}_3$ and with the expansion (3.2). If time axes are parallel $\mathbf{e}_0 = \tilde{\mathbf{e}}_0$, then Lorentz matrix S in (2.3) is reduced to orthogonal matrix $O \in SO(3)$ relating two right orthonormal bases $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\tilde{\mathbf{e}}_1$, $\tilde{\mathbf{e}}_2$, $\tilde{\mathbf{e}}_3$. It has the following blockwise-diagonal shape:

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & O^1_1 & O^1_2 & O^1_3 \\ 0 & O^2_1 & O^2_2 & O^2_3 \\ 0 & O^3_1 & O^3_2 & O^3_3 \end{pmatrix}. \tag{4.1}$$

Thus, in case if $T \parallel \tilde{T}$ two inertial coordinate systems differ only in directions of spatial axes. They do not move with respect to each other.

Now let's consider the case $T \nparallel \tilde{T}$. Hence $\mathbf{e}_0 \neq \tilde{\mathbf{e}}_0$. Let H be linear span of vectors $\mathbf{e}_0$ and $\tilde{\mathbf{e}}_0$. Denote by W intersection of subspaces V and $\tilde{V}$ from (3.1) and (3.2):

$$H = \operatorname{Span}(\mathbf{e}_0, \tilde{\mathbf{e}}_0), \qquad W = V \cap \tilde{V}. \tag{4.2}$$

Lemma 4.1. *Two-dimensional subspaces H and W in (4.2) are perpendicular to each other in Minkowsky metric g. Their intersection is zero: $H \cap W = \{0\}$, while direct sum of these subspaces coincides with the whole Minkowsky space: $H \oplus W = M$.*

PROOF. Subspace H is two-dimensional since it is linear span of two non-collinear vectors. Subspaces V and $\tilde{V}$ are three-dimensional and $V \neq \tilde{V}$. Hence their sum $V + \tilde{V}$ coincides with M, i.e. $\dim(V + \tilde{V}) = 4$. Applying theorem on the dimension of sum and intersection of two subspaces (see [4]), we get

$$\dim(W) = \dim V + \dim \tilde{V} - \dim(V + \tilde{V}) = 3 + 3 - 4 = 2.$$

In order to prove orthogonality of H and W we use orthogonality of T and V in the expansion (3.1) and orthogonality of $\tilde{T}$ and $\tilde{V}$ in (3.2). Let $\mathbf{y}$ be an arbitrary vector in subspace W. Then $\mathbf{y} \in V$. From $V \perp T$ we get $\mathbf{y} \perp \mathbf{e}_0$. Analogously from $\mathbf{y} \in \tilde{V}$ we get $\mathbf{y} \perp \tilde{e}_0$. Now from orthogonality of $\mathbf{y}$ to both vectors $\mathbf{e}_0$ and $\tilde{\mathbf{e}}_0$ we derive orthogonality of $\mathbf{y}$ to their linear span: $\mathbf{y} \perp H$. Since $\mathbf{y}$ is arbitrary vector in W, we have $W \perp H$.

Now let's prove that $H \cap W = \{0\}$. Let's consider an arbitrary vector $\mathbf{x} \in H \cap W$. From $\mathbf{x} \in H$ and $\mathbf{x} \in W$ due to orthogonality of H and W, which is already proved, we get $g(\mathbf{x}, \mathbf{x}) = 0$. But $\mathbf{x} \in W \subset V$, while restriction of Minkowsky metric to subspace V is negative quadratic form of signature $(0, 3)$. Therefore from $g(\mathbf{x}, \mathbf{x}) = 0$ we derive $\mathbf{x} = 0$. Proposition $H \cap W = \{0\}$ is proved.

From $H \cap W = \{0\}$ we conclude that sum of subspaces H and W is direct sum and $\dim(H + W) = 2 + 2 = 4$. Hence $H \oplus W = M$. Lemma is proved. $\square$

Now let's return back to considering pair of inertial coordinate systems with bases $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\tilde{\mathbf{e}}_0$, $\tilde{\mathbf{e}}_1$, $\tilde{\mathbf{e}}_2$, $\tilde{\mathbf{e}}_3$. There is the expansion (3.4) for vector $\tilde{\mathbf{e}}_0$. Let's write it as follows:

$$\tilde{\mathbf{e}}_0 = S^0_0\, \mathbf{e}_0 + \mathbf{v}. \tag{4.3}$$

Here $\mathbf{v} = S^1_0\,\mathbf{e}_1 + S^2_0\,\mathbf{e}_2 + S^3_0\,\mathbf{e}_3 \in V$. Since matrix S is orthochronous and since $\tilde{\mathbf{e}}_0 \neq \mathbf{e}_0$, we have

$$S^0_0 > 1, \qquad\qquad \mathbf{v} \neq 0. \tag{4.4}$$

For any real number $a > 1$ there exists a number $\alpha > 0$ such that $a = \cosh(\alpha)$. Let's apply this observation to S^0_0 in (4.3):

$$S^0_0 = \cosh(\alpha). \tag{4.5}$$

From (4.3), from (4.5), and from orthogonality of vectors $\mathbf{e}_0$ and $\mathbf{v}$ in Minkowsky metric we obtain

$$1 = g(\tilde{\mathbf{e}}_0, \tilde{\mathbf{e}}_0) = (S^0_0)^2\, g(\mathbf{e}_0, \mathbf{e}_0) + g(\mathbf{v}, \mathbf{v}) = \cosh^2(\alpha) - |\mathbf{v}|^2.$$

Using this equality we can find Euclidean length of vector $\mathbf{v}$ in three-dimensional subspace V:

$$|\mathbf{v}| = \sinh(\alpha), \quad \text{where} \quad \alpha > 0. \tag{4.6}$$

Let's replace vector $\mathbf{v}$ by vector of unit length $\mathbf{h}_1 = \mathbf{v}/|\mathbf{v}|$ and rewrite the relationship (4.3) as follows:

$$\tilde{\mathbf{e}}_0 = \cosh(\alpha)\,\mathbf{e}_0 + \sinh(\alpha)\,\mathbf{h}_1. \tag{4.7}$$

Due to (4.7) vector $\mathbf{h}_1$ is linear combination of vectors $\mathbf{e}_0$ and $\tilde{\mathbf{e}}_0$, hence $h_1 \in H$. But $h_1 \in V$ as well. Therefore $h_1 \in V \cap H$. Vectors $\mathbf{e}_0$ and $\mathbf{h}_1$ are perpendicular to each other, they form orthonormal base in two-dimensional subspace H:

$$g(\mathbf{e}_0, \mathbf{e}_0) = 1, \qquad\qquad g(\mathbf{h}_1, \mathbf{h}_1) = -1. \tag{4.8}$$

From (4.8) we conclude that restriction of Minkowsky metric to subspace H is metric with signature $(1, 1)$.

Now we need another vector from subspace H. Let's determine it by the following relationship:

$$\tilde{\mathbf{h}}_1 = \sinh(\alpha)\,\mathbf{e}_0 + \cosh(\alpha)\,\mathbf{h}_1. \tag{4.9}$$

It is easy to check that vectors $\tilde{\mathbf{e}}_0$ and $\tilde{\mathbf{h}}_1$ form another orthonormal base in subspace H. Transition matrix relating these two bases has the following form:

$$S_{\mathrm{L}} = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix}. \tag{4.10}$$

Matrix (4.10) is called the matrix of *Lorentzian rotation.*

There is four-dimensional version of matrix (4.10). Indeed, vector $\mathbf{h}_1 \in V$ is perpendicular to subspace $W \subset V$. Therefore we have the expansion of subspace V as a direct sum:

$$V = \mathrm{Span}(\mathbf{h}_1) \oplus W.$$

Let's choose two vectors $\mathbf{h}_2$ and $\mathbf{h}_3$ forming orthonormal base in subspace W and complementing $\mathbf{h}_1$ up to an orthonormal right base in V. Then four vectors $\mathbf{e}_0$, $\mathbf{h}_1$, $\mathbf{h}_2$ $\mathbf{h}_3$ constitute orthonormal right base in M with time vector $\mathbf{e}_0$ directed to the future. Transition matrix relating this base with the base $\tilde{\mathbf{e}}_0$, $\tilde{\mathbf{h}}_1$, $\mathbf{h}_2$ $\mathbf{h}_3$ has the following blockwise-diagonal form:

$$S_{\mathrm{L}} = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) & 0 & 0 \\ \sinh(\alpha) & \cosh(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.11}$$

Transition from base $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$ $\mathbf{e}_3$ to base $\mathbf{e}_0$, $\mathbf{h}_1$, $\mathbf{h}_2$ $\mathbf{h}_3$ is given by a matrix of the form (4.1). This is because their time vectors do coincide. In a similar way transition from base $\tilde{\mathbf{e}}_0$, $\tilde{\mathbf{h}}_1$, $\mathbf{h}_2$ $\mathbf{h}_3$ to base $\tilde{\mathbf{e}}_0$, $\tilde{\mathbf{e}}_1$, $\tilde{\mathbf{e}}_2$ $\tilde{\mathbf{e}}_3$ is given by a matrix of the same form (4.1). Ultimate change of base $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$ $\mathbf{e}_3$ for another base $\tilde{\mathbf{e}}_0$, $\tilde{\mathbf{e}}_1$, $\tilde{\mathbf{e}}_2$ $\tilde{\mathbf{e}}_3$ then can be done in three steps.

Theorem 4.1. *Each Lorentzian matrix $S \in SO^+(1,3)$ is a product of three matrices $S = S_1\,S_{\mathrm{L}}\,S_2$, one of which S_{L} is a matrix of Lorentzian rotation* (4.11), *while two others S_1 and S_2 are matrices of the form* (4.1).

In order to clarify physical meaning of Lorentz transformations let's first consider transformations with matrix S of the form (4.11). Let $ct = r^0$, r^1, r^2, r^3 be coordinates of some vector $\mathbf{r} \in M$ in the base $\mathbf{e}_0$, $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{h}_3$. By $c\tilde{t} = \tilde{r}^0$, $\tilde{r}^1$, $\tilde{r}^2$, $\tilde{r}^3$ we denote coordinates of the same vector in the base $\tilde{\mathbf{e}}_0$, $\tilde{\mathbf{h}}_1$, $\mathbf{h}_2$ $\mathbf{h}_3$. For matrix S of the form (4.11) formula (2.3) leads to relationships

$$
\begin{aligned}
t &= \cosh(\alpha)\,\tilde{t} + \frac{\sinh(\alpha)}{c}\,\tilde{r}^1,\\
r^1 &= \sinh(\alpha)\,c\,\tilde{t} + \cosh(\alpha)\,\tilde{r}^1,\\
r^2 &= \tilde{r}^2,\\
r^3 &= \tilde{r}^3.
\end{aligned}
\tag{4.12}
$$

Let $\tilde{r}^1$, $\tilde{r}^2$, $\tilde{r}^3$ be coordinates of radius-vector of some point A which is at rest in inertial coordinate system with base $\tilde{\mathbf{e}}_1$, $\tilde{\mathbf{e}}_2$, $\tilde{\mathbf{e}}_3$. Then $\tilde{r}^1$, $\tilde{r}^2$, $\tilde{r}^3$ are constants, they do not depend on time $\tilde{t}$ in this coordinate system. Upon calculating coordinates of this point A in other inertial coordinate system by means of formulas (4.12) its first coordinate r^1 appears to be a function of parameter $\tilde{t}$. We use first relationship (4.12) in order to express parameter $\tilde{t}$

through time variable t in second coordinate system:

$$\tilde{t} = \frac{t}{\cosh(\alpha)} - \frac{\tanh(\alpha)}{c}\,\tilde{r}^1. \tag{4.13}$$

Substituting (4.13) into other three formulas (4.12), we get

$$\begin{aligned} r^1 &= r^1(t) = c\,\tanh(\alpha)\,t + \text{const},\\ r^2 &= r^2(t) = \text{const},\\ r^3 &= r^3(t) = \text{const}\,. \end{aligned} \tag{4.14}$$

From (4.14) we see that in second coordinate system our point A is moving with constant velocity $u = c\,\tanh(\alpha)$ in the direction of first coordinate axis.

In contrast to parameter α in matrix (4.11), parameter u has transparent physical interpretation as magnitude of relative velocity of one coordinate system with respect to another. Let's express components of matrix (4.11) through u:

$$\cosh(\alpha) = \frac{1}{\sqrt{1 - \dfrac{u^2}{c^2}}}, \qquad \sinh(\alpha) = \frac{u}{c}\,\frac{1}{\sqrt{1 - \dfrac{u^2}{c^2}}}.$$

Let's substitute these formulas into (4.12). As a result we get

$$\begin{aligned} t &= \frac{\tilde{t} + \dfrac{u}{c^2}\,\tilde{r}^1}{\sqrt{1 - \dfrac{u^2}{c^2}}}, & r^1 &= \frac{u\tilde{t} + \tilde{r}^1}{\sqrt{1 - \dfrac{u^2}{c^2}}},\\ r^2 &= \tilde{r}^2, & r^3 &= \tilde{r}^3. \end{aligned} \tag{4.15}$$

Denote for a while by $\mathbf{r}$ and $\tilde{\mathbf{r}}$ the following two three-dimensional vectors in subspaces V and $\tilde{V}$:

$$
\begin{aligned}
\mathbf{r} &= r^1\,\mathbf{h}_1 + r^2\,\mathbf{h}_2 + r^3\,\mathbf{h}_3,\\
\tilde{\mathbf{r}} &= \tilde{r}^1\,\tilde{\mathbf{h}}_1 + \tilde{r}^2\,\mathbf{h}_2 + \tilde{r}^3\,\mathbf{h}_3.
\end{aligned}
\tag{4.16}
$$

Then we define linear map $\theta\colon V \to \tilde{V}$ determined by its action upon base vectors $\mathbf{h}_1$, $\mathbf{h}_2$, and $\mathbf{h}_3$:

$$
\theta(\mathbf{h}_1) = \tilde{\mathbf{h}}_1, \qquad \theta(\mathbf{h}_2) = \mathbf{h}_2, \qquad \theta(\mathbf{h}_3) = \mathbf{h}_3.
$$

This map θ is orientation preserving isometry, since it maps right orthonormal base of subspace V to right orthonormal base in subspace $\tilde{V}$. Using above notations (4.16) and the map θ, we can write formulas (4.15) in vectorial form:

$$
\begin{aligned}
t &= \frac{\tilde{t} + \dfrac{\langle\theta\mathbf{u},\,\tilde{\mathbf{r}}\rangle}{c^2}}{\sqrt{1 - \dfrac{|\mathbf{u}|^2}{c^2}}},\\
\theta\mathbf{r} &= \frac{\theta\mathbf{u}\,\tilde{t} + \dfrac{\langle\theta\mathbf{u},\,\tilde{\mathbf{r}}\rangle}{|\mathbf{u}|^2}\,\theta\mathbf{u}}{\sqrt{1 - \dfrac{|\mathbf{u}|^2}{c^2}}} + \tilde{\mathbf{r}} - \frac{\langle\theta\mathbf{u},\,\tilde{\mathbf{r}}\rangle}{|\mathbf{u}|^2}\,\theta\mathbf{u}.
\end{aligned}
\tag{4.17}
$$

Here $\mathbf{u} = u\,\mathbf{h}_1$ is vector of relative velocity of second coordinate system with respect to first one. Formulas (4.17) are irrespective to the choice of bases in subspaces V and $\tilde{V}$. Therefore they are applicable to Lorentz transformations with special matrix of the form (4.11) and to arbitrary Lorentz transformations with matrix $S = S_1\,S_{\text{L}}\,S_2$ (see theorem 4.1).

Very often the sign of map θ realizing isomorphism of subspaces V and $\tilde{V}$ in formulas (4.17) is omitted:

$$
\begin{aligned}
t &= \frac{\tilde{t} + \dfrac{\langle \mathbf{u},\, \tilde{\mathbf{r}} \rangle}{c^2}}{\sqrt{1 - \dfrac{|\mathbf{u}|^2}{c^2}}},\\
\mathbf{r} &= \frac{\mathbf{u}\,\tilde{t} + \dfrac{\langle \mathbf{u},\, \tilde{\mathbf{r}} \rangle}{|\mathbf{u}|^2}\,\mathbf{u}}{\sqrt{1 - \dfrac{|\mathbf{u}|^2}{c^2}}} + \tilde{\mathbf{r}} - \frac{\langle \mathbf{u},\, \tilde{\mathbf{r}} \rangle}{|\mathbf{u}|^2}\,\mathbf{u}.
\end{aligned}
\tag{4.18}
$$

Formulas (4.18) represent “conditionally three-dimensional” understanding of Lorentz transformations when vectors $\mathbf{r}$ and $\tilde{\mathbf{r}}$ treated as vectors of the same three-dimensional Euclidean space, while t and $\tilde{t}$ are treated as scalar parameters. However, according to modern paradigm four-dimensional Minkowsky space is real physical entity, not purely mathematical abstraction convenient for shortening formulas (compare (2.3) and (4.17)).

When writing formulas (4.17) and (4.18) in components we should expand vectors $\mathbf{r}$ and $\mathbf{u}$ in the base of one coordinate system, while vector $\tilde{\mathbf{r}}$ is expanded in the base of another coordinate system. Thereby the difference in the shape of these two formulas completely disappears.

