

## Analysis of Multiphase Non-Darcy Flow in Porous Media

Yu-Shu Wu · Bitao Lai · Jennifer L. Miskimins ·  
Perapon Fakcharoenphol · Yuan Di

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**Abstract** Recent laboratory studies and analyses (Lai et al. Presented at the 2009 Rocky Mountain Petroleum Technology Conference, 14–16 April, Denver, CO, 2009) have shown that the Barree and Conway model is able to describe the entire range of relationships between flow rate and potential gradient from low- to high-flow rates through porous media. A Buckley and Leverett type analytical solution is derived for non-Darcy displacement of immiscible fluids in porous media, in which non-Darcy flow is described using the Barree and Conway model. The comparison between Forchheimer and Barree and Conway non-Darcy models is discussed. We also present a general mathematical and numerical model for incorporating the Barree and Conway model in a general reservoir simulator to simulate multiphase non-Darcy flow in porous media. As an application example, we use the analytical solution to verify the numerical solution for and to obtain some insight into one-dimensional non-Darcy displacement of two immiscible fluids with the Barree and Conway model. The results show how non-Darcy displacement is controlled not only by relative permeability, but also by non-Darcy coefficients, characteristic length, and injection rates. Overall, this study provides an analysis approach for modeling multiphase non-Darcy flow in reservoirs according to the Barree and Conway model.

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Y.-S. Wu (✉) · J. L. Miskimins · P. Fakcharoenphol  
Department of Petroleum Engineering, Colorado School of Mines, Golden, CO, USA  
e-mail: YWu@mines.edu

J. L. Miskimins  
e-mail: jmiskimi@mines.edu

P. Fakcharoenphol  
e-mail: pfakchar@mymail.mines.edu

B. Lai  
Department of Petroleum Engineering, University of Louisiana at Lafayette, Lafayette, LA, USA  
e-mail: bxl5695@louisiana.edu

Y. Di  
Department of Energy and Natural Resources, Peking University, Beijing, China  
e-mail: dyzm@pku.edu.cn

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## List of Symbols

$A$	Cross-section area of flow, m <sup>2</sup>
$A_{ij}$	Common interface area between the connected blocks or nodes $i$ and $j$ , m <sup>2</sup>
$C_\beta$	non-Darcy constant, m <sup>0.25</sup>
$D$	Depth from a datum, m
$D_i$	Distance from the center of block $i$ to the common interface of blocks $i$ and $j$
$f_\beta$	Fractional flow of phase $\beta$ , fraction
flow $_\beta$	mass flux of fluid $\beta$ , kg/s
$g$	Gravitational acceleration constant, m/s <sup>2</sup>
$k_d$	Darcy permeability, m <sup>2</sup>
$k_{\min}$	Minimum permeability at high rate, m <sup>2</sup>
$k_{mr}$	Minimum permeability ratio, relative to Darcy permeability, fraction
$k_{r\beta}$	Relative permeability to fluid $\beta$ , fraction
$N$	Total number of nodes/elements/gridblocks of the grid
$P_\beta$	Pressure of fluid $\beta$ , Pa
$P_{cg0}$	Gas–oil capillary pressure, Pa
$P_{cgw}$	Gas–water capillary pressure, Pa
$P_{cow}$	Oil–water capillary pressure, Pa
$\nabla P$	Pressure gradient, Pa/m
$q$	Injection rate, m <sup>3</sup> /sec
$q_\beta$	Mass sink/source per unit volume for the fluid $\beta$ , kg/m <sup>3</sup>
$Q$	Fluid volumetric flow rate, m <sup>3</sup> /s
$Q_{\beta i}$	Mass sink/source term at element $i$ , for the fluid $\beta$ , kg/m <sup>3</sup>
$R_i$	Residue term of mass balance at element $i$ , kg
$S_\beta$	Saturation of fluid $\beta$ , fraction
$\bar{S}_\beta$	Average saturation of fluid $\beta$ , fraction
$\Delta t$	Time step size, s
$V_i$	Volume of block $i$ , m <sup>3</sup>
$v_\beta$	Velocity of fluid $\beta$ , m/s
$x_{sw}$	Location of the specific saturation, m

## Greeks

$\alpha$	Angle from horizontal plane, Degree
$\beta$	Non-Darcy coefficient, 1/m
$\rho_\beta$	Fluid density of fluid $\beta$ , kg/m <sup>3</sup>
$\mu_\beta$	Viscosity of fluid $\beta$ , Pa s
$\tau$	Characteristic length, 1/m
$\phi$	Effective porosity of the medium, fraction
$\nabla \Phi$	Flow potential gradient, Pa/m

## Subscripts

- eff Effective
- g Gas phase
- n Nonwetting phase
- o Oil phase
- p Iteration at p level
- w Water phase
- wr Irridusible water

## Superscripts

- $n$  Previous time step
- $n + 1$  Next time step

## 1 Introduction

Darcy's Law has been used exclusively in studies of porous-medium flow in reservoirs. However, there is more evidence that high-velocity non-Darcy flow occurs in many subsurface systems, such as in the flow near wells of oil or gas production and liquid waste injection. Darcy's law, describing a linear relationship between volumetric flow rate (Darcy velocity) and pressure (or potential) gradient, has been the fundamental principle in analyzing flow processes in porous media. Any deviation from this linear relation may be defined as non-Darcy flow. In this article, our concern is only with the non-Darcy flow behavior caused by high-flow velocities. Effects of non-Darcy or high-velocity flow regimes in porous media have been observed and investigated for decades (e.g., [Tek et al. 1962](#); [Scheidegger 1972](#); [Katz and Lee 1990](#); [Wu 2002](#)). Earlier studies on non-Darcy flow in porous media focused mostly on single-phase-flow conditions in petroleum engineering ([Tek et al. 1962](#); [Swift and Kiel 1962](#); [Lee et al. 1987](#)). Some investigations have been conducted for non-Darcy flow in fractured reservoirs ([Skjetne et al. 1999](#)) and for non-Darcy flow into highly permeable fractured wells (e.g., [Guppy 1981, 1982](#); [Wu 2002](#)). Other studies have concentrated on finding and validating correlations of non-Darcy flow coefficients (e.g., [Liu et al. 1995](#)).

