An analytical solution for transient radial flow through unsaturated fractured porous media

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Received 13 February 2004; revised 6 December 2004; accepted 13 December 2004; published 26 February 2005.

[1] This paper presents analytical solutions for one-dimensional radial transient flow through a horizontal, unsaturated fractured rock formation. In these solutions, unsaturated flow through fractured media is described by a linearized Richards' equation, while fracture-matrix interaction is handled using the dual-continuum concept. Although linearizing Richards' equation requires a specially correlated relationship between relative permeability and capillary pressure functions for both fractures and matrix, these specially formed relative permeability and capillary pressure functions are still physically meaningful. These analytical solutions can thus be used to describe the transient behavior of unsaturated flow in fractured media under the described model conditions. They can also be useful in verifying numerical simulation results, which as demonstrated in this paper, are otherwise difficult to validate.

Citation: Wu, Y.-S., and L. Pan (2005), An analytical solution for transient radial flow through unsaturated fractured porous media, *Water Resour. Res.*, *41*, W02029, doi:10.1029/2004WR003107.

1. Introduction

[2] In the past few decades, flow through unsaturated fractured rock, a special case of multiphase flow, has received a lot of attention because of subsurface environmental considerations. Quantitative analysis of flow in unsaturated fractured rock is often based on Richards' equation. Because of its nonlinear nature, Richards' equation solutions for general flow through fractured media may be obtained only with a numerical approach. On the other hand, analytical solutions, if available, provide more direct insight into the physics of unsaturated flow phenomena than do numerical or laboratory studies, and they are often needed to examine and verify numerical model schemes or results.

[3] For unsaturated flow through homogeneous singleporosity soils, many analytical solutions, both exact and approximate, have been developed, based on different levels of Richards' equation linearization [e.g., *Pullan*, 1990; *Warrick and Parkin*, 1995; *Basha*, 1999; *Philips*, 1969; *Zimmerman and Bodvarsson*, 1995]. Despite the advances made so far, however, precise analytical solutions to Richards' equation remain intractable under general flow conditions, because of its known nonlinearity. In addition, it becomes more difficult to obtain an analytical solution for flow through unsaturated fractured porous media because of the additional complexity introduced by fracture-matrix interaction.

[4] Recently, we presented a set of new analytical solutions for unsaturated flow within a single matrix block with fracture-matrix interaction [*Wu and Pan*, 2003]. These analytical solutions required a specially correlated relationship between relative permeability and capillary pressure functions. The present work extends our analytical solution approach to the entire fracture-matrix flow system, using a general dual-continuum approach. In this work, we show that it is possible to obtain analytical solutions if the specially correlated relative permeability and capillary pressure functions hold true for both fracture and matrix systems. In addition, we demonstrate that the new analytical solutions are very useful for checking numerical model results for unsaturated flow through fractured porous media.

2. Mathematical Formulation

[5] The problems to be solved are cases of unsaturated radial flow in a horizontal and uniform fracture-matrix formation corresponding to a fully penetrating injection well, with either constant well pressure or constant injection rate. The formation consists of identical cubic matrix blocks separated by a uniform three-dimensional fracture network, as in the Warren and Root model [Warren and Root, 1963]. In this work, the Warren and Root double-porosity model is extended into a general dual-continuum concept to include flow within matrix [e.g., Pruess and Narasimhan, 1985]. In the extended dual-continuum model, an "effective" porous continuum is adapted to approximate these two types of media (fractures and rock matrix), and unsaturated flow in fractured rocks is separately described using a doublet of Richards' equations for the two continua. In particular, flow inside matrix blocks is handled fully transiently using a local spherical coordinate, which is different from the quasisteady state method of the Warren and Root model. Furthermore, we assume that the two sets of capillary pressure and relative permeability functions for fracture and matrix, respectively, are in the form:

$$k_{r\xi}(S_{\xi}) = C_{k\xi}(S_{\xi}^{*})^{\alpha_{\xi}}$$
(1)

and

$$P_{c\xi}(S_{\xi}) \equiv P_{g\xi} - P_{w\xi} = C_{p\xi}(S_{\xi}^{*})^{-\beta_{\xi}}$$