Exercise 4.1. *Using expansions* (4.16) *for vectors* $\mathbf{r}$ *and* $\tilde{\mathbf{r}}$*, derive the following formulas:*

$$
\tilde{r}^1 = \frac{\langle \theta\mathbf{u},\, \tilde{\mathbf{r}} \rangle}{|\mathbf{u}|}, \qquad \tilde{r}^2\,\mathbf{h}_2 + \tilde{r}^3\,\mathbf{h}_3 = \tilde{\mathbf{r}} - \frac{\langle \theta\mathbf{u},\, \tilde{\mathbf{r}} \rangle}{|\mathbf{u}|^2}\,\theta\mathbf{u}.
$$

Combining these formulas with (4.15)*, derive formulas* (4.17)*.*

§ 5. Relativistic law of velocity addition.

Classical law of velocity addition was first consequence that we obtained from Galileo transformations:

$$\mathbf{v} = \tilde{\mathbf{v}} + \mathbf{u}, \tag{5.1}$$

see formulas (1.2). Replacing Galileo transformations by Lorentz transformations, now we should derive new *relativistic* law of velocity addition.

Suppose that vector-function $\tilde{\mathbf{r}}(\tilde{t})$ describes the motion of a point A in inertial coordinate system $(\tilde{\mathbf{r}}, \tilde{t})$ and suppose that this coordinate system moves with velocity $\mathbf{u}$ with respect to other inertial coordinate system $(\mathbf{r}, t)$. For passing to coordinate system $(\mathbf{r}, t)$ we use Lorentz transformation given by formulas (4.18). As a result we get two functions

$$\begin{aligned} &t(\tilde{t}) = \frac{\tilde{t} + \dfrac{\langle \mathbf{u},\, \tilde{\mathbf{r}}(\tilde{t})\rangle}{c^2}}{\sqrt{1 - \dfrac{|\mathbf{u}|^2}{c^2}}},\\ &\mathbf{r}(\tilde{t}) = \frac{\mathbf{u}\,\tilde{t} + \dfrac{\langle \mathbf{u},\, \tilde{\mathbf{r}}(\tilde{t})\rangle}{|\mathbf{u}|^2}\,\mathbf{u}}{\sqrt{1 - \dfrac{|\mathbf{u}|^2}{c^2}}} + \tilde{\mathbf{r}}(\tilde{t}) - \frac{\langle \mathbf{u},\, \tilde{\mathbf{r}}(\tilde{t})\rangle}{|\mathbf{u}|^2}\,\mathbf{u}. \end{aligned} \tag{5.2}$$

Let's calculate first derivatives of functions (5.2):

$$\frac{dt}{d\tilde{t}} = \frac{1 + \dfrac{\langle \mathbf{u},\, \tilde{\mathbf{v}}\rangle}{c^2}}{\sqrt{1 - \dfrac{|\mathbf{u}|^2}{c^2}}}, \tag{5.3a}$$

$$\frac{d\mathbf{r}}{d\tilde{t}} = \frac{\mathbf{u} + \dfrac{\langle\mathbf{u},\,\tilde{\mathbf{v}}\rangle}{|\mathbf{u}|^2}\,\mathbf{u}}{\sqrt{1 - \dfrac{|\mathbf{u}|^2}{c^2}}} + \tilde{\mathbf{v}} - \frac{\langle\mathbf{u},\,\tilde{\mathbf{v}}\rangle}{|\mathbf{u}|^2}\,\mathbf{u}. \tag{5.3b}$$

By $\tilde{\mathbf{v}}$ we denote the velocity of the point A in coordinates $(\tilde{\mathbf{r}}, \tilde{t})$:

$$\tilde{\mathbf{v}} = \dot{\tilde{\mathbf{r}}}(\tilde{t}) = \frac{d\tilde{\mathbf{r}}}{d\tilde{t}}.$$

In a similar way by $\mathbf{v}$ we denote the velocity of this point in other coordinates $(\mathbf{r}, t)$. To calculate $\mathbf{v}$ we divide derivatives:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}(t) = \left(\frac{d\mathbf{r}}{d\tilde{t}}\right) \Big/ \left(\frac{dt}{d\tilde{t}}\right). \tag{5.4}$$

Substituting (5.3a) and (5.3b) into (5.4), we get formula

$$\mathbf{v} = \frac{\mathbf{u} + \dfrac{\langle\mathbf{u},\,\tilde{\mathbf{v}}\rangle}{|\mathbf{u}|^2}\,\mathbf{u}}{1 + \dfrac{\langle\mathbf{u},\,\tilde{\mathbf{v}}\rangle}{c^2}} + \frac{\tilde{\mathbf{v}} - \dfrac{\langle\mathbf{u},\,\tilde{\mathbf{v}}\rangle}{|\mathbf{u}|^2}\,\mathbf{u}}{1 + \dfrac{\langle\mathbf{u},\,\tilde{\mathbf{v}}\rangle}{c^2}}\sqrt{1 - \frac{|\mathbf{u}|^2}{c^2}}. \tag{5.5}$$

Formula (5.5) is relativistic law of velocity addition. It is mach more complicated than classical law given by formula (5.1). However, in the limit of small velocities $|\mathbf{u}| \ll c$ formula (5.5) reduces to formula (5.1).

Exercise 5.1. *Derive relativistic law of velocity addition from formula* (4.17). *Explain why resulting formula differs from* (5.5).

§ 6. World lines and private time.

Motion of point-size material object in arbitrary inertial coordinate system $(\mathbf{r}, t)$ is described by vector-function $\mathbf{r}(t)$, where t

is time variable and $\mathbf{r}$ is radius-vector of point. Four-dimensional radius-vector of this material point has the following components:

$$r^0(t) = ct, \quad r^1(t), \quad r^2(t), \quad r^3(t). \tag{6.1}$$

Vector-function with components (6.1) determines parametric line in Minkowsky space M, this line is called *world line* of material point. Once world line is given, motion of material point is described completely. Let's differentiate four-dimensional radius-vector (6.1) with respect to parameter t. As a result we get four-dimensional vector tangent to world line:

$$\mathbf{K} = (c, \dot{r}^1, \dot{r}^2, \dot{r}^3) = (c, v^1, v^2, v^3). \tag{6.2}$$

Last three components of this vector form velocity vector of material point. Velocity of most material objects is not greater than light velocity: $|\mathbf{v}| < c$. When applied to vector $\mathbf{K}$ in (6.2) this means that tangent-vector of world line is time-like vector:

$$g(\mathbf{K}, \mathbf{K}) = c^2 - |\mathbf{v}|^2 > 0. \tag{6.3}$$

Definition 6.1. Smooth curve in Minkowsky space is called time-like curve if tangent-vector of this curve is time-like vector at each its point.

World lines for most material objects are time-like curves. Exception are world lines of photons (light particles) and world lines of other elementary particles with zero mass. For them $|\mathbf{v}| = c$, hence we get $g(\mathbf{K}, \mathbf{K}) = 0$.

World line have no singular points. Indeed, even if $g(\mathbf{K}, \mathbf{K}) = 0$, tangent vector $\mathbf{K}$ in (6.2) is nonzero since $K^0 = c \neq 0$.

Let's consider world line of material point of nonzero mass.

For this line we have the condition (6.3) fulfilled, hence we can introduce natural parameter on this line:

$$s(t) = \int\limits_{t_0}^{t} \sqrt{g(\mathbf{K}, \mathbf{K})}\, dt. \tag{6.4}$$

Integral (6.4) yields invariant parameter for world lines. For any two points A and B on a given world line the quantity $s(B) - s(A)$ does not depend on inertial coordinate system used for calculating integral (6.4). This quantity is called interval length of the arc AB on world line.

Theorem 6.1. *Straight line segment connecting end points of an arc on smooth time-like curve is a segment of time-like straight line. Its interval length is greater than interval length of corresponding arc.*

Let A and B be two successive events in the "life" of material point of nonzero mass. The answer to the question what time interval separates these two events depend on the choice of inertial coordinate system from which we observe the "life" of this material point. So this answer is relative (not invariant). However, there is invariant quantity characterizing time distance between two events on world line:

$$\tau = \frac{s(B) - s(A)}{c}. \tag{6.5}$$

This quantity τ in formula (6.5) is called interval of *private time* on world line.

Concept of private time determine *microlocal* concept of time in theory of relativity. According to this concept each material point lives according to its own watch, and watches of different material points are synchronized only in very rough way: they count time from the past to the future. This rough synchronization is determined by polarization in Minkowsky space. Exact synchronization of watches is possible only when material points come to immediate touch with each other, i. e. when their world lines intersect. However, even after such exact synchronization in the point of next meeting watches of tho material points will show different times. This difference is due to different "life paths" between two meetings.

Concept of private time is illustrated by so-called twins problem, well-known from science fiction. Suppose that one of twins goes to far-away travel in interstellar spacecraft, while his brother stays on the Earth. Which of them will be older when they meet each other on the Earth in the end of space voyage.

The answer is: that one who stayed on the Earth will be older. World lines of twins intersect twice. Both intersections occur on the Earth, one before travel and other after travel. Its known that Coordinate system associated with the Earth can be taken for inertial coordinate system with high degree of accuracy (indeed, acceleration due to rotation of the Earth around its axis and due to orbital rotation around the Sun is not sensible in our everyday life). Therefore world line of twin stayed on the Earth is straight line. World line of twin in spacecraft is curved. In the beginning of travel he accelerates reaching substantial velocity comparable with light velocity in the middle of the path. Then he experiences backward acceleration in order to brake before reaching target point of his travel. Then he accelerates and brakes again in his back way to the Earth. According to theorem 6.1 interval length of curved world line connecting two events is shorter than interval length of straight world line connecting the same two events. Hence twin stayed on the Earth will be older.

Exercise 6.1. *Remember proof of the fact that the length of curved line connecting two points A and B in Euclidean space is* *greater than the length of straight line segment AB. By analogy to this proof find the proof for theorem 6.1.*

§ 7. Dynamics of material point.

Motion of material point in theory of relativity is described by its world line in Minkowsky space. Let's choose natural parameter on world line and consider four-dimensional tangent vector

$$\mathbf{u}(s) = \frac{d\mathbf{r}(s)}{ds}, \tag{7.1}$$

where $\mathbf{r}(s)$ is four-dimensional radius vector of events on world line. Vector $\mathbf{u}$ is (7.1) is called vector of 4-*velocity*. It is time-like vector and it is unit vector in Minkowsky metric: $g(\mathbf{u}, \mathbf{u}) = 1$. Upon choosing some inertial coordinate system we can write components of 4-velocity vector explicitly:

$$\mathbf{u} = \frac{1}{\sqrt{c^2 - |\mathbf{v}|^2}} \begin{Vmatrix} c \\ v^1 \\ v^2 \\ v^3 \end{Vmatrix}. \tag{7.2}$$

Here v^1, v^2, v^3 are components of three-dimensional velocity vector $\mathbf{v}$. Note that components u^0, u^1, u^2, u^3 of 4-velocity vector are absolute numbers (without measure unit). It is easy to see from (7.2). Upon multiplying $\mathbf{u}$ by scalar mc with the measure unit of momentum we get vector of 4-*momentum*

$$\mathbf{p} = \frac{m}{\sqrt{1 - \dfrac{|\mathbf{v}|^2}{c^2}}} \begin{Vmatrix} c \\ v^1 \\ v^2 \\ v^3 \end{Vmatrix} \tag{7.3}$$

for material point with mass m. Vector $\mathbf{p}$ plays important role in physics since there is fundamental law of nature: *the law of conservation of* 4-*momentum.*

Momentum conservation law. *Vector of* 4*-momentum of material point which do not experience external action remains unchanged.*

According to the law just stated, for particle that do not experience external action we have $\mathbf{p} = \text{const}$. Hence $\mathbf{u} = \text{const}$. Integrating the equation (7.1), for $\mathbf{r}(s)$ we derive

$$\mathbf{r}(s) = \mathbf{r}_0 + \mathbf{u}\,s.$$

Conclusion: in the absence of external action material point moves uniformly along straight line.

External actions causing change of 4-momentum of material point are subdivided into two categories:

(1) continuous;
(2) discrete.

Continuous actions are applied to material particle by external fields. They cause world line to bend making it curved line. In this case $\mathbf{p} \neq \text{const}$. Derivative of 4-momentum with respect to natural parameter s is called *vector of* 4*-force*:

$$\frac{d\mathbf{p}}{ds} = \mathbf{F}(s). \tag{7.4}$$

Vector of 4-force in (7.4) is quantitative characteristic of the action of external fields upon material particle. It is determined by parameters of particle itself and by parameters of external fields at current position of particle as well. We know that vector of 4-velocity $\mathbf{u}$ is unit vector, therefore $g(\mathbf{p}, \mathbf{p}) = m^2\,c^2$. Differentiating this relationship with respect to s and taking into account that components of matrix (2.7) are constant, we find

$$g(\mathbf{u}, \mathbf{F}) = 0. \tag{7.5}$$

The relationship (7.5) means that vector of 4-force is perpendicular to vector of 4-velocity in Minkowsky metric, i. e. force vector is perpendicular to world line of particle.

Choosing some inertial coordinate system, we can replace natural parameter s in (7.5) by time variable t of this coordinate system. Then, taking into account (7.3), from (7.4) we derive

$$\frac{dp^i}{dt}=\sqrt{c^2-|\mathbf{v}|^2}\,F^i,\quad\text{where}\quad i=1,2,3. \tag{7.6}$$

Now, if we denote by $\mathbf{f}$ three-dimensional vector with components $f^i=\sqrt{c^2-|\mathbf{v}|^2}\,F^i$, then for three-dimensional vector of momentum from (7.6) we obtain differential equation

$$\frac{d\mathbf{p}}{dt}=\mathbf{f}. \tag{7.7}$$

The equation (7.7) is treated as relativistic analog of Newton's second law. Instead of classical formula $\mathbf{p}=m\mathbf{v}$ relating momentum and velocity vectors here we have the following relationship:

$$\mathbf{p}=\frac{m\mathbf{v}}{\sqrt{1-\dfrac{|\mathbf{v}|^2}{c^2}}}. \tag{7.8}$$

In order to write (7.8) in classical form we introduce the quantity

$$m_v=\frac{m}{\sqrt{1-\dfrac{|\mathbf{v}|^2}{c^2}}}. \tag{7.9}$$

Constant m is called *mass at rest*, while m_v in (7.8) is called *dynamic mass* of moving particle. Now $\mathbf{p}=m_v\,\mathbf{v}$, and Newton's second law is written as follows:

$$(m_v\,\mathbf{v})'_t=\mathbf{f}. \tag{7.10}$$

Formulas (7.9) and (7.10) are the very ones which are in mind when one says that mass in theory of relativity depends on velocity. It seems to me that such terminology is not so good. In what follows we shall mostly use four-dimensional invariant equation (7.4) and, saying mass, we shall imply mass at rest.

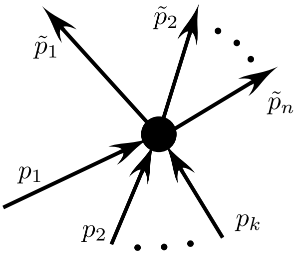

Fig. 7.1

Discrete external actions appear in those situations when 4-momentum of material particle changes abruptly in jump-like manner. Such situation arise in particle collisions, particle confluence, and particle decay. Collision of particles correspond to that point in Minkowsky space where world lines of two or several particles come together. After collision particles can simply fly out from that point. But if these are molecules of ingredients in chemical reaction, then after collision we would have new molecules of reaction products. In a similar way in collisions of atomic nuclei nuclear reactions occur.

Let's consider simultaneous collision of k particles. Denote by $\mathbf{p}_1, \ldots, \mathbf{p}_k$ their 4-momenta just before the collision. Suppose that as a result of collision he have n new particles created from initial ones. Denote by $\tilde{\mathbf{p}}_1, \ldots, \tilde{\mathbf{p}}_n$ 4-momenta of outgoing particles just after the collision. If $k = 1$ this is particle decay process, while if $n = 1$ we have particle confluence into one composite particle.

Momentum conservation law. *Total 4-momentum of ingoing particles before collision is equal to total 4-momentum of outgoing particles after collision:*

$$\sum_{i=1}^{k} \mathbf{p}_i = \sum_{i=1}^{n} \tilde{\mathbf{p}}_i. \tag{7.11}$$

As an example we consider process of frontal collision of two identical particles of mass m leading to creation of one particle of mass M. Suppose that velocities of initial particles are equal by magnitude but opposite to each other:

$$\mathbf{p}_1 = \frac{m}{\sqrt{1 - \dfrac{|\mathbf{v}|^2}{c^2}}} \begin{Vmatrix} c \\ v^1 \\ v^2 \\ v^3 \end{Vmatrix}, \qquad \mathbf{p}_2 = \frac{m}{\sqrt{1 - \dfrac{|\mathbf{v}|^2}{c^2}}} \begin{Vmatrix} c \\ -v^1 \\ -v^2 \\ -v^3 \end{Vmatrix}.$$

For 4-momentum of resulting particle we have

$$\tilde{\mathbf{p}}_1 = \frac{M}{\sqrt{1 - \dfrac{|\mathbf{w}|^2}{c^2}}} \begin{Vmatrix} c \\ w^1 \\ w^2 \\ w^3 \end{Vmatrix}.$$

Applying momentum conservation law (7.11) to this situation, we get $\mathbf{w} = 0$ and additionally we obtain

$$M = \frac{2\,m}{\sqrt{1 - \dfrac{|\mathbf{v}|^2}{c^2}}}. \tag{7.12}$$

From (7.12) we see that mass at rest of resulting composite particle is greater than sum of rest masses of its components: $M > m + m$. Conclusion: the low of mass conservation is fulfilled approximately only in the limit of small velocities $|\mathbf{v}| \ll c$.