In analyzing non-Darcy flow through porous media, the Forchheimer equation (1901) has been exclusively used and has also been extended to multiphase flow conditions ([Evans et al. 1987](#); [Evans and Evans 1988](#); [Liu et al. 1995](#); [Wu 2001, 2002](#)). [Barree and Conway \(2004\)](#), based on experimental results and field observations, proposed a new, physical-based model for describing non-Darcy flow in porous media, which does not rely on the assumption of a constant permeability or a constant Forchheimer- $\beta$  factor. Recent laboratory studies and analyses ([Lopez 2007](#); [Lai et al. 2009](#)) have shown that the Barree and Conway model describes the entire range of relationships between flow rate and potential gradient from low- to high-flow rates through porous media, including those in transitional zones. In particular, these new experimental data show that the Barree and Conway non-Darcy flow model is able to overcome many of the drawbacks with the Forchheimer analysis, while still honoring the basics of Darcy and Forchheimer flow behavior under lower flow rates ([Lai 2010](#)).

In this article, we derive a Buckley and Leverett type analytical solution for one-dimensional non-Darcy displacement of immiscible fluids in porous media according to

the Barree and Conway model. We also present a general numerical model for incorporating the Barree and Conway model to simulate multiphase, multidimensional non-Darcy flow in porous media.

This article represents a continual study of our previous investigation of both single-phase (Lai et al. 2009) and multiphase (Lai 2010; Wu et al. 2009) non-Darcy flow in reservoirs according to the Barree and Conway model. The objective of this study is to develop a mathematical method for quantitative analysis of multiphase non-Darcy flow through heterogeneous porous rocks using the Barree and Conway model. The numerical solution of the proposed mathematical model is based on a discretization scheme using an unstructured grid with regular or irregular meshes for multidimensional simulation. The final discretized nonlinear equations are handled fully implicitly with the Newton iteration. As an application example, we use the analytical solution to verify the numerical solution for and to obtain some insight into one-dimensional non-Darcy displacement of two immiscible fluids according to the Barree and Conway model. Overall, this study provides a quantitative approach for modeling multiphase non-Darcy flow in reservoirs.

## 2 Mathematical Model

A multiphase system in a porous reservoir is assumed to be similar to the black oil model, composed of three phases: oil, gas, and water. For simplicity, the three fluid components, water, oil (or NAPL), and gas are assumed to be present only in their associated phases. Each phase flows in response to pressure, gravitational, and capillary forces according to the multiphase extension of the Barree and Conway model (Barree and Conway 2007) for non-Darcy flow. In an isothermal system containing three mass components, three mass-balance equations are needed to fully describe the system, as described in an arbitrary flow region of a porous or fractured domain for flow of phase  $\beta$  ( $\beta = w$  for water,  $\beta = o$  for oil or NAPL, and  $\beta = g$  for gas),

$$\frac{\partial}{\partial t}(\phi S_\beta \rho_\beta) = -\nabla \cdot (\rho_\beta \mathbf{v}_\beta) + q_\beta \quad (2.1)$$

where  $\rho_\beta$  is the density of fluid  $\beta$ ;  $\mathbf{v}_\beta$  is the volumetric velocity vector of fluid  $\beta$ ;  $S_\beta$  is the saturation of fluid  $\beta$ ;  $\phi$  is the effective porosity of formation;  $t$  is the time; and  $q_\beta$  is the sink/source term of phase (component)  $\beta$  per unit volume of formation, representing mass exchange through injection/production wells or due to fracture and matrix interactions.

Volumetric flow rate (namely, Darcy velocity with Darcy flow) for non-Darcy flow of each fluid may be described using the multiphase extension of the Barree and Conway's model, extended to a vector form for multidimensional flow (see Appendix A):

$$-\nabla \Phi_\beta = \frac{\mu_\beta \mathbf{v}_\beta}{k_d k_{r\beta} \left( k_{mr} + \frac{(1-k_{mr})\mu_\beta \tau}{\mu_\beta \tau + \rho_\beta |\mathbf{v}_\beta|} \right)} \quad (2.2)$$

where  $\nabla \Phi_\beta$  is the flow potential gradient, defined as:

$$\nabla \Phi_\beta = (\nabla P_\beta - \rho_\beta g \nabla D) \quad (2.3)$$

where  $P_\beta$  is the pressure of the fluid;  $g$  is the gravitational acceleration; and  $D$  is the depth from a datum. In Eq. 2.2,  $k_d$  is the constant Darcy or absolute permeability;  $k_{mr}$  is the minimum permeability ratio at high rate, relative to Darcy permeability (fraction);  $k_{r\beta}$  is the relative permeability to fluid  $\beta$ ;  $\mu_\beta$  is the viscosity of fluid  $\beta$ ; and  $\tau$  is the characteristic length.

Equation 2.1, the governing of mass balance for the three fluids, needs to be supplemented with constitutive equations, which express all the secondary variables and parameters as functions of a set of primary thermodynamic variables of interest. The following relationships are used to complete the description of multiphase flow through porous media:

$$S_w + S_o + S_g = 1 \quad (2.4)$$

The capillary pressures relate pressures between the phases. The aqueous- and gas-phase pressures are related by

$$P_w = P_g - P_{cgw}(S_w), \quad (2.5)$$

where  $P_{cgw}$  is the gas–water capillary pressure in a three-phase system and assumed to be a function of water saturation only. The NAPL pressure is related to the gas-phase pressure by

$$P_o = P_g - P_{cgo}(S_w, S_o) \quad (2.6)$$

where  $P_{cgo}$  is the gas–oil capillary pressure in a three-phase system, which is a function of both water and oil saturations. For formations, the wettability order is (1) aqueous phase, (2) oil phase, and (3) gas phase. The gas–water capillary pressure is usually stronger than the gas–oil capillary pressure. In a three-phase system, the oil–water capillary pressure,  $P_{cow}$ , may be defined as

$$P_{cow} = P_{cgw} - P_{cgo} = P_o - P_w \quad (2.7)$$

The relative permeabilities are assumed to be functions of fluid saturations only, i.e., not affected by non-Darcy flow behavior. The relative permeability to the water phase is taken to be described by

$$k_{rw} = k_{rw}(S_w) \quad (2.8)$$

to the oil phase by

$$k_{ro} = k_{ro}(S_w, S_g) \quad (2.9)$$

and to the gas phase by

$$k_{rg} = k_{rg}(S_g) \quad (2.10)$$

The densities and viscosities of water, oil, and gas, as well as porosity are in general treated as functions of pressure in the numerical model, as normally done with multiphase flow simulation in reservoirs.

