#  <br> - Project Review 

Consortium Project on
Seismic Inverse Methods for Complex Structures
Breckenridge, Colorado
May 9-12, 1995
prepared by

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## ACKNOWLEDGMENTS

This project review is prepared for the sponsors of the Consortium Project at the Center for Wave Phenomena. This project provides substantial funding for the overall research and educational program at the Center. We also receive funds from the Department of Energy, the Office of Naval Research, and the Gas Research Institute for research that is integrated into our program and reported on in this document. We are extremely grateful to our consortium sponsors (listed on page 441) and our other sponsors for their support.

## INTRODUCTION

This is the eleventh edition of the Project Review Report on the Consortium Project at the Center for Wave Phenomena. Since our last project review, four new companies have joined - Intevep (Venezuela), Norsk Hydro (Norway), Union Pacific Resources, and PGS Tensor; while two have left - IBM and Advance.

We are pleased to announce that we were on three successful proposals in the Department of Energy ACTI Program, the Advanced Computer Technology Initiative. At the time of this writing, the level of funding for these projects is still uncertain, but we are gratified by our initial success. We are confident that some level of funding will be provided and that we will be able to move forward on our cooperative projects with researchers from the national laboratories.

The feedback that we received after our Project Review meeting last year that is compiled in our "Sponsor Reviews" booklet, was truly gratifying. The comments reinforced our belief that our garrulous interactive style has kept us close to the interests of our sponsors while providing us the resources and opportunity to do basic and deep research in the problems of interest to us all.

During this year, John Anderson, Sr. Geophysical Advisor, Mobil Research and Development Corporation, completed his two-year visit with CWP. This was a wonderful experience for us; we hope to have other industrial visitors in the future. Another visitor during the year was Helle Wagner, from the Niels Bohr Institute for Astronomy, Physics and Geophysics, Copenhagen, Denmark. Helle learned about our approach to inverse problems and related seismic data processing while working on a research project with John Scales. Fabien Bosquet, visiting scientist from Elf-Acquitaine, joined us in November for a sixteen-month stay.

We are very proud of our students who completed their Ph.D. degrees this school year: Omar Uzcategui, Geophysics; Tong Fei, Geophysics; and Zhenyue Liu, Mathematical and Computer Sciences. Since our last meeting, we have distributed twelve CWP reports and reprints of published articles, eight computer codes, and the SU User's Manual. The reports and software, as well as papers submitted for publishing, are listed at the end of this introduction.

We have begun the process of updating our computer environment. We have purchased five Linux-based Pentium 90 PC's, two tapedrives, and two diskdrives. (We have increased our diskspace by 20 gigabytes.) We plan to order more Pentiums in the future to replace our current NeXT network. Furthermore, we are planning the purchase of at least one Silicon Graphics computer, to be used as a compute server. Ultimately, we expect to have 48-64 megabytes on each workstations to make them work optimally in a compute server environment. We are also looking into a high-speed communications network; our own upgrade will likely be integrated into a campus-wide system currently in the design stage and soon to be implemented.

An examination of the Table of Contents in this book will reveal a healthy mix of continuation of last year's work along with the introduction of new projects into our
program. These projects are in anisotropic modeling, DMO, migration and inversion; velocity analysis, object-oriented programming and opitimization; full-waveform inversion; high-frequency inversion integrated with wavelet processing; and analysis of the Berkhout approach to inversion.

As always, we look forward to the Project Review Meeting where we first distribute this report. It is a time to tell our sponsors what we are doing and to get feedback on how we are doing. This year, we have invited some of our ACTI colleagues from the national laboratories to the project review, so that they might also be exposed to the full scope of our research program and meet our industrial sponsors. This interaction is consistent with the objectives of the ACTI program. We hope this new element enriches the experience for everyone.

Norman Bleistein, Director Center for Wave Phenomena

April 1995

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CWP-133P Alkhalifah, T., and K. Larner, 1993, Migration error in transversely isotropic media: Geophysics, 359, no. 9, 1405-1418 (September 1994).
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## Software Distributed

U27R Fei, T., FCT: Flux-Corrected Transport correction applied to the 2D acoustic wave equation (in second order PDE form) for finite difference modeling and migration. Proprietary: $1 / 10 / 97$

U30 Gouveia, W., SUDGAST: Residual statics by a distributed genetic algorithm. Proprietary: 1/6/98

U31 Alkhalifah, T., and Artley, C., SUDMOTIVZ: DMO for transversely isotropic $\mathrm{v}(\mathrm{z})$ media for common-offset gathers. Proprietary: $1 / 6 / 98$

U32 Alkhalifah, T., SUMIGPSTI: Migration by phase shift for TI media with turning rays. Proprietary: 1/6/98

U33 Anderson, John E., TIFOWLER: Constant velocity stack and prestack time migration in constant velocity TI media via the Fowler method. Proprietary: 1/6/98

U34 Anderson, John E., TIHALEDMO: Hale-style DMO codes for TI media.
U35 Fei, T., FCTANIS: Flux corrected transport correction applied to anisotropic elastic finite difference modeling. Proprietary: $1 / 6 / 98$

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# Anisotropy processing in vertically inhomogeneous media 

Tariq Alkhalifah

# Anisotropy processing in vertically inhomogeneous media 

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#### Abstract

Alkhalifah and Tsvankin (1995) show that $P$-wave normal-moveout (NMO) velocity for dipping reflectors in transversely isotropic (TI) media with a vertical symmetry axis, specified in terms of ray parameter, depends just on the zero-dip NMO velocity [ $V_{\text {nmo }}(0)$ ], and a parameter $\eta$ that is a combination of Thomsen's (1986) parameters. Their inversion procedure makes it possible to obtain $\eta$ and reconstruct the NMO velocity as a function of ray parameter using moveout velocities for two different dips. Moreover, $V_{\text {nmo }}(0)$ and $\eta$ determine not only the NMO velocity, but also long-spread (nonhyperbolic) $P$-wave moveout for horizontal reflectors and the time-migration impulse response. This means that inversion of dip-dependent information allows one to perform all time-processing in TI media using only surface $P$-wave data. Such findings have paved the way for constructing a full processing sequence for TI media.

The first and most important step in processing data in TI $v(z)$ media is parameter estimation. Alkhalifah and Tsvankin (1995) generalized the single-layer NMO equation to layered TI media with a dipping reflector. This equation provides the basis for extending TI velocity analysis to vertically inhomogeneous media. The multi-layered NMO equation is based on a root-mean-square (rms) average of modified interval velocities corresponding to a single ray parameter, that of the dipping event. Therefore, modified interval velocity values can be extracted from the stacking velocities using a Dix-type differentiation procedure. In addition, the $\eta$ inversion is performed simultaneously with the interval velocity evaluation in each layer.

Since the moveout for reflections from steep reflectors is small and relatively insensitive to velocity, stacking-velocity estimates can be improved by applying velocity analysis after doing dip moveout correction (DMO), which increases the moveout, and therefore increases the moveout sensitivity to velocity. As a result, a modification to the NMO velocity equation is done to accommodate the application of the DMO operation, which here is based on the assumption of a homogeneous, isotropic medium.

Time migration, like DMO, depends on two parameters in vertically inhomogeneous media, namely the NMO velocity and $\eta$, both of which can vary with depth.


Therefore, the NMO velocity and $\eta$ estimated using the dip dependency of $P$-wave moveout velocity can be used in a TI time migration.

An application of anisotropic processing to seismic data from offshore Africa demonstrates the importance of considering anisotropy, especially as it pertains to focusing dipping events.

## INTRODUCTION

While it is convenient to consider the earth subsurface to be homogeneous, it is at a minimum vertically inhomogeneous. Through the combined action of gravity and sedimentation, velocity variation with depth represents the most important firstorder inhomogeneity in the earth. This is one reason why time migration (based on lateral homogeneity) works well in so many places. Dip moveout (DMO) and migration algorithms that can handle isotropic $v(z)$ media are well established, and even velocity estimation in such media is considered trivial. Nevertheless, problems remain in focusing images, estimating depths, and preserving dipping events in $v(z)$ media. It may be that the problem at this point is the restrictive assumption that the medium is isotropic. Because basic processes that developed the earth's crust (i.e., sedimentation, pressure and gravity) have a preferred direction (vertical in most cases), seismic wave speed can vary with propagation direction in the vertical plane. Otherwise, it is difficult to explain the success of isotropic homogeneous DMO in areas with a clear velocity increase with depth (Gonzalez et al., 1992), knowing that such an increase in velocity causes the dipping events to stack at a lower velocity than the horizontal ones (Artley and Hale, 1994).

The first and most important step in a successful processing sequence for $P$-wave data is to estimate the medium parameters needed to apply the various processing operations. Existing work on anisotropic traveltime inversion of reflection data has been done for laterally homogeneous subsurface models (Byun and Corrigan, 1990; Sena, 1991; Tsvankin and Thomsen, 1995). These inversions, although providing useful information on anisotropy in the subsurface, either use the weak-anisotropy approximation or require $P$-wave data to be supplemented by additional information (e.g., the vertical velocity from check shots or well logs). For example, the inversion method of Tsvankin and Thomsen (1995) requires acquisition of $S$-wave, as well as $P$-wave data, for estimation of anisotropy parameters to be feasible. One reason for the limitations associated with these algorithms is the number of parameters needed to be estimated in transversely isotropic (TI) media. Using Thomsen's (1986) notation, three parameters ( $V_{P 0}, \epsilon$, and $\delta$ ) are needed to characterize the kinematics of $P$-waves in TI media with vertical symmetry axis (VTI). As shown by Tsvankin and Thomsen (1995), $P$-wave moveout from horizontal reflectors is insufficient to recover the three Thomsen's parameters, even if long spreads (twice the reflector depth) are used. In fact, it is impossible to recover these three parameters using any additional surface $P$-wave data including moveout from dipping events (Alkhalifah and Tsvankin, 1995). The reason for this ambiguity is the trade-off between the
vertical velocity and anisotropic coefficients, which cannot be overcome by using any $P$-wave surface seismic information.

Therefore, there is a redundancy in the three-parameter representation that characterizes $P$-wave moveout in VTI media. In fact, Alkhalifah and Tsvankin (1995) demonstrated that, for TI media with vertical symmetry axis (VTI media), just two parameters are sufficient for performing all time-related processing such as NMO correction (including non-hyperbolic moveout correction, if necessary), DMO correction, and prestack and poststack time migration. Taking $V_{h}$ to be the $P$-wave velocity in the horizontal direction, one of these two parameters, $\eta$, is given by

$$
\begin{equation*}
\eta \equiv 0.5\left(\frac{V_{h}^{2}}{V_{\mathrm{nmo}}^{2}(0)}-1\right)=\frac{\epsilon-\delta}{1+2 \delta} \tag{1}
\end{equation*}
$$

and the other, the short-spread normal moveout (NMO) velocity for a horizontal reflector, is given by

$$
\begin{equation*}
V_{\mathrm{nmo}}(0)=V_{P 0} \sqrt{1+2 \delta}, \tag{2}
\end{equation*}
$$

where $V_{P 0}$ is the $P$-wave vertical velocity, and $\epsilon$ and $\delta$ are Thomsen's (1986) dimensionless anisotropy parameters.

These two parameters can also be characterized directly in terms of the elastic coefficients $c_{i j}$ as follows

$$
\eta=\frac{c_{11}\left(c_{33}-c_{44}\right)}{2 c_{13}\left(c_{13}+2 c_{44}\right)+2 c_{33} c_{44}}-\frac{1}{2}
$$

and

$$
V_{\mathrm{nmo}}(0)=\sqrt{\frac{c_{13}\left(c_{13}+2 c_{44}\right)+c_{33} c_{44}}{\left(c_{33}-c_{44}\right)}}
$$

The fact that we cannot uniquely determine the elastic coefficients from $\eta$ and $V_{\text {nmo }}(0)$ does not matter, because time-related processing depends just on $V_{\text {nmo }}(0)$ and $\eta$.

Alkhalifah and Tsvankin (1995) further show that these two parameters, $\eta$ and $V_{\mathrm{nmo}}(0)$, can be obtained solely from surface seismic $P$-wave data, using estimates of stacking velocity for reflections from interfaces having two distinct dips.

The inversion technique discussed by Alkhalifah and Tsvankin (1995) is designed for a homogeneous medium above the reflector, while realistic subsurface models are, at a minimum, vertically inhomogeneous. Therefore, it is appropriate to extend the inversion mechanism of Alkhalifah and Tsvankin (1995) to handle vertically inhomogeneous media.

Alkhalifah (1995b) suggested to invert for $\eta$ and $V_{\text {nmo }}(0)$ using the nonhyperbolic moveout behavior of $P$-wave reflections in vertically varying VTI media. Although this method does not require dipping events to be present, which makes it more flexible than the dip-dependent moveout approach of Alkhalifah and Tsvankin (1995), it is
less stable and depends on having reasonably large offsets to obtain realistic estimates of parameters at greater depths.

A key feature of time-related processing is that the final output is still given in time. Therefore, a reflection from a horizontal reflector at zero-offset (coincident source and receiver) remains in exactly the same position after applying NMO, DMO, and time migration. As a result, all transformations done by these processes are with respect to this zero-offset reflection rather than its depth position. This eliminates the need to specify the depth of the reflection point. In VTI media, such a feature is valuable because it eliminates the need for the vertical velocity when time -related processing are expressed in terms of $V_{\mathrm{nmo}}(0)$ and $\eta$, and therefore, reduces the number of required parameters needed to specify these processes. (Vertical velocity, however, is required in any attempt to convert seismic data from time to depth.)

The bulk of the paper concentrates on the $v(z)$ inversion process. Here, I extend the inversion technique of Alkhalifah and Tsvankin (1995) to handle layered transversely isotropic media based on the fact that NMO velocity for dipping reflectors is a root-mean-square (rms) average of its interval values. Such an rms relation, derived by Alkhalifah and Tsvankin (1995) for transversely isotropic layered media, depends also on only $V_{\text {nmo }}(0)$ and $\eta$. Next, I study the dependence of both DMO and time migration on $V_{\text {nmo }}(0)$ and $\eta$ in vertically inhomogeneous media. Then, I apply the inversion method, as well as anisotropy processing, to a marine data set from offshore Africa.

## NMO VELOCITY FOR DIPPING REFLECTORS IN TI MEDIA

The analysis here is based on the equation for the normal-moveout (short-spread) velocity for dipping reflectors in a homogeneous anisotropic medium derived by Tsrankin (1995):

$$
\begin{equation*}
V_{\mathrm{nmo}}(\phi)=\frac{V(\phi)}{\cos \phi} \frac{\sqrt{1+\frac{1}{V(\phi)} \frac{d^{2} V}{d \theta^{2}}}}{1-\frac{\tan \phi}{V(\phi)} \frac{d V}{d \theta}}, \tag{3}
\end{equation*}
$$

where $V$ is the phase velocity as a function of the phase angle $\theta$ ( $\theta$ is measured from vertical) and $\phi$ is the dip of the reflector; the derivatives are evaluated at the $\operatorname{dip} \phi$. Unfortunately, reflection data do not carry any explicit information about dip; rather, we can count on recovering the ray parameter $p(\phi)$ corresponding to the zero-offset reflection. Therefore, for inversion purposes, formula (3) must be recast in terms of the ray parameter (Alkhalifah and Tsvankin, 1995),

$$
\begin{equation*}
p(\phi)=\frac{1}{2} \frac{d t_{0}}{d x_{0}}=\frac{\sin \phi}{V(\phi)} \tag{4}
\end{equation*}
$$

where $t_{0}\left(x_{0}\right)$ is the two-way traveltime on the zero-offset (or stacked) section, and $x_{0}$ is the midpoint position. In this case, the phase angle $\phi$ and phase velocity $V(\phi)$
corresponding to a given value of $p$ can be obtained from the Christoffel equation and used in formula (3) (Alkhalifah and Tsvankin, 1995).