Let's multiply zeroth component of 4-momentum of material particle by c. Resulting quantity has the measure unit of energy. Let's denote this quantity by E:

$$E = \frac{mc^2}{\sqrt{1 - \dfrac{|\mathbf{v}|^2}{c^2}}}. \tag{7.13}$$

The quantity (7.13) is called *kinetic energy* of moving particle. Writing relationship (7.11) for zeroth components of colliding particles, we get *energy conservation law*:

$$\sum_{i=1}^{k} E_i = \sum_{i=1}^{n} \tilde{E}_i. \tag{7.14}$$

Thus, 4-momentum conservation law for collision includes both energy conservation law (7.14) and the law of conservation for three-dimensional momentum.

Note that for zero velocity $\mathbf{v} = 0$ the above quantity (7.13) does not vanish, but takes nonzero value

$$E = mc^2. \tag{7.15}$$

This quantity is known as *rest energy* of material particle. Formula (7.15) is well-known. It reflects very important fact absent in classical physics: the ability of energy to mass and mass to energy conversion. In practice conversion of energy to mass is realized in particle confluence (see $M > m + m$ in formula (7.12)). Converse phenomenon of particle decay yields mass defect (mass decrease). Lost mass is realized in additional amount of kinetic energy of outgoing particles. Total conversion of mass to energy is also possible. This happens in process of *annihilation*, when elementary particle meets corresponding antiparticle. Large amount of energy released in annihilation is scattered in form of short-wave electromagnetic radiation.

§ 8. Four-dimensional form of Maxwell equations.

Starting from electromagnetic equations $\square\mathbf{E} = 0$ and $\square\mathbf{H} = 0$ in previous sections we have constructed and described Lorentz transformations preserving form of these equations. We also have given geometric and physical interpretation of Lorentz transformations and even have described dynamics of material points on the base of new relativistic notion of space and time. Now time has come to remember that equations $\Box\mathbf{E} = 0$ and $\Box\mathbf{H} = 0$ are not primary equations of electrodynamics, they were derived from Maxwell equations. To have complete picture we should we should write Maxwell equations in four-dimensional form. Let's begin with second pair of these equations containing charges and currents (see equations (1.2) in Chapter II. Let's modify them:

$$\frac{1}{c}\,\frac{\partial\mathbf{E}}{\partial t} - \operatorname{rot}\mathbf{H} = -\frac{4\pi}{c}\,\mathbf{j}, \qquad\qquad -\operatorname{div}\mathbf{E} = -4\pi\rho.$$

Then rewrite these equations in components using Levi-Civita symbol for to express rotor (see [3]):

$$\begin{gathered}\frac{\partial E^p}{\partial r^0} - \sum_{q=1}^{3}\sum_{k=1}^{3}\varepsilon_{pqk}\,\frac{\partial H^k}{\partial r^q} = -\frac{4\pi}{c}\,j^p,\\ -\sum_{q=1}^{3}\frac{\partial E^q}{\partial r^q} = -4\pi\,\rho.\end{gathered}\tag{8.1}$$

Here we used notation $r^0 = ct$ associating time variable with zeroth component of radius-vector in Minkowsky space.

Using Levi-Civita symbol and components of vector $\mathbf{H}$, we can construct skew-symmetric 3×3 matrix with elements

$$F^{pq} = -\sum_{k=1}^{3}\varepsilon_{pqk}\,H^k.\tag{8.2}$$

Due to (8.2) we can easily write explicit form of matrix F:

$$F^{pq}=\begin{pmatrix} 0 & -H^3 & H^2 \\ H^3 & 0 & -H^1 \\ -H^2 & H^1 & 0 \end{pmatrix}. \tag{8.3}$$

Let's complement the above matrix (8.3) with one additional line and one additional column:

$$F^{pq}=\begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -H^3 & H^2 \\ E^2 & H^3 & 0 & -H^1 \\ E^3 & -H^2 & H^1 & 0 \end{pmatrix}. \tag{8.4}$$

Additional line and additional column in (8.4) are indexed by zero, i. e. indices p and q run over integer numbers from 0 to 3. In addition, we complement three-dimensional vector of current density with one more component

$$j^0=\rho c. \tag{8.5}$$

By means of (8.4) and (8.5) we can rewrite Maxwell equations (8.1) in very concise four-dimensional form:

$$\sum_{q=0}^{3}\frac{\partial F^{pq}}{\partial r^q}=-\frac{4\pi}{c}\,j^p. \tag{8.6}$$

Now let's consider first pair of Maxwell equations (see equations (1.1) in Chapter II). In coordinates they are written as

$$\frac{\partial H^p}{\partial r^0}+\sum_{q=1}^{3}\sum_{k=1}^{3}\varepsilon_{pqk}\,\frac{\partial E^k}{\partial r^q}=0, \qquad \sum_{q=1}^{3}\frac{\partial H^q}{\partial r^q}=0. \tag{8.7}$$

The structure of the equations (8.7) is quite similar to that of (8.1). However, their right hand sides are zero and we see slight difference in signs. Main difference is that components of vectors **E** and **H** have exchanged their places. To exchange components of vectors **E** and **H** in matrix (8.4) we need four-dimensional analog of Levi-Civita symbol:

$$\varepsilon_{pqks}=\varepsilon^{pqks}=\begin{cases}0, & \text{if among } p,\ q,\ k,\ s \text{ there are at least two equal numbers;}\\ 1, & \text{if } (p\,q\,k\,s) \text{ is even permutation of numbers } (0\,1\,2\,3);\\ -1, & \text{if } (p\,q\,k\,s) \text{ is odd permutation of numbers } (0\,1\,2\,3).\end{cases}$$

Let's define matrix G by the following formula for its components:

$$G^{pq}=-\frac{1}{2}\sum_{k=0}^{3}\sum_{s=0}^{3}\sum_{m=0}^{3}\sum_{n=0}^{3}\varepsilon^{pqks}\,g_{km}\,g_{sn}\,F^{mn}. \tag{8.8}$$

Here g is matrix (2.7) determining Minkowsky metric. Matrix G with components (8.8) can be expressed in explicit form:

$$G^{pq}=\begin{pmatrix}0 & -H^1 & -H^2 & -H^3\\ H^1 & 0 & E^3 & -E^2\\ H^2 & -E^3 & 0 & E^1\\ H^3 & E^2 & -E^1 & 0\end{pmatrix}. \tag{8.9}$$

The structure of matrix (8.9) enable us to write remaining two

Maxwell equations (8.7) in concise four-dimensional form:

$$\sum_{q=0}^{3}\frac{\partial G^{pq}}{\partial r^q}=0. \tag{8.10}$$

Usage of both matrices F and G in theory is assumed to be too excessive. For this reason equations (8.10) are written as

$$\sum_{q=0}^{3}\sum_{k=0}^{3}\sum_{s=0}^{3}\varepsilon^{pqks}\,\frac{\partial F_{ks}}{\partial r^q}=0. \tag{8.11}$$

Matrix F_{ks} is obtained from F^{mn} by means of standard index lowering procedure using matrix (2.7):

$$F_{ks}=\sum_{m=0}^{3}\sum_{n=0}^{3}g_{km}\,g_{sn}\,F^{mn}. \tag{8.12}$$

Four-dimensional form of Maxwell equations (8.6) and (8.11) gives a hint for proper geometric interpretation of these equations. Matrix (8.4) defines tensor of the type $(2,0)$ in Minkowsky space. This tensor is called *tensor of electromagnetic field*. Tensorial interpretation of matrix (8.4) immediately yields transformation rules, which were lacking so far:

$$F^{pq}=\sum_{m=0}^{3}\sum_{n=0}^{3}S^p_m\,S^q_n\,\tilde F^{mn}. \tag{8.13}$$

These relationships (8.13) determine transformation rules for components of vectors $\mathbf{E}$ and $\mathbf{H}$. Before now we express these rules in undetermined form by the relationships (1.6). For special Lorentz matrices (4.11) vectors of electric and magnetic fields **E** and **H** in two inertial coordinate systems are related as follows:

$$E^1 = \tilde{E}^1, \qquad E^2 = \frac{\tilde{E}^2 + \dfrac{u}{c}\,\tilde{H}^3}{\sqrt{1 - \dfrac{u^2}{c^2}}}, \qquad E^3 = \frac{\tilde{E}^3 - \dfrac{u}{c}\,\tilde{H}^2}{\sqrt{1 - \dfrac{u^2}{c^2}}},$$

$$H^1 = \tilde{H}^1, \qquad H^2 = \frac{\tilde{H}^2 - \dfrac{u}{c}\,\tilde{E}^3}{\sqrt{1 - \dfrac{u^2}{c^2}}}, \qquad H^3 = \frac{\tilde{H}^3 + \dfrac{u}{c}\,\tilde{E}^2}{\sqrt{1 - \dfrac{u^2}{c^2}}}.$$

According to theorem 4.1, general Lorentz matrix is a product of special Lorentz matrix of the form (4.11) and two matrices of spatial rotation in tree-dimensional space. The latter ones can be excluded if one writes Lorentz transformation in “conditionally three-dimensional” vectorial form:

$$\begin{aligned}
&\mathbf{E} = \frac{\langle\mathbf{u},\,\tilde{\mathbf{E}}\rangle}{|\mathbf{u}|^2}\,\mathbf{u} + \frac{\tilde{\mathbf{E}} - \dfrac{\langle\mathbf{u},\,\tilde{\mathbf{E}}\rangle}{|\mathbf{u}|^2}\,\mathbf{u} - \dfrac{1}{c}\,[\mathbf{u},\,\tilde{\mathbf{H}}]}{\sqrt{1 - \dfrac{|\mathbf{u}|^2}{c^2}}},\\
&\mathbf{H} = \frac{\langle\mathbf{u},\,\tilde{\mathbf{H}}\rangle}{|\mathbf{u}|^2}\,\mathbf{u} + \frac{\tilde{\mathbf{H}} - \dfrac{\langle\mathbf{u},\,\tilde{\mathbf{H}}\rangle}{|\mathbf{u}|^2}\,\mathbf{u} + \dfrac{1}{c}\,[\mathbf{u},\,\tilde{\mathbf{E}}]}{\sqrt{1 - \dfrac{|\mathbf{u}|^2}{c^2}}}.
\end{aligned}\tag{8.14}$$

From (8.13) we derive the following rule for transforming covariant components of the tensor of electromagnetic field:

$$F_{pq} = \sum_{m=0}^{3}\sum_{n=0}^{3} T^m_p\,T^n_q\,\tilde{F}_{mn}.\tag{8.15}$$

This relationship (8.15) provides invariance of the form of Maxwell equations (8.11) under Lorentz transformation (2.3). In order to verify this fact it is sufficient to apply relationships (2.8) for transforming derivatives and then remember well-known property of four-dimensional Levi-Civita symbol ε^{pqks}:

$$\sum_{a=0}^{3}\sum_{b=0}^{3}\sum_{c=0}^{3}\sum_{d=0}^{3} T^p_a\,T^q_b\,T^k_c\,T^s_d\,\varepsilon^{abcd} = \det T\,\varepsilon^{pqks}. \tag{8.16}$$

The condition of invariance of Maxwell equations (8.6) with respect to Lorentz transformations leads to the following transformation rule for components of four-dimensional current density:

$$j^p = \sum_{m=0}^{3} S^p_m\,\tilde{j}^m. \tag{8.17}$$

In (8.17) it is easy to recognize the transformation rule for components of four-dimensional vector. In case of special Lorentz matrix of the form (4.11), taking into account (8.5), one can write the above relationship (8.17) as follows:

$$\begin{aligned}
&\rho = \frac{\tilde{\rho} + \dfrac{u}{c^2}\,\tilde{j}^1}{\sqrt{1 - \dfrac{u^2}{c^2}}}, && j^1 = \frac{u\tilde{\rho} + \tilde{j}^1}{\sqrt{1 - \dfrac{u^2}{c^2}}},\\
&j^2 = \tilde{j}^2, && j^3 = \tilde{j}^3.
\end{aligned} \tag{8.18}$$

Remember that here $u = c\tanh(\alpha)$ is a magnitude of relative velocity of one inertial coordinate system with respect to another.

In vectorial form relationships (8.18) are written as

$$\rho = \frac{\tilde{\rho} + \dfrac{\langle \mathbf{u}, \tilde{\mathbf{j}} \rangle}{c^2}}{\sqrt{1 - \dfrac{|\mathbf{u}|^2}{c^2}}},$$

(8.19)

$$\mathbf{j} = \frac{\mathbf{u}\,\tilde{\rho} + \dfrac{\langle \mathbf{u}, \tilde{\mathbf{j}} \rangle}{|\mathbf{u}|^2}\,\mathbf{u}}{\sqrt{1 - \dfrac{|\mathbf{u}|^2}{c^2}}} + \tilde{\mathbf{j}} - \frac{\langle \mathbf{u}, \tilde{\mathbf{j}} \rangle}{|\mathbf{u}|^2}\,\mathbf{u}.$$

In such form they give transformation rule for charge density ρ and three-dimensional current density $\mathbf{j}$ under Lorentz transformations with arbitrary Lorentz matrix.

Exercise 8.1. *Prove the relationship* (8.16), *assuming T to be an arbitrary 4×4 matrix.*

Exercise 8.2. *Using* (2.12), *derive the relationship* (8.15) *from* (8.12) *and* (8.13).

Exercise 8.3. *Using* (8.15), (8.16) *and* (2.8), *transform Maxwell equations* (8.11) *from one inertial coordinate system to another. Verify that the form of these equations is invariant.*

Exercise 8.4. *Using* (8.13), (8.17) *and* (2.8), *transform Maxwell equations* (8.6) *from one inertial coordinate system to another. Verify that the form of these equations is invariant.*

§ 9. Four-dimensional vector-potential.

Due to special structure of Maxwell equations one can introduce vector-potential $\mathbf{A}$ and scalar potential φ. This was done in

§ 3 of Chapter II. Here are formulas for components of **E** and **H**:

$$
\begin{aligned}
&E^p = -\frac{\partial\varphi}{\partial r^p} - \frac{1}{c}\,\frac{\partial A^p}{\partial t},\\
&H^p = \sum_{q=1}^{3}\sum_{k=1}^{3} \varepsilon^{pqk}\,\frac{\partial A^k}{\partial r^q},
\end{aligned}
\tag{9.1}
$$

(see formulas (3.4) in Chapter II). Denote $A^0 = \varphi$ and consider four-dimensional vector **A** with components A^0, A^1, A^2, A^3. This is *four-dimensional vector-potential* of electromagnetic field. By lowering index procedure we get covector **A**:

$$
A_p = \sum_{q=0}^{3} g_{pq}\,A^q.
\tag{9.2}
$$

Taking into account relationships (2.7) for components of matrix g_{pq}, from formula (9.2) we derive

$$
\begin{aligned}
&A_0 = A^0, &&A_1 = -A^1,\\
&A_2 = -A^2, &&A_3 = -A^3.
\end{aligned}
\tag{9.3}
$$

Moreover, let's write explicitly covariant components for the tensor of electromagnetic field:

$$
F_{pq} = \begin{pmatrix}
0 & E^1 & E^2 & E^3\\
-E^1 & 0 & -H^3 & H^2\\
-E^2 & H^3 & 0 & -H^1\\
-E^3 & -H^2 & H^1 & 0
\end{pmatrix}.
\tag{9.4}
$$

Due to (9.3) and (9.4) first relationship (9.1) can be written as

$$F_{0q} = \frac{\partial A_q}{\partial r^0} - \frac{\partial A_0}{\partial r^q}. \tag{9.5}$$

In order to calculate other components of tensor F_{pq} let's apply (8.2) and second relationship (9.1). Thereby let's take into account that $F_{pq} = F^{pq}$ and $A_p = -A^p$ for $p, q = 1, 2, 3$:

$$F_{pq} = -\sum_{k=1}^{3} \varepsilon_{pqk}\, H^k = \sum_{k=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3} \varepsilon_{pqk}\, \varepsilon^{kmn} \frac{\partial A_n}{\partial r^m}. \tag{9.6}$$

Further transformation of (9.6) is based on one of the well-known contraction identities for Levi-Civita symbol:

$$\sum_{k=1}^{3} \varepsilon_{pqk}\, \varepsilon^{kmn} = \delta^m_p \delta^n_q - \delta^m_q \delta^n_p. \tag{9.7}$$

Applying (9.7) to (9.6), we get

$$F_{pq} = \sum_{m=1}^{3}\sum_{n=1}^{3} (\delta^m_p \delta^n_q - \delta^m_q \delta^n_p)\, \frac{\partial A_n}{\partial r^m} = \frac{\partial A_q}{\partial r^p} - \frac{\partial A_p}{\partial r^q}. \tag{9.8}$$

Combining (9.8) and (9.5), we obtain the following formula for all covariant components of the tensor of electromagnetic field:

$$F_{pq} = \frac{\partial A_q}{\partial r^p} - \frac{\partial A_p}{\partial r^q}. \tag{9.9}$$

In essential, formula (9.9) is four-dimensional form of the relationships (9.1). It unites these two relationships into one.