### 3 Numerical Solution

Equations 2.1 and 2.2, as described by the Barree and Conway model, for multiphase non-Darcy flow of gas, oil, and water in porous media, are highly nonlinear and in general need to be solved numerically. In this study, the methodology for using a numerical approach to simulate the non-Darcy flow consists of the following three steps: (1) spatial discretization of the mass conservation equation; (2) time discretization; and (3) iterative approaches to solve the resulting nonlinear, discrete algebraic equations. A mass-conserving discretization scheme, based on the integral finite-difference method (Pruess et al. 1999), is used and discussed here. Specifically, non-Darcy flow equations, as discussed in Sect. 2, have been implemented into a general-purpose, three-phase reservoir simulator, the MSFLOW code (Wu 1998). As

implemented in the code, Eq. 2.1 can be discretized in space using an integral finite-difference or control-volume finite-element scheme for a porous medium. The time discretization is carried out with a backward, first-order, finite-difference scheme. The discrete nonlinear equations for water, oil, and gas flow at node  $i$  are written as follows:

$$\left\{ (\phi S_\beta \rho_\beta)_i^{n+1} - (\phi S_\beta \rho_\beta)_i^n \right\} \frac{V_i}{\Delta t} = \sum_{j \in \eta_i} (\text{flow}_\beta)_{ij}^{n+1} + Q_{\beta i}^{n+1} \quad (3.1)$$

where  $n$  denotes the previous time level;  $n + 1$  is the current time level;  $V_i$  is the volume of element  $i$  ( $i = 1, 2, 3, \dots, N$ ,  $N$  being the total number of elements of the grid);  $\Delta t$  is the time step size;  $\eta_i$  contains the set of neighboring elements ( $j$ ), porous block, to which element  $i$  is directly connected; and “ $\text{flow}_\beta$ ” is a mass flow term between elements  $i$  and  $j$  for fluid  $\beta$ , defined by Eq. 2.2 implicitly. For flow between two grid blocks, the mass flow term, “ $\text{flow}_\beta$ ”, can be evaluated directly (See Appendix A) as,

$$\text{flow}_{\beta,ij} = A_{ij} \rho_\beta v_\beta = \frac{A_{ij}}{2\mu_\beta} \left\{ -[\mu_\beta^2 S_\beta \tau - \Delta \Phi_{\beta,ij} k_d k_r \rho_\beta] \right. \\ \left. + \sqrt{[\mu_\beta^2 S_\beta \tau - \Delta \Phi_{\beta,ij} k_d k_r \rho_\beta]^2 + 4\mu_\beta \rho_\beta \Delta \Phi_{\beta,ij} k_d k_r \mu_\beta S_\beta \tau} \right\} \quad (3.2)$$

where  $A_{ij}$  is the common interface area between connected elements  $i$  and  $j$ ; all the parameters, such as permeability, relative permeability, density, and viscosity need proper averaging or weighting of properties at the interface between the two elements  $i$  and  $j$ ; and the discrete flow potential gradient is defined in an integral finite difference as,

$$\Delta \Phi_{\beta,ij} = \frac{(P_{\beta,i} - \rho_{\beta,ij+1/2g} D_i) - (P_{\beta,j} - \rho_{\beta,ij+1/2g} D_j)}{D_i + D_j} \quad (3.3)$$

Note that Eq. 3.1 has the same form regardless of the dimensionality of the model domain, i.e., it applies to one-, two-, or three-dimensional analyses of multiphase non-Darcy flow through porous media.

In the model formulation, Darcy permeability, relative permeability, and other non-Darcy flow parameters, such as minimum permeability ratio,  $k_{mr}$ , and characteristic length,  $\tau$ , are all considered as flow properties of the porous media and need to be averaged between connected elements in calculating the mass flow terms. In general, weighting approaches used are that absolute permeability is harmonically weighted along the connection between elements  $i$  and  $j$ , relative permeability is upstream weighted, and non-Darcy flow coefficients are arithmetically averaged.

In Eq. 3.1, the mass sink/source term at element  $i$ ,  $Q_{\beta i}$  for phase  $\beta$ , is defined as

$$Q_{\beta i} = q_{\beta i} V_i \quad (3.4)$$

Newton/Raphson iterations are used to solve Eq. 3.1. For a three-phase flow system,  $3 \times N$  coupled nonlinear equations must be solved, including three equations at each element for the three mass-balance equations of water, oil, and gas, respectively. The three primary variables ( $x_1, x_2, x_3$ ) selected for each element are oil pressure, oil saturation, and gas saturation with the mixed formulation (Wu and Forsyth 2001). In terms of the three primary variables, the Newton/Raphson scheme gives rise to

$$\sum_m \frac{\partial R_i^{\beta,n+1}(x_{m,p})}{\partial x_m} (\delta x_{m,p+1}) = -R_i^{\beta,n+1}(x_{m,p}) \quad \text{for } m = 1, 2, \text{ and } 3 \quad (3.5)$$

where index  $m = 1, 2$ , and  $3$  indicates the primary variable  $1, 2$ , or  $3$ , respectively;  $p$  is the iteration level; and  $i = 1, 2, 3, \dots, N$ , the nodal index. The primary variables are updated after each iteration:

$$x_{m,p+1} = x_{m,p} + \delta x_{m,p+1} \quad (3.6)$$

A numerical method is used to construct the Jacobian matrix for Eq. 3.5, as outlined by Forsyth et al. (1995).

### 3.1 Boundary Condition

Similarly to Darcy flow handling, first-type or Dirichlet boundary conditions denote constant or time-dependent phase pressure, and saturation conditions. These types of boundary conditions can be treated using the large-volume or inactive-node method (Pruess 1991), in which a constant pressure/saturation node may be specified with a huge volume while keeping all the other geometric properties of the mesh unchanged. However, caution should be taken in (1) identifying phase conditions when specifying the “initial condition” for the large-volume boundary node and (2) distinguishing upstream/injection from downstream/production nodes. Once specified, primary variables will be fixed at the big-volume boundary nodes, and the code handles these boundary nodes exactly like any other computational nodes.

Flux-type or Neuman boundary conditions are treated as sink/source terms, depending on the pumping (production) or injection condition, which can be directly added to Eq. 3.1. This treatment of flux-type boundary conditions is especially useful for a situation where flux distribution along the boundary is known, such as dealing with a single-node well. More general treatment of multilayered well boundary conditions is discussed in Wu et al. (1996) and Wu (2000).

## 4 Buckley–Leverett Analytical Solution for Two-Phase Non-Darcy Displacement

Buckley and Leverett (1942) established the fundamental principle for Darcy displacement of immiscible fluids through porous media in their classical study of fractional flow theory. Their solution involves the noncapillary displacement process of two incompressible, immiscible fluids in a one-dimensional, homogeneous system. The Buckley–Leverett fractional flow theory has been applied and generalized to study enhanced oil recovery (EOR) problems (e.g., Patton et al. 1971; Hirasaki and Pope 1974; Pope 1980; Larson 1978; Hirasaki 1981). An extension to more than two immiscible phases dubbed “coherence theory” was described by Helfferich (1981). The more recent example in the development of the Buckley–Leverett theory is the extension to non-Newtonian fluid flow and displacement (Wu et al. 1991 and Wu et al. 1992), and non-Darcy displacement according to the Forchheimer model (Wu 2001).