## VELOCITY ANALYSIS IN $V(Z)$ MEDIA

Inversion in layered VTI media can be implemented through a layer-stripping algorithm where the parameters of a certain layer (or interval) are estimated by removing the influence of the overlying layers. The layer-stripping portion of the inversion is similar to what Dix (1955) used to estimate interval velocities from stacking velocities based on a small-offset approximation.

NMO velocity equation for dipping reflectors in $v(z)$ media
For horizontal layers, whether the media are isotropic or VTI, the NMO velocity at a certain zero-offset time, $t_{0}$, (equivalent to the migrated time, for horizontal layers) is given by an rms relation (Hake et al., 1984; Tsvankin and Thomsen, 1994) as follows

$$
\begin{equation*}
V_{\mathrm{nmo}}^{2}\left(t_{0}\right)=\frac{1}{t_{0}} \int_{0}^{t_{0}} v_{\mathrm{nmo}}^{2}(\tau) d \tau \tag{5}
\end{equation*}
$$

where $v_{\text {nmo }}(\tau)$ are "interval NMO velocities" given by

$$
v_{\mathrm{nmo}}(\tau)=v(\tau) \sqrt{1+2 \delta(\tau)}
$$

and $v(\tau)$ is the interval vertical velocity.
For dipping reflectors, when expressed in terms of ray parameter $p$, NMO velocity is also given by a similar rms relation (Alkhalifah and Tsvankin, 1995).

$$
\begin{equation*}
V_{\mathrm{nmo}}^{2}\left[p, t_{0}(p)\right]=\frac{1}{t_{0}(p)} \int_{0}^{t_{0}(p)} v_{\mathrm{nmo}}^{2}\left[p, t_{m}(\tau)\right] d \tau \tag{6}
\end{equation*}
$$

where $v_{\text {nmo }}\left[p, t_{m}\right]$ is the interval NMO velocity as a function of vertical time (migrated time), $t_{m}$, and $t_{0}(p)$ is zero-offset time for a single ray parameter, $p$. This ray parameter corresponds to the reflection from the dipping reflector at time $t_{0}(p)$ used to measure $V_{\text {nmo }}^{2}\left[p, t_{0}(p)\right]$, where $t_{0}(0)=t_{m}$ corresponds to the two-way traveltime to a horizontal reflector; i.e., migrated time. As demonstrated in equation (4), the ray parameter can be determined from the slope of the reflection in the zero-offset domain.

The integral in equation (6) can be expressed in terms of migrated time, $t_{m}$, as follows

$$
\begin{equation*}
V_{\mathrm{nmo}}^{2}\left[p, t_{0}(p)\right]=\frac{1}{t_{0}(p)} \int_{0}^{t_{\mathrm{m}}} v_{\mathrm{nmo}}^{2}(p, \tau) \frac{d t_{0}(p)}{d \tau} d \tau \tag{7}
\end{equation*}
$$

This equation reduces to equation (5) for horizontal reflectors $(p=0)$, where $\frac{d t_{0}(p)}{d \tau}=1$. Further, $v_{\text {nmo }}(p, \tau)$ depends only on the interval values $v_{\text {nmo }}(0, \tau)$ and $\eta(\tau)$ in each
layer or time sample. Alkhalifah and Tsvankin (1995) show that $t_{0}(p)$ is a function of the medium parameters $v_{\text {nmo }}(0)$ and $\eta$, as well as the vertical time, given by

$$
\begin{equation*}
t_{0}(p)=t_{m} f\left[\eta, v_{\mathrm{nmo}}(0), p\right] \tag{8}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\frac{d t_{0}(p)}{d t_{m}}=f\left[\eta, v_{\mathrm{nmo}}(0), p\right] \tag{9}
\end{equation*}
$$

where $f$ is the operator that relates the vertical time to the zero-offset time, which can be obtained through ray tracing. As a result, $V_{\mathrm{nmo}}\left[p, t_{0}(p)\right]$ based on equation (7) depends on $\eta$ and $v_{\text {nmo }}(0)$ in each layer. For isotropic media, $\eta=0$, and

$$
f\left[v_{\mathrm{nmo}}(0), p\right]=\frac{1}{\sqrt{1-p^{2} v_{\mathrm{nmo}}^{2}(0)}}
$$

Equation (6), when expressed in terms of discrete layers, is given by

$$
\begin{equation*}
\left[V_{\mathrm{nmo}}^{(n)}(p)\right]^{2}=\frac{1}{t_{0}(p)} \sum_{i=1}^{n} \Delta t_{0}^{(i)}(p)\left[v_{\mathrm{nmo}}^{(i)}(p)\right]^{2}, \tag{10}
\end{equation*}
$$

where $\Delta t_{0}^{(i)}(p)$ is the two-way zero-offset traveltime through layer $i$ for ray parameter $p$.

To obtain the NMO interval velocity in any layer $i$ (including the one immediately above the reflector), we apply the Dix formula (Dix, 1955) to the NMO velocities at the top $\left[V_{\text {nmo }}^{(i-1)}\right]$ and bottom $\left[V_{\text {nmo }}^{(i)}\right]$ of the layer:

$$
\begin{equation*}
\left[v_{\mathrm{nmo}}^{(i)}(p)\right]^{2}=\frac{t_{0}^{(i)}(p)\left[V_{\mathrm{nmo}}^{(i)}(p)\right]^{2}-t_{0}^{(i-1)}(p)\left[V_{\mathrm{nmo}}^{(i-1)}(p)\right]^{2}}{t_{0}^{(i)}(p)-t_{0}^{(i-1)}(p)} \tag{11}
\end{equation*}
$$

where $t_{0}^{(i-1)}(p)$ and $t_{0}^{(i)}(p)$ are the two-way traveltimes to the top and bottom of the layer, respectively, calculated along the ray given by the ray parameter $p$ for normalincidence reflection from the dipping reflector, used in measuring stacking velocity; all NMO velocities here correspond to a single ray-parameter value $p$. Suppose, we wish to use equation (11) to obtain the normal moveout velocity $\left[v_{\text {nmo }}^{(n)}(p)\right]$ in the medium immediately above the reflector to use as an input value in the inversion algorithm discussed above. Clearly, from equation (10), the recovery of $v_{\text {nmo }}^{(n)}(p)$ requires obtaining the moveout velocities in the overlying medium for the same value of the ray parameter. However, as we will see later, such a problem can be solved by using an interpolation procedure.

## Inversion in $v(z)$ media

When interval NMO velocity values, $v_{\text {nmo }}^{(n)}(p)$, are obtained for at least two distinct dips, the problem reduces within each layer (or time sample, if the inversion was based
on the integral form) to a homogeneous inversion that can be solved in the same way described by Alkhalifah and Tsvankin (1995). Therefore, interval values $v_{\mathrm{nmo}}^{(n)}(p)$ for two distinct dips in each layer (or each time sample) are used to estimate $\eta(\tau)$ and $v_{\mathrm{nmo}}(0, \tau)$. Since estimating $v_{\mathrm{nmo}}^{(n)}(p)$ using equation (11) depends on obtaining $v_{\mathrm{nmo}}^{(i)}(p)$ for previous layers at the same ray parameter, estimating $\eta(\tau)$ and $v_{\text {nmo }}(0, \tau)$ must be done simultaneously with the layer-stripping process for $v_{\text {nmo }}^{(i)}(p)$.

First, I use the values $V_{\mathrm{nmo}}^{(1)}\left(p_{1}\right)$ and $V_{\mathrm{nmo}}^{(1)}\left(p_{2}\right)$, which correspond to the first interval, to estimate $\eta^{(1)}$ and $v_{\mathrm{nmo}}^{(1)}(0)$ using the inversion of Alkhalifah and Tsvankin (1995) for a homogeneous medium, where $p_{1}$ and $p_{2}$ are ray parameters of the dipping reflectors in this first interval (one of these reflectors could be horizontal). Each interval is considered homogeneous. Then, I use the estimated $\eta^{(1)}$ and $v_{\mathrm{nmo}}^{(1)}(0)$ to obtain $V_{\mathrm{nmo}}^{(1)}\left(p_{3}\right)$ and $V_{\text {nmo }}^{(1)}\left(p_{4}\right)$, as well as $\frac{d t_{0}\left(p_{3}\right)}{d t_{0}}$ and $\frac{d t_{0}\left(p_{4}\right)}{d t_{0}}$, in the first interval. Ray parameters $p_{3}$ and $p_{4}$ correspond to the dipping (or horizontal) reflectors in the second interval. Using equation (11), I then obtain the interval values $v_{\mathrm{nmo}}^{(2)}\left(p_{3}\right)$ and $v_{\mathrm{nmo}}^{(2)}\left(p_{4}\right)$, which corresponds to the second interval, from $V_{\mathrm{nmo}}^{(1)}\left(p_{3}\right)$ and $V_{\mathrm{nmo}}^{(1)}\left(p_{4}\right)$, and in turn use them to obtain $\eta^{(2)}$ and $v_{\text {nmo }}^{(2)}(0)$, and so on.

Although the method requires NMO velocities measured at two different dips in each interval, one can define interval thicknesses depending on the available reflectors. Specifically, each interval is chosen to include two dips, no matter how large that interval gets. A better and more practical approach is to fit a piecewise-linear, continuous interval velocity models for each of the ray parameters of the dipping reflections used to measure the stacking velocities. These models satisfy these measured stacking velocities based on equation (7). Specifically, the interval velocities are taken as continuous at the times of the measured stacking velocities and linear in between. Eventually, we must obtain at least two continuous interval velocities corresponding to two distinct dips. As a result, the homogeneous inversion is applied at each time sample to obtain $v_{\text {nmo }}(0, \tau)$ and $\eta(\tau)$. A detailed description of the inversion is given in Appendix A.

As with isotropic media, intermediate interval values (i.e., values between measured ones) can be estimated using any interpolation technique between measured values. The sole requirement is that interval values yield the measured stacking velocities based on equation (7). For example, we could consider the measured values to be constant in each layer. Here, however, the application is based on a linear interpolation that keeps the inverted values continuous. This continuity is important for various ray tracing applications.

Errors in the inverted interval values of $\eta$ can arise from the linear interpolation of velocities used in the layer-stripping process, and from the inversion in each homogeneous interval used to obtain $\eta$. The interpolation errors are similar to those encountered in layer-stripping applications for isotropic media. Errors associated with the homogeneous inversion, as described in detail by Alkhalifah and Tsvankin (1995), depend mainly on the accuracy of the measured quantities, primarily the stacking velocities.

## Stacking-velocity measurements

As is well known, the stacking velocity for steep reflectors $\left(=\frac{V_{\text {nmo }}(0)}{\cos (\theta)}\right.$ in isotropic homogeneous media, where $\theta$ is the reflector dip) is large; therefore, the moveout is small and insensitive to velocity. Specifically, the curvature of reflection moveout, $d t^{2} / d^{2} X\left[\propto 1 / V_{\text {nmol }}^{2}\right.$ ], where $X$ is the source-receiver offset, decreases with increase in velocity. As a result, the resolution of velocity analyses is poor, causing problems in picking the appropriate stacking velocities corresponding to dipping reflectors.

One way to avoid this problem is to pick the stacking velocity after applying isotropic homogeneous DMO to the data. The DMO operation reduces the stacking velocity of dipping reflectors (approximately equivalent to multiplying by $\cos \theta$ ), and therefore, increases the sensitivity of moveout to velocity. As a result, I modify the NMO velocity equation in TI media to account for the isotropic homogeneous DMO operation. This is accomplished by including the traveltime shifts that correspond to the DMO operation in the NMO equation for dipping reflectors.

$$
t^{2}(p, X)=t_{0}^{2}(p)+\left(\frac{1}{V_{\mathrm{nmo}}^{2}(p)}+p^{2}\right) X^{2}=t_{0}^{2}(p)+\frac{X^{2}}{V_{s t k}^{2}(p)}
$$

where $t$ is the two-way traveltime as a function of offset, $X$. Therefore, the NMO velocity for a dipping reflector after isotropic homogeneous DMO is given by

$$
\begin{equation*}
V_{s t k}(p)=\frac{V_{\mathrm{nmo}}(p)}{\sqrt{1+p^{2} V_{\mathrm{nmo}}^{2}(p)}} \tag{12}
\end{equation*}
$$

Equation (12) can, therefore, be used to replace the $V_{\text {nmo }}(p)$ function in inverting for $\eta$ and $V_{\text {nmo }}(0)$.

There is an additional advantage in applying the DMO operation prior to velocity analysis in inverting for $\eta$. Specifically, we can verify the presence of anisotropy by comparing the NMO velocity of the sloping event (after DMO) to that of a horizontal event (or any other distinct slope). If the velocity of the sloping event is higher, then, in most cases, anisotropy is present, and $\eta$ is positive. If the medium is also vertically inhomogeneous, then the anisotropy is even more significant, because inhomogeneity tends to reduce the influence of anisotropy on the isotropic homogeneous DMO operation (Alkhalifah, 1995a). If the velocity of the sloping event is lower than that of the horizontal event after applying a homogeneous isotropic DMO, then there are two possibilities: the first is that the medium is vertically inhomogeneous (Artley and Hale, 1994), and the second is that the medium is anisotropic with a negative $\eta$, which is unlikely (Thomsen, 1986; Alkhalifah and Tsvankin, 1995).

If the NMO velocities of the sloping and horizontal reflections are equal after applying homogeneous isotropic DMO (which is a goal of applying the DMO) then the medium may be isotropic and homogeneous, in concurrence with the type of operation used. However, if velocity analysis implies vertical inhomogeneity (which is typically the case), then anisotropy is present and has the same size (with an opposite
sign) influence as the vertical inhomogeneity on the DMO operation for these two dips (Gonzalez et al., 1992; Alkhalifah, 1995a). However, although the homogeneous isotropic DMO focussed these two reflections (the sloping and horizontal) at the same stacking velocity, it might not focus as well other reflections (with other slopes), because the isotropic $v(z)$ DMO impulse response is not identical to the anisotropic one (Alkhalifah, 1995a). Here, I have tried to outline the main possibilities. The presence of strong lateral inhomogeneity would introduce further complications.

