

# Special relative permeability functions with analytical solutions for transient flow into unsaturated rock matrix

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[1] This paper presents a class of analytical solutions for transient flow into unsaturated rock matrix. These analytical solutions are derived using specially correlated, physically meaningful relative permeability and capillary functions. The transient flow processes in unsaturated rock matrix blocks are generally described by the Richards' equation. The analytical solutions describe the full transient behavior of flow into unsaturated matrix blocks and have proven (through various examples) to be useful for verifying numerical model results. **INDEX TERMS:** 1875 Hydrology: Unsaturated zone; 1829 Hydrology: Groundwater hydrology; 1832 Hydrology: Groundwater transport; 1866 Hydrology: Soil moisture; **KEYWORDS:** Richards' equation, unsaturated zone, relative permeability, capillary pressure, analytical solutions, fracture-matrix interactions

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## 1. Introduction

[2] Fluid flow through variably saturated fractured porous media occurs in many subsurface systems related to petroleum-reservoir engineering, vadose zone hydrology, and soil sciences. Quantitative analysis of such flow in unsaturated soil or rock is fundamentally based on Richards' equation. However, because of its nonlinear nature, Richards' equation solutions for general unsaturated flow may be obtained only with a numerical approach. As a result, great research effort and significant progress have been made in numerical modeling of unsaturated flow and infiltration since the late 1950s. On the other hand, analytical approaches still prove to be irreplaceable. Analytical solutions, if available, provide more direct insight into the physics of unsaturated flow phenomena than numerical or laboratory studies, especially when dealing with effects of various parameters. Moreover, analytical solutions are often needed to examine and verify numerical schemes or results.

[3] In the past few decades, a considerable amount of effort has been devoted in groundwater hydrology and soil science to mathematical modeling of steady state and transient Richards' flow through unsaturated porous media [see, e.g., Milly, 1988; Pullan, 1990; Bodvarsson *et al.*, 2000]. Many exact and approximate analytical solutions have been developed. In general, the analytical solutions derived for Richards' equation are dependent upon the level of the applied linearizations or approximations. These existing, closed-form analytical solutions may be divided into three classes: (1) steady state solutions using the exponential hydraulic conductivity model [Gardner, 1958] and quasi-linear approximations [Pullan, 1990]; (2) transient infiltration solutions using special forms of soil retention curves [e.g., Warrick *et al.*, 1990, 1991; Hills and Warrick, 1993; Warrick and Parkin, 1995; Chen *et al.*, 2001] or using linearization and the Kirchhoff transformation [e.g., Basha,

1999]; and (3) approximate and asymptotic solutions [e.g., Philip, 1969; Zimmerman and Bodvarsson, 1989, 1990, 1995].

[4] Despite the advances made so far, exact analytical solutions to Richards' equation remain intractable under general flow conditions because of the known nonlinearity of Richards' equation. This explains why so much research effort has been devoted to finding new solutions over the past half century. The present work is motivated by our modeling studies to characterize fluid flow and tracer transport in the unsaturated zone of Yucca Mountain, Nevada, a potential repository site for storing high-level radioactive waste. The dual-continuum numerical modeling method has been used in those studies to handle fracture-matrix flow in unsaturated fractured tuffs at the site. Verifying the accuracy of numerical schemes for fracture-matrix interactions motivated us to resort to analytical solutions to examine numerical model results. In this work, we present a class of analytical solutions for unsaturated flow within a matrix block to be used in examining numerical solutions for fracture-matrix interactions. These analytical solutions are derived from a linearized Richards' equation, which requires a specially correlated relationship between relative permeability and capillary-pressure functions.

## 2. Linearization of Richards' Equation

[5] Consider the flow of an incompressible liquid in a homogeneous, isothermal, incompressible, and isotropic porous medium, such as an unsaturated rock matrix. Ignoring air dynamics and gravity, the flow is commonly described by Richards' equation:

$$\nabla \left( \frac{k k_{rw}}{\mu_w} \nabla P_w \right) = \phi \frac{\partial}{\partial t} (S_w), \quad (1)$$

where  $k$  is the absolute permeability,  $k_{rw}$  is the relative permeability to the water phase,  $\mu_w$  is the viscosity of the

water phase,  $P_w$  is the pressure in the water phase,  $\phi$  is the effective porosity of the formation, and  $S_w$  is the water saturation.

[6] To find analytical solutions for equation (1) while keeping  $P_c$  and  $k_{rw}$  as nonlinear functions of  $S_w$ , we select a relative permeability in the form

$$k_{rw}(S_w) = C_k(S_w^*)^\alpha \quad (2)$$

and capillary pressure in the form

$$P_c(S_w) \equiv P_g - P_w = C_p(S_w^*)^{-\beta}, \quad (3)$$

where  $P_g$  is a constant air (or gas) pressure,  $C_k$  and  $C_p$  (Pa) are coefficients,  $\alpha$  and  $\beta$  are exponential constants, respectively, of relative permeability and capillary-pressure functions, and  $S_w^*$  is the effective water saturation,

$$S_w^* = \frac{S_w - S_{wr}}{1 - S_{wr}}, \quad (4)$$

with  $S_{wr}$  as the residual water saturation. Note that if Brooks and Corey's capillary function is used [Brooks and Corey, 1964], the coefficient  $C_p$  in equation (3) becomes the air entry pressure  $P_b$  and  $\beta = 1/\lambda$ , with  $\lambda$  being an index of pore-size distribution [Honarpour et al., 1986].