Remember that vectorial and scalar potentials of electromagnetic field are not unique. They are determined up to a gauge transformation (see formula (4.1) in Chapter II). This uncertainty could be included into transformation rule for component of four-dimensional potential **A**. However, if we assert that A^0, A^1, A^2, A^3 are transformed as components of four-dimensional vector

$$A^p = \sum_{q=0}^{3} S^p_q\,\tilde A^q, \tag{9.10}$$

and A_0, A_1, A_2, A_3 are obtained from them by index lowering procedure (9.2), then we find that quantities F_{pq} defined by formula (9.9) are transformed exactly by formula (8.15), as they actually should.

From (9.10) one can easily derive explicit transformation formulas for scalar potential φ and for components of three-dimensional vector-potential **A**. For special Lorentz transformations with matrix (4.11) they are written as follows:

$$\begin{aligned} &\varphi = \frac{\tilde\varphi + \dfrac{u}{c}\,\tilde A^1}{\sqrt{1 - \dfrac{u^2}{c^2}}}, &&A^1 = \frac{\dfrac{u}{c}\,\tilde\varphi + \tilde A^1}{\sqrt{1 - \dfrac{u^2}{c^2}}},\\ &A^2 = \tilde A^2, &&A^3 = \tilde A^3. \end{aligned} \tag{9.11}$$

Note that one can rederive transformation rules for components of electric and magnetic fields (see § 8 above). However, we shall not do it now.

In case of Lorentz transformations with arbitrary Lorentz matrix the relationships (9.11) should be written in vectorial form:

$$\varphi = \frac{\tilde\varphi + \dfrac{\langle \mathbf{u}, \tilde{\mathbf{j}}\rangle}{c}}{\sqrt{1 - \dfrac{|\mathbf{u}|^2}{c^2}}}, \tag{9.12a}$$

$$\mathbf{A}=\frac{\dfrac{\mathbf{u}}{c}\,\tilde{\varphi}+\dfrac{\left\langle\mathbf{u},\,\tilde{\mathbf{A}}\right\rangle}{|\mathbf{u}|^2}\,\mathbf{u}}{\sqrt{1-\dfrac{|\mathbf{u}|^2}{c^2}}}+\tilde{\mathbf{A}}-\frac{\left\langle\mathbf{u},\,\tilde{\mathbf{A}}\right\rangle}{|\mathbf{u}|^2}\,\mathbf{u}. \tag{9.12b}$$

Theorem 9.1. *Each skew-symmetric tensor field* $\mathbf{F}$ *of type* $(0,2)$ *in four-dimensional space satisfying differential equations* (8.11) *is determined by some covector field* $\mathbf{A}$ *according to the above formula* (9.9)*.*

Proof. Each skew-symmetric tensor field $\mathbf{F}$ of type $(0,2)$ in four-dimensional space can be identified with pair of three-dimensional vector fields $\mathbf{E}$ and $\mathbf{H}$ depending on additional parameter $r^0 = ct$. In order to do this one should use (9.4). Then equations (8.11) are written as Maxwell equations for $\mathbf{E}$ and $\mathbf{H}$:

$$\operatorname{div}\mathbf{H}=0, \qquad\qquad \operatorname{rot}\mathbf{E}=-\frac{1}{c}\,\frac{\partial\mathbf{H}}{\partial t}.$$

Further construction of covector field $\mathbf{A}$ is based on considerations from § 3 of Chapter II, where three-dimensional vector-potential and scalar potential were introduced. Then we denote $A^0 = \varphi$ and thus convert three-dimensional vector-potential into four-dimensional vector. And the last step is index lowering procedure given by formula (9.2). □

Choice of vector field $\mathbf{A}$ in formula (9.9), as we noted above, has gauge uncertainty. In four-dimensional formalism this fact is represented by gauge transformations

$$A_k \to A_k + \frac{\partial\psi}{\partial r^k}, \tag{9.13}$$

where ψ — is some arbitrary scalar field. Formula (9.13) is four-dimensional version of gauge transformations (4.1) considered in Chapter II. It is easy to verify that gauge transformations (9.13) do not break transformation rules (9.10) for contravariant components of vector **A**.

Exercise 9.1. *Prove theorem 9.1 immediately in four-dimensional form without passing back to three-dimensional statements and constructions.*

§ 10. The law of charge conservation.

Earlier we have noted that charge conservation law can be derived from Maxwell equations (see § 1 in Chapter II). To prove this fact in four-dimensional formalism is even easier. Let's differentiate the relationship (8.6) with respect to r^p and add one more summation with respect to index p:

$$\sum_{p=0}^{3}\sum_{q=0}^{3}\frac{\partial^2 F^{pq}}{\partial r^p\,\partial r^q}=-\frac{4\pi}{c}\sum_{p=0}^{3}\frac{\partial j^p}{\partial r^p}. \tag{10.1}$$

Double differentiation in (10.1) is symmetric operation, while tensor of electromagnetic field F^{pq}, to which it is applied, is skew-symmetric. Therefore the expression under summation in left hand side of (10.1) is skew-symmetric with respect to indices p and q. This leads to vanishing of double sum in left hand side of formula (10.1). Hence we obtain

$$\sum_{p=0}^{3}\frac{\partial j^p}{\partial r^p}=0. \tag{10.2}$$

The equality (10.2) is four-dimensional form of charge conservation law. If we remember that $j^0 = c\rho$ and $r^0 = ct$, we see that this equality coincides with (5.4) in Chapter I.

Conservation laws for scalar quantities (those like electric charge) in theory of relativity are expressed by equations analogous to (10.2) in form of vanishing of four-dimensional divergencies for corresponding four-dimensional currents. For vectorial quantities corresponding current densities are tensors. Thus the law of conservation of 4-momentum for fields is represented by the equation

$$\sum_{p=0}^{3}\frac{\partial T^{qp}}{\partial r^p}=0. \tag{10.3}$$

Tensor T^{qp} in (10.3) playing the role of current density of 4-momentum is called *energy-momentum tensor.*

Theorem 10.1. *For any vector field* $\mathbf{j}$ *in n-dimensional space ($n\geqslant 2$) if its divergency is zero*

$$\sum_{p=1}^{n}\frac{\partial j^p}{\partial r^p}=0, \tag{10.4}$$

then there is skew-symmetric tensor field $\boldsymbol{\psi}$ *of type* $(2,0)$ *such that*

$$j^p=\sum_{q=1}^{n}\frac{\partial \psi^{pq}}{\partial r^q}. \tag{10.5}$$

Proof. Choosing some Cartesian coordinate system, we shall construct matrix ψ^{pq} of the following special form:

$$\psi^{pq}=\begin{pmatrix} 0 & \dots & 0 & \psi^{1n}\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & 0 & \psi^{n-1\,n}\\ -\psi^{1n} & \dots & -\psi^{n-1\,n} & 0\end{pmatrix}. \tag{10.6}$$

Matrix (10.6) is skew-symmetric, it has $(n-1)$ independent components. From (10.5) for these components we derive

$$\begin{aligned}&\frac{\partial\psi^{kn}}{\partial r^n}=j^k,\quad\text{where}\quad k=1,\,\ldots\,,n-1,\\&\sum_{k=1}^{n-1}\frac{\partial\psi^{kn}}{\partial r^k}=-j^n.\end{aligned}\tag{10.7}$$

Let's define functions ψ^{kn} in (10.7) by the following integrals:

$$\begin{aligned}\psi^{kn}=&\int\limits_0^{r^n} j^k(r^1,\ldots\,,r^{n-1},y)\,dy+\\&+\frac{1}{n-1}\int\limits_0^{r^k} j^n(r^1,\ldots\,,y,\ldots\,,r^{n-1},0)\,dy.\end{aligned}\tag{10.8}$$

It is easy to verify that functions (10.8) satisfy first series of differential equations (10.7). Under the condition (10.4) they satisfy last equation (10.7) as well. Thus, theorem is proved. □

Theorem 10.1 can be easily generalized for arbitrary tensorial currents. Its prove thereby remains the same in most.

Theorem 10.2. *For any tensorial field* $\mathbf{T}$ *of type* (m,s) *in the space of dimension* $n\geqslant 2$ *if its divergency is zero*

$$\sum_{p_m=1}^{n}\frac{\partial T^{p_1\,\ldots\,p_m}_{q_1\,\ldots\,q_s}}{\partial r^{p_m}}=0,$$

then there is tensorial field $\boldsymbol{\psi}$ *of type* $(m+1, s)$ *skew-symmetric in last pair of upper indices and such that*

$$T^{p_1 \dots p_m}_{q_1 \dots q_s} = \sum_{p_{m+1}=1}^{n} \frac{\partial \psi^{p_1 \dots p_m \, p_{m+1}}_{q_1 \dots q_s}}{\partial r^{p_{m+1}}}.$$

Exercise 10.1. *Verify that the equation* (10.4) *provides last equation* (10.7) *to be fulfilled for the functions* (10.8).

Exercise 10.2. *Clarify the relation of theorem 10.1 and theorem on vortex field in case of dimension* $n = 3$.

§ 11. Note on skew-angular and curvilinear coordinates.

In previous three sections we have managed to write in four-dimensional form all Maxwell equations, charge conservation law, and the relation of **E**, **H** and their potentials. The relationships (8.6), (8.11), (9.9), (9.13), (10.2), which were obtained there, preserve their shape when we transfer from one rectangular Cartesian coordinate system to another. Such transitions are interpreted as Lorentz transformations, they are given by Lorentz matrices. However, all these relationships (8.6), (8.11), (9.9), (9.13), (10.2) possess transparent tensorial interpretation. Therefore they can be transformed to any skew-angular Cartesian coordinate system as well. Thereby we would have minor differences: the shape of matrix g would be different and instead of ε^{pqks} in (8.11) we would require volume tensor with components

$$\omega^{pqks} = \pm\sqrt{-\det \hat{g}}\; \varepsilon^{pqks}. \tag{11.1}$$

Matrix g_{pq} in skew-angular coordinate system is not given by formula (2.7), here it is arbitrary symmetric matrix determining quadratic form of signature $(1, 3)$. Therefore differential equations $\square\mathbf{E} = 0$ and $\square\mathbf{H} = 0$, which we are started from, have not their initial form. They are written as $\square F^{pq} = 0$, where d'Alambert operator is given by formula (2.6) with non-diagonal matrix g^{ij}.

In arbitrary skew-angular coordinate system none of axes should have time-like direction. Therefore none of them can be interpreted as time axis. Three-dimensional form of electrodynamics equations, even if we could write them, would not have proper physical interpretation in such coordinate system. In particular, interpretation of components of tensor F^{pq} as components of electric and magnetic fields in formula (8.4) would not be physically meaningful.

Tensorial form of four-dimensional electrodynamics equations enables us to make one more step toward increasing arbitrariness in the choice of coordinate system: we can use not only skew-angular, but curvilinear coordinates as well. To make this step we need to replace partial derivatives by covariant derivatives:

$$\frac{\partial}{\partial r^p} \to \nabla_p \tag{11.2}$$

(see [3] for more details). Connection components required for passing to covariant derivatives (11.2) are determined by components of metric tensor. The latter ones in curvilinear coordinate system do actually depend on r^0, r^1, r^2, r^3:

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{s=0}^{3} g^{ks} \left(\frac{\partial g_{sj}}{\partial r^i} + \frac{\partial g_{is}}{\partial r^j} - \frac{\partial g_{ij}}{\partial r^s} \right). \tag{11.3}$$

No we give list of all basic equations, which we derived above, in covariant form. Maxwell equations are written as follows:

$$
(11.4)\qquad
\begin{gathered}
\sum_{q=0}^{3} \nabla_q F^{pq} = -\frac{4\pi}{c}\, j^p,\\
\sum_{q=0}^{3}\sum_{k=0}^{3}\sum_{s=0}^{3} \omega^{pqks}\, \nabla_q F_{ks} = 0.
\end{gathered}
$$

Here components of volume tensor ω^{pqks} are given by formula (11.1). Tensor of electromagnetic field is expressed through four-dimensional vector-potential by formula

$$
(11.5)\qquad F_{pq} = \nabla_p A_q - \nabla_q A_p,
$$

while gauge uncertainty in the choice of vector-potential itself is described by the relationship

$$
(11.6)\qquad A_k \to A_k + \nabla_k \psi,
$$

where ψ is arbitrary scalar field. Charge conservation law in curvilinear coordinates is written as

$$
(11.7)\qquad \sum_{p=0}^{3} \nabla_p j^p = 0.
$$

Instead of formula (2.6) for D'Alambert operator here we have

$$
(11.8)\qquad \Box = \sum_{i=0}^{3}\sum_{j=0}^{3} g^{ij}\, \nabla_i \nabla_j.
$$

Dynamics of material point of nonzero mass $m \neq 0$ is described by ordinary differential equations of Newtonian type:

$$\dot{\mathbf{r}} = \mathbf{u}, \qquad \nabla_s \mathbf{u} = \frac{\mathbf{F}}{mc}. \tag{11.9}$$

Here dot means standard differentiation with respect to natural parameter s on world line, while ∇_s is covariant derivative with respect to the same parameter.

Exercise 11.1. *Using symmetry of Christoffel symbols* (11.3) *with respect to lower pair of indices i and j, show that the relationship* (11.5) *can be brought to the form* (9.9) *in curvilinear coordinate system as well.*

CHAPTER IV

LAGRANGIAN FORMALISM IN THEORY OF RELATIVITY

§ 1. Principle of minimal action for particles and fields.

Dynamics of material points in theory of relativity is described by their world lines. These are time-like lines in Minkowsky space.

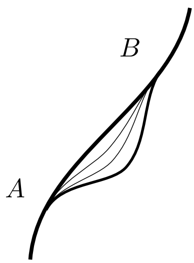

Fig. 1.1

Let's consider some world line corresponding to real motion of some particle under the action of external fields. Let's fix two points A and B on this world line not too far from each other. Then consider small deformation of world line in the range between these two points A and B. Suppose that we have some coordinate system in Minkowsky space (either Cartesian, or curvilinear, no matter). Then our world line is given in parametric form by four functions

$$r^0(s),\quad r^1(s),\quad r^2(s),\quad r^3(s), \tag{1.1}$$

where s is natural parameter. Then deformed curve can be given by the following four functions:

$$\hat{r}^i(s) = r^i(s) + h^i(\varepsilon, s),\ i = 0, \ldots, 3. \tag{1.2}$$

Here s is original natural parameter on initial non-deformed world line (1.1), while $h^i(\varepsilon, s)$ are smooth functions which are nonzero only within the range between points A and B. Note that functions $h^i(\varepsilon, s)$ in (1.2) depend on additional parameter ε which is assumed to be small. Moreover, we shall assume that

$$h^i(\varepsilon, s) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{1.3}$$

Thus, in (1.2) we have whole family of deformed lines. This family of lines is called *variation* of world line (1.1). Due to (1.3) we have the following Taylor expansion for $h^i(\varepsilon, s)$:

$$h^i(\varepsilon, s) = \varepsilon\, h^i(s) + \ldots\,. \tag{1.4}$$

Under the change of one curvilinear coordinate system for another quantities $h^i(s)$ are transformed as components of four-dimensional vector. This vector is called *vector of variation* of world line, while quantities

$$\delta r^i(s) = \varepsilon h^i(s) \tag{1.5}$$

are called *variations of point coordinates.* It is clear that they also are transformed as components of four-dimensional vector. Due to formulas (1.4) and (1.5) parametric equations of deformed curves (1.2) are written as follows:

$$\hat{r}^i(s) = r^i(s) + \delta r^i(s) + \ldots\,. \tag{1.6}$$

By this formula we emphasize that terms other than linear with respect to small parameter ε are of no importance.

By varying functions $h^i(\varepsilon, s)$ in (1.2) and by varying parameter ε in them we can surround segment of initial world line by a swarm of its variations. Generally speaking, these variations do not describe real dynamics of points. However, they are used in statement of *minimal action principle.* Within framework of Lagrangian formalism *functional of action* S is usually introduced, this is a map that to each line connecting two points A and B put into correspondence some real number S.

Principle of minimal action for particles. *World line connecting two points A and B describes real dynamics of material point if and only if action functional S reaches local minimum on it among other lines being its small variations.*

Action functional S producing number by each line should depend only on that line (as geometric set of points in M), but it should not depend on coordinate system (r^0, r^1, r^2, r^3) in M. By tradition this condition is called *Lorentz invariance*, though changes of one curvilinear coordinate system by another form much broader class of transformations than Lorentz transformations relating two rectangular Cartesian coordinate systems in Minkowsky space.

Action functional in most cases is integral. For single point of mass m in electromagnetic field with potential $\mathbf{A}$ it is written as

$$S = -mc \int\limits_{s_1}^{s_2} ds - \frac{q}{c} \int\limits_{s_1}^{s_2} g(\mathbf{A}, \mathbf{u})\, ds. \tag{1.7}$$

Here q is electric charge of particle, while $\mathbf{u} = \mathbf{u}(s)$ is vector of its 4-velocity (unit tangent vector of world line). First integral in (1.7) yields action for free particle (in the absence of external fields), second integral describes interaction of particle with electromagnetic field.