## TIME-RELATED PROCESSING

The main argument used to show the dependence of time-related processing (e.g., DMO and time migration) on only $V_{\text {nmo }}(0)$ and $\eta$ in homogeneous VTI media is that such time-related processing become independent of the vertical velocity $V_{P 0}$ when expressed in terms of $V_{\text {nmo }}(0)$ and $\eta$. That is, it does not matter what values of $V_{P 0}$, $\epsilon$, and $\delta$ are used; only $V_{\text {nmo }}(0)$ and $\eta$ need to be specified. To prove such an assertion, $V_{P 0}, \epsilon$ and $\delta$ are varied from one test to another while keeping $V_{\text {nmo }}(0)$ and $\eta$ the same, and changes in impulse responses (such as migration impulse responses or diffraction curves) are then observed. Alkhalifah and Tsvankin (1995) used such an argument for homogeneous media. Here I will apply it to vertically inhomogeneous media.


Fig. 1. Parameter variation as a function of vertical time. The parameters here correspond to the interval vertical velocity ( $V_{P 0}$ ), the interval NMO velocity for horizontal reflectors $\left[v_{\text {nmo }}(0)\right.$ ], and the anisotropy parameter $\eta$. Different combinations of these parameters result in different models.

Figure 1 shows parameter variations as a function of vertical time that I use below to generate impulse responses. The vertical velocity ( $V_{P 0}$ ) given by the solid black curve is the same as the $v_{\text {nmo }}(0)$ curve, and, therefore, $\delta$ for this model equals zero. When combined with $v_{\text {nmo }}(0)$, the other two $V_{P 0}$ curves correspond to $\delta$ values that do not equal zero [see equation (2)]. The dashed curve (vertical velocity is a constant, $1500 \mathrm{~m} / \mathrm{s}$ ), when combined with $v_{\text {nmo }}(0)$, results in $\delta$ reaching values as large as the unrealistic value of 2. Therefore, in terms of Thomsen's (1986) parameters, the difference between the model given by the solid black $V_{P 0}$ curve and the model given the dashed curve is large, but the parameters have been chosen such that $\eta$ is nevertheless the same.

## Dip-moveout correction

As mentioned above, Alkhalifah and Tsvankin (1995) showed that the NMO velocity for dipping reflectors depends on only two medium parameters in homogeneous VTI media, namely $V_{\text {nmo }}(0)$ and $\eta$. Alkhalifah (1995a) further demonstrates that the DMO operation itself, as well as its impulse response, depends solely on these two parameters. This result holds as well for $\eta(\tau)$, as we see next.

Figure 2 shows four DMO impulse responses generated using the anisotropic DMO algorithm described by Alkhalifah (1995a). The first of these responses (Figure 2a) corresponds to the parameters given by the solid black curves in Figure 1 for $V_{P 0}$, $v_{\text {nmo }}(0)$ and $\eta$. Note how different the DMO impulse response in \TI media are from the elliptical shape we have grown accustomed to for isotropic media. The responses in Figure 2b and 2c correspond to using the gray and the dashed curves of $\mathrm{V}^{1} \mathrm{p}_{0}$ in Figure 1, respectively, while keeping the values of $v_{\text {nmo }}(0)$ and $\eta$ the same as those used in Figure 2a (the solid black curves). The three DMO impulse responses look exactly the same; that is, they are independent of the value of $V_{P_{0}}$. in support of the result that was partially suggested by equation (6), a small-offset approximation of the moveout. (Recall that for the response in Figure $2 \mathrm{c}, \delta$ reaches values of about 2!) On the other hand, if we change $\eta$, using the gray curve in Figure 1 instead of the black one, the response changes dramatically, implying that it is highly dependent on $\eta$.

## Time migration

Alkhalifah (1995b) showed that the nonhyperbolic moveout based on a Taylor's series expansion in vertically inhomogeneous VTI media is dependent on only $v_{\text {nmo }}(0, \tau)$ and $\eta(\tau)$. Again, such a moveout equation represents a small-dip approximation of a time-migration diffraction curve.

Figure 3 shows four time-migration impulse responses generated using an anisotropic phase-shift time migration (Kitchenside, 1991). The first of these responses (Figure 3a) corresponds to the parameters given by the solid black curves in Figure 1 for $V_{P 0}, v_{\text {nmo }}(0)$ and $\eta$. On the other hand, the responses in Figure 3 b and 3 c correspond to using the gray and the dashed curves of vertical velocity ( $V_{P 0}$ ) from


Fig. 2. DMO impulse responses for an impulse at time 2.1 s and offset 1.5 km using (a) the parameters represented by solid black curves in Figure 1, (b) the vertical velocity given by the gray curve in Figure 1 while keeping the other parameters the same as (a), (c) the vertical velocity given by the dashed curve in Figure 1 while keeping the other parameters the same as (a), and (d) the $\eta$ values represented by the gray curve in Figure 1 while keeping $V_{P 0}$ and $v_{\text {nmo }}(0)$ the same as (a).


Fig. 3. Zero-offset time-migration impulse responses for an impulse at time 2.1 s , using (a) the parameters in Figure 1 represented by the solid black curves, (b) the vertical velocity given by the gray curve in Figure 1 while keeping the other parameters the same as (a), (c) the vertical velocity given by the dashed curve in Figure 1 while keeping the other parameters the same as (a), and (d) the $\eta$ values represented by the gray curve in Figure 1 while keeping $V_{P 0}$ and $v_{\text {nmo }}(0)$ the same as (a).

Figure 1, respectively, while keeping $v_{\text {nmo }}(0)$ and $\eta$ the same as those used in Figure 3a. The three time migration impulse responses look identical. Given the large difference between the Thomsen's parameters used to generate Figure 3a from those used to generate Figure 3c, the similarity of the responses that are based on the exact traveltime calculation (within the frame work of ray theory) is remarkable. Therefore, time migration in VTI media is also independent of vertical velocity when expressed in terms of $v_{\text {nmo }}(0)$ and $\eta$. However, if the gray $\eta$ curve in Figure 1 is used, differences begin to appear. Specifically, note that, because of the overall lower $\eta$, the response in this case is slightly squeezed (see arrows). Although the time migration responses appear to have less variation with change in $\eta$ than do the DMO responses, note that the scales at which the responses in the case of DMO and time migration are plotted are not the same. The conclusion in any event is that migration will be less sensitive to ignoring anisotropy than DMO, at least for modest dip. This is consistent with the results of Alkhalifah and Larner (1994).

## FIELD-DATA EXAMPLE

Figure 4 shows a stacked seismic section, from offshore Africa provided by Chevron Overseas Petroleum, Inc., that contains reflections from a large number of dipping faults. The section was processed using a sequence of conventional NMO and DMO without taking anisotropy into account. While horizontal and mildly sloping reflections are imaged well, as we will see below, steep fault-plane reflections have been weakened because anisotropy was ignored. The predominant velocity variation in the section is vertical. In fact, in the area between CMP locations 400 and 800 and up to vertical time 3 s , the lateral variation of velocity is small.

The arrows in Figure 4 point to the sloping reflections used to measure the stacking velocities. Likewise, $V_{\text {nmo }}(0)$ measurements are based on the horizontal events. Although the sloping reflections used in the inversion seem to span the whole 5 s of data, the actual parameter information stops at about 3.5 s - the vertical (migrated) time corresponding to the deepest reflection used in the measurement of stacking velocity. This difference follows from the relation between the vertical time $\left[t_{m}\right]$ and the zero-offset time $\left[t_{0}(p)\right]$. In addition to the picked reflections, $\eta$ at the surface is constrained to equal zero since these are marine data and the water layer is isotropic.

Carrying out the inversion process described in Appendix A, using the measured values of stacking velocities and corresponding ray parameters, I obtain the functions $v_{\text {nmo }}(\tau)$ and $\eta(\tau)$ shown in Figure 5. The inversion assumes no lateral velocity variation in the region of the picks; mild lateral velocity variation, however, should not be a problem for this DMO-based inversion: most DMO algorithms, while based on lateral homogeneity, still produce practical results where lateral velocity variation is smooth. The continuous representation shown in Figure 5 is a direct result of fitting a piecewise linear velocity model, as mentioned in Appendix A, for both the mildly dipping reflectors (for simplicity I refer to them as horizontal reflectors) and the faults. In the water layer, $v_{\text {nmo }}$ is equal to $1.5 \mathrm{~km} / \mathrm{s}$ and $\eta$, as mentioned earlier,


Fig. 4. Stacked section from offshore Africa, after applying NMO and isotropic homogeneous DMO. The arrows point to the sloping reflections used in the inversion.


Fig. 5. Interval values $v_{\text {nmo }}$ and $\eta$ as a function of vertical time.
is equal to zero. The accuracy of these estimated curves of $v_{\text {nmo }}$ and $\eta$ depends on the accuracy of the stacking velocity estimates for both dipping and horizontal reflectors (Alkhalifah and Tsvankin, 1995). Based on the locations of the measured stacking velocities (Figure 4), as well as the extent of the lateral homogeneity, these inverted values can be considered representative of the area between CMP location 500 and 900.

The interval values of $\eta$ in Figure 5 show more detail than can be reliably trusted considering the many uncertainties associated with the few events picked in these data and the particular assumption used for interpolating interval NMO velocities. However, we can still trust the general trend of the $\eta$ curve, which implies an overall increase in the anisotropy with vertical time up to about 3 s . The $\eta$ values after time equal 3.5 s were constrained to equal zero because no $\eta$ information is present for these times using this inversion. The region above 3 s , which exhibits positive values of $\eta$, corresponds to a shale formation. Shale is often transversely isotropic and may thus be the major source of anisotropy in the data.


Fig. 6. Stacked section after $v(z)$ anisotropic DMO using the parameters in Figure 5. The NMO correction is based on the velocities obtained from the conventional velocity analysis. Compare with Figure 4.

Next, I apply a DMO algorithm that uses the derived functions $v_{\mathrm{nmo}}(\tau)$ and $\eta(\tau)$ in Figure 5. Figure 6 shows the result of TI DMO applied to the data, based on the ray-
tracing DMO algorithm of Alkhalifah (1995a). Relative to the result of isotropic DMO given in Figure 4, this section is much improved. Note, in particular, the reflections from the faults. The improvements extend throughout the whole section, and includes reflections not used in the inversion. This implies that the lateral variation in $\eta$, especially prior to 2 s , is small.

Figure 7 shows representative VTI DMO operators used for these data. The shapes are far from the isotropic ellipse or even a stretched version of it. Therefore, we should expect the result from the anisotropic DMO to be different from that of the isotropic DMO, and so it is.


Fig. 7. VTI DMO impulses response for the parameters in Figure 5. The offset is 1.5 km , and the apex is at (a) 1.8 s , and (b) 2.5 s .

Figure 8 shows CMP gathers at CMP location 700 after (a) homogeneous isotropic DMO, and (b) $v(z)$ VTI DMO using the parameters in Figure 5. The same NMO correction, based on the stacking velocities obtained from conventional semblance velocity analysis, was used in both DMO examples. The arrows point to reflections from some of the dipping faults present in this highly faulted portion of the data. Note that the maximum offset is large (up to $X / D=2$ ). Clearly, for the isotropic DMO result, the reflections from the dipping faults are not aligned. They have deviations caused by an NMO velocity that is smaller than what is needed for this anisotropic medium. Such deviations in reflection traveltimes are proportional to $X^{2}$. Even the reflections from the horizontal events are not aligned. The misalignment for the horizontal reflections, however, is caused by the nonhyperbolic moveout associated with VTI media. Therefore, the deviations in this case start at larger offsets $X / D>1$ (Tsvankin and Thomsen, 1994, Alkhalifah, 1995b), and are proportional to the nonhyperbolic term $X^{4}$. This implies that the horizontal reflectors, as well as the dipping event, are less focused in Figure 4 than in Figure 6. Both horizontal and dipping events are better aligned after application of the ray-tracing anisotropic DMO based on the inverted parameters. Close comparison of Figures 4 and 6 reveals improvement in the horizontal features as a result of anisotropic processing.


Fig. 8. CMP gathers for CMP location 700 after (a) homogeneous isotropic DMO, and (b) $v(z)$ anisotropic DMO. The NMO correction, based on the velocities obtained from velocity analysis, is the same for both examples.

Figure 9a shows the result of conventional processing: phase-shift, isotropic time migration was applied to the zero-offset section obtained by the isotropic homogeneous DMO. For comparison, Figure 9b shows the data imaged with phase-shift anisotropic time migration (using the inverted parameters of Figure 5) applied to the stack obtained from the $v(z)$ VTI DMO algorithm. This comparison gives a clear picture of the benefit of taking anisotropy into account in DMO prior to doing migration. The improvements here are numerous and significant. One example is the fault located at CMP location 870, between 2.5 and 3 seconds. An interpreter using the isotropic processing result can easily extend the reflections across this fault ignoring it or suggest a minor subsidence to the left of the fault. However, the imaged result of the anisotropic processing (as well as the inverted values of $\eta$ ) suggests the extension of the shales up to 3 seconds under CMP location 800 , and probably a larger subsidence has occurred. Another example is the region of the nearly horizontal events near CMP location 500 , at 2.5 s . The improved continuity of the gently dipping events likely is a result of non-hyperbolic moveout correction in the anisotropic processing.

Although most of the reflections here correspond to features within or near the 2-D plane that contains the sources and receivers, some events may represent out-of-plane reflections, requiring $3-\mathrm{D}$ processing. Ignoring the three-dimensionality can cause mispositioning in some areas, especially where the fault reflections cross what seem to be continuous horizontal reflections. However, based on examination of parallel lines in the same area, most reflections are in the dip plane of the section.

Lynn et al. (1991) observe that problems of mis-focussing of dipping faults are encountered in many data sets from around the world and such problems can not be attributed to use of 2-D as opposed to 3-D processing, lateral velocity variations, or statics problems. Their assessment is that such problems are caused by the presence of anisotropy. They also state that isotropic prestack migration often gives poorer results than does isotropic poststack processing applied to DMO-processed data sets. Whereas full prestack migration seems to be the ideal way to process data, it is intolerant of any shortcomings of the model or the data.

## DISCUSSION AND CONCLUSIONS

Although the inversion described here cannot resolve the vertical velocity and anisotropic coefficients $\epsilon$ and $\delta$ individually, it makes it possible to obtain the parameters needed to apply time-related processing (including NMO, DMO, and time migration) in vertically inhomogeneous media. These parameters are the zero-dip NMO velocity $v_{\text {nmo }}(0, \tau)$ and the anisotropy parameter $\eta(\tau)$.

The inversion algorithm described by Alkhalifah and Tsvankin (1995) was developed for a homogeneous, transversely isotropic medium above the reflector. To extend the method to vertically inhomogeneous media, the inversion must be applied using the NMO equation of Alkhalifah and Tsvankin (1995) for layered anisotropic media above a dipping reflector. The influence of a stratified isotropic or anisotropic


Fig. 9. Time migrated section using (a) isotropic phase-shift migration of the data shown in Figure 4, and (b) anisotropic phase-shift migration of the data shown in Figure 6 using the parameters shown in Figure 5.
overburden on moveout velocity can be stripped through a Dix-type differentiation procedure.