[7] If the condition

$$\alpha = \beta + 1 \quad (5)$$

is satisfied, the Richards equation (1) can be readily linearized as follows:

$$D \left[ \frac{\partial^2 S_w}{\partial x^2} + \frac{\partial^2 S_w}{\partial y^2} + \frac{\partial^2 S_w}{\partial z^2} \right] = \frac{\partial S_w}{\partial t}, \quad (6)$$

where  $D$ , called soil water or moisture diffusivity [Philip, 1969], is defined by

$$D = \frac{k k_{rw}}{\phi \mu_w} \frac{\partial P_w}{\partial S_w} = \frac{k C_k C_p \beta}{\phi \mu_w (1 - S_{wr})} \quad (7)$$

with a dimension of  $m^2/s$ .

[8] As long as  $D$  is a constant, analytical solutions to equation (6) are easy to obtain. Note that the linearization expressed in equations (2)–(7) is different from simply assuming a constant moisture diffusivity,  $D$  [e.g., Philip, 1969], because relative permeability and capillary pressure are still nonlinear functions of saturation in equations (2) and (3), which are particularly useful in assessing the accuracy of numerical approaches that solve the nonlinear Richards equation (1).

### 3. Analytical Solutions

[9] In this work, we are primarily interested in a cubic shape of matrix blocks: a rock matrix cube surrounded by a three-dimensional (3-D) orthogonal fracture network.

Because fluid flow or pressure propagation in the highly permeable fractures is usually much more rapid than in the low-permeability matrix, we can assume that the water pressure at the surface of the matrix cube is constant everywhere at any given time. To facilitate analytical solutions, we will define the physical problem with the linearized equation (6) associated with the initial and boundary conditions for both imbibition (adsorption) and drainage (desorption) processes as follows:

[10] Initial condition within matrix:

$$S_w = S_i \quad (8)$$

at  $t = 0$  within the matrix block. The boundary conditions at the matrix surface (e.g., half the dimension of a 3-D matrix block) are

$$S_w = S_b \quad (9)$$

on a matrix surface for  $t > 0$ , where  $S_i$  and  $S_b$  are constant initial and boundary saturations, respectively.

#### 3.1. Exact 3-D Solution

[11] Let us first introduce the following dimensionless variables. The dimensionless distances are defined as

$$X = \frac{x}{2a}, Y = \frac{y}{2a}, Z = \frac{z}{2a} \quad (10)$$

and the dimensionless time is

$$\tau = \frac{Dt}{a^2}, \quad (11)$$

where  $a$  is half the dimension of the 3-D cube. The normalized (or scaled) water saturation is

$$S_D = \frac{S_w - S_i}{S_b - S_i}. \quad (12)$$

[12] Under these transformations, the unsaturated water flow problem is mathematically equivalent to the heat transfer problem solved by Carslaw and Jaeger [1959, equation (11), p. 185]. Therefore the solution, in terms of the normalized saturation, can be expressed as

$$\begin{aligned} S_D(X, Y, Z, \tau) = 1 - \frac{64}{\pi^3} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{1+m+n}}{(2l+1)(2m+1)(2n+1)} \\ \cdot \cos[(2l+1)\pi X] \cdot \cos[(2m+1)\pi Y] \\ \cdot \cos[(2n+1)\pi Z] e^{-\pi^2((2l+1)^2 + (2m+1)^2 + (2n+1)^2)\tau/4} \end{aligned} \quad (13)$$

[13] The rate of mass flow into or out of the cube through the matrix surface can be derived by differentiating the total mass of water within the matrix block with respect to time, given by

$$q(t) = 2aD\rho(S_b - S_i)q_D(\tau), \quad (14)$$

where  $\rho$  is the water density and  $q_D(\tau)$  is called dimensionless mass flow rate, defined as the mass flow

rate normalized by the maximum mobile water mass  $[8a^3\phi\rho(S_b - S_i)]$ , divided by a time constant  $[(4a^2/D)]$ :

$$q_D(\tau) = \frac{1536}{\pi^4} \left\{ \sum_{n=0}^{\infty} e^{-\pi^2(2n+1)^2\tau/4} \right\} \left\{ \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} e^{-\pi^2(2m+1)^2\tau/4} \right\}^2. \quad (15)$$

[14] The cumulative mass flow into or out of the cube is given by the difference between the total mass and initial mass,

$$Q(t) = 8a^3\phi\rho(S_b - S_i)Q_D(\tau), \quad (16)$$

where  $Q_D(\tau)$  is a dimensionless cumulative mass exchange, defined as the cumulative mass normalized by the maximum mobile water mass under the initial and boundary conditions,

$$Q_D(\tau) = 1 - \frac{512}{\pi^6} \left\{ \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} e^{-\pi^2(2m+1)^2\tau/4} \right\}^3. \quad (17)$$

### 3.2. Solution for 1-D Spherical Flow

[15] In most modeling studies of fracture-matrix flow, 3-D interflow within matrix blocks is approximated as 1-D spherical flow when using a double porosity concept [Warren and Root, 1963] or a multiple interacting continua (MINC) concept [Pruess and Narasimhan, 1985]. We use the term 1-D “spherical” flow because the governing equation of such flow can be shown to be identical to that of radially symmetric 1-D spherical flow under the MINC approximation, i.e., thermodynamic variables (pressure, temperature, concentration, etc.) are the same spatially at an equal distance from the matrix surface [Pruess and Narasimhan, 1985]. The advantage of the 1-D flow approximation is that it significantly reduces the total numbers of grid blocks for discretizing matrix blocks. In practice, the 1-D flow approximation is perhaps the most commonly used, and an analytical solution for such a 1-D spherical flow problem is very useful.