This article presents a Buckley–Leverett type analytical solution describing the displacement mechanism of non-Darcy multiphase flow in porous media according to the Barree and Conway model. The analysis approach follows upon the work for multiphase non-Newtonian fluid flow and displacement in porous media (Wu et al. 1991 and Wu et al. 1992; Wu 2001) and results in an analytical solution that includes effects of non-Darcy multiphase displacement. The details on deriving the two-phase displacement solution are given in Appendix B. Also discussed in Appendix B is a practical procedure for evaluating the behavior of the analytical solution, which is similar to the graphic method by Welge (1952) for solving

the Buckley–Leverett problem. The analytical solution and the resulting procedure can be regarded as an extension of the Buckley–Leverett theory to analyzing the Barree and Conway non-Darcy flow problem of two-phase immiscible fluids in porous media.

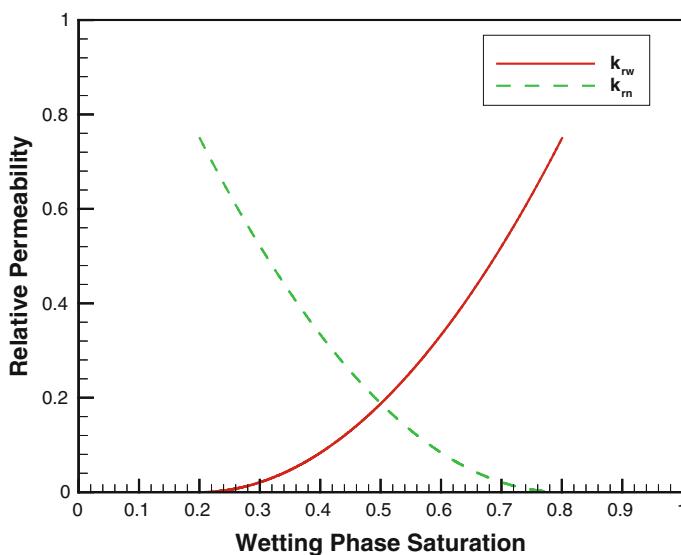
The analytical solution of Appendix B reveals that the saturation profile and displacement efficiency are controlled not only by relative permeabilities, as in the Buckley–Leverett solution for Darcy displacement, but also by the parameters of non-Darcy flow equations as well as injection rates.

## 5 Application

In this section, the Buckley–Leverett analytical solution is used to (1) discuss insight into non-Darcy flow and displacement phenomena and (2) verify the numerical formation. The physical flow system: a one-dimensional linear porous medium, which is at first saturated uniformly with a nonwetting fluid ( $S_0 = 0.8$ ) and a wetting fluid ( $S_w = S_{wr} = 0.2$ ). A constant volumetric injection rate of the wetting fluid is imposed at the inlet ( $x = 0$ ), starting from  $t = 0$ . The relative permeability curves used for all the calculations in this article are shown in Fig. 1. The properties of the rock and fluids used are listed in Table 1.

### 5.1 Non-Darcy Displacement Processes

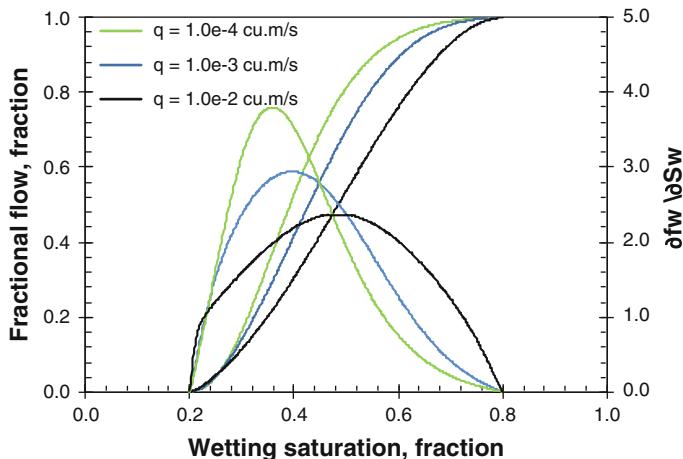
For a given displacement system with constant injection rate, the solution (B.12) shows that non-Darcy fluid displacement in a porous medium is characterized not only by relative permeability data, as in Buckley–Leverett displacement, but also by non-Darcy flow parameters of the two fluids, as introduced in Eqs. 2.2, B.9, and B.10. Using the results from the analytical solution, some fundamental aspects of non-Darcy fluid displacement are established. Figure 2, determined using Eqs. B.9 and B.10 for the flow system, shows that both fractional



**Fig. 1** Relative permeability curves used in analytical and numerical solutions for Barree and Conway non-Darcy displacement and comparison between the two non-Darcy flow and Darcy flow models

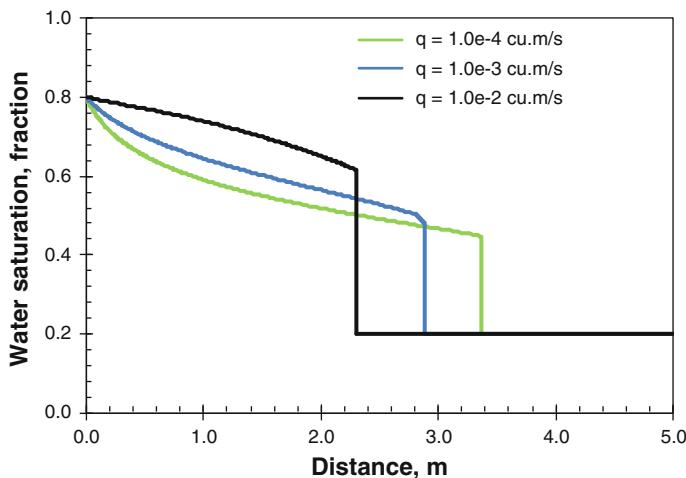
**Table 1** Parameters for the non-Darcy displacement example