Using sloping reflections to extract velocity information in $v(z)$ media requires, among other things, positioning the reflections at their true (migrated) position. This is accomplished by relating the zero-offset time to the vertical (migrated) time, and therefore positioning the extracted interval velocities at their true times (relative depths). Although this concept is beneficial in isotropic media, it is exceptionally important in anisotropic media, where such velocities are compared with those extracted from horizontal events, and then used to invert for anisotropy information, specifically $\eta$. This inversion process is based on the rms assumption of stacking velocities for a given ray parameter. Such a relation, for horizontal reflectors, reduces to the familiar Dix (1955) expression. The idea underlying the inversion is that the $v_{\text {nmo }}(\tau)$ and $\eta(\tau)$ functions obtained from the inversion are those that best focus reflections from the dipping fault and the horizontal reflectors at the same stacking (or NMO) velocity, for each vertical time at which the velocity measurements are made.

Analysis of dip moveout and time-migration impulse responses shows that these processes depend solely on two parameters $v_{\text {nmo }}(0)$ and $\eta$ in vertically inhomogeneous media. Therefore, the results of the inversion [values of $v_{\text {nmo }}(0)$ and $\eta$ ] can be used to apply NMO, DMO, and time migration. To an extent, time migration can be used to evaluate the performance of the inversion in data that include reflectors with known positions (i.e., fault traces as delimited by terminations of sedimentary bedding). Specifically, the results of the inversion for $\eta$ can be checked by inspecting the quality of images generated by poststack migration using the same inverted parameters. If the image indicates undermigration, the true $\eta$ overall is higher than the estimated values.

As we saw in the field example, isotropic DMO cannot properly focus dipping reflectors where the inversion results indicate that the medium is anisotropic. On the other hand, $v(z)$ VTI DMO based on the inverted values of $v_{\text {nmo }}(\tau)$ and $\eta(\tau)$ did focus such reflectors, and, because it also takes non-hyperbolic moveout into account, it can even improve the focussing of horizontal reflections, as well. In addition, anisotropic time migration based on the inverted parameters $\left[v_{\text {nmo }}(\tau)\right.$ and $\eta(\tau)$ ] placed the steep reflections at their true time migrated position, while the isotropic migration, which used only the values of $v_{\text {nmo }}(\tau)$, mispositioned the sloping features relative to the horizontal ones.

The cost of anisotropic processing is close to that of its isotropic counterpart. In fact, the processing algorithms needed for both types of media run in about the same time. For example, although slower than the typical log-stretched DMO techniques, the DMO algorithm used here (Alkhalifah, 1995) is as efficient as Artley and Hale's (1994) isotropic $v(z)$ DMO. The difference in computation effort for the isotropic and anisotropic algorithms in phase-shift time migration is negligible. The true additional cost of the anisotropic processing arises from the time needed to measure stacking velocities, as well as ray parameters, for the sloping reflections.

Applying a general anisotropic processing sequence, therefore, is appropriate for all data. If the medium is isotropic, then the lack of anisotropy will be reflected in the small values for the inverted parameter $\eta(\eta \approx 0)$. However, if $\eta$ departs from zero by a substantial amount (i.e., $\eta>0.05$ ), then it is best to take anisotropy into account. Practically, typical performance of isotropic DMO suggests anisotropy in data. In particular, the fact that an isotropic homogeneous DMO works better than isotropic $v(z)$ DMO in a vertically inhomogeneous medium suggests the presence of anisotropy because this anisotropy counters the influence of an increase in velocity with depth. Nevertheless, the fact that isotropic constant-velocity DMO often works better than the $v(z)$ DMO does not imply that the result is optimum. The DMO process can further benefit from an added degree of freedom, in our case $\eta$, which can be calculated and has a physical basis, specifically anisotropy. Because it has this physical basis this same parameter provides the added degree of freedom needed in migration as well.

## ACKNOWLEDGMENTS

I thank Ken Larner and Ilya Tsvankin for reviewing the manuscript. I also thank my peers Gabriel F. Alvarez and Abdulfattah Al-Dajani for their review of the paper. I wish to thank Tagir Galikeev for loading and preprocessing the data, and John Toldi and Chris Dale of Chevron Overseas Petroleum, Inc. for providing the field data. I also wish to acknowledge the financial support from KACST (Saudi Arabia).

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## APPENDIX A: VELOCITY ANALYSIS IN LAYERED MEDIA

The first step of the inversion process involves estimating stacking velocities as a function of zero-offset traveltime from $P$-wave reflection data. These velocities are commonly considered to equal the NMO velocity. Measuring stacking velocities is common practice in isotropic processing, but here we must estimate such stacking velocities for dipping, as well as horizontal reflections. In addition, we must measure the ray parameters (slopes) corresponding to these reflections.

The inversion method can be applied using any number of dips using a leastsquares approach. For simplicity, I constrain the description here to the model given in Figure A-1, where we have only two distinct dips (horizontal reflectors and a dipping fault). The medium is considered to be laterally homogeneous above the fault. Note that, because it is dipping, this single fault provides velocity information at several zero-offset times that can be used to extract vertical parameter variations with depth.

After obtaining stacking-velocity information as a function of ray parameter and zero-offset time, we need to construct an interval-velocity model that satisfies the measured stacking velocities based on equation (6). As mentioned in the text, the


Fig. A-1. Depth model consisting of a fault and a number of horizontal layers. The rays drawn correspond to the measured stacking velocities ( $V_{\text {noo }}\left[p_{i}, t_{i}\left(p_{i}\right)\right]$ and $\left.V_{\text {nmo }}\left[p_{i+1}, t_{i+1}\left(p_{i+1}\right)\right]\right)$ described in the Appendix. Such rays illustrate the relation between the zero-offset time and the vertical time for the dipping fault.
velocity model that I use is continuous with linear increases in modified interval velocity (a quantity that depends on ray parameter) between the measured values of stacking velocities. For the horizontal reflectors ( $p=0$ ), construction of such a velocity model is straightforward, following the familiar method of Dix (1955). However, for the dipping fault, the problem is much more complicated because the ray parameter along the fault reflection varies with recording time due to the variation of velocity with depth. Therefore, the measured stacking velocities for the dipping fault at different vertical times correspond to different ray parameters.

Suppose we want to fit a linear interval-velocity model between the measured stacking velocities $V_{\mathrm{nmo}}\left[p_{i}, t_{i}\left(p_{i}\right)\right]$ and $V_{\mathrm{nmo}}\left[p_{i+1}, t_{i+1}\left(p_{i+1}\right)\right]$, where $p_{i}$ and $t_{i}$ are the ray parameter and zero-offset time of the fault reflection used in measuring the stacking velocities. This linear interval velocity will correspond to a ray parameter $p_{i+1}$ and should be continuous with the calculated interval velocities prior to time $t_{i}(0)$ at this same ray parameter $p_{i+1}$. Here, $t_{i}(0)$ is the two-way vertical traveltime to the reflection recorded at time $t_{i}\left(p_{i}\right)$, as shown in Figure A-1. Therefore, the initial velocity for the linear model between $V_{\text {nmo }}\left[p_{i}, t_{i}\left(p_{i}\right)\right]$ and $V_{\text {nmo }}\left[p_{i+1}, t_{i+1}\left(p_{i+1}\right)\right]$ is $v_{\text {nmo }}\left[p_{i+1}, t_{i}(0)\right]$ calculated at vertical time $t_{i}(0)$ using the values of $v_{\text {nmo }}(0, \tau)$ and $\eta(\tau)$ at $\tau=t_{i}(0)$. The interval values in between the two measured stacking velocities are given by

$$
\begin{equation*}
v_{\mathrm{nmo}}\left(p_{i+1}, \tau\right)=v_{\mathrm{nmo}}\left[p_{i+1}, t_{i}(0)\right]+a_{i+1}\left[t\left(p_{i+1}, \tau\right)-t_{1}\left(p_{i+1}\right)\right], \tag{A-1}
\end{equation*}
$$

where $a_{i+1}$ is the constant velocity gradient between vertical time $t_{i}\left(p_{i+1}\right)$ and $t_{i+1}\left(p_{i+1}\right)$, and $t\left(p_{i+1}, \tau\right)$ is the zero-offset two-way traveltime calculated as follows

$$
\begin{equation*}
t\left(p_{i+1}, \tau\right)=\int_{0}^{\tau} f\left[\eta\left(\tau_{1}\right), v_{\mathrm{nmo}}\left(\tau_{1}\right), p_{i+1}\right] d \tau_{1} \tag{A-2}
\end{equation*}
$$

where $f$, as mentioned in the text, is the operator that relates zero-offset time to vertical time. Here, $\tau$ corresponds to the two-way vertical time, and $t_{i}\left(p_{i+1}\right)$ is the two-way zero-offset traveltime computed, using equation (A-2) by setting $\tau=t_{i}(0)$. For $i=0$ (corresponding to the earth's surface), $t_{0}(p)=0$, and the interval velocities are estimated either by considering the medium to be homogeneous up to time $t_{1}(0)$ ( $a_{1}=0$ ), or by using a value for the velocity at the surface that satisfies a certain condition (i.e., for marine data, velocity at the surface is usually set to $1.5 \mathrm{~km} / \mathrm{s}$ ). Therefore, the only unknowns in equation (A-1) as we progress from the top to the bottom of the seismic section are the velocity gradients $a_{i}$.

Using the expressions of stacking velocities and traveltimes given above, equation (6) can be written as follows

$$
\begin{equation*}
V_{\mathrm{nmo}}^{2}\left[p_{i+1}, t_{i+1}\left(p_{i+1}\right)\right] t_{i+1}\left(p_{i+1}\right)=\int_{0}^{t_{i}\left(p_{i+1}\right)} v_{\mathrm{nmo}}^{2}\left(p_{i+1}, \tau\right) d \tau+\int_{t_{i}\left(p_{i+1}\right)}^{t_{i+1}\left(p_{i+1}\right)} v_{\mathrm{nmo}}^{2}\left(p_{i+1}, \tau\right) d \tau \tag{A-3}
\end{equation*}
$$

The first term on the right hand side can be calculated from the estimated values of $\eta$ and $v_{\text {nmo }}(0)$ prior to $t_{i}(0)$. Let us assume that it equals $f_{1}$. If we are trying to determine $a_{1}$ corresponding to the region between the surface and the first measurement, then $f_{1}$ equals zero because $t_{0}\left(p_{i+1}\right)=0$.

Substituting equation (A-1) into the second term of equation (A-3) results in a quadratic equation in $a_{i+1}$. Solving equation (A-3) for $a_{i+1}$ involves solving the quadratic equation, and therefore

$$
a_{i+1}=0.5\left(\sqrt{\frac{4 V_{\mathrm{nm}}^{2}\left[p_{i+1}, t_{i+1}\left(p_{i+1}\right)\right] t_{i+1}\left(p_{i+1}\right)-v_{\mathrm{nm}}^{2}\left[p_{i+1}, t_{i}\left(p_{i+1}\right)\right] t_{d}}{t_{d}}-4 f_{1}}\right.
$$

where $t_{d}=t_{i+1}\left(p_{i+1}\right)-t_{i}\left(p_{i+1}\right)$.
Each time a new velocity gradient is obtained, for example $a_{i+1}$, it is directly used to compute the interval velocities using equation (A-1) in the region between $t_{i}(0)$ and $t_{i+1}(0)$. Then, these interval velocities, which correspond to the dipping fault, along with the horizontal interval NMO velocities, are used to invert - one sample at time - for $v_{\mathrm{nmo}}(0, \tau)$ and $\eta(\tau)$ based on the homogeneous DMO inversion of Alkhalifah and Tsvankin (1995). We continue to invert for $v_{\text {nmo }}(0, \tau)$ and $\eta(\tau)$ as a function of vertical time until we reach the time $t_{i+1}(0)$. Then a new velocity gradient, $a_{i+2}$, for the region between $t_{i+1}(0)$ and $t_{i+2}(0)$ is calculated in the same way.


# On the Berkhout approach to modeling and inversion of seismic inversion 

Norman Bleistein

# On the Berkhout approach to modeling and inversion of seismic data 

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#### Abstract

A major feature of the Berkhout approach to migration/inversion is early discretization of the forward modeling problem for seismic data surveys. Thus, Green's functions are replaced by propagator matrices and ordinary reflection coefficients are replaced by a poorly understood reflectivity matrix. One of the advantages of this method is that many special features of the underlying model can be characterized by further matrix multiplications. Migration/inversion becomes a matter of determining the reflectivity matrix. In this approach, this determination is reduced to a cascade of matrix inversions. On the other hand, the reflectivity matrix remains elusive to the exploration geophysics community. In my own research, I am developing a continuum analog for the Berkhout formalism. This allows an interpretation of the reflectivity matrix in terms of the more well understood geometrical optics reflection coefficient. It also reveals the inaccesibility of an exact inverse of the propagator matrices, hence the need for using the adjoint as an approximate "well-conditioned" inverse. The analysis also reveals that the " $W R W$ " characterization of the underlying model is conceptual, rather than exact, unless one makes certain compromises on the meaning of $W$. For anything but the horizontal reflector, $W$ is a dyadic operator. While imaging has been confirmed for the planar reflector by this approach, it is at best onlyindicated by the current state of analysis for the curved reflector case. Further, extraction of the reflection coefficient from the continuum analog of the reflectivity matrix remains undone for the curved reflector case. Berkhout inversion is a fixed frequency process, with stacking over frequency effecting image enhancement. This leads to the possibility of accounting for dispersion by using different background velocities for different frequencies. This aspect of Berkhout inversion has yet to be exploited.


## INTRODUCTION

This is a report on in-progress research. I presented some preliminary results for the previous project review, put the problem aside for a while and then picked it up
again. This research actually began with a discussion with Samuel Gray at the time of Svenfest, but is also based on an ongoing desire by each of us gain insight into the Berkhout approach to migration/inversion. (Berkhout and van Wulfften Palthe, 1979; Berkhout, 1984, 1985, 1992; Berkhout and Wapenaar, 1989; Fokkema et al., 1993; Wapenaar and Berkhout, 1985a, 1985b 1989, 1993; Wapenaar and Jaimé, 1990; Wapenaar, 1993a, 1993b, 1994)

A fundamental problem with gaining an understanding of this method is that the Berkhout school discretizes the forward scattering problem. In this approach, the Green's function is replaced by a propagator matrix and the geometrical optics reflection coefficient is replaced by a reflectivity matrix in the spatial-frequency domain. The spatial transform of this result provides information about the angularly dependent reflection coefficient at all incidence angles consistent with the aperture of the data. Another subtlety of the method is the introduction of the dipole response in the forward model, rather than the more traditional impulse response; hence, the use of $W$ for a "propagator" rather than $G$. Combining these changes into a succinct descriptor for the Berkhout approach, I use the terminology, " $W R W$ model" for a shorthand descriptor of this form of the forward problem. Here the right-most $W$ is a matrix propagator representing downward propagation, $R$ is a matrix characterizing scattering, so that $R W$ is the initial state of upward scattering at depth. Finally, the left-most $W$ is the upward propagator, making $W R W$ the upward scattered field. Thus, determination of the reflectivity matrix, $R$, amounts to finding (approximate) inverses to the $W$-matrices.