$$(2)$$

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This paper is not subject to U.S. copyright. Published in 2005 by the American Geophysical Union. where subscript ξ is an index for fracture ($\xi = F$) or matrix ($\xi = M$); $P_{g\xi}$ is constant air (or gas) pressure in fractures or the matrix; $P_{w\xi}$ is liquid water pressure in fractures or the matrix, respectively; $C_{k\xi}$ and $C_{p\xi}$ (Pa) are coefficients, α_{ξ} and β_{ξ} are exponential constants of relative permeability and capillary pressure functions, respectively, for fracture or matrix systems; and S_{ξ}^{*} is the effective or normalized fracture or matrix water saturation,

$$S_{\xi}^{*} = \frac{S_{\xi} - S_{\xi r}}{1 - S_{\xi r}}$$
(3)

with $S_{\xi r}$ being the residual water saturation in fracture or matrix systems.

If the following condition:

$$\alpha_{\xi} = \beta_{\xi} + 1 \tag{4}$$

is satisfied for both fractures and matrix, Richards' equation can be readily linearized for flow through both fractures or the matrix [*Wu and Pan*, 2003]. In this work, however, the linearization of the continuity in capillary pressure on the matrix surface requires the exponential constants of capillary $\beta_{\xi} = 1$, leading to $\alpha_{\xi} = 2$, for both fracture and matrix media. The linearized governing equation for unsaturated radial flow (ignoring the gravity effect and the compressibility of water and rock) through the fractures can be derived by combining a mass balance on a control volume with the dual-continuum concept (see Appendix A), as follows:

$$\frac{\partial^2 S_F}{\partial r^2} + \frac{1}{r} \frac{\partial S_F}{\partial r} - \frac{6 \phi_M}{B \phi_F} \frac{D_M}{D_F} \frac{\partial S_M}{\partial x} \bigg|_{x=B/2} = \frac{1}{D_F} \frac{\partial S_F}{\partial t}$$
(5)

where ϕ_M and ϕ_F are porosity for matrix or fractures (if not described, the symbols for variables and parameters in equation (5) or in the following equations are defined in the appendixes.) The third term on the left-hand side of equation (5) represents flow exchange terms on the local matrix interface between fracture and matrix systems, describing the continuity in mass flux.

[6] For flow inside the matrix, we use a 1-D spherical flow approximation, because the 1-D spherical flow within matrix blocks is the most commonly used in the literature for estimating matrix flow. In addition, the different shape and flow geometry of matrix blocks are found to have an insignificant effect on the fracture-matrix interaction of water and oil in fractured petroleum reservoirs [*Wu and Pruess*, 1988]. One-dimensional spherical unsaturated flow inside a cubic matrix block is then governed by [*Wu and Pan*, 2003]

$$\frac{\partial^2 S_M}{\partial x^2} + \frac{2}{x} \frac{\partial S_M}{\partial x} = \frac{1}{D_M} \frac{\partial S_M}{\partial t}.$$
 (6)

The initial conditions within fractures and matrix systems are uniform:

$$\mathbf{S}_{\xi}\big|_{t=0} = \mathbf{S}_{\xi r} \tag{7}$$

Note here that for simplicity, initial saturations in fractures and the matrix are set to their residual values. [7] The first inner boundary condition is that the wellbore be specified with constant saturation:

$$\mathbf{S}_{\mathrm{F}}(\mathbf{r} = \mathbf{r}_{\mathrm{w}}, \mathbf{t}) = \mathbf{S}_{0} \tag{8}$$

and the second is that the injection rate q be:

$$\left. \frac{2\pi r_w h k_F C_{kF} C_{pF}}{\mu_w} \frac{\partial S_F^*}{\partial r} \right|_{r=r_w} = q \tag{9}$$

Far from the well, the saturation in the fractures remains at its initial value:

$$S_{\rm F}(r=\infty,t) = S_{\rm Fr} \tag{10}$$

At the matrix surface, continuity in pressure or capillary pressure is enforced:

$$P_{cF}(r,t) = P_{cM}(x = B/2,t;r)$$
 (11)

while at the matrix block center, a zero-gradient condition is maintained for symmetry:

$$\frac{\partial S_{M}(x=0,t;r)}{\partial x} = 0 \tag{12}$$

The linearized equation system above is similar to the gradient flow model for single-phase flow through fractured reservoirs [*Streltsova*, 1983].