[16] With the 1-D spherical flow MINC approximation, the unsaturated flow toward the center of a cubic matrix block can be generally described by [e.g., Lai et al., 1983]

$$D \left[ \frac{\partial^2 S_w}{\partial x^2} + \frac{2}{x} \frac{\partial S_w}{\partial x} \right] = \frac{\partial S_w}{\partial t}, \quad (18)$$

where  $x$  is the distance from a nested cross sectional surface within the matrix block (having an equal distance to the matrix surface) to the center of the cube.

[17] Using the same dimensionless variables, defined by equations (10)–(12), the analytical solution of equation (18), subject to equations (8) and (9), is given by Carslaw and Jaeger [1959, equation (4), p. 233] as

$$S_D(X, \tau) = 1 + \frac{1}{\pi X} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(2n\pi X) e^{-n^2\pi^2\tau}. \quad (19)$$

[18] The rate of mass flow into or out of the cube through the matrix surface, and the cumulative mass of flow into or

out of the cube, are given by equations (14) and (16), respectively, while the dimensionless mass flow rate is given as

$$q_D(\tau) = 24 \sum_{n=1}^{\infty} e^{-n^2\pi^2\tau} \quad (20)$$

[19] The dimensionless cumulative mass exchange is given as

$$Q_D(\tau) = 1 - \frac{6}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n^2\pi^2\tau}. \quad (21)$$

## 4. Discussion and Application

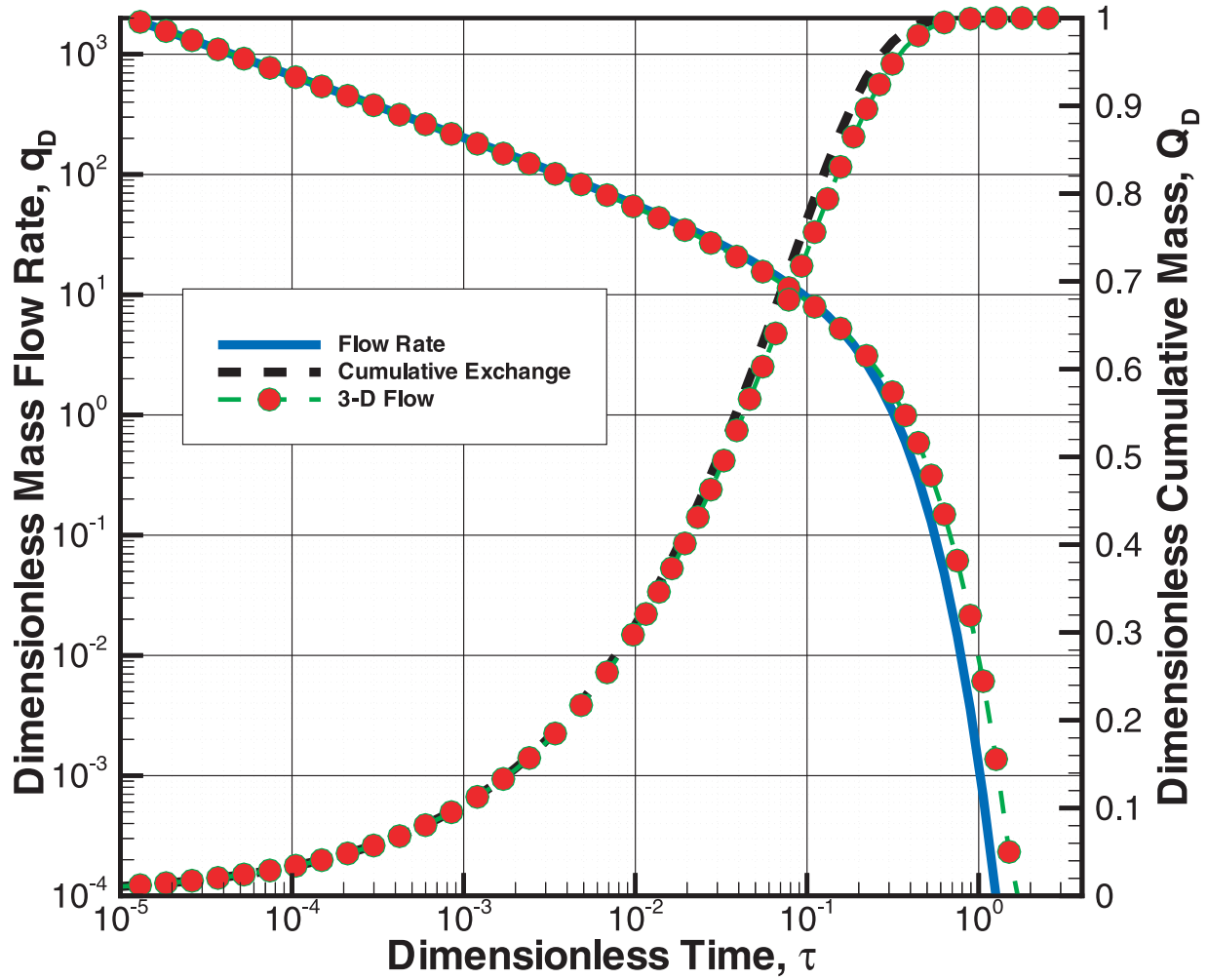
[20] With the proposed linearization of Richards' equation in section 3, many more analytical solutions can be easily derived for 1-D, 2-D, and 3-D problems in a finite, semi-finite, or infinite flow domain (e.g., by analogy with the corresponding heat conduction problems [Carslaw and Jaeger, 1959]). Note that the assumption (equation (5)) used in deriving such analytical solutions will impose limitations on their applicability. Specifically, the requirements of specially correlated relations between relative permeability and capillary functions and no gravity effects are critical to deriving analytical solutions. These assumptions are the necessary conditions to linearizing the flow-governing equation or for the existence of analytical solutions. Despite these restrictions, the relative permeability and capillary functions of equations (2) and (3) are among the most widely used relations [Honarpour et al., 1986]. On the other hand, in most field studies, capillary forces in the matrix system are generally dominant in controlling fracture-matrix flow relative to the effect of gravity, as long as matrix block sizes are relatively small. For these reasons, gravity effects have been ignored in almost all dual-continuum models for studies of multiphase or unsaturated flow in fractured reservoirs.