The matrix approach allows for relatively easy incorporation of features of the seismic experiment that are more cumbersome to model in a continuum (Kirchhoff, exploding reflector) integral approach. Of course, in the final analysis all approaches must have an equivalence, but that does not mean that it has to be equally difficult or easy to achieve the same results from different approaches.

On the other hand, there are features of the continuum theory that are not easily accessible to the discrete method. Indeed, any analysis based on applying the method of stationary phase to integral operators are not available in the discrete case. (Although a recent paper by Keller and Knessl, 1993.) suggests an approach to asymptotic evaluation of oscillatory sums.)

The discrete approach has no analog of the 2.5D theory, which allows for processing of a line of data while accounting for out-of-plane spreading. The reason is that 2.5 D processing results from applying the method of stationary phase to the integral operator representing 3D migration/inversion under the assumption that there is no out-of-plane variation in the medium (and, hence, in the data).

In the discrete approach, one would have to estimate the out-of-plane operator sum under the same conditions-possible with the Keller/Knessl theory, but as yet undone.

The Berkhout formalism assumes infinite aperture in sources and receivers, as do all other methods. However, again, the availability of the method of stationary
phase provides a mechanism for analyzing finite aperture effects with integral operators, not possible for discrete inversion. For example, one can show in Kirchhoff migration/inversion that diffraction smiles are characterized by the passage of a stationary point through the endpoint of integration; the visible smile occurs when the stationary point of the integral over sources and receivers is outside of the interval of integration and the dominant term of the asymptotic expansion of the integral arises from the endpoint of integration.

Using the stationary phase principle again, one can estimate all sorts of geometrical attributes-parameters associated with the specularity of the rays that actually dominate the data that images a reflector. Among these attributes are the incidence angle, source or receiver coordinate for the specular rays, and travel time along the specular ray path, among others. See for example, Geoltrain (1991).

There is another subtle difference between Berkhout inversion and other approaches presently in use. I will explain with the aid of three figures. In Figure 1, we depict a data cube with coordinate axes being midpoint, offset and time/frequency. In Figure 2, we show a typical slice used for a common offset migration/inversion.


Fig. 1. The elementary data cube.
That is, we take a fixed offset and process over midpoints and time or frequency to obtain an image and parameter estimates. In this approach, we would use data in the orthogonal (horizontal) direction for velocity analysis and, later, to stack for image enhancement. Common shot migration/inversion could be represented by a different vertical slice through the data, again with velocity analysis and stacking being carried out with the use of parallel vertical slices through the data.


FIG. 2. Data slice for a common offset migration/inversion.


FIG. 3. Data slice for Berkhout migration/inversion.

In contrast, in Figure 3 we show the type of data slice used in Berkhout migration/inversion. Note that all sources and receivers are used in the inversion process, leaving the orthogonal direction-time or frequency-for velocity analysis and image enhancement.

I think of alternative approaches as constituting a tradeoff of time for depth to obtain a preliminary reflector map and parameter estimates; the Berkhout approach would seem to be a tradeoff of offset for depth. It seems to me that it would take a "lot" of offset to provide significant depth information. Where that is not available, this method would seem to be much more dependent on the redundancy of data to overcome the artifacts of limited aperture.

On the other hand, an inversion at fixed frequency has an important feature that alternative methods lack, namely, the ability to use different velocities for different frequencies. That is, it would seem that this method can accommodate dispersion more easily than other methods. I do not believe that this aspect of the Berkhout approach has been exploited to date. It also suggests the possiblity of developing a velocity analysis scheme in which one examines residual moveout as a function of frequency, thereby developing a frequency dependent background velocity-essentially, deriving a dispersion relationship from residual moveout in frequency.

Analytically, I determine the properties of an inversion operator by applying the operator to Kirchhoff data for a curved reflector. It is this approach that led to an interpretation of our original inversion in terms of the geometrical optics reflection coefficient. This followed from the analysis of Kirchhoff data, despite the fact that the basic inversion formalism was based on the Born approximation-seemingly precluding large changes in medium parameters across an interface and precluding wide offset, near critical reflection and beyond. See, for example, Bleistein (1987).

It is this same type of analysis that I am trying to carry out for the Berkhout formalism.

To date, I have been able to derive the results described below.

1. I have produced a straightforward continuum analog of the $W R W$ model for the case of a horizontal reflector in a constant background medium. As part of this result, I find a continuum analog for the reflectivity matrix for this model.
2. This derivation provides a motivation for the idea that inversion amounts to finding inverse operators for the two $W$ 's of the theory. However, I argue that an exact inverse cannot exist because the forward operator has evanescent modes whose rate of exponential decay approaches infinity with $k_{1}$, the transverse wave number. Of course, for finite values of $k_{1}>\omega / c$, an approximate inverse leads to an ill-conditioned operator, still not satisfactory. (Here, $\omega$ is frequency and $c$ is propagation speed.)
I then show that the adjoint operator, $W^{*}$, is a reasonable inverse, in the sense that it inverts the propagating modes and attenuates evanescent modes. (Note
this means in an amplitude-consistent sense in addition to a kinematic sense.)
3. I confirm that adjoint processing provides a means of imaging and a means of estimating the angularly dependent geometrial optics reflection coefficient. This result is achieved by applying an operator based on $W^{*}$ to an exact representation of the upward scattered wave and calculating the resulting integrals exactly.
4. The simple $W R W$ model does not work for the dipping planar reflector or the curved reflector in a constant background medium. However, I am able to carry through the asymptotic analysis of applying the same adjoint operators to a model of data for the dipping plane. Here, it is necessary to introduce the first approximations in the analysis. The dipping reflector is assumed to emerge at the upper surface "far" from the processing region and I neglect these endpoint effects. I consider this to be a not-to-serious problem in the sense that we are not really interested, in practice, in reflectors that emerge at the upper surface.
I show again how one obtains an image of the reflector and an estimate of the angularly dependent reflection coefficient. However, the latter now requires an accurate estimate of the angle of dip of the reflector. Of course, having an image of the reflector allows one to estimate the dip angle-easy for a planar reflector, harder for a curved reflector. Here, I expect that the stationary phase principal applied to integral versions of the Berkhout inversion will provide a means of obtaining a numerical estimate of the dip of the reflector. This extension has not yet been carried out.
5. Using a recent thesis by a von Vronhoven, a student of Fokkema's, (Fokkema et al, 1993) as a point of departure, I am able to obtain the form $W R W$ for the upward propagating field from a single reflector. The difference between my work and van Vroonhoven's is that her $W$-operators require knowledge of the normal direction all along the reflector. In that case, the $W$ 's are no longer propagators; propagators should only depend on the source mechanism and the background medium. In order to overcome this shortcoming, I find it necessary to introduce $1 \times 4$ dyadic propagator operators operating on a $4 \times 4$ dyadic of reflectivity matrices. Furthermore, the von Vroonhoven result applies to the impulse response and not to the dipole response. I believe that to obtain a result for the dipole response will require one to fall back to the asymptotic approximation of the upward scattered field and not try to work directly on an exact representation.
6. For the curved reflector, applying the same adjoint operators produces a more complicated result from which imaging is apparent, but estimation of the reflection coefficient is not yet completed.

This is the present status of this research project. In the following sections, the results are presented in the same order as in the enumerated list.

## THE HORIZONTAL REFLECTOR

In this section, I develop some elementary ideas in the context of the simplest reflection problem, namely a point source in two dimensions over a single horizontal reflector. We introduce the exact representation of the upward scattered solution and confirm an assertion in de Bruin [1992] that for the source and receiver on the reflector, this response is just the Fourier transform of the angularly dependent reflection coefficient. We then derive a continuum analog of the solution representation form, $W_{g} R W_{s}$, and use this as motivation for derivation of an approximate inversion operator, $W^{*}$, for each of the forward modeling operators, $W_{g}$ and $W_{s}$. Finally, I check the application of this approximate inverse operator on our exact solution representation and show that it really provides information about the angularly dependent reflection coefficient and also provides a means of imaging the reflector.

## The Forward Model

Let us consider the simple example of a point source over a horizontal interface across which only the propagation speed changes. See Figure 4. We are interested in


Fig. 4. Source and receiver over a horizontal reflector.
the upward scattered wave. The solution to this problem in two dimensions can be found in many books including Bleistein [1984] eq. (8.1.1);

$$
\begin{gather*}
u\left(x_{g}, z_{g}, x_{s}, z_{s}, \omega\right)=-\frac{1}{4 \pi i} \int_{C} R\left(k_{1}, \omega\right) \exp \{i \Phi\} \frac{d k_{1}}{k_{3}} \\
z_{s}>H, \quad z_{g}>H \tag{1}
\end{gather*}
$$

Here,

$$
\begin{equation*}
R\left(k_{1}, \omega\right)=\frac{k_{3}\left(k_{1}, \omega\right)-k_{4}\left(k_{1}, \omega\right)}{k_{3}\left(k_{1}, \omega\right)+k_{4}\left(k_{1}, \omega\right)} \tag{2}
\end{equation*}
$$

and

$$
\begin{align*}
\Phi & =k_{1}\left[x_{g}-x_{s}\right]+k_{3}\left[2 H-\left(z_{g}+z_{s}\right)\right]  \tag{3}\\
& =k_{1} s+k_{3} n
\end{align*}
$$

In these equations, as seen in Figure $4,\left(x_{s}, z_{s}\right)$ are the source coordinates, $\left(x_{g}, z_{g}\right)$ are the receiver coordinates, $H$ is the normal distance from the source point to the reflector, and $c_{1}$ and $c_{2}$ are the propagation speeds above and below the reflector, respectively.

The variables, $s$ and $n$ are shown in Figure 5. They are the projected distancesparallel and normal to the reflector-from the image of the source point. Equivalently, they are the parallel and normal distances from the source point to the receiver point along the specular ray path. As defined, $n$ is a signed quantity that is positive for the source and the receiver above the reflector. In modeling, this will always be the case; in downward continuation of the sources and receivers, this need not be the case, since the downward continuation is a mathematical process that is not limited by the physical constraints of the problem.

The functions, $k_{3}$ and $k_{4}$ are given by

$$
\begin{equation*}
k_{3}=\sqrt{\omega^{2} / c_{1}^{2}-k_{1}^{2}}, \quad k_{4}=\sqrt{\omega^{2} / c_{2}^{2}-k_{1}^{2}} \tag{4}
\end{equation*}
$$

with both square roots defined to be positive at $k_{1}=0$. Also, the contour $C$ on the


Fig. 5. Rays and normal and parallel coordinates.
right side of (1) extends from $-\infty$ to $\infty$, passing over the branch points of $k_{3}$ and $k_{4}$ in the left half $k_{1}$-plane and under the branch points in the right half $k_{1}$-plane, This is a classic technique in the method of complex variables for defining the square roots so that they are real and positive for $\omega^{2} / c_{1}^{2}>k_{1}^{2}$, and $\omega^{2} / c_{2}^{2}>k_{1}^{2}$, respectively, and they have a positive imaginary part for $\omega^{2} / c_{1}^{2}<k_{1}^{2}$, and $\omega^{2} / c_{2}^{2}<k_{1}^{2}$, respectively.

This solution is the response to an impulsive point source at $\left(x_{s}, z_{s}\right)$. In the absence of the reflector, the solution is the free space Green's function, which, in the wave number domain, is given by

$$
\begin{equation*}
G\left(k_{1}, z, x_{s}, z_{s}, \omega\right)=-\frac{1}{2 i k_{3}} \exp \left\{-i k_{1} x_{s}+i k_{3}\left|z-z_{s}\right|\right\} \tag{5}
\end{equation*}
$$

We are, instead, interested in the response to a dipole source. In particular, to make the constants come out nicely, introduce the solution $W$ to the equation,

$$
\begin{equation*}
\nabla^{2} W+\frac{\omega^{2}}{c_{1}^{2}} W=2 \delta^{\prime}\left(z-z_{s}\right) \delta\left(x-x_{s}\right) \tag{6}
\end{equation*}
$$

In the $k_{1}$-plane, the solution to this equation is

$$
\begin{equation*}
W\left(k_{1}, z, x_{s}, z_{s}, \omega\right)=\operatorname{sign}\left(z-z_{s}\right) \exp \left\{-i k_{1} x_{s}+i k_{3}\left|z-z_{s}\right|\right\} \tag{7}
\end{equation*}
$$

obtained from the previous result by taking the derivative with respect to $z$ and multiplying by -2 . The representation of the dipole response in the space-frequency domain is obtained by making a corresponding adjustment in (1), namely, multiplication by $-2 i k_{3}$. This function will still be called $u$, because there will be no further need of the former result in the discussion to follow. Thus,

$$
\begin{equation*}
u\left(x_{g}, z_{g}, x_{s}, z_{s}, \omega\right)=\frac{1}{2 \pi} \int_{C} R\left(k_{1}, \omega\right) d k_{1} \exp \{i \Phi\} \tag{8}
\end{equation*}
$$

In this application, the dipole response is evaluated at $z>z_{s}$ so that the sign- function in $W$ is positive and can be omitted.

## Reflectivity

Our point of departure for comparison with the Berkhout approach is a discussion in Cees de Bruin's thesis [102]. He points out that the dipole response with source and receiver on the reflecting surface is just the Fourier transform ${ }^{1}$ of the angularly dependent reflection coefficient. Here, we will verify that observation for the exact field representation, (8). All we need to do is to take the limit in that equation as $z_{s}$ and $z_{g}$ both approach $H$. See Figure 6. It is necessary to think of this as a limiting process from above rather than a direct evaluation because there is a different limit of the "scattered field" from below, namely, the transmitted wave. It is the limit from above that we define as the field on the reflector for this discussion. The result is

$$
\begin{equation*}
u\left(x_{g}, H, x_{s}, H, \omega\right)=\frac{1}{2 \pi} \int_{C} R\left(k_{1}, \omega\right) d k_{1} \exp \left\{i k_{1} s\right\} \tag{9}
\end{equation*}
$$

[^0]

Fig. 6. Source and receiver on the horizontal reflector.
Indeed, this result is just the Fourier transform of the reflection coefficient (2) expressed as a function of $k_{1}$ and $\omega$. For $k_{1}^{2} \leq \omega^{2} / c^{2}$, we can introduce the incidence angle $\gamma$ with respect to the normal by

$$
\begin{equation*}
k_{1}=\frac{\omega}{c_{1}} \sin \gamma \tag{10}
\end{equation*}
$$

and rewrite the reflection coefficient in the form

$$
\begin{equation*}
R\left(\left(\omega / c_{1}\right) \sin \gamma, \omega\right)=\frac{\cos \gamma / c_{1}-\sqrt{1 / c_{2}^{2}-\sin ^{2} \gamma / c_{1}^{2}}}{\cos \gamma / c_{1}+\sqrt{1 / c_{2}^{2}-\sin ^{2} \gamma / c_{1}^{2}}} . \tag{11}
\end{equation*}
$$

or, in terms of the ray parameter $q$ with $k_{1}=\left(\omega / c_{1}\right) q$,

$$
\begin{equation*}
R\left(\omega q / c_{1}, \omega\right)=\frac{\sqrt{1-q^{2}} / c_{1}-\sqrt{1 / c_{2}^{2}-q^{2} / c_{1}^{2}}}{\sqrt{1-q^{2}} / c_{1}+\sqrt{1 / c_{2}^{2}-q^{2} / c_{1}^{2}}} . \tag{12}
\end{equation*}
$$

These latter two forms prove useful for integration (summation) over all $\omega$ for fixed incidence direction.