3. Analytical Solutions

[8] In the following dimensionless variables, the dimensionless distances and time are defined as

$$r_{D} = \frac{r}{r_{w}}, \quad x_{D} = \frac{2x}{B}, \text{ and } t_{D} = \frac{D_{F}t}{(B/2)^{2}}$$
 (13)

The normalized (or scaled) water saturation is defined as the same as the effective saturation of equation (3).

[9] In terms of these dimensionless variables, the solution for the normalized matrix saturation in Laplace space is given by (Appendix B):

$$\bar{S}_{MD} = A_4 \frac{\bar{S}_{FD}}{\sqrt{x_D}} \frac{I_{1/2}(\sigma x_D)}{I_{1/2}(\sigma)} = A_4 \frac{\bar{S}_{FD}}{x_D} \frac{\sinh(\sigma x_D)}{\sinh(\sigma)}$$
(14)

where $\sigma = \sqrt{A_3 p}$ and $I_{1/2}$ is the modified Bessel function of the first kind.

[10] \bar{S}_{FD} is the solution of the normalized fracture saturation in Laplace space, defined differently for the two well boundary conditions (Appendix B). The solution \bar{S}_{FD} with constant water saturation at the wellbore, in equation (8), is given by

$$\bar{S}_{FD} = \frac{S_{0D}}{p} \frac{K_0(\sqrt{x_2}r_D)}{K_0(\sqrt{x_2})}$$
(15)

where $x_2 = A_1A_4[\sigma \operatorname{coth} \sigma - 1] + A_2p$ and $A_4 = C_{pM}/C_{pF}$. For the case of constant flow rate as defined in equation (9), the solution \bar{S}_{FD} is

$$\bar{S}_{FD} = \frac{q_D}{p} \frac{K_0(\sqrt{x_2} r_D)}{\sqrt{x_2} K_1(\sqrt{x_2})}$$
(16)

Parameter	Matrix	Fracture	Units
Matrix dimension	B = 1		m
Porosity	$\phi_{M} = 0.30$	$\phi_{\rm F} = 0.001$	
Permeability	$k_{\rm M} = 1.0 \times 10^{-15}$	$k_{\rm F} = 1.0 \times 10^{-12}$	m^2
Residual/initial saturation	$S_{Mr} = 0.2$	$S_{Fr} = 0.01$ (Figures 1 and 2) or 0.2 (Figure 3)	
Coefficient of relative permeability	$C_{kM} = 1.0$	$C_{kF} = 0.1$ (Figures 1 and 2) or 1.0 (Figure 3)	
Coefficient of capillary pressure	$C_{pM} = 1.0 \times 10^4$	$C_{nF} = 1.0 \times 10^3$	Ра
Saturation at well	$S_0 = 1.0$	$S_0^{r} = 1.0$	
Injection rate at well	$q = 1.57 \times 10^{-4}$	$q = 1.57 \times 10^{-4}$	m ³ /s
Fluid viscosity	$\mu_{\rm w} = 1.0 \times 10^{-3}$	$\mu_{\rm w} = 1.0 \times 10^{-3}$	Pa s
Fluid density	$\rho = 1000$	$\rho = 1000$	kg/m ³
Wellbore radius	$r_{w} = 0.1$	$r_w = 0.1$	m

Table 1. Parameters for the Type Curves of Fracture Liquid Saturation

In the solutions above, K_0 and K_1 are the modified Bessel functions of the second kind of zero and first order, respectively.