### 4.1. Type Curves of Transient Fracture-Matrix Flow

[21] The analytical solutions presented above can be used to generate several type curves for transient flow into rock matrix. In terms of the dimensionless variables for saturation, flow rate, and cumulative mass exchanges, the type curves are independent of the size of matrix blocks or the specific sets of rock or fluid properties. In addition, the type curves give us some insight into transient-flow processes through unsaturated fractured rock and can be used directly to verify simulation results of numerical models for particular applications.

[22] Spatial distributions of normalized water saturation within the matrix, as a function of dimensionless time and distance, are identical to Figure 29 of Carslaw and Jaeger [1959, p. 234] if calculated using equation (19), i.e., the 1-D flow approximation solution. These types curves cover the entire transient flow period of imbibition as well as drainage processes between fracture and matrix systems.

[23] Figure 1 presents type curves of dimensionless flow rate and cumulative mass exchange, respectively, using both 3-D solutions (equations (15) and (17)) and 1-D approximations (equations (20) and (21)). Figure 1 shows that the



**Figure 1.** Type curves of dimensionless flow rate and cumulative mass exchanges on matrix surface versus dimensionless time.

cumulative dimensionless mass exchange rate approaches 1 or the maximum mobile volume is filled (or vacuumed) at dimensionless time of 1, which gives a physical definition of the dimensionless time. Note the very small difference between 3-D and the 1-D solutions in describing flow into matrix blocks. The comparison of 3-D and 1-D inside-matrix flow results indicates that the simplified 1-D, MINC-type approximation may be accurate enough for estimating mass exchanges between fracture and matrix systems in practical applications.

#### 4.2. Evaluation of Numerical Modeling Results

[24] Numerical modeling approaches are widely used for simulating interactions between fractures and rock matrix in variably saturated fractured porous media. The key issue in numerical modeling efforts is how to handle fracture-matrix interactions under different flow conditions. Among the commonly used methods for dealing with such interactions is the dual-continuum method, including double-porosity and multiporosity models [e.g., Barenblatt *et al.*, 1960; Warren and Root, 1963; Kazemi, 1969; Pruess and Narasimhan, 1985; Wu and Pruess, 1988]. In a double-porosity concept, a flow domain is composed of matrix blocks with low permeability embedded in a network of interconnected