Parameter	Value	Unit
Effective porosity	$\phi = 0.30$	Fraction
Darcy permeability	$k_d = 10$	Darcy
Minimum permeability	$k_{mr} = 0.01; 0.05; 0.10$	Fraction
Characteristic length	$\tau = 1,000; 2,000; 5,000$	$m^{-1}$
Wetting phase density	$\rho_w = 1,000$	$kg/m^3$
Wetting phase viscosity	$\mu_w = 1.0 \times 10^{-3}$	Pas
Nonwetting phase density	$\rho_n = 800$	$kg/m^3$
Nonwetting phase viscosity	$\mu_n = 5.0 \times 10^{-3}$	Pas
Equivalent non-Darcy flow constant of wetting phase	$C_w = 4.18 \times 10^{-7}$	$m^{3/2}$
Equivalent non-Darcy flow constant of nonwetting phase	$C_n = 4.53 \times 10^{-7}$	$m^{3/2}$
Injection rate	$q = 1.0 \times 10^{-2}; 1.0 \times 10^{-3}; 1.0 \times 10^{-4}$	$m^3/s$
Cross-section area	$A = 1.0$	$m^2$

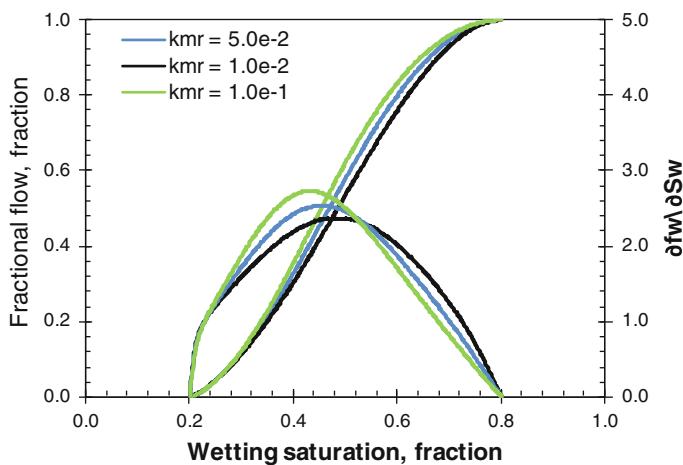
**Fig. 2** Fractional flow and their derivative curves with respect to wetting phase saturation with different injection rates

flow and its derivative curves change significantly with a change in injection rates for the horizontal displacement system, which is entirely different from Darcy displacement behavior. Saturation profiles after  $0.36 m^3$  injection for the three injection rates are shown in Fig. 3, indicating very different displacement efficiency within the flooded zones. The higher the injection rate, the better the displacement efficiency, because of the large flow resistance due to the non-Darcy flow effect.

Figures 4, 5, 6, and 7 present results for sensitivity of non-Darcy flow parameters:  $k_{mr}$  and  $\tau$ . The resulting fractional flow and its derivative curves are shown in Figs. 4 and 6. As shown in the two figures, fractional flow curves change also with the non-Darcy model parameters under the same saturation. Saturation profiles of non-Darcy displacement with varying  $k_{mr}$



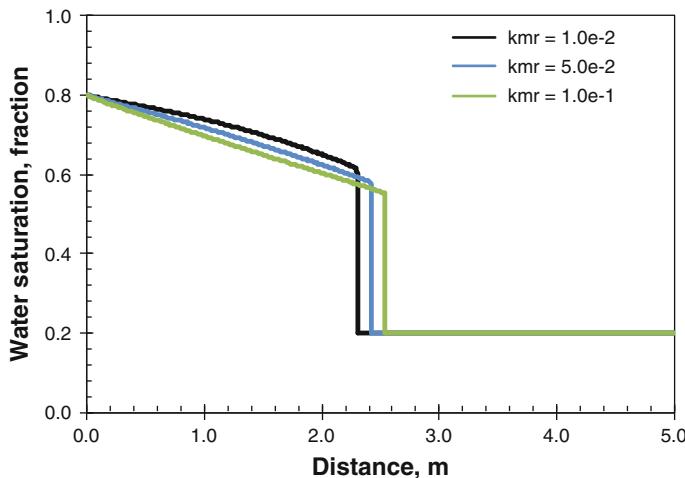
**Fig. 3** Displacement saturation distribution of the non-Darcy displacement system with different injection rates



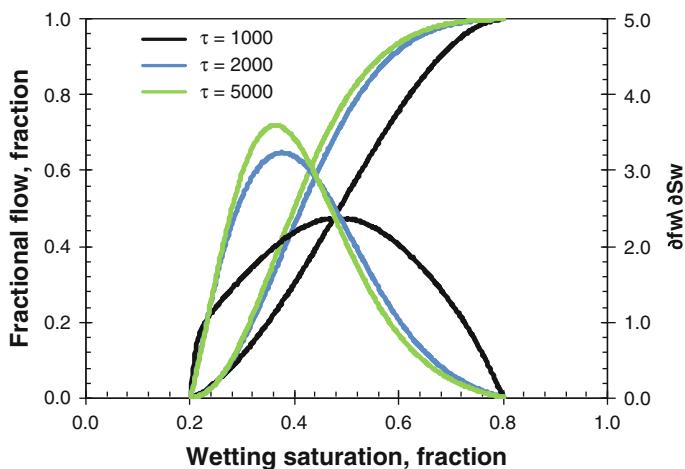
**Fig. 4** Fractional flow and their derivative curves with respect to wetting phase saturation with different minimum permeability ratios

and  $\tau$  are plotted in Figs. 5 and 7, showing typical behavior of non-Darcy displacement according to the Barree and Conway model with different model parameters. Figures 5 and 7 also indicate that the model results in terms of saturation profiles are more sensitive to the parameter,  $\tau$ , than  $k_{mr}$ , from the parameters selected.

Figure 8 shows a comparison among Darcy, non-Darcy Forchheimer, and non-Darcy Barree and Conway models. Note that to compare the results from the two models, equivalent non-Darcy flow parameters (derived in Appendix C) are used for the Forchheimer and Barree and Conway models. As shown in Fig. 8, the two non-Darcy models, when using the equivalent non-Darcy flow parameters, present very similar behavior where non-Darcy effect decreases the frontal velocity, and the slight discrepancy between the two non-Darcy



**Fig. 5** Displacement saturation distribution of the non-Darcy displacement system with different minimum permeability ratio

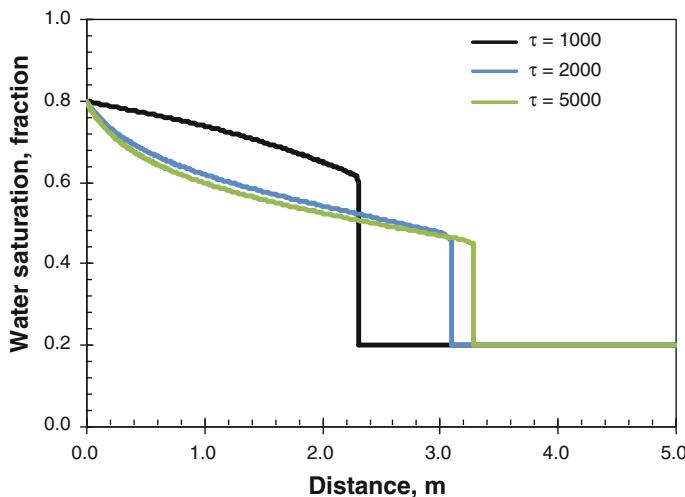


**Fig. 6** Fractional flow and their derivative curves with respect to wetting phase saturation with different characteristic lengths