[11] Equations (14), (15), and (16) constitute the solutions in Laplace space for normalized fracture and matrix saturations under two types of inner well boundary conditions, which depend on four parameters (A₁, A₂, A₃, and A₄), relating to matrix size and ratios of porosity, absolute permeability, and relative permeability and capillary pressure coefficients for fracture and matrix systems, in addition to the dimensionless spatial and time variables. Analysis of these dimensionless variables and their interrelations indicates that the dimensionless solutions are characterized mainly by the ratios of permeability, storage, and capillary pressure coefficients of fractures and matrix, respectively. This shows that the analytical solutions are a little more complicated that the Warren and Root solution for singlephase flow, which is characteristic of only two parameters, λ and ω .

4. Application

[12] To apply these solutions, we use the *Stehfest* [1970] algorithm to invert these solutions from Laplace space to real space. First, the analytical solutions of equations (15) and (16) were used to calculate type curves for transient flow through a fully penetrating well into a uniform, horizontal fractured formation, which is 10 m thick and radially infinite. The fractured formation consists of uniform, identical $1 \times 1 \times 1$ m cubes of matrix blocks, surrounded by a uniform, 3-D fracture network, identical to the Warren and Root conceptual model. The basic fracture-matrix and fluid parameters used for the example are listed in Table 1. A numerical inversion algorithm [Stehfest, 1970] was used in calculating the analytical solutions. During the numerical Laplace inversion, Stehfest's method was found to work well, and the inverted solutions were verified by comparison with numerical simulations and asymptotic solutions through many numerical experiments.

[13] The results are depicted as normalized liquid saturation in fractures versus the dimensionless radial distance from the well at different dimensionless times in Figures 1 (a constant saturation at the wellbore) and 2 (a constant injection rate at well), respectively. These two dimensionless type curves can be used for examining numerical model results for the same fracture-matrix flow system.

[14] The analytical solutions were then used to assess the performance of the double-porosity (i.e., the Warren and

Root model) and MINC approaches in simulating fracturematrix interaction in unsaturated fracture rock. Note that numerical implementation of the double-porosity model is equivalent to representing the matrix by one single grid block, whereas the MINC concept subdivides a matrix block into many single or multidimensional or nested cells [*Pruess and Narasimhan*, 1985]. Numerical simulations are performed using a numerical reservoir simulator [*Wu et al.*, 1996]. Note that the governing equation solved in numerical modeling is still the original Richards' equation, instead of the linearized forms of equations (5) and (6). The same relative permeability and capillary functions as those of equations (1) and (2) in the analytical solutions are input to the numerical model.

[15] In the second application, the radial single-well flow system is basically the same as that in the first example. Rock and fluid properties are given in Table 1. In the numerical model, however, the uniform, horizontal, radially infinite formation is represented by a finite radial system with an outer boundary radius of 1000 m, large enough such that the wetting front does not reach the boundary at the end of simulation. Two numerical grids, a double-porosity mesh and a MINC, are generated for



Figure 1. Type curves of normalized liquid saturation in fractures versus dimensionless radial distance at different dimensionless times, with $S_F = 1$ at well.



Figure 2. Type curves of normalized liquid saturation in fractures versus dimensionless radial distance at different dimensionless times, with constant injection dimensionless rate $q_D = 0.25$ at well.

the radial symmetric formation. The double-porosity grid represents the matrix system by one mesh locally [*Warren and Root*, 1963], equivalent to one-block approximation of the matrix continuum in the MINC approach [*Wu and Pruess*, 1988], while the MINC grid subgrids each matrix block with seven nested cells, for better numerical accuracy in estimating fracture-matrix flow [*Pruess and Narasimhan*, 1985].

[16] Figure 3 shows the saturation distribution along the fractures in the radial direction at time of 0.1 and 10 days, respectively, simulated by the analytical, doubleporosity, and MINC modeling results. Note that the physical process simulated in this example is extremely nonlinear and dynamic. The initial liquid saturations are at residual values for both fracture and matrix systems. At the beginning, the boundary saturation for fractures at the wellbore jumps to one (thus flow rate at the well becomes infinitely large). Once imbibed into the fractures, the liquid will be competed by two forces in two directions, one for continuous flow along fractures away from the well, and the other sucked into dry matrix blocks.