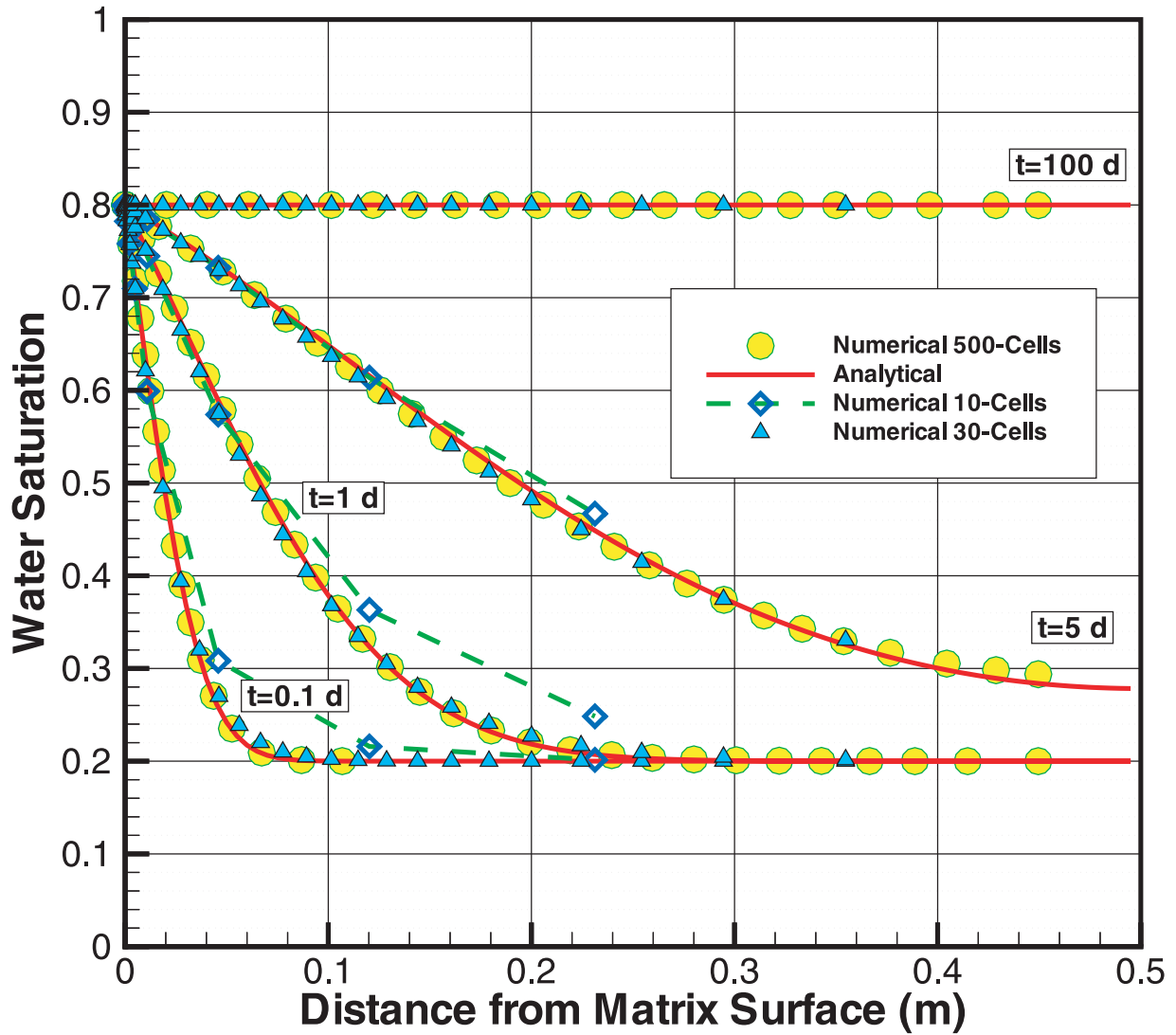
fractures. Global flow in the formation occurs only through the fracture system, conceptualized as an effective continuum, and matrix blocks are treated as spatially distributed source/sink terms, based on a quasi steady state assumption of interporosity flow.

[25] As a generalization of the Warren-Root model, a more rigorous dual-continuum method, the MINC concept

**Table 1.** Parameters for the Comparison Problem

Parameter	Value	Unit
Half dimension of the matrix cube	$a = 0.5$	m
Effective porosity	$\phi = 0.30$	
Matrix permeability	$k = 1.0 \times 10^{-15}$	m <sup>2</sup>
Water density	$\rho = 1000$	kg/m <sup>3</sup>
Water viscosity	$\mu_w = 1.0 \times 10^{-3}$	Pa s
Residual saturation	$S_{wr} = 0.2$	
Initial saturation	$S_i = 0.8$ and $0.2$	
Saturation on matrix surface	$S_b = 0.2$ and $0.8$	
Coefficient of permeability function	$C_k = 1.0$	
Exponential of permeability function	$\alpha = 2.0$	
Coefficient of capillary function	$C_p = 1.0 \times 10^4$	Pa
Exponential of capillary function	$\beta = 1.0$	





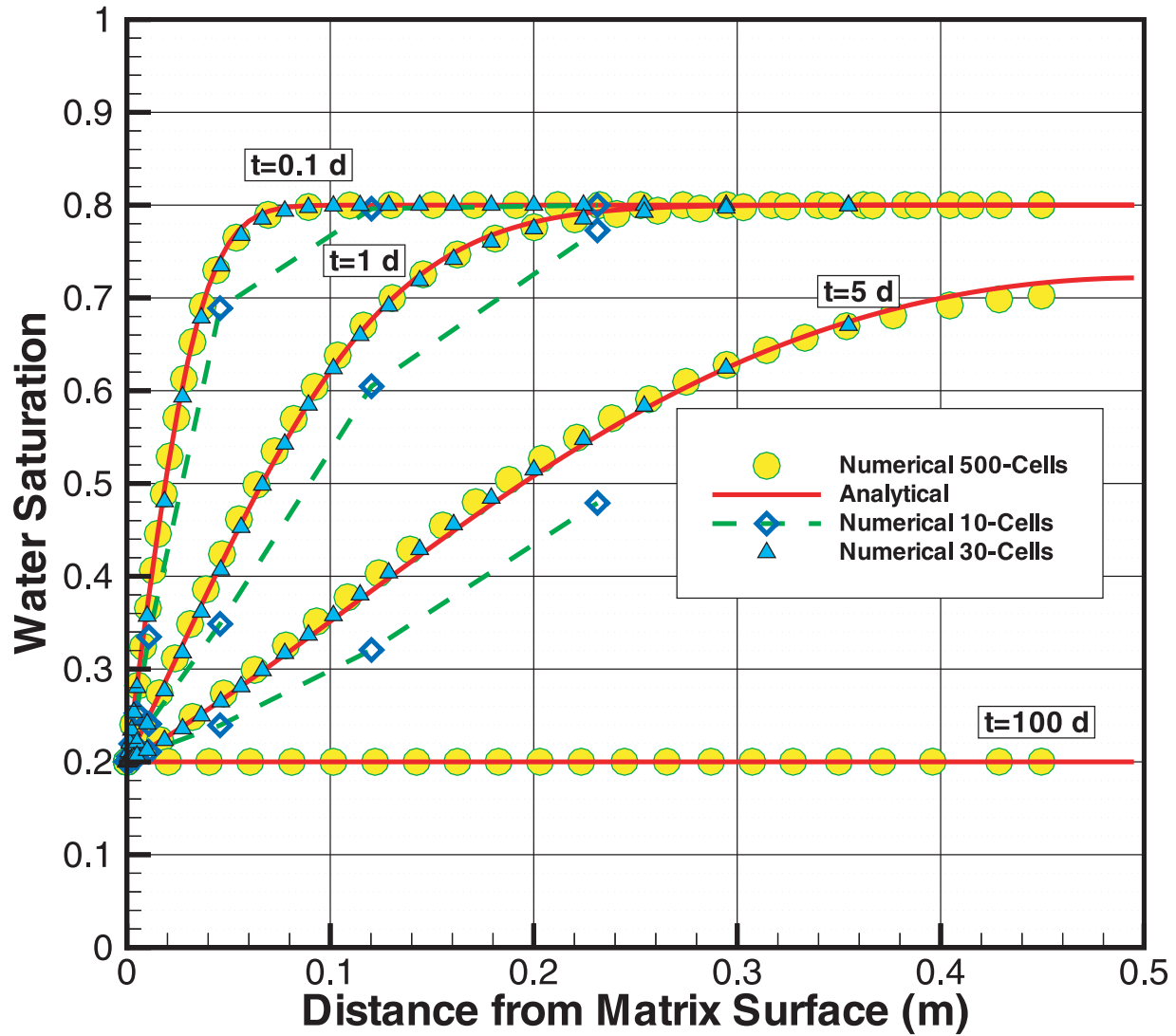
**Figure 2.** Comparison of calculated saturation distributions from analytical and numerical solutions for imbibition into a cubic matrix block.