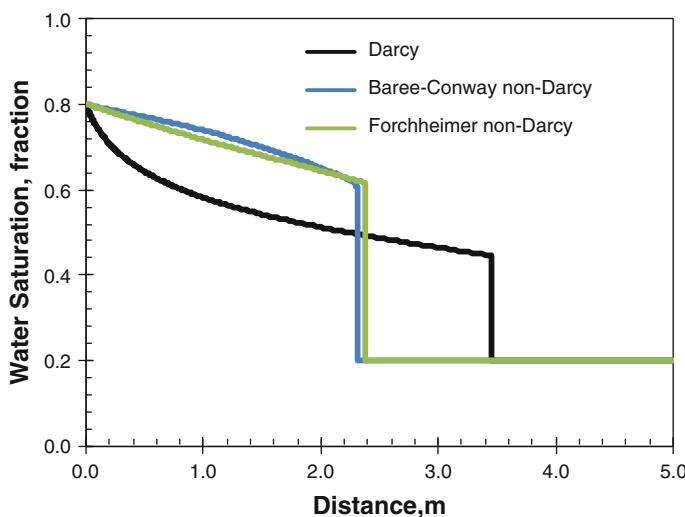
models appear to be minimal. However, the difference between the non-Darcy and Darcy displacement seems large.

## 5.2 Verification of Numerical Model

As another application example, the analytical solution is used to examine the validity of the numerical method, as discussed in Sect. 3, which is implemented in a general-purpose, three-phase reservoir simulator, the MSFLOW code (Wu 1998) for modeling multiphase non-Darcy flow and displacement processes according to the Barree and Conway model. To reduce the effects of discretization on numerical simulation results, very fine, uniform mesh spacing ( $\Delta x = 0.01$  m) is chosen. A one-dimensional 5 m linear domain is discretized



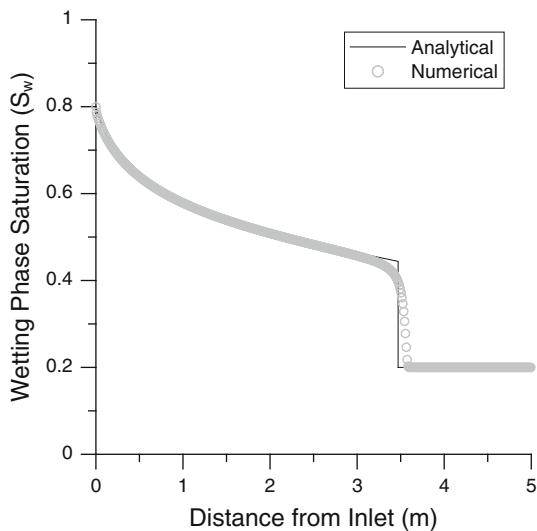
**Fig. 7** Displacement saturation distribution of the non-Darcy displacement system with different characteristic lengths



**Fig. 8** Comparison of the results from Darcy, non-Darcy Forchheimer, and non-Darcy Barree and Conway models

into 500 one-dimensional uniform grid blocks. The flow description and the parameters for this problem are identical to those, in Table 1, except the injection rate is  $10^{-5}\text{m}^3/\text{s}$  in this example. The comparison between the analytical and numerical solutions is shown in Fig. 9. The figure indicates that the numerical results are in excellent agreement with the analytical prediction of the non-Darcy displacement for the entire wetting-phase sweeping zone. Except at the shock, advancing saturation front, the numerical solution deviates only slightly from the analytical solution, resulting from a typical “smearing front” phenomenon of numerical dispersion effects when matching the Buckley–Leverett solution using numerical results

**Fig. 9** Comparison between displacement saturation profiles calculated from analytical and numerical solutions after 10 h of injection



(Aziz and Settari 1979). The comparison between the analytical and numerical solutions is shown in Fig. 9.

## 6 Summary and Conclusions

This article presents a general mathematical model and numerical approach for incorporating the Barree and Conway model to simulate multiphase, multidimensional non-Darcy flow in porous media. Both analytical and numerical approaches are discussed in this study. In particular, we derive a Buckley and Leverett type analytical solution for one-dimensional non-Darcy displacement of immiscible fluids in porous media with the Barree and Conway non-Darcy flow model. In the numerical solution, the multiphase non-Darcy flow formulation is implemented into a general-purpose reservoir simulator.

The analytical solution for non-Darcy displacement is based on the assumptions, similar to those used for the classical Buckley–Leverett solution. The analytical solution provides some insight into the physics of displacement involving non-Darcy flow, a more complicated process than the Darcy displacement, as described by the Buckley–Leverett solution. Multiphase non-Darcy flow and displacement are controlled not only by relative permeability curves, such as in Darcy displacement, but also by non-Darcy flow relations and model parameters as well as injection or flow rates. The comparison between Forchheimer and Barree and Conway models indicates minimal differences in describing non-Darcy displacement, if equivalent parameters are used. As an example of application, the analytical solution is applied to verify the numerical formulation of a numerical simulator for modeling multiphase non-Darcy flow.

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## Appendix A. Relationship of One-Dimensional Flow Rate versus Pressure Gradient

The Barree and Conway equation (Barree and Conway 2007) presents a one-dimensional model for pressure gradient versus multiphase flow rate. In the two-phase non-Darcy flow model, the pressure gradient of each phase (e.g., gas phase) is written as:

$$\left(\frac{\partial P}{\partial L}\right)_g = \left(\frac{\mu v}{k_{g\_eff}}\right)_g \quad (\text{A.1})$$

where the effective permeability of gas,  $k_{g\_eff}$ , can be written as:

$$k_{g\_eff} = k_d k_{rg} k_{rm} + \frac{(1 - k_{rm}) k_d k_{rg}}{\left(1 + \frac{\rho_g v_g}{\mu_g S_g \tau}\right)} \quad (\text{A.2})$$

Substituting (A.2) into (A.1), we have the one-dimensional form of the Barree and Conway non-Darcy flow equation, pressure gradient as a function of flow velocity as,

$$\left(\frac{\partial P}{\partial L}\right)_g = \frac{\mu_g v_g}{k_d k_{rg} k_{rm} + \frac{(1 - k_{rm}) k_d k_{rg}}{\left(1 + \frac{\rho_g v_g}{\mu_g S_g \tau}\right)}} \quad (\text{A.3})$$

If we replace pressure gradient by potential gradient and extend one-dimensional velocity in (A.3) to a multidimensional vector, we have a general form of Eq. 2.2 for correlating flow potential gradient and flow rate for the Barree and Conway model.