[17] The numerical model results, shown in Figure 3, indicate that the MINC model does a much better job in matching the analytical solutions than the double-porosity model. This implies that in this case, the MINC concept better captures these physical processes by considering capillary gradients at the matrix surface and inside matrix blocks. However, Figure 3 shows that even the MINC simulations with seven cells cannot match the analytical results very well, because the matrix surface is subject to dynamic boundary conditions (i.e., varying fracture saturation or capillary forces, which occurs at the upstream boundary for initializing imbibition into the matrix). Using a dual-continuum numerical approach [*Wu and Pan*, 2003], extremely refined spatial discretization (30 MINC cells) is

required to model imbibing processes accurately under dynamic upstream boundary conditions.

[18] Although the analytical solutions presented above are obtained under very strict assumptions (i.e., specially correlated relations between relative permeability and capillary functions for both fractures and the matrix, as well as negligible gravity effects), the relative permeability and capillary functions of equations (1) and (2) are not only physically meaningful, but also among the most widely used relations [*Honarpour et al.*, 1986]. Since the treatment of fracture-matrix interaction in this analytical model is based rigorously on the dual-continuum concept, the analytical solutions obtained can be useful in evaluating numerical model results from dual-continuum models, as shown above.

5. Concluding Remarks

[19] This paper shows that with specially formed capillary pressure and relative permeability functions, it is possible to obtain analytical solutions for transient unsaturated flow in fracture-matrix systems using the commonly used dual-continuum concept. With the analytical solutions provided in Laplace space, analytical solutions in real space can be readily obtained using numerical inversion techniques. The analytical solution approach of this work can be readily extended to other boundary conditions or different flow geometries, such as linear and multidimensional unsaturated flow through fractured porous formations.

[20] The analytical solutions, though limited by the assumptions for their applications, can be used to obtain some insight into the physics of transient flow processes related to fracture-matrix interactions. As demonstrated in this work, these analytical solutions are very useful in verifying numerical models and their results for describing flow through unsaturated fractured rock, especially the flow



Figure 3. Comparison of fracture liquid saturations simulated using the analytical solution and numerical results with the double-porosity and MINC modeling approaches, with $S_F = 1$ at well.

through a fracture-matrix interface, which is otherwise very difficult to evaluate.

Appendix A: Derivation of Governing Equations

[21] Let us consider the situation of unsaturated flow in a horizontal and uniform fracture-matrix formation using the Warren and Root conceptual model for fracture network and matrix blocks. Furthermore, gravity effects are ignored and incompressible liquid flows through a single well into a radial infinite system. The governing equations of unsaturated radial flow through such a fracture-matrix system can be derived by combining a mass balance on the control volume with the dual-continuum concept [*Lai et al.*, 1983]. In the radial system, a control (bulk) volume at a radial distance of r from the well is defined as

$$V_n = \pi \Bigl((r+dr)^2 - r^2 \Bigr) h \approx 2 \pi r h dr \eqno(A1)$$

where h is the thickness of formation.

[22] The interface area A_c between rock matrix blocks and surrounding fractures within the control volume, when the volume fraction of fractures can be ignored, V_n is written as

$$A_{c} = 6B^{2} \left(\sqrt[V_{n}]{B^{3}} \right) \tag{A2}$$

where B is the dimension of matrix cubes.

[23] Mass balance for the incompressible fluid for the fracture system within the control volume requires that:

$$q_{r}A_{r} - \left[q_{r}A_{r} + \frac{\partial}{\partial r}(q_{r}A_{r})dr\right] + \left[q_{x}A_{c}\right]|_{x=B/2} = \frac{\partial(V_{n}\varphi_{F}S_{F})}{\partial t} \quad (A3)$$

where x is the distance from a nested cross-sectional surface within the matrix block (having an equal distance to the matrix surface) to the center of the cube (i.e., a onedimensional spherical coordinate with its center within the matrix block), and q_r and q_x are Darcy's velocity along r and x directions, respectively, calculated as

$$q_{r} = -\frac{k_{F}k_{rF}}{\mu_{w}}\frac{\partial P_{wF}}{\partial r} = \frac{k_{F}k_{rF}}{\mu_{w}}\frac{\partial P_{cF}}{\partial r} = -\frac{k_{F}C_{kF}C_{pF}}{\mu_{w}(1 - S_{Fr})}\frac{\partial S_{F}}{\partial r} \quad (A4)$$