[Pruess and Narasimhan, 1985] takes into account gradients of pressures, temperatures, and concentrations between fractures and matrix by appropriate subgridding of the matrix blocks. This approach provides a better approximation to transient fracture-matrix interactions than the one-block representation of fractures or the matrix in a double-porosity model. In comparison, however, the double-porosity model may produce inaccurate modeling results when gradients of pressures or moisture conditions are large or changing rapidly at or near fracture-matrix interfaces. This section presents efforts to quantify such numerical errors introduced by the dual-continuum conceptual model in handling fracture-matrix flow.

[26] We use the analytical solution for 1-D spherical flow into a cubic matrix to examine the numerical simulation results. The test problem concerns both imbibition (flow into matrix) and drainage (flow out of matrix). The numerical simulations were performed using a numerical reservoir simulator [Pruess, 1991; Wu *et al.*, 1996]. Note that the governing equation solved in numerical modeling is still the

original Richards' equation (1) instead of the linearized forms of equations (6) or (18).

[27] The example problem deals with transient flow processes into a  $1 \times 1 \times 1$  m cube of matrix, which is discretized into 2, 5, 10, 30, and 500 nested cells, respectively, using volume fractions of the MINC concept for 1-D spherical flow toward or from the matrix center. The basic parameters used for the example are listed in Table 1. The matrix surface is maintained at  $S_w = 0.8$  with a uniform condition of  $S_i = 0.2$ . Saturation distributions within the matrix at four different times, calculated using the analytical and numerical solutions of the matrix imbibition problem, are displayed in Figure 2, with discretizations of 10, 30, and 500 cells, respectively. As shown in Figure 2, the numerical results with refined grids (30 and 500) cells are in excellent agreement with the analytical solution during the entire transient imbibing period. In contrast, the simulation using the coarser 10-cell grid cannot (in general) match the analytical solution well, except near the matrix surface.