For incorporation of the Barree and Conway model into the continuity or mass conservation equation 2.1, it is more convenient to express flow rate in terms of pressure or potential gradient. Under one-dimensional flow condition along  $x$ -direction, solving the flow velocity from (2.2) in terms of flow potential gradient leads to,

$$v_\beta = \frac{-[\mu_\beta^2 S_\beta \tau - \left(-\frac{\partial \Phi_\beta}{\partial x}\right) k_d k_{r\beta} k_{rm} \rho_\beta] + \sqrt{[\mu_\beta^2 S_\beta \tau - \left(-\frac{\partial \Phi_\beta}{\partial x}\right) k_d k_{r\beta} k_{rm} \rho_\beta]^2 + 4\mu_\beta \rho_\beta \left(-\frac{\partial \Phi_\beta}{\partial x}\right) k_d k_{r\beta} \mu_\beta S_\beta \tau}}{2\mu_\beta \rho_\beta} \quad (\text{A.4})$$

Note that Eq. A.4 is used in this article to replace Darcy's law for correlating flow rate and flow potential gradient according to the Barree and Conway non-Darcy flow model.

## Appendix B. Derivation of Buckley–Leverett Analytical Solution

For the derivation of the analytical solution, we assume the following Buckley–Leverett flow conditions for one-dimensional non-Darcy flow of two immiscible fluids:

- Both fluids and the porous medium are incompressible.
- Capillary pressure gradient is negligible.
- Gravity segregation effect is negligible (i.e., stable displacement exists near the displacement front).
- One-dimensional flow and displacement is along the  $x$ -coordinate of a semi-infinite linear flow system with a constant cross-sectional area ( $A$ ).

Among these assumptions, incompressibility of fluids and formation is critical to deriving the Buckley–Leverett solution. This assumption provides a good approximation to displacement processes of two liquids (e.g., oil and water) in porous media, because of the small compressibilities of the two fluids.

Under the Buckley–Leverett flow conditions, Eq. 2.1 can then be changed for describing two-phase displacement of one wetting ( $\beta = w$ ) and one nonwetting phase ( $\beta = n$ ) as follows:

$$-\frac{\partial v_\beta}{\partial x} = \phi \frac{\partial S_\beta}{\partial t} \quad (\text{B.1})$$

For the one-dimensional flow,  $v_\beta$  can be determined from Eqs. 2.2 or A.4 as,

$$\begin{aligned} v_\beta = & \frac{1}{2\mu_\beta\rho_\beta} \left( -[\mu_\beta^2 S_\beta \tau - \left( \frac{\partial P}{\partial x} + \rho_\beta g \sin(\alpha) \right) k_d k_r \beta k_{rm} \rho_\beta] + \right. \\ & \left. + \frac{1}{2\mu_\beta\rho_\beta} \left( \sqrt{[\mu_\beta^2 S_\beta \tau - \left( \frac{\partial P}{\partial x} + \rho_\beta g \sin(\alpha) \right) k_d k_r \beta k_{rm} \rho_\beta]^2 + 4\mu_\beta\rho_\beta \left( \frac{\partial P}{\partial x} + \rho_\beta g \sin(\alpha) \right) k_d k_r \beta \mu_\beta S_\beta \tau} \right) \right) \end{aligned} \quad (\text{B.2})$$

where  $(\frac{\partial P}{\partial x})$  is a component of the pressure gradient along the  $x$ -coordinate—the same for the wetting or nonwetting phase;  $g$  is the gravitational acceleration constant, and  $\alpha$  is the angle between the horizontal plane and the flow direction (the  $x$ -coordinate).

To complete the mathematical description of the physical problem, the initial and boundary conditions must be specified. For simplicity in derivation, the system is initially assumed to be uniformly saturated with both wetting and nonwetting fluids. The wetting phase is at its residual saturation, and a nonwetting fluid, such as oil or gas, is at its maximum saturation in the system, as follows:

$$S_n(x, t = 0) = 1 - S_{wr} \quad (\text{B.3})$$

where  $S_{wr}$  is the initial, residual wetting-phase saturation. Wetting fluid, such as water, is continuously being injected at a known rate  $q$ . Therefore, the boundary conditions at the inlet ( $x = 0$ ) are:

$$v_w(x = 0, t) = \frac{q}{A} \quad (\text{B.4})$$

where  $v_w$  is the flow rate or flux of water across a unit area of the one-dimensional system; and  $A$  is the cross-sectional area of the one-dimensional flow system and

$$v_n(x = 0, t) = 0 \quad (\text{B.5})$$

The derivation of the analytical solution follows the study by Wu et al. (1991), in which the fractional flow concept is used to simplify the governing Eq. B.1 in terms of saturation only. The fractional flow of a fluid phase is defined as a volume fraction of the phase flowing at a location  $x$  and time  $t$  to the total volume of the flowing phases (Willhite 1986). The fractional flow can be written as

$$f_\beta = \frac{v_\beta}{v_w + v_n} = \frac{v_\beta}{v} \quad (\text{B.6})$$

where the total flow flux is

$$v = v_w + v_n \quad (\text{B.7})$$

From volume balance due to incompressibility of the system, we have

$$f_w + f_n = 1 \quad (\text{B.8})$$

The fractional flow function for the wetting phase may be written in the following form:

$$f_w = \frac{1}{1 + \frac{v_n}{v_w}} \quad (\text{B.9})$$

when the flux  $v_w$  and  $v_n$  for wetting and nonwetting phases are defined in Eq. B.2.