and

$$q_{x} = -\frac{k_{M}k_{rM}}{\mu_{w}}\frac{\partial P_{wM}}{\partial x} = \frac{k_{M}k_{rM}}{\mu_{w}}\frac{\partial P_{cM}}{\partial x} = -\frac{k_{M}C_{kM}C_{pM}}{\mu_{w}(1 - S_{Mr})}\frac{\partial S_{M}}{\partial x}$$
(A5)

where μ_w is dynamic water viscosity. Substituting (A4) and (A5), as well as, radial cross area $A_r = 2\pi rh$ into (A3), yields

$$\begin{aligned} \frac{k_F C_{kF} C_{pF}}{\mu_w (1 - S_{Fr})} \left(\frac{\partial^2 S_F}{\partial r^2} + \frac{1}{r} \frac{\partial S_F}{\partial r} \right) &- \frac{6}{B} \frac{k_M C_{kM} C_{pM}}{\mu_w (1 - S_{Mr})} \frac{\partial S_M}{\partial x} \Big|_{x=B/2} \\ &= \varphi_F \frac{\partial S_F}{\partial t} \end{aligned}$$
(A6)

If we define D_{ξ} as soil water or moisture diffusivity [*Wu and Pan*, 2003],

$$D_{\xi} = \frac{k_{\xi}k_{rx}}{\varphi_{\xi}\mu_{w}} \frac{\partial P_{wx}}{\partial S_{w}} = \frac{k_{\xi}C_{kx}C_{px}}{\varphi_{\xi}\mu_{w}\left(1 - S_{\xi r}\right)}$$
(A7)

with a dimension of m^2/s . We thus derive the linearized flow governing equation (5) for flow through the fractures.

Appendix B: Derivation of Analytical Solutions

[24] In terms of the dimensionless variables (equations (3) and (13)), the two governing equations become:

$$\frac{\partial^2 S_{FD}}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial S_{FD}}{\partial r_D} - A_1 \frac{\partial S_{MD}}{\partial x_D} \bigg|_{x_D = 1} = A_2 \frac{\partial S_{FD}}{\partial t_D}$$
(B1)

and

$$\frac{\partial^2 S_{MD}}{\partial x_D^2} + \frac{2}{x_D} \frac{\partial S_{MD}}{\partial x_D} = A_3 \frac{\partial S_{MD}}{\partial t_D}$$
(B2)

where

$$A_1 = \frac{12D_M \varphi_M r_w^2}{D_F \varphi_F B^2} \frac{1 - S_{Mr}}{1 - S_{Fr}}, A_2 = \frac{4r_w^2}{B^2}, \text{ and } A_3 = \frac{D_F}{D_M}$$
(B3)

The initial condition becomes

$$\mathbf{S}_{\xi \mathbf{D}}\big|_{\mathbf{t}_{\mathbf{D}}=\mathbf{0}} = \mathbf{0} \tag{B4}$$

The boundary condition of constant saturation at the well becomes

$$S_{FD}(r_D=1,t_D) = \frac{S_0 - S_{Fr}}{1-S_{Fr}} = S_{0D} \eqno(B5)$$

The constant rate condition turns into

$$\frac{\partial S_{FD}}{\partial r_D}\Big|_{r_D=1} = \frac{q\mu_w}{2\pi h k_F C_{kF} C_{pF}} = q_D \tag{B6}$$

Far from the well,

$$S_{FD}(r_D=\infty,t_D)=0 \tag{B7}$$

At the matrix surface, the continuity in pressure becomes

$$S_{MD}(x_{D} = 1, t_{D}; r_{D}) = \frac{C_{pM}}{C_{pF}} S_{FD}(r_{D}, t_{D})$$
(B8)