If we consider system of N particles, then we should write integral (1.7) for each of them and we should add all these integrals. And finally, in order to get the action functional for total system of field and particles we should add integral of action for electromagnetic field itself:

$$S=\sum_{i=1}^{N}\left(-m_i\,c\int\limits_{s_1(i)}^{s_2(i)}ds-\frac{q_i}{c}\int\limits_{s_1(i)}^{s_2(i)}g(\mathbf{A},\mathbf{u})\,ds\right)-\\ -\frac{1}{16\,\pi\,c}\int\limits_{V_1}^{V_2}\sum_{p=0}^{3}\sum_{q=0}^{3}F_{pq}\,F^{pq}\sqrt{-\det g}\;d^4r. \tag{1.8}$$

Last integral in (1.8) deserves special consideration. This is four-dimensional volume integral over the domain enclosed between two three-dimensional hypersurfaces V_1 and V_2. Hypersurfaces V_1 and V_2 are space-like, i. e. their normal vectors are time-like vectors. These hypersurfaces determine the fissure between the future and the past, and over this fissure we integrate in (1.8). Thereby change of field functions (here these are components of vector-potential $\mathbf{A}$) when passing from V_1 to V_2 reflects evolution of electromagnetic field from the past to the future.

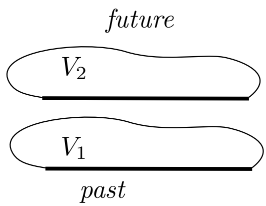

Fig. 1.2

Electromagnetic field is described by field functions. Therefore variation of field is defined in other way than that of particles. Suppose that Ω is some restricted four-dimensional domain enclosed between hypersurfaces V_1 and V_2. Let's consider four smooth functions $h^i(\varepsilon,\mathbf{r})=h^i(\varepsilon,r^0,r^1,r^2,r^3)$ being identically zero outside the domain Ω and vanishing for $\varepsilon = 0$. Let's define

$$\hat{A}^i(\mathbf{r}) = A^i(\mathbf{r}) + h^i(\varepsilon, \mathbf{r}) \tag{1.9}$$

and consider Taylor expansion of h^i at the point $\varepsilon = 0$:

$$h^i(\varepsilon, \mathbf{r}) = \varepsilon\, h^i(\mathbf{r}) + \ldots\,. \tag{1.10}$$

The following functions determined by linear terms in the above Taylor expansions (1.10)

$$\delta A^i(\mathbf{r}) = \varepsilon\, h^i(\mathbf{r}) \tag{1.11}$$

are called *variations of field functions* for electromagnetic field. Deformation of vector-potential (1.9) now can be written as

$$\hat{A}^i(\mathbf{r}) = A^i(\mathbf{r}) + \delta A^i(\mathbf{r}) + \ldots\,. \tag{1.12}$$

Principle of minimal action for fields. *Field functions determine actual configuration of physical fields if and only if they realize local minimum of action functional S in class of all variations with restricted support*.*

The condition of minimum of action for actual field configuration and for actual world lines of particles, as a rule, is not used. In order to derive dynamical equations for fields and particles it is sufficient to have extremum condition (no matter minimum, maximum, or saddle point). For this reason minimal action principle often is stated as *principle of extremal action.*

Exercise 1.1. *Verify that $h^i(s)$ in* (1.4) *are transformed as components of vector under the change of coordinates.*

---

* Variations with restricted support are those which are identically zero outside some restricted domain Ω.

Exercise 1.2. *Prove that under gauge transformations* (11.6) *from Chapter III action functional* (1.8) *is transformed as follows:*

$$S \to S - \sum_{i=1}^{N} \left(\frac{q_i}{c}\,\psi(\mathbf{r}(s_2(i))) - \frac{q_i}{c}\,\psi(\mathbf{r}(s_1(i)))\right). \tag{1.13}$$

Explain why terms added to action functional in (1.13) *are not sensitive to variation of word lines* (1.2).

§ 2. Motion of particle in electromagnetic field.

In order to find world line of relativistic particle in external electromagnetic field we shall apply particle version of extremal action principle to functional (1.8). Let's choose one of N particles in (1.8) and consider deformation (1.6) of its world line. When we substitute deformed world line into (1.8) in place of initial non-deformed one the value of last integral remains unchanged. Thereby in first term containing sum of integrals only one summand changes its value, that one which represent the particle we have chosen among others. Therefore writing extremity condition for (1.8) we can use action functional in form of (1.7). The value of (1.7) for deformed world line is calculated as follows:

$$S_{\text{def}} = -mc\int\limits_{s_1}^{s_2} \sqrt{g(\mathbf{K},\mathbf{K})}\,ds - \frac{q}{c}\int\limits_{s_1}^{s_2} g(\mathbf{A},\mathbf{K})\,ds. \tag{2.1}$$

Formula (2.1) visually differs from formula (1.7) because s is natural parameter on initial world line, but it is not natural parameter on deformed line. Here tangent vector

$$\mathbf{K}(s) = \frac{d\hat{\mathbf{r}}(s)}{ds} = \mathbf{u}(s) + \varepsilon\,\frac{d\hat{\mathbf{h}}(s)}{ds} + \ldots \tag{2.2}$$

is not unit vector. Therefore first integral (1.7) is rewritten as length integral (see (6.4) in Chapter III). In second integral (1.7) unit tangent vector is replaced by vector **K**.

Let's write in coordinate form both expressions which are under integration in (2.1) taking into account that we deal with general curvilinear coordinate system in Minkowsky space:

$$\begin{aligned}
&\sqrt{g(\mathbf{K},\mathbf{K})}=\sqrt{\sum_{i=0}^{3}\sum_{j=0}^{3}g_{ij}(\hat{r}(s))\,K^i(s)\,K^j(s)},\\
&g(\mathbf{A},\mathbf{K})=\sum_{i=0}^{3}A_i(\hat{\mathbf{r}}(s))\,K^i(s).
\end{aligned}\tag{2.3}$$

Let's substitute (2.2) into (2.3) and take into account (1.2) and the expansion (1.4). As a result for the expressions (2.3) we get the following power expansions with respect to small parameter ε:

$$\begin{aligned}
\sqrt{g(\mathbf{K},\mathbf{K})}&=\sqrt{g(\mathbf{u},\mathbf{u})}+\frac{\varepsilon}{\sqrt{g(\mathbf{u},\mathbf{u})}}\left(\sum_{i=0}^{3}u_i(s)\,\frac{dh^i(s)}{ds}+\right.\\
&\left.+\frac{1}{2}\sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3}\frac{\partial g_{ij}}{\partial r^k}\,u^i(s)\,u^j(s)\,h^k(s)\right)+\ldots,\\
g(\mathbf{A},\mathbf{K})&=g(\mathbf{A},\mathbf{u})+\varepsilon\sum_{i=0}^{3}A_i(\mathbf{r}(s))\,\frac{dh^i(s)}{ds}+\\
&+\varepsilon\sum_{i=0}^{3}\sum_{k=0}^{3}\frac{\partial A_i}{\partial r^k}\,u^i(s)\,h^k(s)+\ldots.
\end{aligned}$$

When substituting these expansions into (2.1) we should remember that $\mathbf{u}$ is unit vector. Then for S_{def} we get

$$S_{\text{def}} = S - \varepsilon \int\limits_{s_1}^{s_2} \sum_{k=0}^{3} \left(m\,c\,u_k(s) + \frac{q}{c}\,A_k(\mathbf{r}(s))\right)\,\frac{dh^k(s)}{ds}\,ds\,-$$

$$-\,\varepsilon \int\limits_{s_1}^{s_2} \sum_{k=0}^{3} \left(\frac{q}{c}\sum_{i=0}^{3}\frac{\partial A_i}{\partial r^k}\,u^i + \frac{mc}{2}\sum_{i=0}^{3}\sum_{j=0}^{3}\frac{\partial g_{ij}}{\partial r^k}u^i u^j\right) h^k(s)\,ds + \ldots\,.$$

Let's apply integration by parts to first integral above. As a result we get the expression without derivatives of functions $h^k(s)$:

$$S_{\text{def}} = S - \varepsilon \sum_{k=0}^{3} \left(m\,c\,u_k(s) + \frac{q}{c}\,A_k(\mathbf{r}(s))\right)\,h^k(s)\,\Biggr|_{s_1}^{s_2} +$$

$$+\,\varepsilon \int\limits_{s_1}^{s_2} \sum_{k=0}^{3} \frac{d}{ds}\left(m\,c\,u_k(s) + \frac{q}{c}\,A_k(\mathbf{r}(s))\right)\,h^k(s)\,ds\,-$$

$$-\,\varepsilon \int\limits_{s_1}^{s_2} \sum_{k=0}^{3} \left(\frac{q}{c}\sum_{i=0}^{3}\frac{\partial A_i}{\partial r^k}\,u^i + \frac{mc}{2}\sum_{i=0}^{3}\sum_{j=0}^{3}\frac{\partial g_{ij}}{\partial r^k}u^i u^j\right) h^k(s)\,ds + \ldots\,.$$

Remember that function $h^k(s)$ vanish at the ends of integration path $h^k(s_1) = h^k(s_2) = 0$ (see § 1). This provides vanishing of non-integral terms in the above formula for S_{def}.

Now in order to derive differential equations for world line of particle we apply extremity condition for S. It means that term linear with respect to ε in power expansion for S_{def} should be identically zero irrespective to the choice of functions $h^k(s)$:

$$\begin{aligned}&\frac{d}{ds}\left(m\,c\,u_k(s)+\frac{q}{c}\,A_k(\mathbf{r}(s))\right)=\\ &=\frac{q}{c}\sum_{i=0}^{3}\frac{\partial A_i}{\partial r^k}\,u^i+\frac{mc}{2}\sum_{i=0}^{3}\sum_{j=0}^{3}\frac{\partial g_{ij}}{\partial r^k}\,u^i\,u^j.\end{aligned}\tag{2.4}$$

Let's calculate derivative in left hand side of (2.4). Then let's rearrange terms so that those with $m\,c$ factor are in left hand side, while others with q/c factor are in right hand side:

$$mc\left(\frac{du_k}{ds}-\frac{1}{2}\sum_{i=0}^{3}\sum_{j=0}^{3}\frac{\partial g_{ij}}{\partial r^k}\,u^i\,u^j\right)=\frac{q}{c}\sum_{i=0}^{3}\left(\frac{\partial A_i}{\partial r^k}-\frac{\partial A_k}{\partial r^i}\right)u^i.$$

Now in right hand side of this equation we find tensor of electromagnetic field (see formula (9.9) in Chapter III). For transforming left hand side of this equation we use formula (11.3) from Chapter III. As a result we get the following equation for world line:

$$m\,c\left(\frac{du_k}{ds}-\sum_{i=0}^{3}\sum_{j=0}^{3}\Gamma^i_{kj}\,u_i\,u^j\right)=\frac{q}{c}\sum_{i=0}^{3}F_{ki}\,u^i.\tag{2.5}$$

In left hand side of the equation (2.5) we find covariant derivative with respect to parameter s along world line:

$$m\,c\,\nabla_s u_k=\frac{q}{c}\sum_{i=0}^{3}F_{ki}\,u^i.\tag{2.6}$$

Comparing (2.6) with the equations (11.9) from Chapter III, we get formula for the vector of four-dimensional force acting on a particle with charge q in electromagnetic field:

$$F_k = \frac{q}{c} \sum_{i=0}^{3} F_{ki}\, u^i. \tag{2.7}$$

Suppose that we have rectangular Cartesian coordinate system in Minkowsky space. Then we can subdivide $\mathbf{F}$ into spatial and temporal parts and can calculate components of three-dimensional force vector: $f^i = \sqrt{c^2 - |\mathbf{v}|^2}\, F^i$ (see formula (7.6) in Chapter III). Upon easy calculations with the use of formulas (7.2) and (9.4) from Chapter III for force vector $\mathbf{f}$ we get

$$\mathbf{f} = q\,\mathbf{E} + \frac{q}{c}\,[\mathbf{v},\,\mathbf{H}]. \tag{2.8}$$

This formula (2.8) is exactly the same as formula for Lorentz force (see (4.4) in Chapter I). Thus formula (2.7) is four-dimensional generalization of formula for Lorentz force. Orthogonality condition for 4-force and 4-velocity (see (7.5) in Chapter III) for (2.7) is fulfilled due to skew symmetry of tensor of electromagnetic field.

Exercise 2.1. *Prove that gauge transformation of action functional* (1.13) *does not change dynamic equations of material point in electromagnetic field* (2.6).

Exercise 2.2. *Verify that the relationship* (7.5) *from Chapter III holds for Lorentz force.*

§ 3. Dynamics of dust matter.

Differential equation (2.6) describes motion of charged particles in electromagnetic field. If the number of particles is not large, then we can follow after the motion of each of them. When describing extremely large number of particles continual limit is used, particles are replaced by continuous medium modeling their collective behavior. Simplest model describing large number of non-colliding particles is a model of *dust cloud*. In this model

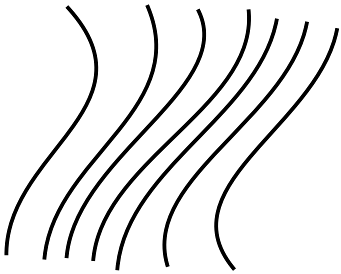

Fig. 3.1

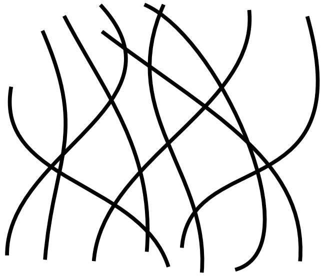

Fig. 3.2

particles of cloud move regularly (not chaotically). Their world lines can be modeled by regular family of lines filling the whole space (see Fig. 3.1).

Another model is a model of *ideal gas*. Here particles also do not collide each other, i. e. their world lines do not intersect. However, their motion is chaotic (see Fig. 3.2). Therefore if we fill the whole space with their world lines, they would intersect.

Besides two models considered mentioned above, there are models describing liquids and solid materials. Points of liquid and solid media move regularly (as on Fig. 3.1). However, in these media interaction of particles is essential. Therefore when describing such media one should either use detailed microscopic analysis and get macroscopic parameters by statistical averaging, or should use some heuristic assumptions based on experiment.

In this book we consider only most simple model of dust cloud. In this case one should assume Minkowsky space to be filled by regular family of world lines. Some of them are world lines of real particles, others are imaginary ones obtained by extrapolation in continual limit. Therefore at each point of M we have unit vector $\mathbf{u}$, this is tangent vector to world line passing through this point. This means that dynamics of dust cloud can be described by vector field $\mathbf{u}(\mathbf{r})$.

Apart from vector field $\mathbf{u}$, below we need scalar parameter $\nu(\mathbf{r})$ which means the density of dust cloud. We define it as follows. Let's choose some small fragment of three-dimensional hypersurface in M orthogonal to vector $\mathbf{u}(\mathbf{r})$ at the point $\mathbf{r}$. The number of dust particles whose world lines cross this fragment is proportional to its three-dimensional volume: $N = \nu(\mathbf{r})\,V$, parameter $\nu(\mathbf{r})$ is coefficient of proportionality. Parameter $\nu(\mathbf{r})$ has measure unit of concentration, it can be treated as concentration of particles in small fragment of dust cloud near the point $\mathbf{r}$ measured in that inertial coordinate system for which particles of this small fragment are at rest. By means of $\nu(\mathbf{r})$ and $\mathbf{u}(\mathbf{r})$ we compose new four-dimensional vector

$$\boldsymbol{\eta}(\mathbf{r}) = c\,\nu(\mathbf{r})\,\mathbf{u}(\mathbf{r}). \tag{3.1}$$

Vector (3.1) is called four-dimensional *flow density* for particles in cloud. If we choose some inertial coordinate system, then η^0/c is interpreted as concentration of particles in dust cloud, while other three components of four-dimensional vector $\boldsymbol{\eta}$ form three-dimensional vector of flow density.

Suppose that dust cloud is formed by identical particles with mass m and charge q. Then four-dimensional current density vector can be represented as follows:

$$\mathbf{j}(\mathbf{r}) = q\,\boldsymbol{\eta}(\mathbf{r}). \tag{3.2}$$

By analogy with (3.2) one can define *mass flow density vector*:

$$\boldsymbol{\mu}(\mathbf{r}) = m\,\boldsymbol{\eta}(\mathbf{r}). \tag{3.3}$$

Total number of particles in cloud is fixed. This conservation law is written as the following equality for $\boldsymbol{\eta}$:

$$\sum_{p=0}^{3} \nabla_p \eta^p = 0. \tag{3.4}$$

From (3.4) and (3.2) one can derive charge conservation law in form of the relationship (11.7) from Chapter III. Taking into account (3.3), we get rest mass conservation law:

$$\sum_{p=0}^{3} \nabla_p \mu^p = 0. \tag{3.5}$$

Rest mass conservation law here is fulfilled due to the absence of collisions when heavy particles can be produced from light ones (see § 7 in Chapter III).