Returning to the form (9), we define the function,

$$
\begin{align*}
R\left(x_{g}, x_{s}, \omega\right) & =u\left(x_{g}, x_{s}, \omega\right) \\
& =\frac{1}{2 \pi} \int_{C} d k_{1} R\left(k_{1}, \omega\right) \exp \left\{i k_{1} s\right\} \tag{13}
\end{align*}
$$

In this particular case, $R\left(x_{g}, x_{s}, \omega\right)$ is actually a function of the difference, $x_{g}-x_{s}$ and is just the dipole response, as predicted. However, in more general examples, it should be anticipated that the result will be somewhat more complicated than this, but still related to the reflection coefficient. If the coordinates $x_{g}$ and $x_{s}$ are discretized through $x_{g}=j \Delta \xi$, and $x_{s}=i \Delta \xi$, then $R$ becomes a matrix at the depth, $H$,

$$
\begin{equation*}
R_{i j}=u(i \Delta \xi, j \Delta \xi, \omega) . \tag{14}
\end{equation*}
$$

This is very close to Berkhout's reflectivity matrix, except that this matrix is defined by an integral in (13), whereas in Berkhout's derivation, it would be the corresponding finite discrete sum for discrete values of $\omega$. With this caveat in mind, below, the function, $R\left(x_{g}, x_{s}, \omega\right)$, in which both depth variables are evaluted at the same depth, will be called the reflectivity matrix, even in its continuous form. The term, reflectivity function, will be reserved for the case in which the two depth-arguments are independent. This will be discussed below.


Fig. 7. Reflection coefficient as a function of $q$, equation (12).
Figure 7 is a Mathematica rendition of the absolute value of the reflection coefficient as a function of $q$ in the range ( $-2,2$ ), which includes the range of real incident angles and a portion of the range of evanescent modes. Note that this is a complex valued even function of $q$ and that its Fourier transform, the reflectivity matrix, even at spatial position $x_{g}-x_{s}=0$, is complex valued. For the purpose of imaging, then, it makes sense to examine the absolute value of the Fourier transform of $R$. Figure 8 is the absolute value of a 128 point Fourier transform of $R$ in the previous figure, as a function of $x_{g}-x_{s}$ with the origin at the center position. The 128X128 discretized version (13) has this plot as its 65th row. This is a Zoepritz matrix, with each row just a shift of this one. In this case, a single row carries all of the information about $R$. Equivalently, in a laterally homogeneous medium, a single shot and double spread of receivers contains all information of all translates of this configuration, equivalent to the translates of the central row of the reflectivity matrix. These figures should be compared with Figure 3.4 in de Bruin [102].

These observations suggest an approach to inversion, namely, that one create a dipole response at depth from the ensemble of observations of dipole response experiments. This is a form of downward continuation of the data. Before we proceed to do this, we will examine the structure of the solution (8) with an eye towards the matrix form of the Berkhout approach and with an eye towards writing an operator


Fig. 8. Fourier transform of the reflection coefficient with zero position as the center point of the plot. In a 128 X 128 matrix representation of the reflectivity matrix for this example, this is the 65th row (or column).
form for the observed field that lends itself to the process of creating an operator inversion that achieves the intended goal.

## The Form WRW

It will be shown here that the integral representation (8) has the form of the inverse Fourier transform of three functions, namely, two dipole Green's functions and the reflectivity in the $k_{1}$-domain. To be more specific, let us define the spatial-frequency domain dipole response in the absence of the reflector as the (distributional) inverse Fourier transform of the result (7), namely,

$$
\begin{equation*}
W(x, z, \xi, \zeta, \omega)=\frac{\operatorname{sign}(z-\zeta)}{2 \pi} \int_{C} \exp \left\{i k_{1}[x-\xi]+i k_{3}|z-\zeta|\right\} d k_{1} \tag{15}
\end{equation*}
$$

Furthermore, let us define a reflectivity function $\mathcal{R}$ of four spatial variables and frequency,

$$
\begin{equation*}
\mathcal{R}\left(\xi_{g}, \zeta_{g}, \xi_{s}, \zeta_{s}, \omega\right)=R\left(\xi_{g}, \xi_{s}, \omega\right) \delta\left(\zeta_{s}-H\right) \delta\left(\zeta_{g}-H\right) \tag{16}
\end{equation*}
$$

In this particular example, the reflectivity function of four spatial variables is defined in such a manner as to restrict it to the only depth where reflection data is nonzero, namely, on the reflector at depth $H$. The reflectivity matrix-the Fourier transform of the reflection coefficient-becomes the weighting function of the distributions that restrict this particular reflector to the depth $H$. This is the continuum mechanism for creating the reflection data with nonzero values only on the given reflector.

Now, we consider the integral,

$$
\begin{align*}
I= & \int W\left(x_{g}, z_{g}, \xi_{g}, \zeta_{g}, \omega\right) \mathcal{R}\left(\xi_{g}, \zeta_{g}, \xi_{s}, \zeta_{a}, \omega\right) \\
& \cdot W\left(\xi_{s}, \zeta_{s}, x_{s}, z_{s}, \omega\right) d \xi_{s} d \xi_{g} d \zeta_{s} d \zeta_{g} \\
= & \int W\left(x_{g}, z_{g}, \xi_{g}, H, \omega\right) R\left(\xi_{g}, H, \xi_{s}, H, \omega\right) W\left(\xi_{s}, H, x_{s}, z_{s}, \omega\right) d \xi_{s} d \xi_{g} \tag{17}
\end{align*}
$$

By using the Fourier representations for $W$ (15) and for $R(13), I$ can be recast as

$$
\begin{align*}
I=\frac{1}{(2 \pi)^{3}} & \int d \xi_{s} d \xi_{g} d k_{1}^{\prime} d k_{1}^{\prime \prime} d k_{1} \\
& \cdot \exp \left\{i k_{1}^{\prime}\left[x_{g}-\xi_{g}\right]+i k_{3}\left(k_{1}^{\prime}\right)\left[H-z_{g}\right]\right\}  \tag{18}\\
& \cdot R\left(k_{1}, \omega\right) \exp \left\{i k_{1}\left[\xi_{g}-\xi_{s}\right]\right\} \\
& \cdot \exp \left\{i k_{1}^{\prime \prime}\left[\xi_{s}-x_{s}\right]+i k_{3}\left(k_{1}^{\prime \prime}\right)\left[H-z_{s}\right]\right\}
\end{align*}
$$

The integrals over $\xi_{s}$ and $\xi_{g}$ yield delta functions in the wavenumber variables, which allow us to carry out those integrals, as well, leaving only the integral in $k_{1}$. In particular, we use the result

$$
\frac{1}{2 \pi} \int_{-\infty}^{\infty} d \xi_{g} \exp \left\{i\left[k_{1}-k_{1}^{\prime}\right] \xi_{g}\right\}=\delta\left(k_{1}-k_{1}^{\prime}\right)
$$

with a similar result in $\xi_{s}$ and $k_{1}^{\prime \prime}$. Consequently, the function $I$ in (17) becomes just the wavefield $u$ as given by (8):

$$
\begin{equation*}
I=u\left(x_{g}, z_{g}, x_{s}, z_{s}, \omega\right) \tag{19}
\end{equation*}
$$

If we think of the operations of multiplication by the functions $W\left(x_{g}, z_{g}, \xi_{g}, H, \omega\right)$ and $W\left(\xi_{s}, H, x_{s}, z_{s}, \omega\right)$ and integration over $\xi_{g}$ and $\xi_{s}$ as operators, $W_{g}$ and $W_{s}$, then (17) and (19) can be rewritten as the operator equation

$$
\begin{equation*}
u\left(x_{g}, z_{g}, x_{s}, z_{s}, \omega\right)=W_{g} R W_{s} \tag{20}
\end{equation*}
$$

Of course, in operator form, this result can be thought of as the integrations that were used here, or, in discrete form, as matrix multiplications, which is the Berkhout,et al representation.

This representation describes the response only from the reflector at depth $H$. In the discrete approach, one must then sum over all depths. Berkhout, et al would
tie the depth variable $\zeta_{s}$ to the depth variable $\zeta_{g}$ and then carry out one summation. Here, I prefer to leave those two variables separate as in (16) and think of a double sum or double integral representing separate propagation in sources and receivers. Using the notation $\mathcal{W}_{s}$ and $\mathcal{W}_{g}$ to include the integrations over depth in (17), I introduce the integral operator notation,

$$
\begin{equation*}
u\left(x_{g}, z_{g}, x_{s}, z_{s}, \omega\right)=\mathcal{W}_{g} \mathcal{R} \mathcal{W}_{s}, \tag{21}
\end{equation*}
$$

as an operator form of this result in terms of all of the integrations that appear in (17).

## ...of Inverses and Adjoints

For the purpose of migration or inversion, we seek a method of propagating the data from the source/receiver surface to the reflecting surface, in such a manner that we obtain (or reasonably approximate) the reflectivity matrix on that surface. Thus, it is our objective to transform the data as described by (8) to data at points $\left(\xi_{s}, \zeta_{s}\right),\left(\xi_{g}, \zeta_{g}\right)$, with $\zeta_{s}>z_{s}$ and $\zeta_{g}>z_{g}$.

In terms of our operators, $W_{s}$ and $W_{g}$, (or $\mathcal{W}_{s}$ and $\mathcal{W}_{g}$, we seek inverse operators, say, $\left[W_{s}\right]^{-1}$ and $\left[W_{g}\right]^{-1}$. That is, we seek something that will operate on these operators and produce a delta function at depth.

First, let me describe something that does not achieve this end. One might think to multiply the observed data by functions $W_{s}^{-}$and $W_{g}^{-}$-given by (15) with the exponent replaced by its negative-and then integrate over $x_{s}$ and $x_{g}$, essentially carrying out a convolution over sources and receivers. Unfortunately, the integral representations for these two new functions do not converge because the integrand in each is exponentially growing for imaginary $k_{3}$. Disregarding this for the moment, the formal integrations over all sources and receivers produce delta functions in the $k_{1}$-variables, just as in the analysis of the integral (18), which, in turn allow for "easy" calculation of the $k$-domain integrals to obtain the spatial domain reflectivity, $R\left(\xi_{g}, \zeta_{g}, \xi_{g}, \zeta_{g}, \omega\right)$. Indeed, these functions $W_{s}^{-}$and $W_{g}^{-}$are an attempt to downward propagate the propagating modes and to simultaneously compensate for the attenuation of the evanescent modes. However, they are not valid.

What is happening here is that the attenuation rate approaches infinity as $\left|k_{1}\right|$ does. Thus, an inversion operator that compensates for attenuation would have to have an exponential growth rate that becomes infinite with $k_{1}$. One cannot create such an inverse operator.

What does seem to work, is to base the operator inversion on the incoming Green's function. Indeed, if one tried to represent $u$ at some point between the reflecting surface and the source receiver surface in terms of Green's functions, one would find that the Green's function of choice is the incoming delta function response. The reason is that we have to form the integral of $u \mathcal{L} G^{*}-G^{*} \mathcal{L} u$ over a domain bounded by the observation surface and some lower surface below the output point where
$u$ is to be evaluated and above the reflector. Here $\mathcal{L}$ is the wave operator. Using Green's theorem transforms this integral into a surface integral over the bounding surfaces with integrand, $u \partial G^{*} / \partial n-G^{*} \partial u / \partial n$. On the upper surface, the value of $u$ is the observed field and its normal derivative is easily determined in the $k_{1}$-domain under the assumption that it is an upward propagating wave. On the lower surface, we do not know the fields. Thus, we need to choose $G^{*}$ so that this integral is zero. Since $u$ is upward propagating on this lower surface, this can be guaranteed by choosing a Green's function with that property. The Green's function of choice is the inward propagating Green's function. Here, "inward" means towards the point at depth where $u$ is to be evaluated. On the upper surface, that Green's function is also inward propagating or now downward propagating, while $u$, itself, is still upward propagating. Hence the integral over the upper surface is nonzero, in general.

The inward propagating Green's function is given by

$$
\begin{equation*}
G^{*}(x, z, \xi, \zeta)=\frac{1}{4 \pi i} \int_{C} \frac{d k_{1}}{k_{3}^{*}\left(k_{1}\right)} \exp \left\{-i k_{1}(x-\xi)-i k_{3}^{*}\left(k_{1}\right)|z-\zeta|\right\} \tag{22}
\end{equation*}
$$

In this equation, $k_{3}^{*}\left(k_{1}\right)$ is the complex conjugate of the function $k_{3}\left(k_{1}\right)$. Thus, on the path of integration, this function agrees with $k_{3}$ when they are both real but is the negative of $k_{3}$ when they are imaginary. This assures that in the latter domain, $-i k_{3}^{*}\left(k_{1}\right)$ has a negative real part and the integral converges. It is an exercise in asymptotics to show that $G^{*}=O\left(\exp \left\{-i \omega r / c_{1}\right\} / \sqrt{ } \omega r / c_{1}\right)$, with $r$ being radial distance between source and receiver. For the temporal Fourier transform that I use, this is the incoming Green's function.