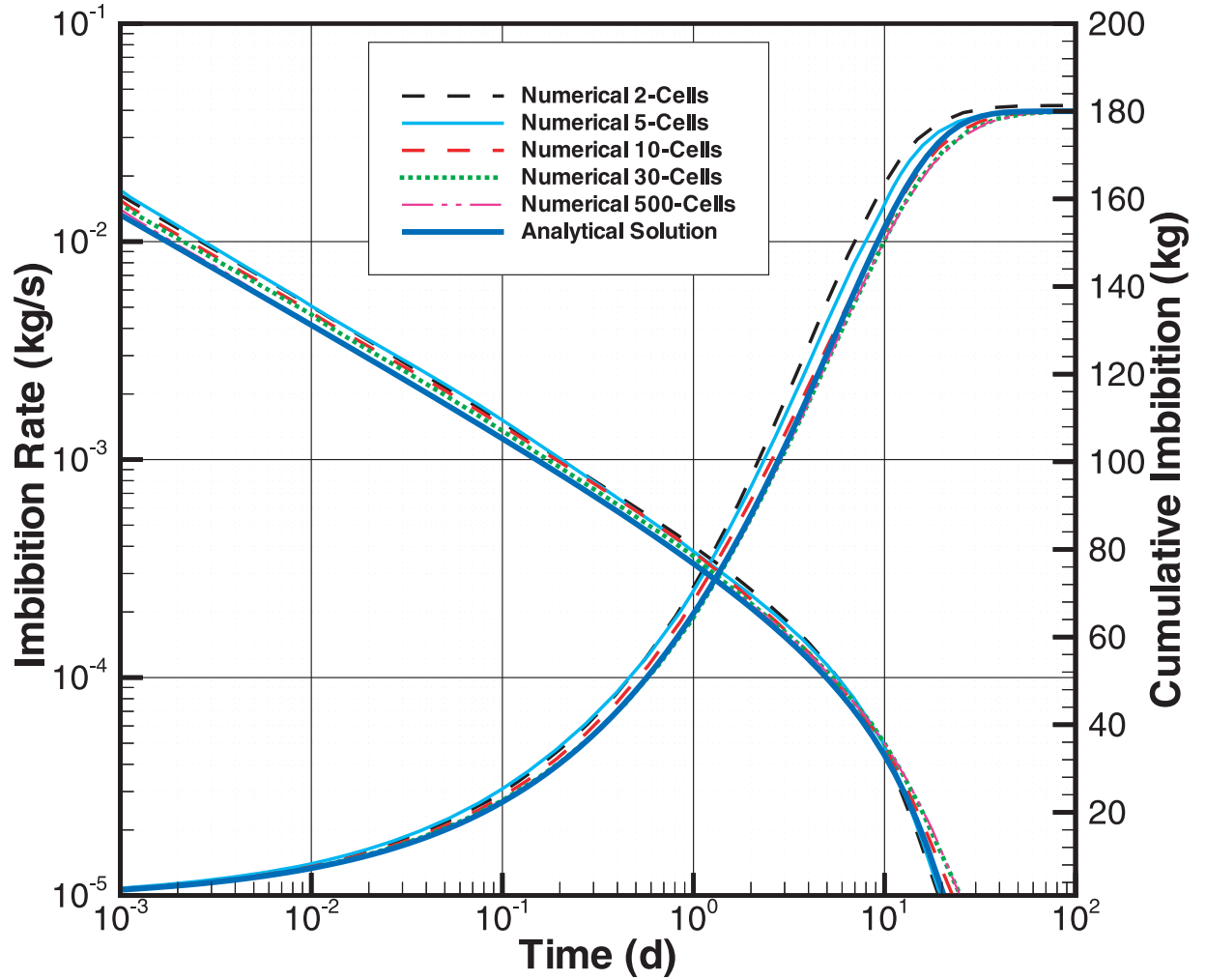


**Figure 3.** Comparison of calculated saturation distributions from analytical and numerical solutions for drainage from a cubic matrix block.

[28] Note that the MINC subgridding of the matrix normally involves a set of volume fractional values, leading to nested cells with approximately equal volumes. In general, an equal-volume mesh results in smaller grid spacings near matrix surface or fractures and thus gives better numerical accuracy for estimating fracture-matrix interactions [Pruess and Narasimhan, 1985]. However, it creates larger grid spacing at or near the matrix center because of the requirement of equal mesh volume. This explains why the results using a 10-cell grid cannot well match the analytical solution inside the matrix block. On the other hand, a two-cell (or double-porosity) model using only one-gridblock average to represent the matrix system cannot match saturation distribution at all. Only after a long time (100 days for this case) do all the numerical and analytical solutions converge to a steady state solution of  $S_w = 0.8$ . Note that in most applications, a MINC-type approach is intended to give the correct cumulative flux (and hence the correct mass balance in the block). So the following comparison of cumulative flux is

more meaningful than the saturation profiles shown in Figure 2.

[29] Figure 3 shows the saturation distribution within the matrix for a case of water drainage from the matrix, simulated using the same matrix subgriddings as in the imbibition case. Similarly, the model results using refined grids of 30 and 500 cells match well with the analytical solution. However, the 10-cell model results are in worse agreement with the analytical solution than the imbibition case (Figure 2). This is primarily because of the change in flow directions in the two cases, since the identical griddings were used for the same cell grids. In the previous imbibition case, the upstream of flow at the matrix surface (or fracture), specified with fixed pressure/saturation flow condition, is simulated using relatively refined subgrids. By comparison, for the drainage situation, only the downstream condition at the matrix surface is physically fixed. The upstream of the drainage flow is located at the center of the matrix block, which is very transient, with pressure head declining rapidly with time. The temporal discretization



**Figure 4.** Comparison of calculated water imbibing rates and cumulative imbibition from analytical and numerical solutions into a cubic matrix block.

errors caused by the fully implicit time-stepping scheme are worsened by coarser gridding near the matrix center, leading to large numerical errors.