Equation B.9, as well as (B.2), indicates that the fractional flow  $f_w$  of the wetting phase is a function of both saturation and pressure gradient. However, for a given injection rate and for given fluid and rock properties of a porous material, the pressure gradient at a given time can be shown by the following to be a function of saturation only under the Buckley–Leverett flow conditions:

$$\begin{aligned} q = & \frac{A}{2\mu_w\rho_w} \left( -[\mu_w^2 S_w \tau - \left( \frac{\partial P}{\partial x} + \rho_w g \sin(\alpha) \right) k_d k_{rw} k_{rm} \rho_w] + \right. \\ & \left. - \frac{1}{2\mu_w\rho_w} \left( \sqrt{[\mu_w^2 S_w \tau - \left( \frac{\partial P}{\partial x} + \rho_w g \sin(\alpha) \right) k_d k_{rw} k_{rm} \rho_w]^2 + 4\mu_w\rho_w \left( \frac{\partial P}{\partial x} + \rho_w g \sin(\alpha) \right) k_d k_{rw} \mu_w S_w \tau}} \right) \right. \\ & \left. - \frac{A}{2\mu_n\rho_n} \left( -[\mu_n^2 S_n \tau - \left( \frac{\partial P}{\partial x} + \rho_n g \sin(\alpha) \right) k_d k_{rn} k_{rm} \rho_n] + \right. \right. \\ & \left. \left. - \frac{1}{2\mu_n\rho_n} \left( \sqrt{[\mu_n^2 S_n \tau - \left( \frac{\partial P}{\partial x} + \rho_n g \sin(\alpha) \right) k_d k_{rn} k_{rm} \rho_n]^2 + 4\mu_n\rho_n \left( \frac{\partial P}{\partial x} + \rho_n g \sin(\alpha) \right) k_d k_{rn} \mu_n S_n \tau}} \right) = 0 \right) \end{aligned} \quad (\text{B.10})$$

Equation B.10 shows that the pressure gradient and the saturation are inter-dependent on each other for this particular displacement system of Buckley–Leverett non-Darcy flow. Therefore, Eq. B.10 implicitly defines the pressure gradient in the system as a function of saturation.

The governing equation, Eq. B.1, subject to the boundary and initial conditions described in Eqs. B.3–B.5 can be solved for the frontal advance equation (see Appendix A in Wu et al. 1991):

$$\left( \frac{dx}{dt} \right)_{S_w} = \frac{q(t)}{\phi A} \left( \frac{\partial f_w}{\partial S_w} \right)_t \quad (\text{B.11})$$

Note that (B.11) has the same form as the Buckley–Leverett equation. However, the dependence of the fractional flow  $f_w$  for the non-Darcy displacement on saturation is different. The fractional flow,  $f_w$ , is related to saturation not only through the relative permeability functions, as in the case of Buckley and Leverett solution, but also through the pressure gradient, as described by Eq. B.10.

Equation B.11 shows that, for a given time and a given injection rate, a particular wetting fluid saturation profile propagates through the porous medium at a constant velocity. As in the Buckley–Leverett theory, the saturation for a vanishing capillary pressure gradient will in general become a triple-valued function of distance near the displacement front (Cardwell 1959). Equation B.11 will then fail to describe the velocity of the shock saturation front, since  $(\partial f_w / \partial S_w)$  does not exist on the front because of the discontinuity in  $S_w$  at that point. The location  $x_{S_w}$  of any saturation  $S_w$  traveling from the inlet at time  $t$  can be determined by integrating Eq. B.11 with respect to time, yielding

$$x_{S_w} = \frac{q \times t}{\phi A} \left( \frac{\partial f_w}{\partial S_w} \right)_{S_w} \quad (\text{B.12})$$

Direct use of Eq. B.12, given  $x$  and  $t$ , will result in a multiple-valued saturation distribution, which can be handled by a mass-balance calculation, as in the Buckley–Leverett solution. An alternative graphic method of Welge (1952) can be shown (Wu et al. 1991) to apply to calculating the above solution in this case. The only additional step in applying this method is to take into account the contribution of the pressure gradient dependence to the non-Darcy displacement, using a fractional flow curve. Therefore, the wetting-phase saturation at the

displacement saturation front may be determined by

$$\left( \frac{\partial f_w}{\partial S_w} \right)_{S_F} = \frac{(f_w)_{S_F} - (f_w)_{S_{wr}}}{S_F - S_{wr}} \quad (\text{B.13})$$

The average saturation in the displaced zone is given by

$$\left( \frac{\partial f_w}{\partial S_w} \right)_{S_F} = \frac{1}{\bar{S}_w - S_{wr}} \quad (\text{B.14})$$

where  $\bar{S}_w$  is the average saturation of the wetting phase in the swept zone behind the sharp displacement front (see Fig. 6 in Wu et al. 1991). Then, the complete saturation profile can be determined using Eq. B.12 for a given non-Darcy displacement problem with constant injection rate according to the Barree and Conway model.

### Appendix C. Derivation of the Equivalent Forchheimer non-Darcy Parameters

The equivalent Forchheimer non-Darcy parameters can be calculated from Barree and Conway input parameters. Wu (2001) describes the Forchheimer model for multiphase flow system as follows,

$$-\nabla \Phi_\beta = \frac{\mu_\beta \mathbf{v}_\beta}{k_d k_{r\beta}} + \beta_\beta \rho_\beta \mathbf{v}_\beta |\mathbf{v}_\beta| \quad (\text{C.1})$$

where  $\nabla \Phi_\beta$  is the flow potential gradient fluid  $\beta$ ;  $\rho_\beta$  is the density of fluid  $\beta$ ;  $\mathbf{v}_\beta$  is the volumetric velocity vector of fluid  $\beta$ ;  $\mu_\beta$  is the viscosity of fluid  $\beta$ ;  $k_d$  is the Darcy permeability;  $k_{r\beta}$  is the relative permeability of fluid  $\beta$ ; and  $\beta_\beta$  is the non-Darcy coefficient, where it is defined as

$$\beta_\beta = \frac{C_\beta}{(k_d k_{r\beta})^{1.25} (\phi (S_\beta - S_{\beta r}))^{0.75}} \quad (\text{C.2})$$

where  $C_\beta$  is the non-Darcy constant;  $S_\beta$  is the saturation of fluid  $\beta$ ; and  $S_{\beta r}$  is the irreducible saturation of fluid  $\beta$ .

From Eqs. 2.2 and C.1, we can solve non-Darcy coefficient ( $\beta$ ) in term of Barree–Conway input parameters.

$$\beta_\beta = \frac{\mu_\beta (1 - k_{mr})}{k k_{r\beta} (k_{mr} \rho_\beta |\mathbf{v}_\beta| + \mu_\beta \tau)} \quad (\text{C.3})$$

And we can calculate non-Darcy constant as

$$C_\beta = \frac{\mu_\beta (1 - k_{mr}) (k k_{r\beta})^{0.25} (\phi (S_\beta - S_{\beta r}))^{0.75}}{(k_{mr} \rho_\beta |\mathbf{v}_\beta| + \mu_\beta \tau)} \quad (\text{C.4})$$

where  $k_{mr}$  is the minimum permeability ratio at high rate, relative to Darcy permeability (fraction); and  $\tau$  is the characteristic length.

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