Note that this linear relationship for the continuity in pressure on the matrix surface assumes the exponential constants of capillary $\beta_{\xi} = 1$ for both fracture and matrix media. At the matrix block center,

$$\frac{\partial S_{MD}(x_D=0,t_D;r_D)}{\partial x_D} = 0 \tag{B9}$$

Applying the Laplace transformation to equations (B1) and (B2) and incorporating the initial condition (B4) yield

$$\frac{\partial^2 \bar{S}_{FD}}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial \bar{S}_{FD}}{\partial r_D} - A_1 \frac{\partial \bar{S}_{MD}}{\partial x_D} \bigg|_{x_D = 1} - p A_2 \bar{S}_{FD} = 0$$
 (B10)

and

$$\frac{\partial^2 S_{MD}}{\partial x_D^2} + \frac{2}{x_D} \frac{\partial S_{MD}}{\partial x_D} - pA_3 \bar{S}_{MD} = 0$$
(B11)

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where \bar{S}_{ED} is the Laplace transformed normalized saturation, and p is the Laplace variable. Then the solutions of normalized matrix and fracture saturations, subject to the Laplace transformed boundary conditions from equations (B5)-(B9), are given by equations (14), (15) and (16).

[25] Acknowledgments. The authors would like to thank Yingqi Zhang and Dan Hawkes for their reviews of this manuscript. The authors would also like to thank their colleague, George Moridis, for providing the computing program for numerical Laplace inversion. This work was supported in part by the Director, Office of Civilian Radioactive Waste Management, U.S. Department of Energy, through Memorandum Purchase Order QA-B004220RB3X between Bechtel SAIC Company, LLC and the Ernest Orlando Lawrence Berkeley National Laboratory (Berkeley Lab). The support is provided to Berkeley Lab through the U.S. Department of Energy contract DE-AC03-76SF00098.

References

- Basha, H. A. (1999), Multidimensional linearized nonsteady infiltration with prescribed boundary conditions at the soil surface, Water Resour. Res., 35(1), 75-83.
- Honarpour, M., L. Koederitz, and A. H. Harvey (1986), Relative Permeability of Petroleum Reservoirs, CRC Press, Boca Raton, Fla.
- Lai, C. H., G. S. Bodvarsson, C. F. Tsang, and P. A. Witherspoon (1983), A new model for well test data analysis for naturally fractured reservoirs, SPE paper 11688 presented at the 1983 California Regional Meeting, Soc. of Petrol. Eng., Ventura, California, March. Philips, J. R. (1969), Theory of Infiltration, *Adv. Hydrosci.*, *5*, 215–296.

- Pruess, K., and T. N. Narasimhan (1985), A practical method for modeling fluid and heat flow in fractured porous media, Soc. Pet. Eng. J., 25, 14 - 26
- Pullan, A. J. (1990), The quasilinear approximation for unsaturated porous media flow, Water Resour. Res., 26(6), 1219-1234.
- Stehfest, H. (1970), Numerical inversion of Laplace transforms, Commun. ACM, 13(1), 47-49.
- Streltsova, T. D. (1983), Well pressure behavior of a naturally fractured reservoir, Soc. Pet. Eng. J., Oct., 769-780.
- Warren, J. E., and P. J. Root (1963), The behavior of naturally fractured reservoirs, Soc. Pet. Eng. J., Sept., 245-255.
- Warrick, A. W., and G. W. Parkin (1995), Analytical solution for onedimensional drainage: Burgers' and simplified forms, Water Resour. Res., 31(11), 2891-2894.
- Wu, Y. S., and L. Pan (2003), Special relative permeability functions with analytical solutions for transient flow into unsaturated rock matrix, Water Resour. Res., 39(4), 1104, doi:10.1029/2002WR001495.
- Wu, Y. S., and K. Pruess (1988), A multiple-porosity method for simulation of naturally fractured petroleum reservoirs, SPE Reservoir Eng., 3, 327-336.
- Wu, Y. S., C. F. Ahlers, P. Fraser, A. Simmons, and K. Pruess (1996), Software qualification of selected TOUGH2 modules, Rep. LBNL-39490, Lawrence Berkeley Natl. Lab., Berkeley, Calif.
- Zimmerman, R. W., and G. S. Bodvarsson (1995), Effective block size for imbibition or adsorption in dual-porosity media, Geophys. Res. Lett., 22(11), 1461-1464.
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