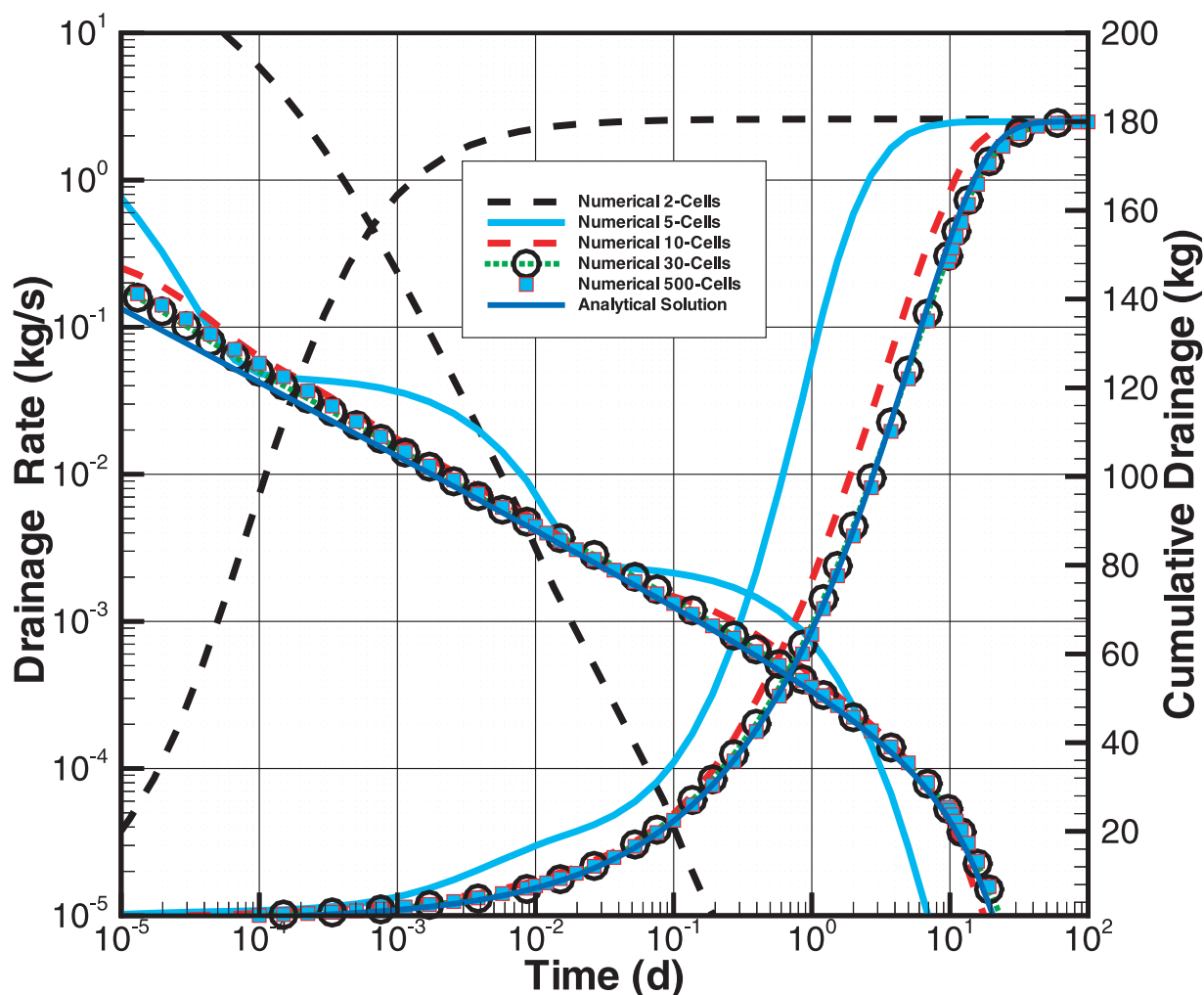
[30] Figure 4 shows the change of water-imbibing rate at the matrix surface and cumulative imbibition mass into the matrix over time. In this case, all the numerical results are in good agreement with the analytical results. In contrast, simulated drainage flow rates and cumulative mass exchanges in Figure 5 show very different results for different matrix subgriddings. In this case, for the same reasons as for saturation distributions, two-cell or double-porosity and five-cell discretizations give extremely large errors to estimated drainage rates, compared with the results from 30- or 500-cell models or analytical solution. Furthermore, the “humps” in drainage rate versus time curves, for 5-cell and even 10-cell discretizations, reflect numerical errors of the coarse-grid models in estimating potential or saturation gradients near the matrix surface, which is time-dependent. Note that several commonly used mobility-weighting schemes were tested, giving results similar to those shown in Figure 5. This indicates that the accuracy in modeling drainage processes is highly dependent on matrix grid resolutions and may require more refined grids than modeling imbibition processes,

because of the more transient nature at the upstream flow condition. This example demonstrates the usefulness of the linearization approach and the resulting analytical solutions in assessing numerical methods that solve the nonlinear Richards’ equation.

## 5. Concluding Remarks

[31] This paper presents a new linearization scheme to Richards’ equation, based on the assumptions of (1) negligible gravitational effect and (2) special forms of capillary pressure and relative permeability functions. Using such simplifications, we can derive a set of analytical solutions for transient flow into unsaturated matrix. The analytical solution approach of this work can be easily extended to other boundary conditions and different flow geometries, such as cylindrical, radial, and other multidimensional unsaturated flow.

[32] The analytical solutions, though limited by the assumptions for their applications, can be used to obtain some insight into the physics of transient imbibition and drainage processes related to fracture-matrix interactions. In particular, several dimensionless type curves, specifically spatial saturation distributions and flow rates for



**Figure 5.** Comparison of calculated water drainage rates and cumulative drainage from analytical and numerical solutions from a cubic matrix block.

mass exchange crossing fracture matrix interfaces, are discussed. These type curves are independent of matrix block size and specific parameters of fluid and rock and can be useful in verifying numerical models and their results for flow through unsaturated fractured rock, using a dual-continuum approach. Note that under the linearization approach proposed here, the relative permeability and capillary pressure remain nonlinear functions of saturation. This feature has proved to be useful in assessing numerical methods that solve the highly nonlinear Richards equation.

[33] As an application example, analytical solutions were used to examine numerical solutions for modeling both transient imbibition and drainage flow processes within rock matrix. The test indicated that numerical approaches, in particular the errors associated with temporal and spatial discretization, generally need to be checked before their application to field studies of unsaturated flow in fractured rock. The 1-D MINC-type flow approximation can provide a good approximation to unsaturated flow inside the matrix, but it may require detailed grid refinements to accurately model transient drainage flow behavior in fractured rocks.

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