The incoming dipole response, $W^{*}$, corresponding to this Green's function is

$$
\begin{equation*}
W^{*}(x, z, \xi, \zeta)=\frac{\operatorname{sign}(z-\zeta)}{2 \pi} \int_{C} d k_{1} \exp \left\{-i k_{1}(x-\xi)-i k_{3}^{*}\left(k_{1}\right)|z-\zeta|\right\} \tag{23}
\end{equation*}
$$

Now consider the integral,

$$
\begin{equation*}
I=\int d x_{s} W\left(x, z, x_{s}, z_{s}, \omega\right) W^{*}\left(\xi, \zeta, x_{s}, z_{s}, \omega\right) \tag{24}
\end{equation*}
$$

with both $z$ and $\zeta$ greater than $z_{3}$. We use the Fourier representations (15) and (23) to rewrite this integral as

$$
\begin{align*}
I=\frac{1}{(2 \pi)^{2}} & \int d x_{s} d k_{1} d k_{1}^{\prime} \\
& \cdot \exp \left\{i k_{1}\left[x-x_{s}\right]+i k_{3}\left(k_{1}\right)\left[z-z_{s}\right]\right\}  \tag{25}\\
& \cdot \exp \left\{-i k_{1}^{\prime}\left[\xi-x_{s}\right]-i k_{3}^{*}\left(k_{1}^{\prime}\right)\left[\zeta-z_{s}\right]\right\}
\end{align*}
$$

As above, the integral over $x_{s}$ is a delta function, which allow us to calculate the integral over $k_{1}^{\prime}$ as well. The result is

$$
I=\frac{1}{(2 \pi)} \int d k_{1} \exp \left\{i k_{1}[x-\xi]+i k_{3}\left(k_{1}\right)\left(z-z_{s}\right)-i k_{3}^{*}\left(k_{1}\right)\left(\zeta-z_{s}\right)\right\}
$$

$$
\begin{align*}
= & \frac{1}{2 \pi} \int_{\omega^{2} / c_{1}^{2} \geq k_{1}^{2}} d k_{1} \exp \left\{i k_{1}[x-\xi]+i k_{3}[z-\zeta]\right\}  \tag{26}\\
& +\frac{1}{2 \pi} \int_{\omega^{2} / c_{1}^{2}<k_{1}^{2}} d k_{1} \exp \left\{i k_{1}[x-\xi]+i k_{3}\left(k_{1}\right)\left[z+\zeta-2 z_{s}\right]\right\}
\end{align*}
$$

We are more interested in the Fourier transform here than in the function, $I$, itself.

$$
\tilde{I}=\left[\begin{array}{l}
\exp \left\{i k_{3}\left(k_{1}\right)[z-\zeta]\right\}, \quad \omega^{2} / c_{1}^{2} \geq k_{1}^{2}  \tag{27}\\
\left.\exp \left\{i k_{3}\left(k_{1}\right)\left[z+\zeta-2 z_{s}\right)\right]\right\}, \quad \omega^{2} / c_{1}^{2}<k_{1}^{2}
\end{array}\right.
$$

In particular, when the two depths agree, $z=\zeta$,

$$
\tilde{I}=\left[\begin{array}{l}
1, \quad \omega^{2} / c_{1}^{2} \geq k_{1}^{2}  \tag{28}\\
\exp \left\{i k_{3}\left(k_{1}\right)\left[2 z-\left(z_{s}+z_{g}\right)\right]\right\}, \quad \omega^{2} / c_{1}^{2}<k_{1}^{2}
\end{array}\right.
$$

That is, for $z=\zeta, \tilde{I}$ is equal to unity in the propagating range and attenuates to zero outside that range, with the rate of attenuation increasing with depth and increasing with $\left|k_{1}\right|$. If $\tilde{I}$ were identically equal to unity at $z=\zeta$, then $I$ would be a delta function, $\delta\left(x_{g}-x_{s}\right)$. Thus, we can think of $I$ as an approximate delta function that at least behaves "right" in the range of propagating wave numbers.

Note that if we were to discretize the operator $I$ as defined by (24) then the indicated product would become a matrix multiplication of $W$ with the transpose of $W^{*}$. Since $W^{*}$ is the complex conjugate of $W$, this is a multiplication of $W$ with its adjoint. In that sense, I view the operation of multiplication by $W^{*}$ and integration as being the continuous analog of matrix multiplication with the adjoint of the matrix $W$. Thus, I view the operator that I propose here as being the continuous analog of the Berkhout school's multiplication with the adjoint matrices associated with the downward propagator matrices.

Of course, this is just the simplest constant background case and we have yet to see what the effect of this operator is on more complicated reflectors, such as dipping planes and arbitrary curved reflectors. However, I believe this is a good start.

## Downward Continuation

We are now prepared to test our adjoint operators as approximate inverse operators by applying one in source variables and one in receiver variables to our representation (9) of the observed field. Thus, we consider the integral

$$
\begin{equation*}
I=\int d x_{s} d x_{g} W^{*}\left(x_{g}, z_{g}, \xi_{g}, \zeta_{g}, \omega\right) u\left(x_{g}, z_{g}, x_{s}, z_{s}, \omega\right) W^{*}\left(\xi_{s}, \zeta_{s}, x_{s}, z_{s}, \omega\right) \tag{29}
\end{equation*}
$$

Physically, the integral over $x_{g}$ downward propagates all the responses for a fixed source to the position $\left(\xi_{g}, \zeta_{g}\right)$. Correspondingly, by reciprocity, the role of sources and receivers can be interchanged. Viewing all source points as receiver locations for a fixed geophone position now viewed as the source, we can perform exactly the same downward propagation of sources. This is the integral over $x_{s}$.

As previously, the integrations here will be carried out by using the Fourier representations of the three functions under the integral sign, with each $W^{*}$ given by (23) and the observed field given by (9). Thus, the previous equation is replaced by

$$
\begin{align*}
I=\frac{1}{8 \pi^{3}} \int & d x_{s} d x_{g} d k_{1}^{\prime} d k_{1}^{\prime \prime} d k_{1} \\
& \cdot \exp \left\{-i k_{1}^{\prime}\left[x_{g}-\xi_{g}\right]-i k_{3}^{*}\left(k_{1}^{\prime}\right)\left[\zeta_{g}-z_{g}\right]\right\} \\
& \cdot R\left(k_{1}, \omega\right) \exp \left\{i k_{1}\left[x_{g}-x_{s}\right]+i k_{3}\left(k_{1}\right)\left[2 H-\left(z_{g}+z_{s}\right)\right]\right\}  \tag{30}\\
& \cdot \exp \left\{-i k_{1}^{\prime \prime}\left[\xi_{s}-x_{s}\right]-i k_{3}^{*}\left(k_{1}^{\prime \prime}\right)\left[\zeta_{s}-z_{s}\right]\right\}
\end{align*}
$$

We calculate these integrals as above. Namely, the integrals in $x_{s}$ and $x_{g}$ are delta functions that allows us to determine the integrals in $k_{1}^{\prime}$ and $k_{1}^{\prime \prime}$, yielding the result,

$$
\begin{align*}
I= & \frac{1}{2 \pi} \int d k_{1} R\left(k_{1}, \omega\right) \\
& \cdot \exp \left\{i k_{1}\left[\xi_{g}-\xi_{s}\right]+i k_{3}\left(k_{1}\right)\left[2 H-\left(z_{g}+z_{s}\right)\right]-i k_{3}^{*}\left(k_{1}\right)\left[\zeta_{g}+\zeta_{s}-\left(z_{g}+z_{s}\right)\right]\right\} \tag{31}
\end{align*}
$$

or

$$
\begin{align*}
I=\frac{1}{2 \pi} & \int_{\omega^{2} / c_{1}^{2} \geq k_{1}^{2}} d k_{1} R\left(k_{1}, \omega\right) \exp \left\{i k_{1}\left[\xi_{g}-\xi_{s}\right]+i k_{3}\left(k_{1}\right)\left[2 H-\left(\zeta_{g}+\zeta_{s}\right)\right]\right\} \\
+ & \frac{1}{2 \pi} \int_{\omega^{2} / c_{1}^{2}<k_{1}^{2}} d k_{1} R\left(k_{1}, \omega\right)  \tag{32}\\
& \cdot \exp \left\{i k_{1}\left[\xi_{g}-\xi_{s}\right]+i k_{3}\left(k_{1}\right)\left[2 H+\zeta_{g}+\zeta_{s}-2\left(z_{g}+z_{s}\right)\right]\right\}
\end{align*}
$$

This equation can be equivalently written as

$$
\begin{align*}
I=\frac{1}{2 \pi} & \int_{\omega^{2} / c_{1}^{2} \geq k_{1}^{2}} d k_{1} R\left(k_{1}, \omega\right) \exp \left\{i k_{1} s+i k_{3}\left(k_{1}\right) n\right\} \\
+ & \frac{1}{2 \pi} \int_{\omega^{2} / c_{1}^{2}<k_{1}^{2}} d k_{1} R\left(k_{1}, \omega\right)  \tag{33}\\
& \cdot \exp \left\{i k_{1} s+i k_{3}\left(k_{1}\right)[n+2 d]\right\}
\end{align*}
$$

Here, $s$ and $n$ are as in Figure 5, except that now they are measured from the downward continued source and receiver positions, $\left(\xi_{s}, \zeta_{s}\right)$ and $\left(\xi_{g}, \zeta_{g}\right)$. The distance, $d$ is the sum of the normal distances from the original source and receiver positions to the propagated source and receiver positions, respectively; that is, $d=\zeta_{s}-z_{s}+\zeta_{g}-z_{g}$.

We evaluate the result (32) at $\zeta_{s}=\zeta_{g}=H$ to obtain

$$
\begin{align*}
I= & \frac{1}{2 \pi} \int_{\omega^{2} / c_{1}^{2} \geq k_{1}^{2}} d k_{1} R\left(k_{1}, \omega\right) \exp \left\{i k_{1}\left[\xi_{g}-\xi_{s}\right]\right\}  \tag{34}\\
& +\frac{1}{2 \pi} \int_{\omega^{2} / c_{1}^{2}<k_{1}^{2}} d k_{1} R\left(k_{1}, \omega\right) \exp \left\{i k_{1}\left[\xi_{g}-\xi_{s}\right]+2 i k_{3}\left(k_{1}\right)\left[2 H-\left(z_{g}+z_{s}\right)\right]\right\}
\end{align*}
$$

The Fourier transform of this result is

$$
\tilde{I}=\left[\begin{array}{l}
R\left(k_{1}, \omega\right), \quad \omega^{2} / c_{1}^{2} \geq k_{1}^{2}  \tag{35}\\
R\left(k_{1}, \omega\right) \exp \left\{2 i k_{3}\left(k_{1}\right)\left[2 H-\left(z_{g}+z_{s}\right)\right]\right\}, \quad \omega^{2} / c_{1}^{2}<k_{1}^{2}
\end{array}\right.
$$

We see that this result produces the reflection coefficient for $\omega^{2} / c_{1}^{2} \geq k_{1}^{2}$. Furthermore, for $k_{1}^{2}>\omega^{2} / c_{1}^{2}$, it produces the reflection coefficient multiplied by a factor that decays exponentially with increasing $k_{1}^{2}$ and with increasing depth $H$. In fact, as noted above, the multiplier of $i k_{3}$ in the second case is twice the sum of the normal distances from the source and the receiver to the reflector. It will be seen below that this is the form that the decay rate takes in the case of the dipping reflector, as well.

Figure 9 shows the function $\tilde{I}$ in (35) for the case $2 \omega\left(2 H-z_{g}-z_{s}\right) / c_{1}=8 \pi$. This latter evaluation effects the decay rate in the second line in (35). The choice of constant made here occurs for $f\left[H-\left(z_{g}+z_{s}\right) / 2\right] / c_{1}=1$, with $f$ being frequency in Hz . For example, this corresponds to a reflector at 500 m depth, with source and receiver at depth zero, at a frequency of 10 Hz and a propagation speed of $5 \mathrm{Km} / \mathrm{s}$. If we were to decrease the propagation speed or increase the frequency to values more realistic for this depth, then the decay rate would be even greater. Similarly, if we were to increase the depth to something more consistent with this frequency and propagation speed, the same thing would happen. The point here is that for relatively conservative choices of parameters the exponential decay in the evanescent range of $k_{1}$ acts effectively like a bandpass filter eliminating all evanescent information from the output.


Fig. 9. Plot of the function in equation (26).


Fig. 10. Fourier transform of the function of Figure 3.

Figure 10 shows the transform of this result, as a function of a counting index $n$. This result is scaled by $\omega / c_{1}$. As with the exact reflection coefficient, this result peaks at zero argument, which is the center position of the 128 point transform, with peak value that is actually higher, $\approx 2.6$, than for the exact reflectivity function in Figure 8 , $\approx 2$. .

## Imaging

In order for the integral $I$ in (33) to be an effective integral for imaging the reflector, we would want this integral to peak in the $z$-direction, as well. It is clear that the integral will peak if both $s$ and $n$ are zero. If we are interested in peaking only in the $z$-direction, then we must set $s=0$. The way to do this is to make the source and receiver coincident. We propose this as the method of imaging for this approach to migration and inversion. In the next section, we show that this provides an image for the case of the dipping reflector, as well. Indeed, for $s=0$, the representation (33) assures us that the output will peak only for $n=0$.

Thus, we consider the evaluation of $I$ for $\xi_{g}=\xi_{s}$, but for $\zeta_{s}=\zeta_{g} \neq H$. The result is predominantly given by the first line in (33) with $\xi_{g}=\xi_{g}$. In Figure 11,


Fig. 11. I in (24) for coincident source receiver but variable depth.
we show this output. The units represent approximately 32 m . Again we see a sharp peak. That is, the operations represented by (29), which I can symbollically write as $W_{g}^{*} u W_{s}^{*}$, provides an output that peaks at the reflector depth for coincident source and receiver. This is what we would want an imaging operator to do. However, in addition to that, the Fourier transform of the downward continued data is the reflection coefficent as a function of $k_{1}$ and $\omega$. Of course, the lateral invariance of this problem assures us that all vertical lines will look alike, as long as we process the data
for coincident source and receiver. Note that this also makes the output a function of two spatial variables which is necessary for an image of a reflector in two dimensions.

In three dimensions, the three coordinates of the source and receiver point would be equal in pairs, making the output a function of three spatial variables as it should be for that case.

Below, when we consider the dipping reflector, we will display a two-dimensional imaging output for this operator inversion. Here, there is no point in showing more than Figure 11 because all vertical lines will be identical.

## Asymptotic Analysis

It is worthwhile to examine $I$, as defined by (33), asymptotically in order to gain some idea about whether or not processing for $I$ will reveal the presence of the reflector, that is, whether this function peaks when $\zeta_{g}=\zeta_{s}=H$. In carrying out this analysis, a change of variable of integration from $k_{1}$ to $\gamma$ through $k_{1}=\left(\omega / c_{1}\right) \sin \gamma$, or to $q$ through $k_{1}=\left(\omega / c_{1}\right) q$. In either case, an overall multiplier of $\omega / c_{1}$ will appear in the representation of $I$ and will appear in our discussion of the asymptotic results, below. Indeed, this factor also carries the dimension of $I$, namely, inverse length or wave number.

The two summands of $I$ as represented in (33) behave quite differently. In the second term, $k_{3}$ is purely imaginary and has a length scale multiplier in depth, $2 \mathrm{H}+$ $\zeta_{g}+\zeta_{s}-2\left(z_{g}+z_{s}\right)$, that is never zero. Furthermore, $k_{3}$ is singular at the endpoints of integration with an infinite derivative there. These are the dominant critical points in the asymptotic analysis of these integrals. In fact, this second line can be shown to be

$$
\frac{\omega}{2 \pi c_{1}} O\left[\left(\omega\left[2 H+\zeta_{g}+\zeta_{s}-2\left(z_{g}+z_{s}\right)\right] / c_{1}\right)^{-2}\right]
$$

which is less than the order of the first term in (33), to be derived below.
As might be expected, the analysis of this first term is sensitive to the value of the relative depth, $2 H-\left(\zeta_{g}+\zeta_{s}\right)$. It should be noted that when this variable is zero, as in (34), the phase is linear in $k_{1}$; that is, there are no stationary points in this integral. When this variable is nonzero, the integrand has a stationary point. In this case, one can show that this first line and, hence, $I$ can be estimated by

$$
\begin{equation*}
I=\frac{\omega}{2 \pi c_{1}} O\left[\left(\omega r / c_{1}\right)^{-1 / 2}\left[2 H-\left(\zeta_{g}+\zeta_{s}\right)\right] / r\right] \tag{36}
\end{equation*}
$$

Here,

$$
\begin{equation*}
r=\sqrt{\left[\xi_{g}-\xi_{s}\right]^{2}+\left[2 H-\left(\zeta_{g}+\zeta_{s}\right)\right]^{2}} \tag{37}
\end{equation*}
$$

This result is not valid when $2 H-\left(\zeta_{g}+\zeta_{s}\right)=0$ because the second derivative of the phase becomes infinite there. That is the case where the phase is linear, as noted above. Thus, this should be viewed as a qualitative estimate only for this quantity
bounded away from zero. When $2 H-\left(\zeta_{g}+\zeta_{0}\right)=0, I$ is given by (33). In particular, when, in addition, $\xi_{g}-\xi_{s}=0$,

$$
I=\frac{\omega}{2 \pi c_{1}} O(1)
$$

clearly larger than the result at other depths and for non-conincident source and receiver at depth $H$.