Let's consider dynamics of particles composing dust cloud. Vector field $\mathbf{u}$ is constituted by tangent vectors to world lines of dust particles. Therefore these world lines can be determined as integral curves of vector field $\mathbf{u}$, i. e. by solving the following system of ordinary differential equations:

$$\frac{dr^i}{ds} = u^i(\mathbf{r}(s)), \ i = 0, \ldots, 3. \tag{3.6}$$

Having determined world line of particle from differential equations, we know vector of its 4-velocity $\mathbf{u}(s)$. Now let's calculate covariant derivative of vector $\mathbf{u}(s)$ with respect to parameter s:

$$\nabla_s u^p = \frac{du^p(s)}{ds} + \sum_{k=0}^{3} \sum_{n=0}^{3} \Gamma^p_{nk}\, u^k(s)\, u^n(s). \tag{3.7}$$

Calculating derivative du^p/ds in (3.7) we take into account (3.6) and the equality $\mathbf{u}(s) = \mathbf{u}(\mathbf{r}(s))$. As a result we get

$$\frac{du^p(s)}{ds} = \sum_{k=0}^{3} u^k \, \frac{\partial u^p}{\partial r^k}. \tag{3.8}$$

Substituting (3.8) into (3.7), we derive the following formula:

$$\nabla_s u^p = \sum_{k=0}^{3} u^k \, \nabla_k u^p. \tag{3.9}$$

Right hand side of (3.9) is covariant derivative of vector field $\mathbf{u}(\mathbf{r})$ along itself (see more details in [3]). Substituting (3.9) into the equations of the dynamics of material point, we get:

$$\nabla_{\mathbf{u}} \mathbf{u} = \frac{\mathbf{F}}{mc}. \tag{3.10}$$

Here $\mathbf{F} = \mathbf{F}(\mathbf{r}, \mathbf{u})$ is some external force field acting on particles of dust matter. For example in the case of charged dust in electromagnetic field the equation (3.10) looks like

$$\sum_{k=0}^{3} u^k \, \nabla_k u_p = \frac{q}{mc^2} \sum_{k=0}^{3} F_{pk} \, u^k. \tag{3.11}$$

In contrast to the equations (11.9) from Chapter III, which describe dynamics of separate particle, here (3.10) are partial differential equations with respect to components of vector field $\mathbf{u}(\mathbf{r})$. They describe dynamics of dust cloud in continual limit. The equation for scalar field $\nu(\mathbf{r})$ is derived from conservation law

(3.4) for the number of particles. Combining these two equations, we obtain a system of differential equations:

$$\begin{aligned}&\sum_{k=0}^{3} u^k\,\nabla_k u_p = \frac{F_p}{mc},\\ &\sum_{k=0}^{3} u^k\,\nabla_k \nu = -\nu\sum_{k=0}^{3}\nabla_k u^k.\end{aligned}\tag{3.12}$$

System of partial differential equations (3.12) yields complete description for the dynamics of dust cloud.

Model of dust matter cam be generalized a little bit. We can consider mixture of particles of different sorts. For each sort of particles we define its own vector field $\mathbf{u}(i,\mathbf{r})$ and its own scalar field of concentration $\nu(i,\mathbf{r})$. Then formulas (3.2) and (3.3) for $\mathbf{j}$ and $\boldsymbol{\mu}$ are generalized as follows:

$$\mathbf{j}(\mathbf{r}) = \sum_{i=1}^{n} q(i)\,\boldsymbol{\eta}(i,\mathbf{r}), \qquad \boldsymbol{\mu}(\mathbf{r}) = \sum_{i=1}^{n} m(i)\,\boldsymbol{\eta}(i,\mathbf{r}).$$

Here $\boldsymbol{\eta}(i,\mathbf{r}) = c\,\nu(i,\mathbf{r})\,\mathbf{u}(i,\mathbf{r})$. Each pair of fields $\mathbf{u}(i,\mathbf{r})$ and $\nu(i,\mathbf{r})$ satisfies differential equations (3.12). We can derive mass and charge conservation laws from these equations.

§ 4. Action functional for dust matter.

Let's study the dynamics of dust matter in electromagnetic field within framework of Lagrangian formalism. Fort this purpose we need to pass to continual limit in action functional (1.8). For the sake of simplicity we consider dust cloud with identical particles. Omitting details of how it was derived, now we write ultimate formula for action functional (1.8) in continual limit:

$$\begin{aligned} S = -m\int\limits_{V_1}^{V_2}\sqrt{g(\boldsymbol\eta,\boldsymbol\eta)}\,\sqrt{-\det g}\;d^4r\,-\\ -\frac{q}{c^2}\int\limits_{V_1}^{V_2} g(\boldsymbol\eta,\mathbf A)\sqrt{-\det g}\;d^4r\,-\\ -\frac{1}{16\pi\,c}\int\limits_{V_1}^{V_2}\sum_{p=0}^{3}\sum_{k=0}^{3}F_{pk}\,F^{pk}\sqrt{-\det g}\;d^4r. \end{aligned}\tag{4.1}$$

Instead of deriving formula (4.1) from (1.8) we shall verify this formula indirectly. For this purpose we shall derive dynamical equation (3.11) from principle of extremal action applied to action functional (4.1).

For describing dust matter in (4.1) we have chosen vector field $\boldsymbol\eta(\mathbf r)$ defined in (3.1). Other two fields $\mathbf u(\mathbf r)$ and $\nu(\mathbf r)$ can be expressed though vector field $\boldsymbol\eta(\mathbf r)$:

$$c\,\nu=|\boldsymbol\eta|=\sqrt{g(\boldsymbol\eta,\boldsymbol\eta)}\,,\qquad\qquad \mathbf u=\frac{\boldsymbol\eta}{c\,\nu}.\tag{4.2}$$

Dealing with variation of vector field $\boldsymbol\eta(\mathbf r)$ we should always remember that components of this field are not independent functions. They satisfy differential equation (3.4). In order to resolve tis equation (3.4) we use slightly modified version of theorem 10.1 from Chapter III.

Theorem 4.1. *Let M be some n-dimensional manifold, where $n\geq 2$, equipped with metric g_{ij}. For each vector field $\boldsymbol\eta$ with zero* *divergency with respect to metric connection*

$$\sum_{p=1}^{n}\nabla_p\eta^p=0 \tag{4.3}$$

there is skew-symmetric tensor field $\boldsymbol{\varphi}$ *of type* $(2,0)$ *such that the following relationships are fulfilled*

$$\eta^p=\sum_{q=1}^{n}\nabla_q\varphi^{pq}. \tag{4.4}$$

Proof. Writing relationships (4.3), we use well-known formula for components of metric connection, see formula (11.3) in Chapter III. As a result we get

$$\begin{aligned}\sum_{p=1}^{n}\nabla_p\eta^p&=\sum_{p=1}^{n}\frac{\partial\eta^p}{\partial r^p}+\sum_{p=1}^{n}\sum_{s=1}^{n}\Gamma^p_{ps}\,\eta^s=\sum_{p=1}^{n}\frac{\partial\eta^p}{\partial r^p}+\\&+\frac{1}{2}\sum_{p=1}^{n}\sum_{s=1}^{n}\sum_{k=1}^{n}g^{pk}\left(\frac{\partial g_{pk}}{\partial r^s}+\frac{\partial g_{ks}}{\partial r^p}-\frac{\partial g_{ps}}{\partial r^k}\right)\eta^s.\end{aligned}$$

Note that last two derivatives of metric tensor in round brackets are canceled when we sum over indices p and k. This is because g^{pk} is symmetric. Hence

$$\begin{aligned}\sum_{p=1}^{n}\nabla_p\eta^p&=\sum_{p=1}^{n}\frac{\partial\eta^p}{\partial r^p}+\frac{1}{2}\sum_{p=1}^{n}\sum_{s=1}^{n}\sum_{k=1}^{n}g^{sk}\,\frac{\partial g_{ks}}{\partial r^p}\eta^p=\\&=\sum_{p=1}^{n}\frac{\partial\eta^p}{\partial r^p}+\frac{1}{2}\sum_{p=1}^{n}\operatorname{tr}\left(g^{-1}\,\frac{\partial g}{\partial r^p}\right)\eta^p.\end{aligned} \tag{4.5}$$

For further transforming of this expression (4.5) we use well known formula for logarithmic derivative of determinant:

$$\frac{\partial \ln|\det g|}{\partial r^p} = \operatorname{tr}\left(g^{-1}\,\frac{\partial g}{\partial r^p}\right). \tag{4.6}$$

Substituting (4.6) into (4.5), we transform (4.5) so that

$$\sum_{p=1}^{n} \nabla_p \eta^p = \frac{1}{\sqrt{|\det g|}} \sum_{p=1}^{n} \frac{\partial(\eta^p \sqrt{|\det g|})}{\partial r^p}. \tag{4.7}$$

Let's carry out analogous calculations for right hand side of (4.4) taking into account skew symmetry of the field φ^{pq} and symmetry of connection components Γ^k_{pq}. These calculations yield

$$\sum_{q=1}^{n} \nabla_q \varphi^{pq} = \frac{1}{\sqrt{|\det g|}} \sum_{q=1}^{n} \frac{\partial(\varphi^{pq} \sqrt{|\det g|})}{\partial r^q}. \tag{4.8}$$

Denote $j^p = \sqrt{|\det g|}\,\eta^p$ and $\psi^{pq} = \sqrt{|\det g|}\,\varphi^{pq}$. Now on the base of (4.7) and (4.8) it is easy to understand that proof of theorem 4.1 is reduced to theorem 10.1 from Chapter III. □

Remark. Generally speaking, theorem 10.2 has no direct generalization for the case of spaces with metric. It is generalized only for metric spaces with zero curvature tensor $R^s_{kpq} = 0$.

Let's define deformation of the field $\boldsymbol{\eta}$ in a way similar to that we used for vector-potential $\mathbf{A}$ in § 1:

$$\hat{\eta}^p(\mathbf{r}) = \eta^p(\mathbf{r}) + \varepsilon\,\zeta^p(\mathbf{r}) + \ldots. \tag{4.9}$$

Both fields $\hat{\boldsymbol{\eta}}$ and $\boldsymbol{\eta}$ satisfy differential equation (3.4). Hence vector field $\boldsymbol{\zeta}$ defined in (4.9) also satisfy this equation. Let's apply theorem 4.1 to vector field $\boldsymbol{\zeta}$:

$$\zeta^p=\sum_{k=0}^{3}\nabla_k\varphi^{pk}. \tag{4.10}$$

Theorem 4.1 does not specify tensor field φ^{pk} in (4.10), this can be any skew-symmetric tensor field. However, we choose it in very special form as follows:

$$\varphi^{pk}=\eta^p\,h^k-h^p\,\eta^k. \tag{4.11}$$

This choice can be motivated by the following theorem.

Theorem 4.2. *For any two vector fields $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$, where $\boldsymbol{\eta}\neq 0$, both satisfying differential equation* (3.4) *there is vector field* $\mathbf{h}$ *such that vector field $\boldsymbol{\zeta}$ is given by formula*

$$\zeta^p=\sum_{k=0}^{3}\nabla_k(\eta^p\,h^k-h^p\,\eta^k).$$

Our choice (4.11) leads to the following expression for the field $\hat{\boldsymbol{\eta}}$:

$$\hat{\eta}^p(\mathbf{r})=\eta^p(\mathbf{r})+\varepsilon\sum_{k=0}^{3}\nabla_k(\eta^p\,h^k-h^p\,\eta^k)+\ldots\,. \tag{4.12}$$

Quantities $h^i(\mathbf{r})$ in (4.12) are chosen to be smooth functions being nonzero only within some restricted domain Ω in Minkowsky space.

When substituting (4.12) into action functional (4.1) we use the following expansion for $\sqrt{g(\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\eta}})}$:

$$\sqrt{g(\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\eta}})}=\sqrt{g(\boldsymbol{\eta},\boldsymbol{\eta})}+\frac{\varepsilon}{\sqrt{g(\boldsymbol{\eta},\boldsymbol{\eta})}}\sum_{p=0}^{3}\sum_{q=0}^{3}\eta_p\,\nabla_k\varphi^{pk}+\ldots\,.$$

We have analogous power expansion for the expression under second integral in formula (4.1):

$$g(\hat{\boldsymbol{\eta}},\mathbf{A})=g(\boldsymbol{\eta},\mathbf{A})+\varepsilon\sum_{p=0}^{3}\sum_{k=0}^{3}A_p\,\nabla_k\varphi^{pk}+\ldots\,.$$

Substituting these two expansions into (4.1), we take into account (4.2). For the action S_{def} this yields

$$\begin{aligned}S_{\text{def}}=S-\varepsilon\,m\int\limits_{\Omega}\sum_{p=0}^{3}\sum_{k=0}^{3}u_p\,\nabla_k\varphi^{pk}\sqrt{-\det g}\,d^4r-\\-\frac{\varepsilon\,q}{c^2}\int\limits_{\Omega}\sum_{p=0}^{3}\sum_{k=0}^{3}A_p\,\nabla_k\varphi^{pk}\sqrt{-\det g}\,d^4r+\ldots\,.\end{aligned}\tag{4.13}$$

Further in order to transform the above expression (4.13) we use Ostrogradsky-Gauss formula. In the space equipped with metric this formula is written as follows:

$$\int\limits_{\Omega}\sum_{k=0}^{3}\nabla_k z^k\sqrt{-\det g}\,d^4r=\int\limits_{\partial\Omega}g(\mathbf{z},\mathbf{n})\,dV.\tag{4.14}$$

Here z^0, z^1, z^2, z^3 are components of smooth vector field $\mathbf{z}$, while $\mathbf{n}$ is unit normal vector for the boundary of the domain Ω. In order to transform first integral in formula (4.13) we take $z^k=\sum_{p=0}^{3}u_p\,\varphi^{pk}$. Then in right hand side of (4.14) we obtain

$$\sum_{k=0}^{3}\nabla_k z^k=\sum_{p=0}^{3}\sum_{k=0}^{3}u_p\,\nabla_k\varphi^{pk}+\sum_{p=0}^{3}\sum_{k=0}^{3}\nabla_k u_p\,\varphi^{pk}.$$

Right hand side of (4.14) vanishes since φ^{pk} do vanish on the boundary of Ω. Hence we have the equality

$$\int\limits_{\Omega}\sum_{p=0}^{3}\sum_{k=0}^{3} u_p\,\nabla_k\varphi^{pk}\sqrt{-\det g}\,d^4r=$$
$$-\int\limits_{\Omega}\sum_{p=0}^{3}\sum_{k=0}^{3}\nabla_k u_p\,\varphi^{pk}\sqrt{-\det g}\,d^4r.$$

In a similar way we transform second integral in (4.13). In whole for the action S_{def} we get the following expression

$$\begin{aligned} S_{\text{def}}=S+\varepsilon\,m\int\limits_{\Omega}\sum_{p=0}^{3}\sum_{k=0}^{3}\nabla_k u_p\,\varphi^{pk}\sqrt{-\det g}\,d^4r\,+\\ +\,\frac{\varepsilon\,q}{c^2}\int\limits_{\Omega}\sum_{p=0}^{3}\sum_{k=0}^{3}\nabla_k A_p\,\varphi^{pk}\sqrt{-\det g}\,d^4r+\ldots\,. \end{aligned}\tag{4.15}$$

Extremity of action S means that linear part with respect to ε in formula (4.15) should vanish:

$$\int\limits_{\Omega}\sum_{p=0}^{3}\sum_{k=0}^{3}\Bigl(m\nabla_k u_p+\frac{q}{c^2}\nabla_k A_p\Bigr)\,\varphi^{pk}\sqrt{-\det g}\,d^4r=0.$$

Let's substitute formula (4.11) for φ^{pk} into the above equality. Then it is transformed to the following one

$$\int\limits_{\Omega}\sum_{p=0}^{3}\sum_{k=0}^{3}\Bigl(m\nabla_k u_p+\frac{q}{c^2}\nabla_k A_p\Bigr)\,\eta^p h^k\,\sqrt{-\det g}\,d^4r=$$

$$= \int\limits_{\Omega} \sum_{p=0}^{3} \sum_{k=0}^{3} \left(m\nabla_k u_p + \frac{q}{c^2}\nabla_k A_p\right) \eta^k h^p \sqrt{-\det g}\, d^4 r.$$

Let's exchange indices k and p in second integral. Thereafter integrals can be united into one integral:

$$\int\limits_{\Omega} \sum_{k=0}^{3} \sum_{p=0}^{3} \left(m\nabla_k u_p - m\nabla_p u_k + \frac{q}{c^2}\nabla_k A_p - \right.$$
$$\left. - \frac{q}{c^2}\nabla_p A_k\right) \eta^p h^k \sqrt{-\det g}\, d^4 r = 0. \tag{4.16}$$

Now let's take into account that in resulting equality $h^k = h^k(\mathbf{r})$ are arbitrary smooth functions vanishing outside the domain Ω. Therefore vanishing of integral (4.16) means vanishing of each summand in sum over index k in the expression under integration:

$$\sum_{p=0}^{3} \left(m\,\nabla_k u_p - m\,\nabla_p u_k + \frac{q}{c^2}\,F_{kp}\right) \eta^p = 0. \tag{4.17}$$

Here we used the relationship (11.5) from Chapter III. It relates tensor of electromagnetic field and four-dimensional vector-potential.

In order to bring the equation (4.17) just derived to its ultimate form we use the relationships (4.2), which relate vector field $\boldsymbol{\eta}$ and vector field $\mathbf{u}$: $\eta^p = c\,\nu\,u^p$. Since $\mathbf{u}$ is unit vector, we have

$$\sum_{p=0}^{3} u^p \nabla_k u_p = 0. \tag{4.18}$$

Taking into account (4.18), we bring (4.17) to the following form:

$$\sum_{p=0}^{3} u^p \nabla_p u_k = \frac{q}{mc^2} \sum_{p=0}^{3} F_{kp}\, u^p. \tag{4.19}$$

Now it is easy to see that (4.19) exactly coincides with the equation (3.11), which we have derived earlier. This result approves the use of the action (4.1) for describing charged dust matter in electromagnetic field.

Exercise 4.1. *Prove that for any skew-symmetric tensor field φ^{pq} vector field $\boldsymbol{\eta}$ determined by formula* (4.4) *has zero divergency, i. e. is satisfies differential equation* (3.4).

Exercise 4.2. *Prove theorem 4.2. For this purpose use the following fact known as theorem on rectification of vector field.*

Theorem 4.3. *For any vector field $\boldsymbol{\eta} \neq 0$ there exists some curvilinear coordinate system r^0, r^1, r^2, r^3 such that $\eta^0 = 1$, $\eta^1 = 0$, $\eta^2 = 0$, $\eta^3 = 0$ in this coordinate system.*

Exercise 4.3. *Prove theorem 4.3 on rectification of vector field.*

Exercise 4.4. *Derive Ostrogradsky-Gauss formula* (4.14) *for the space equipped with metric on the base of the following integral relationship in standard space $\mathbb{R}^n$:*

$$\int\limits_{\Omega} \frac{\partial f(\mathbf{r})}{\partial r^i} d^n r = \int\limits_{\partial\Omega} f(\mathbf{r})\, dr^1 \ldots dr^{i-1} dr^{i+1} \ldots dr^n.$$

§ 5. Equations for electromagnetic field.

In this section we continue studying action functional (4.1). This functional describes dust cloud composed of particles with mass m and charge q in electromagnetic field. In previous section we have found that applying extremal action principle to S with respect to the field $\boldsymbol{\eta}$ one can derive dynamical equations for velocity field in dust cloud. Now we shall apply extremal action principle to S with respect to vector-potential $\mathbf{A}$. Deformation of vector-potential is defined according to (1.9), (1.10), (1.11), (1.12):

$$\hat{A}_i(\mathbf{r}) = A_i(\mathbf{r}) + \varepsilon h_i(\mathbf{r}) + \ldots . \tag{5.1}$$

For components of tensor of electromagnetic field we derive

$$\hat{F}_{ij} = F_{ij} + \varepsilon\,(\nabla_i h_j - \nabla_j h_i) + \ldots . \tag{5.2}$$

When substituting (5.2) into action functional (4.1) we carry out the following calculations:

$$\begin{aligned}\sum_{p=0}^{3}\sum_{k=0}^{3}\hat{F}_{pk}\,\hat{F}^{pk} &= \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{p=0}^{3}\sum_{k=0}^{3}\hat{F}_{pk}\,\hat{F}_{ij}\,g^{pi}\,g^{kj} = \\ &= \sum_{p=0}^{3}\sum_{k=0}^{3}F_{pk}\,F^{pk} + 2\,\varepsilon\sum_{p=0}^{3}\sum_{k=0}^{3}F^{pk}(\nabla_p h_k - \nabla_k h_p) + \ldots .\end{aligned}$$

Taking into account skew symmetry of tensor F^{pk}, this expansion can be simplified more and can be brought to the form

$$\sum_{p=0}^{3}\sum_{k=0}^{3}\hat{F}_{pk}\,\hat{F}^{pk} = \sum_{p=0}^{3}\sum_{k=0}^{3}F_{pk}\,F^{pk} + 4\,\varepsilon\sum_{p=0}^{3}\sum_{k=0}^{3}F^{pk}\nabla_p h_k + \ldots .$$

Analogous calculations in substituting (5.1) into (4.1) yield

$$g(\boldsymbol{\eta}, \hat{\mathbf{A}}) = g(\boldsymbol{\eta}, \mathbf{A}) + \varepsilon\sum_{k=0}^{3}\eta^k\,h_k + \ldots .$$

As a result for deformation of action functional (4.1) we get

$$S_{\text{def}} = S - \frac{\varepsilon q}{c^2} \int\limits_{\Omega} \sum_{k=0}^{3} \eta^k \, h_k \sqrt{-\det g} \, d^4 r -$$

$$- \frac{\varepsilon}{4\pi \, c} \int\limits_{\Omega} \sum_{p=0}^{3} \sum_{k=0}^{3} F^{pk} \nabla_p h_k \sqrt{-\det g} \, d^4 r + \ldots .$$

Let's transform second integral in the above expansion for S_{def} by means of Ostrogradsky-Gauss formula (4.14). For this purpose let's choose $z^p = \sum_{k=0}^{3} F^{pk} h_k$ and take into account vanishing of h_k on the boundary of the domain Ω. Then for S_{def} we get

$$S_{\text{def}} = S + \varepsilon \int\limits_{\Omega} \sum_{k=0}^{3} \left(-\frac{q\eta^k}{c^2} + \sum_{p=0}^{3} \frac{\nabla_p F^{pk}}{4\pi \, c} \right) h_k \sqrt{-\det g} \, d^4 r + \ldots .$$

Extremal action principle means that linear in ε part of the above expansion for S_{def} should vanish. Note also that Ω is an arbitrary domain and $h_k(\mathbf{r})$ are arbitrary functions within Ω. This yield the following equations for the tensor of electromagnetic field:

$$\sum_{p=0}^{3} \nabla_p F^{pk} = \frac{4\pi \, q}{c} \, \eta^k. \tag{5.3}$$

Remember that $\boldsymbol{\eta}(\mathbf{r})$ is related to current density by means of (3.2). Then (5.3) can be written as

$$\sum_{p=0}^{3} \nabla_p F^{kp} = -\frac{4\pi}{c} \, j^k. \tag{5.4}$$

It is easy to see that (5.4) are exactly Maxwell equations written in four-dimensional form (see (11.4) in Chapter III). Another pair of Maxwell equations written in four-dimensional form

$$\sum_{q=0}^{3}\sum_{k=0}^{3}\sum_{s=0}^{3}\omega^{pqks}\,\nabla_q F_{ks}=0$$

is a consequence of the relationship $F_{pq} = \nabla_p A_q - \nabla_q A_p$ (see formula (11.5) in Chapter III).

Exercise 5.1. *Which form will have differential equations* (5.3) *if we consider dust cloud composed by particles of several sorts with masses* $m(1), \ldots, m(N)$ *and charges* $q(1), \ldots, q(N)$ *? Will this change differential equations* (5.4) *?*

CHAPTER V

GENERAL THEORY OF RELATIVITY

§ 1. Transition to non-flat metrics and curved Minkowsky space.

Passing from classical electrodynamics to special theory of relativity, in previous two chapters we have successively geometrized many basic physical concepts. Having denoted $r^0 = ct$ and combining r^0 with components of three-dimensional radius-vector in inertial coordinate system, we have constructed four-dimensional space of events (Minkowsky space). This space appears to be equipped with metric of signature $(1, 3)$, which is called Minkowsky metric. Thereby inertial coordinate systems are interpreted as orthonormal bases in Minkowsky metric.

In four-dimensional formalism dynamics of material point is described by vectorial differential equations, while Maxwell equations for electromagnetic field are written in tensorial form. Due to this circumstance in previous two chapters we managed to include into consideration skew-angular and even curvilinear coordinate systems in Minkowsky space. Thereby we got explicit entries of metric tensor components g_{ij}, metric connection components Γ^k_{ij}, and covariant derivatives ∇_i in all our equations.

Next step in this direction is quite natural. One should keep the shape of all equations and pass from flat Minkowsky metric to metric of signature $(1,3)$ with nonzero curvature tensor:

$$R^k_{qij} = \frac{\partial \Gamma^k_{jq}}{\partial r^i} - \frac{\partial \Gamma^k_{iq}}{\partial r^j} + \sum_{s=0}^{3} \Gamma^k_{is}\,\Gamma^s_{jq} - \sum_{s=0}^{3} \Gamma^k_{js}\,\Gamma^s_{iq}. \tag{1.1}$$

This crucial step was first made by Einstein. Theory he had discovered in this way later was called *Einstein's theory of gravitation* or *general theory of relativity.*

Definition 1.1. Four-dimensional affine space equipped with orientation, polarization, and with metric of signature $(1,3)$ and nonzero curvature (1.1) is called *curved Minkowsky space.*

In non-flat Minkowsky space we loose some structures available in flat case. In such space there are no coordinates for which Minkowsky metric is given by matrix (2.7) from Chapter III, i. e. here we have no inertial coordinate systems. This is substantial loss, but it is not catastrophic since dynamic equation for material points and Maxwell equations rewritten in vectorial and tensorial form are not bound to inertial coordinate systems.

Geodesic lines in curved Minkowsky space do not coincide with affine straight lines. Therefore affine structure becomes excessive restriction in general relativity. As appears, one can give up topologic structure of flat space $\mathbb{R}^4$ as well. Even in two-dimensional case, as we know, apart from deformed (curved) plain, there are surfaces with more complicated topology: sphere, torus and sphere with several handles glued to it (see [5]). In multidimensional case these objects are generalized in concept of *smooth manifold* (see details in [2], [5], and [6]).

Smooth manifold M of dimension n is a topologic space each point of which has a neighborhood (*a chart*) identical to some neighborhood of a point in $\mathbb{R}^n$. In other words M is covered by a family of charts U_α, each of which is diffeomorphic to some open set V_α in $\mathbb{R}^n$. Such chart maps (chart diffeomorphisms) define local curvilinear coordinate systems within their chart domains U_α. At those points of manifold M, where two chart domains are overlapping, transition functions arise. They relate one curvilinear coordinate system with another:

$$\begin{aligned} &\tilde{r}^i = \tilde{r}^i(r^1, \ldots, r^n), \quad \text{where} \quad i = 1, \ldots, n, \\ &r^i = r^i(\tilde{r}^1, \ldots, \tilde{r}^n), \quad \text{where} \quad i = 1, \ldots, n. \end{aligned} \tag{1.2}$$

According to definition of smooth manifold, transition functions (1.2) are smooth functions (of class C^∞). Transition functions determine transition matrices S and T:

$$T^i_j = \frac{\partial \tilde{r}^i}{\partial r^j}, \qquad\qquad S^i_j = \frac{\partial r^i}{\partial \tilde{r}^j}. \tag{1.3}$$

Presence of transition matrices (1.3) lead to full-scale theory of tensors, which is almost literally the same as theory of tensors for curvilinear coordinates in $\mathbb{R}^n$ (see [3]). The only difference is that here we cannot choose Cartesian coordinates at all. This is because in general there is no smooth diffeomorphic map from manifold M to $\mathbb{R}^n$.

Definition 1.1. Four-dimensional smooth manifold equipped with orientation, polarization, and with metric of signature $(1, 3)$ is called *generalized Minkowsky space* or *Minkowsky manifold*.

§ 2. Action for gravitational field. Einstein equation.

Space of events in general relativity is some smooth Minkowsky manifold M. This circumstance provides additional arbitrariness consisting in choosing M and in choosing metric on M. Nonzero curvature described by tensor (1.1) is interpreted as *gravitational field*. Gravitational field acts upon material bodies and upon electromagnetic field enclosed within M. This action is due to the presence of covariant derivatives in dynamic equations. The magnitude of gravitational field itself should be determined by presence of matter in M in form of massive particles or in form of electromagnetic radiation, i. e. we should have backward relation between geometry of the space and its content.

In order to describe backward relation between gravitational field and other physical fields we use Lagrangian formalism and extremal action principle. Let's start from action functional (4.1) in Chapter IV. It is sum of three integrals:

$$S = S_{\text{mat}} + S_{\text{int}} + S_{\text{el}}. \tag{2.1}$$

First integral S_{mat} is responsible for material particles in form of dust cloud, second integral describes interaction of dust cloud and electromagnetic field, third term in (2.1) describes electromagnetic field itself. In order to describe gravitational field one more summand in (2.1) is added:

$$S = S_{\text{gr}} + S_{\text{mat}} + S_{\text{int}} + S_{\text{el}}. \tag{2.2}$$

This additional term is chosen in the following form:

$$S_{\text{gr}} = -\frac{c^3}{16\pi\gamma} \int\limits_{V_1}^{V_2} R\,\sqrt{-\det g}\; d^4r. \tag{2.3}$$

Here γ gravitational constant same as in Newton's universal law of gravitation (see formula (1.11) in Chapter I). Scalar quantity R in (2.3) is *scalar curvature* determined by curvature tensor (1.1) according to the following formula:

$$R = \sum_{q=0}^{3}\sum_{k=0}^{3}\sum_{j=0}^{3} g^{qj}\,R^k_{qkj}. \tag{2.4}$$

Ricci tensor is an intermediate object relating curvature tensor (1.1) and scalar quantity (2.4). Here are its components:

$$R_{qj} = \sum_{k=0}^{3} R^{k}_{qkj}. \tag{2.5}$$

Ricci tensor is symmetric (see [3]). Scalar curvature R is obtained by contracting Ricci tensor and metric tensor g^{qj} with respect to both indices q and j. This fact is obvious due to (2.5) and (2.4).

Note that sometimes in the action for gravitational field (2.3) one more constant parameter Λ is added:

$$S_{\text{gr}} = -\frac{c^3}{16\pi\gamma} \int\limits_{V_1}^{V_2} (R + 2\,\Lambda)\,\sqrt{-\det g}\;d^4 r.$$

This parameter is called *cosmological constant*. However, according to contemporary experimental data the value of this constant is undetectably small or maybe is exactly equal to zero. Therefore further we shall use action S_{gr} in form of (2.3).

Note also that metric tensor describing gravitational field enters in implicit form into all summand in (2.2). Therefore we need not add special terms describing interaction of gravitational field with material particles and electromagnetic field. Moreover, such additional terms could change the form of dynamical equations for matter and form of Maxwell equations for electromagnetic field thus contradicting our claim that these equations are the same in general and in special relativity.

Now let's begin with deriving dynamical equations for gravitational field. For this purpose we consider deformation of components of metric tensor given by the following relationship:

$$\hat{g}^{ij}(\mathbf{r}) = g^{ij}(\mathbf{r}) + \varepsilon\, h^{ij}(\mathbf{r}) + \ldots\,. \tag{2.6}$$

Functions $h^{ij}(\mathbf{r})$ in (2.6) are assumed to be smooth functions vanishing outside some restricted domain $\Omega \subset M$. Deformation of matrix g^{ij} lead to deformation of inverse matrix g_{ij}:

$$\begin{aligned}\hat{g}_{ij} &= g_{ij} - \varepsilon\, h_{ij} + \ldots = \\ &= g_{ij} - \varepsilon \sum_{p=0}^{3} \sum_{q=0}^{3} g_{ip}\, h^{pq}\, g_{qj} + \ldots .\end{aligned} \tag{2.7}$$

Let's differentiate the relationship (2.7) and let's express partial derivatives through covariant derivatives in resulting formula:

$$\begin{aligned}\frac{\partial \hat{g}_{ij}}{\partial r^k} = \frac{\partial g_{ij}}{\partial r^k} - \varepsilon\, \frac{\partial h_{ij}}{\partial r^k} + \ldots = \frac{\partial g_{ij}}{\partial r^k} - \varepsilon\, \nabla_k h_{ij} + \\ + \varepsilon \sum_{p=0}^{3} \Gamma^p_{ki}\, h_{pj} + \varepsilon \sum_{p=0}^{3} \Gamma^p_{kj}\, h_{ip} + \ldots .\end{aligned} \tag{2.8}$$

In (2.8) we used covariant derivatives corresponding to non-deformed metric g_{ij}. Now on the base of (2.8) we calculate the following combination of derivatives:

$$\begin{aligned}\frac{\partial \hat{g}_{kj}}{\partial r^i} + \frac{\partial \hat{g}_{ik}}{\partial r^j} - \frac{\partial \hat{g}_{ij}}{\partial r^k} = \frac{\partial g_{kj}}{\partial r^i} + \frac{\partial g_{ik}}{\partial r^j} - \frac{\partial g_{ij}}{\partial r^k} - \\ - \varepsilon \left(\nabla_i h_{kj} + \nabla_j h_{ik} - \nabla_k h_{ij} - 2 \sum_{p=0}^{3} \Gamma^p_{ij}\, h_{pk} \right) + \ldots .\end{aligned} \tag{2.9}$$

Let's use the relationships (2.6) and (2.8) in calculating deformation of connection components. For this purpose let's apply well-known formula to $\hat{\Gamma}^p_{ij}$ (see formula (11.3) in Chapter III):

$$\hat{\Gamma}^p_{ij} = \Gamma^p_{ij} + \frac{\varepsilon}{2} \sum_{k=0}^{3} g^{pk} \left(\nabla_i h_{kj} + \nabla_j h_{ik} - \nabla_k h_{ij} \right) + \ldots .$$

This expansion for $\hat{\Gamma}^p_{ij}$ can be written in symbolic concise form

$$\hat{\Gamma}^p_{ij} = \Gamma^p_{ij} + \varepsilon\, Y^p_{ij} + \ldots \tag{2.10}$$

by introducing the following quite natural notation:

$$Y^p_{ij} = \frac{1}{2}\sum_{k=0}^{3} g^{pk}\left(\nabla_i h_{kj} + \nabla_j h_{ik} - \nabla_k h_{ij}\right). \tag{2.11}$$

Now let's substitute the expansion (2.10) into the formula (1.1) for curvature tensor. This yields

$$\hat{R}^k_{qij} = R^k_{qij} + \varepsilon\,\left(\nabla_i Y^k_{jq} - \nabla_j Y^k_{iq}\right) + \ldots\,. \tag{2.12}$$

Upon contracting (2.12) with respect to one pair of indices we get similar expansion for deformation of Ricci tensor:

$$\hat{R}_{qj} = R_{qj} + \varepsilon\sum_{k=0}^{3}\left(\nabla_k Y^k_{jq} - \nabla_j Y^k_{kq}\right) + \ldots\,. \tag{2.13}$$

We multiply (2.13) by g_{qj} using formula (2.6). Then we carry out complete contraction with respect to both indices q and j. This yields deformation of scalar curvature:

$$\hat{R} = R + \varepsilon\sum_{j=0}^{3}\sum_{q=0}^{3}\left(R_{qj}\,h^{qj} + \sum_{k=0}^{3} g^{qj}(\nabla_k Y^k_{jq} - \nabla_j Y^k_{kq})\right) + \ldots\,.$$

Let's introduce vector field with the following components:

$$Z^k = \sum_{j=0}^{3}\sum_{q=0}^{3}\left(Y^k_{jq} g^{qj} - Y^j_{jq} g^{qk}\right).$$

Then we can rewrite deformation of scalar curvature $\hat{R}$ as

$$\hat{R} = R + \varepsilon \sum_{j=0}^{3} \sum_{q=0}^{3} R_{qj}\, h^{qj} + \varepsilon \sum_{k=0}^{3} \nabla_k Z^k + \ldots\,. \tag{2.14}$$

When substituting (2.14) into action integral (2.3) we should note that second sum in (2.14) is exactly covariant divergency of vector field $\mathbf{Z}$. Components of $\mathbf{Z}$ are smooth functions vanishing outside the domain Ω. Therefore integral of such sum is equal to zero:

$$\int\limits_{\Omega} \sum_{k=0}^{3} \nabla_k Z^k \sqrt{-\det g}\, d^4 r = \int\limits_{\partial\Omega} g(\mathbf{Z}, \mathbf{n})\, dV = 0.$$

This follows from Ostrogradsky-Gauss formula (see (4.14) in Chapter IV). Hence for deformation of S_{gr} we get

$$S_{\text{def}} = S_{\text{gr}} - \frac{\varepsilon\, c^3}{16\pi\gamma} \int\limits_{\Omega} \sum_{j=0}^{3} \sum_{q=0}^{3} \left(R_{qj} - \frac{R}{2} g_{qj}\right) h^{qj} \sqrt{-\det g}\, d^4 r + \ldots\,.$$

In deriving this formula we also used the following expansion:

$$\sqrt{-\det \hat{g}} = \sqrt{-\det g} \left(1 - \varepsilon \sum_{j=0}^{3} \sum_{q=0}^{3} \frac{g_{qj}\, h^{qj}}{2}\right) + \ldots\,. \tag{2.15}$$

It follows from (2.6). Now we shall not calculate deformations of other three terms in (2.2) in explicit form. This will be done in § 4 and § 5 below. However, we introduce notation

$$S_{\text{m.f.}} = S_{\text{mat}} + S_{\text{int}} + S_{\text{el}}. \tag{2.16}$$

Here $S_{\text{m.f.}}$ denotes overall action for all material fields other than gravitation. The number of terms in the sum (2.16) could be much more than three, if one consider more complicated models for describing matter. But in any case action of gravitational field is excluded from this sum since gravitational field plays exceptional role in general relativity. Now we shall write deformation of the action (2.16) in the following conditional form:

$$S_{\text{def}} = S_{\text{m.f.}} + \frac{\varepsilon}{2c} \int\limits_{\Omega} \sum_{q=0}^{3} \sum_{j=0}^{3} T_{qj}\, h^{qj} \sqrt{-\det g}\; d^4 r + \ldots . \tag{2.17}$$

Then extremity condition for total action (2.2) is written as

$$R_{qj} - \frac{R}{2} g_{qj} = \frac{8\pi\gamma}{c^4} T_{qj}. \tag{2.18}$$

This equation (2.18) is known as *Einstein equation.* It is basic equation describing dynamics of metric tensor g_{ij} in general theory of relativity.

Exercise 2.1. *Derive the relationships* (2.7) *and* (2.15) *from the expansion* (2.6) *for deformation of tensor* g^{ij}*.*

§ 3. Four-dimensional momentum conservation law for fields.

Tensor $\mathbf{T}$ in right hand side of Einstein equation (2.18) is called energy-momentum tensor for material fields. It is determined by the relationship (2.17) and comprises contributions from all material fields and their interactions. In the model of dust matter in electromagnetic field tensor T is composed of three parts (see formula (2.16)).

Energy-momentum tensor is related with 4-momentum conservation law for material fields. In order to derive this conservation law we use well-known Bianchi identity:

$$\nabla_k R^p_{sij} + \nabla_i R^p_{sjk} + \nabla_j R^p_{ski} = 0. \tag{3.1}$$

More details concerning Bianchi identity (3.1) can be found in [2] and [6]. Let's contract this identity with respect to i and p:

$$\nabla_k R_{sj} + \sum_{p=0}^{3} \nabla_p R^p_{sjk} - \nabla_j R_{sk} = 0. \tag{3.2}$$

Here we used skew symmetry of curvature tensor with respect to last pair of indices (see [3]). Let's multiply (3.2) by g^{sj} and contract it with respect to double indices s and j. Upon slight transformation based on skew symmetry $R^{ps}_{ij} = -R^{sp}_{ij}$ we get

$$\sum_{s=0}^{3} \nabla_s R^s_k - \frac{1}{2}\nabla_k R = 0. \tag{3.3}$$

Now let's raise index j in the equation (2.18), then apply covariant differentiation ∇_j and contract with respect to double index j:

$$\sum_{j=0}^{3} \nabla_j R^j_q - \frac{1}{2}\nabla_q R = \frac{8\pi\gamma}{c^4}\sum_{j=0}^{3} \nabla_j T^j_q. \tag{3.4}$$

Comparing (3.3) and (3.4), we get the following equation for energy-momentum tensor of material fields:

$$\sum_{j=0}^{3} \nabla_j T^j_q = 0. \tag{3.5}$$

The equation (3.5) expresses 4-*momentum conservation law* for the whole variety of material fields. It is usually written in the following form with raised index q:

$$\sum_{j=0}^{3} \nabla_j T^{qj} = 0. \tag{3.6}$$

Energy-momentum tensor is symmetric therefore the order of indices q and j in (3.6) is unessential.

§ 4. Energy-momentum tensor for electromagnetic field.

Energy-momentum tensor for whole variety of material fields is defined by the relationship (2.17). By analogy with (2.17) we define energy-momentum tensor for electromagnetic field:

$$S_{\text{def}} = S_{\text{el}} + \frac{\varepsilon}{2c} \int\limits_{\Omega} \sum_{q=0}^{3} \sum_{j=0}^{3} T_{qj}\, h^{qj} \sqrt{-\det g}\; d^4r + \ldots . \tag{4.1}$$

Basic fields in the action S_{el} are covariant components of vector-potential $A_i(\mathbf{r})$. Covariant components of tensor of electromagnetic field are defined by formula

$$F_{ij} = \nabla_i A_j - \nabla_j A_i = \frac{\partial A_j}{\partial r^i} - \frac{\partial A_i}{\partial r^j} \tag{4.2}$$

(see also formula (11.5) in Chapter III). Ultimate expression in right hand side of (4.2) has no entry of connection components Γ^k_{ij}. Therefore covariant components F_{ij} are not changed by deformation of metric (2.6). Upon raising indices we get

$$\hat{F}^{pk} = \sum_{i=0}^{3} \sum_{j=0}^{3} \hat{g}^{pi}\, \hat{g}^{kj}\, F_{ij}$$

and, using this formula, for contravariant components F^{pq} of tensor of electromagnetic field we derive the expansion

$$\hat{F}^{pk} = F^{pk} + \varepsilon \sum_{i=0}^{3} \sum_{j=0}^{3} (h^{pi}\, g^{kj} + g^{pi}\, h^{kj})\, F_{ij} + \ldots . \tag{4.3}$$

Substituting $\hat{F}^{pk}$ and $\hat{g}$ into action functional S_{el}, we get

$$S_{\text{def}} = -\frac{1}{16\pi\, c}\int\limits_{V_1}^{V_2}\sum_{p=0}^{3}\sum_{k=0}^{3} F_{pk}\,\hat{F}^{pk}\sqrt{-\det\hat{g}}\;d^4r$$

Then, taking into account (4.3) and (2.15), we derive formula

$$S_{\text{def}} = S_{\text{el}} - \frac{\varepsilon}{16\pi\, c}\int\limits_{\Omega}\sum_{q=0}^{3}\sum_{j=0}^{3}\left(\sum_{p=0}^{3}\sum_{i=0}^{3} 2\,F_{pq}\,g^{pi}\,F_{ij} - \right.$$
$$\left. -\frac{1}{2}\sum_{p=0}^{3}\sum_{i=0}^{3} F_{pi}\,F^{pi}\,g_{qj}\right) h^{qj}\sqrt{-\det g}\;d^4r + \ldots\,.$$

Comparing this actual expansion with expected expansion (4.1) for S_{def}, we find components of energy-momentum tensor for electromagnetic field in explicit form:

$$T_{qj} = -\frac{1}{4\pi}\sum_{p=0}^{3}\sum_{i=0}^{3}\left(F_{pq}\,g^{pi}\,F_{ij} - \frac{1}{4}F_{pi}\,F^{pi}\,g_{qj}\right). \tag{4.4}$$

Raising indices q and j in (4.4), for contravariant components of energy-momentum tensor $\mathbf{T}$ we derive

$$T^{qj} = -\frac{1}{4\pi}\sum_{p=0}^{3}\sum_{i=0}^{3}\left(F^{pq}\,g_{pi}\,F^{ij} - \frac{1}{4}F_{pi}\,F^{pi}\,g^{qj}\right). \tag{4.5}$$

By means of formula (4.5) one can calculate covariant divergency for energy-momentum tensor of electromagnetic field:

$$\sum_{s=0}^{3}\nabla_s T^{ps} = -\frac{1}{c}\sum_{s=0}^{3} F^{ps}\,j_s. \tag{4.6}$$

Formula (4.6) shows that 4-momentum conservation law for separate electromagnetic field is not fulfilled. This is due to momentum exchange between electromagnetic field and other forms of matter, e. g. dust matter.

Exercise 4.1. *Verify the relationship* (4.6). *For this purpose use well-known formula for commutator of covariant derivatives*

$$(\nabla_i \nabla_j - \nabla_i \nabla_j) A_k = -\sum_{s=0}^{3} R^s_{kij}\, A_s$$

and properties of curvature tensor (see details in [3]*).*

Exercise 4.2. *Calculate components of energy-momentum tensor* (4.5) *in inertial coordinate system for flat Minkowsky metric. Compare them with components of Maxwell tensor, with density of energy, and with vector of energy flow for electromagnetic field (see formulas* (2.5) *and* (2.15) *in Chapter II).*

§ 5. Energy-momentum tensor for dust matter.

Let's consider energy-momentum tensor related with last two terms S_{mat} and S_{int} in the action (2.16). They contain entries of vector field $\boldsymbol{\eta}$ whose components satisfy differential equation

$$\sum_{p=0}^{3} \nabla_p \eta^p = 0, \tag{5.1}$$

see (3.4) in Chapter IV. This circumstance differs them from components of vector-potential $\mathbf{A}$. Metric tensor g_{ij} enters differential equation (5.1) through connection components Γ^k_{ij} of metric connection. Therefore by deformation of metric $g_{ij} \to \hat{g}_{ij}$ one cannot treat η^p as metric independent quantities.

In order to find truly metric independent variables for dust matter we use formula (4.7) from Chapter IV and rewrite differential equation (5.1) as follows:

$$\sum_{p=0}^{3}\frac{\partial(\eta^p\sqrt{-\det\hat{g}\,})}{\partial r^p}=0.$$

Denote $\hat{\eta}^p=\eta^p\sqrt{-\det g}$. These quantities $\hat{\eta}^p$ can be treated as metric independent ones since differential constraint for them is written in form of the equation that does not contain metric:

$$\sum_{p=0}^{3}\frac{\partial\hat{\eta}^p}{\partial r^p}=0. \tag{5.2}$$

Expressing η^p through $\hat{\eta}^p$, for action functional S_{int} describing interaction of dust matter and electromagnetic field we get

$$S_{\text{int}}=-\frac{q}{c^2}\int\limits_{V_1}^{V_2}\sum_{p=0}^{3}\hat{\eta}^p\,A_p\,d^4r. \tag{5.3}$$

It is easy to see that integral (5.3) does not depend on metric tensor. Therefore action functional S_{int} makes no contribution to overall energy-momentum tensor.

Now let's express η^p through $\hat{\eta}^p$ in action functional S_{mat} for dust matter. As a result we get formula

$$S_{\text{mat}}=-m\int\limits_{V_1}^{V_2}\sqrt{\sum_{p=0}^{3}\sum_{q=0}^{3}g_{pq}\,\hat{\eta}^p\,\hat{\eta}^q}\;d^4r. \tag{5.4}$$

The dependence of this functional on metric tensor is completely determined by explicit entry of g_{pq} under square root sign in right hand side of (5.4). Therefore power extension for S_{mat} is easily calculated on the base of the expansion (2.7):

$$S_{\text{def}} = S_{\text{mat}} + \frac{\varepsilon}{2} \int\limits_{\Omega} \left(\sum_{p=0}^{3} \sum_{q=0}^{3} \frac{m\,\eta_p\,\eta_q}{\sqrt{g(\boldsymbol{\eta},\boldsymbol{\eta})}} \right) h^{pq} \sqrt{-\det g}\, d^4r + \ldots .$$

Let's compare this expansion with expected expansion for S_{def}:

$$S_{\text{def}} = S_{\text{mat}} + \frac{\varepsilon}{2c} \int\limits_{\Omega} \sum_{p=0}^{3} \sum_{q=0}^{3} T_{pq}\, h^{pq} \sqrt{-\det g}\, d^4r + \ldots .$$

By this comparison we find explicit formula for components of energy-momentum tensor for dust matter:

$$T_{pq} = \frac{mc\;\eta_p\,\eta_q}{\sqrt{g(\boldsymbol{\eta},\boldsymbol{\eta})}} = mc\,\sqrt{g(\boldsymbol{\eta},\boldsymbol{\eta})}\;\,u_p\,u_q. \tag{5.5}$$

Contravariant components of energy-momentum tensor (5.5) are obtained by raising indices p and q:

$$T^{pq} = \frac{mc\;\eta^p\,\eta^q}{\sqrt{g(\boldsymbol{\eta},\boldsymbol{\eta})}} = mc\,\sqrt{g(\boldsymbol{\eta},\boldsymbol{\eta})}\;\,u^p\,u^q. \tag{5.6}$$

Using collinearity of vectors $\mathbf{u}$ and $\boldsymbol{\eta}$ (see formula (3.1) in Chapter IV) and recalling that $\mathbf{u}$ is unit vector, we can bring formula (5.6) to the following simple form:

$$T^{pk} = mc\,u^p\,\eta^k. \tag{5.7}$$

Formula (5.7) is convenient for calculating covariant divergency of energy-momentum tensor for dust matter:

$$\sum_{s=0}^{3} \nabla_s T^{ps} = \frac{q}{c} \sum_{s=0}^{3} F^{ps}\,\eta_s. \tag{5.8}$$

Now, applying formula (3.2) from Chapter IV, we can transform formula (5.8) and write it as follows:

$$\sum_{s=0}^{3} \nabla_s T^{ps} = \frac{1}{c} \sum_{s=0}^{3} F^{ps}\, j_s. \tag{5.9}$$

Let's compare (5.9) with analogous formula (4.6) for energy-momentum tensor of electromagnetic field. Right hand sides of these two formulas differ only in sign. This fact has transparent interpretation. It means that in our model the overall energy-momentum tensor for matter

$$\mathbf{T}_{\text{m.f.}} = \mathbf{T}_{\text{mat}} + \mathbf{T}_{\text{el}}$$

satisfies differential equation (3.6). This fact is in complete agreement with 4-momentum conservation law.

Another important conclusion, which follows from of (4.6) and (5.9), is that 4-momentum conservation law for the whole variety of material fields can be derived from dynamical equations for these fields. Therefore this law is valid also in special relativity, where Einstein equation (2.18) is not considered and where in general case for flat Minkowsky metric it is not fulfilled.

Exercise 5.1. *Derive the relationship* (5.8) *on the base of equations* (3.4) *and* (4.19) *from Chapter IV.*

§ 6. Concluding remarks.

Event space in general theory of relativity is some Minkowsky manifold M with Minkowsky metric of signature $(1, 3)$. This metric is determined by material content of the space according to Einstein equation (2.18). However, topology of the manifold M has great deal of arbitrariness. This manifold can have local singularities at the points with extremely high concentration of matter. Such objects are called *black holes*. Moreover, global topology of M also can be nontrivial (other than topology of $\mathbb{R}^4$). In contemporary physics most popular models of M include *big bang* in the very beginning of times. According to these models in far past times our Universe M was extremely small, while density of matter in it was extremely high. In further evolution our Universe was expanding up to its present size. Will this expansion last infinitely long or it will change for contraction? This problem is not yet solved. The answer to this question depends on estimates of total amount of matter in the Universe.

In this book we cannot consider all these fascinating problems of modern astrophysics and cosmology. However, I think the above theoretical material makes sufficient background for to continue studying these problems e. g. in books [2], [7], and [8]. I would like also to recommend the book [9] of popular genre, where these problems are discussed in commonly understandable and intriguing manner.

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