It seems to me that the operations indicated in (29) provide a reasonable approximation to the reflectivity function we seek, since they at least provide an estimate of the reflection coefficient in the $k_{1}$-domain in the propagating range, $-\omega / c_{1}<k_{1}<$ $\omega / c_{1}$. Therefore, with acknowledgement of the error of the approximation demonstrated here, I define

$$
\begin{equation*}
R\left(\xi_{g}, \zeta_{g}, \xi_{g}, \zeta_{s}, \omega\right)=\int d x_{s} d x_{g} W^{*}\left(x_{g}, z_{g}, \xi_{g}, \zeta_{g}, \omega\right) u\left(\xi_{g}, \zeta_{g}, x_{s}, z_{s}, \omega\right) W^{*}\left(\xi_{s}, \zeta_{s}, x_{s}, z_{s}, \omega\right) \tag{38}
\end{equation*}
$$

## Recapitulation

In summary, we have started with the model of a dipole source over a horizontal reflector and verified a basic premise for this example that the response is the Fourier transform of the reflection coefficient when the source and receiver are moved down to the reflector. We have also verified the structure $W R W$ for the field representation and used that as a motivation for a mechanism of inverting data to produce an image of the reflecting interface and an estimate of the reflection coefficient. We showed that an exact inverse of the forward propagators is not available because the attenuation rate of the forward propagators approaches infinity with $k_{1}$, requiring that the exponential growth rate of the inverter would have to do the same.

We then introduced the adjoint operator as an approximate inverse operator and showed that it acted as an inverse operator for the propagating range of $k_{1}, \omega^{2} / c_{1}^{2}>k_{1}^{2}$, and $\omega^{2} / c_{2}^{2}>k_{1}^{2}$. When these operators in source and geophone coordinates were applied to the data, the result was, indeed, an approximate inverse of the data. Imaging was achieved by plotting the output of this inversion for coincident source and receiver coordinates. Examination of the output at the depth where peaking occurred, yielded the spatial Fourier transform of the reflection coefficient restricted to the propagating range of $k_{1}$ with some error in the evanescent range. However, Fourier inversion of that spatial output yields the reflection coefficient for the propagating modes or, equivalently, for all real angles of incidence on the reflector.

## Contrast with Conventional Migration/Inversion

I do not use the word "conventional" for alternatve approaches to imply something perjorative about the method being discussed here, but only as a reference for discussion and comparison.

Note that in this method, the frequency has been held fixed throughout the discussion; imaging and parameter estimation were achieved for fixed frequencies by using data from all sources and receivers simultaneously. Subsequently, one could use summation over frequency for image enhancement through stacking, or summing over fixed ratio, $c k_{1} / \omega$ for stacking at varying incidence angle. Crudely speaking, one uses all sources and receivers for imaging, then frequency for stacking.

Conventional migration approaches the problem by migrating data from one fixed offset or one fixed source and all frequencies and then sums over offsets (first case) or source locations (second case) for stacking purposes.

It is my view that the method considered here reverses the role of summing over frequency and stacking over sources or midpoints when compared to other migration/inversion methods such as wave equation migration, Fourier migration/inversion, or Kirchhoff migration/inversion.

It is this reversal of roles that allows for the possibility of processing with a frequency dependent velocity, essentially by changing the velocity for each separate inversion. Thus, dispersion could be built into this approach to inversion more easily than in the conventional approaches.

## THE DIPPING REFLECTOR

Here, we extend the discussion to the case of a dipole source over a dipping


Fig. 12. Point source over a dipping reflector
reflector as shown in Figure 12. The objective of this discussion is to extend the ideas of the previous section to this case in order to examine the effect of reflector dip on the results of the previous discussion. In summary, the results are not so clean. The modeling formula does not lead to as clean a result as (8), nor is the function $\mathcal{R}\left(\xi_{s}, \zeta_{s}, \xi_{g}, \zeta_{g}, \omega\right)$ as clean. On the other hand the inversion operations $W_{g}^{*} u W_{s}^{*}$ does produce as simple a result as in the previous case. We can achieve imaging and we can achieve parameter estimation. It is all just a little more difficult than in the case of the horizontal reflector.

## The Forward Model

We begin be presenting the solution for the reflection from the dipping reflector in Figure 12. This solution can be derived from the previous one by first solving for the impulse response in a rotated coordinate system in which the $x^{\prime}$-axis is parallel to the reflector. In this new system, the impulse response is given by (1). Then, a change of variables back to the original coordinates yields the solution to the present problem. By taking $-2 \partial / \partial z$ of this solution, we obtain the dipole response solution.

The rotation of coordinates is given by

$$
\begin{align*}
x^{\prime} & =\left(x-x_{s}\right) \cos \phi+\left(z-z_{s}\right) \sin \phi \\
z^{\prime} & =-\left(x-x_{s}\right) \sin \phi+\left(z-z_{s}\right) \cos \phi \\
H^{\prime} & =\left(x_{s}-x_{0}\right) \sin \phi-z_{s} \cos \phi \tag{39}
\end{align*}
$$

In the last equation, $x_{0}$ is the point where the dipping reflector emerges at the upper surface, $z=0$ and $H^{\prime}$ is the normal distance from the source point to the reflector. In fact, by setting $H^{\prime}=0$ in this last equation, we obtain the equation of the reflector,

$$
\begin{equation*}
\left(x-x_{0}\right) \sin \phi-z \cos \phi=0 \tag{40}
\end{equation*}
$$

The dipole response solution for the dipping reflector is given by

$$
\begin{align*}
& u\left(x_{g}, z_{g}, x_{s}, z_{s}, \omega\right)= \frac{1}{2 \pi} \int_{C} R\left(k_{1}, \omega\right) \frac{k\left(-k_{1}\right) \cdot \hat{n}}{k_{3}\left(k_{1}\right)} d k_{1} \exp \{i \Phi\}, \\
& z_{s}<\left(x_{s}-x_{0}\right) \tan \phi, \quad z_{g}<\left(x_{g}-x_{0}\right) \tan \phi \tag{41}
\end{align*}
$$

In this equation, $R\left(k_{1}, \omega\right)$ is again given by (2); $\hat{\boldsymbol{n}}=(-\sin \phi, \cos \phi)$ is the downward pointing unit normal to the reflector and

$$
\begin{equation*}
k\left(k_{1}\right)=\left(k_{1}, k_{3}\left(k_{1}\right)\right), \quad k\left(-k_{1}\right)=\left(-k_{1}, k_{3}\left(k_{1}\right)\right) \tag{42}
\end{equation*}
$$

with the second equation arising from the symmetry of $k_{3}$.

Finally,

$$
\begin{align*}
\Phi= & k_{1}\left[\left(x_{g}-x_{s}\right) \cos \phi+\left(z_{g}-z_{s}\right) \sin \phi\right] \\
& \quad+k_{3}\left(k_{1}\right)\left[\left(x_{s}+x_{g}-2 x_{0}\right) \sin \phi-\left(z_{g}+z_{s}\right) \cos \phi\right]  \tag{43}\\
& =k_{1} s\left(x_{g}, z_{g}, x_{s}, z_{s}\right)+k_{3}\left(k_{1}\right)\left[n\left(x_{s}, z_{s}\right)+n\left(x_{g}, z_{g}\right)\right] \\
s\left(x_{g}, z_{g}, x_{s}, z_{s}\right)= & \left(x_{g}-x_{s}\right) \cos \phi+\left(z_{g}-z_{s}\right) \sin \phi \\
n(x, z)= & \left.\left(x-x_{0}\right) \sin \phi\right)-z \cos \phi
\end{align*}
$$

The functions, $k_{3}\left(k_{1}\right)$ and $k_{4}\left(k_{1}\right)$ are given by (4). We will have need of functions $k_{3}$ and $k_{4}$ of other arguments here; hence, the introduction of their explicit argument in this discussion.

Although the phase is much more complicated here, its representation in terms of $s$ and $n$ is just as it was in (3) for the flat reflector case. ${ }^{2}$ A new feature here is the ratio $k\left(-k_{1}\right) \cdot \hat{n} / k_{3}\left(k_{1}\right)$. This factor is equal to unity when the reflector is flat; hence, it was not present in the discussion of the previous section.

We take the point of view that $x_{0}$ is far to the left of the domain of interest, so that the reflector is actually more than a "few wavelengths" deep. In fact, one can obtain the previous solution from the present one by allowing $\phi \rightarrow 0$ while $x_{0} \rightarrow \infty$ in such a manner that $-x_{0} \sin \phi \rightarrow H$. This assumption that $x_{0}$ is "far away" is essentially a high frequency assumption. Below, it will be necessary to make approximations based on this.

One can see from the solution representation, (41), (43), that the dipole response can no longer be the Fourier transform of the reflection coefficient, itself. What we can say here is that for the source and the receiver both on the reflector, the dipole response is the Fourier transform of the reflection coefficient multiplied by this new factor, $\boldsymbol{k}\left(-k_{1}\right) \cdot \hat{n} / k_{3}\left(k_{1}\right)$; that is,

$$
\begin{align*}
u\left(x_{g},\left(x_{g}-x_{0}\right) \tan \phi, x_{s},\left(x_{s}-x_{0}\right) \tan \phi, \omega\right) & =\frac{1}{2 \pi} \int_{C} R\left(k_{1}, \omega\right) \frac{k\left(-k_{1}\right) \cdot \hat{\boldsymbol{n}}}{k_{3}\left(k_{1}\right)} d k_{1} \exp \left\{i k_{1} s\right\} \\
s & =\left(x_{g}-x_{s}\right) \sec \phi \tag{44}
\end{align*}
$$

On the other hand, we should expect that the reflectivity matrix will be just (41) evaluated at equal depth values, $z_{g}=z_{s}$, as a function of the two variables, $x_{g}$ and

[^1]$x_{s}$. That is for each depth, $z$,
\[

$$
\begin{align*}
R_{i j}(z, \omega)= & \frac{1}{2 \pi} \int_{C} R\left(k_{1}, \omega\right) \frac{k\left(-k_{1}\right) \cdot \hat{\boldsymbol{n}}}{k_{3}\left(k_{1}\right)} d k_{1} \exp \{i \Phi\} \\
\Phi= & k_{1}\left(x_{g}-x_{s}\right) \cos \phi+k_{3}\left(k_{1}\right)\left[\left(x_{s}+x_{g}-2 x_{0}\right) \sin \phi-2 z \cos \phi\right] \\
& \text { continuous; }  \tag{45}\\
= & k_{1}[(i-j) \Delta x \cos \phi]+k_{3}\left(k_{1}\right)\left[\left\{(i+j) \Delta x-2 x_{0}\right\} \sin \phi-2 z \cos \phi\right]
\end{align*}
$$
\]

discrete.
As a check on this result, let us consider its Fourier transform with respect to $x_{g}$, defined by

$$
\begin{equation*}
\tilde{R}_{i j}(z, \omega)=\int R_{i j}(z, \omega) d x_{g} \exp \left\{-i \kappa_{1}\left[x_{g}-x_{s}\right]\right\} \tag{46}
\end{equation*}
$$

Integration with respect to $x_{g}$ yields a Dirac delta function, namely, $\delta\left[f\left(k_{1}, \kappa_{1}\right)\right]$, with

$$
\begin{equation*}
f\left(k_{1}, \kappa_{1}\right)=k_{1} \cos \phi+k_{3}\left(k_{1}\right) \sin \phi-\kappa_{1} \tag{47}
\end{equation*}
$$

Thus, the integral in $k_{1}$ can now be carried out, by determining the zero of this equation. We need the results,

$$
\begin{align*}
k_{1}\left(\kappa_{1}\right) & =\kappa_{1} \cos \phi-k_{3}\left(\kappa_{1}\right) \sin \phi \\
k_{3}\left(k_{1}\left(\kappa_{1}\right)\right) & =\kappa_{1} \sin \phi+k_{3}\left(\kappa_{1}\right) \cos \phi  \tag{48}\\
\frac{\partial f\left(k_{1}, \kappa_{1}\right)}{\partial k_{1}} & =\frac{k\left(k_{1}\right) \cdot \hat{n}}{k_{3}\left(k_{1}\right)}
\end{align*}
$$

By using these results in (46), we find that

$$
\begin{align*}
\tilde{R}_{i j}(z, \omega) & =\frac{\boldsymbol{k}\left(-k_{1}\right) \cdot \boldsymbol{n}}{\boldsymbol{k}\left(k_{1}\right) \cdot n} R\left(k_{1}\left(\kappa_{1}\right), \omega\right) \exp \left\{2 i k_{3}\left(k_{1}\left(\kappa_{1}\right)\right) n\left(x_{s}, z\right)\right\} \\
& =\frac{\kappa_{1} \sin 2 \phi+k_{3}\left(\kappa_{1}\right) \cos 2 \phi}{k_{3}\left(\kappa_{1}\right)} R\left(k_{1}\left(\kappa_{1}\right), \omega\right) \exp \left\{2 i k_{3}\left(k_{1}\left(\kappa_{1}\right)\right) n\left(x_{s}, z\right)\right\} \tag{49}
\end{align*}
$$

Thus, we see that after Fourier transform, we obtain the reflection coefficient at a wave number that would seem to be "rotated" through the negative of the dip angle, multiplied by a factor which also is tied to the dip. However, note that when $n=0$, that is, when the source point is moved to the reflector at the depth $z, R\left(k_{1}\left(\kappa_{1}\right), \omega\right)$ is the Fourier transform of the reflection coefficient, but at a wavenumber that is "corrected" for dip.

## Downward Continuation

Let us go directly now to applying the operators $W_{g}^{*}$ and $W_{g}^{*}$ to the solution representation (41). That is, we repeat the operator analysis, (29), with $u$ replaced by the solution (41). Our objective is to see how close the result is to claimed reflectivity matrix, (45).

In place of (30), we now obtain

$$
\begin{align*}
I=\frac{1}{8 \pi^{3}} & \int_{x_{s}, x_{g}>x_{0}} d x_{s} d x_{g} d k_{1}^{\prime} d k_{1}^{\prime \prime} d k_{1} \\
& \cdot \exp \left\{-i k_{1}^{\prime}\left[x_{g}-\xi_{g}\right]-i k_{3}^{*}\left(k_{1}^{\prime}\right)\left[\zeta_{g}-z_{g}\right]\right\} \\
& \cdot R\left(k_{1}, \omega\right) \frac{k\left(-k_{1}\right) \cdot \hat{n}}{k_{3}\left(k_{1}\right)} \exp \left\{i k_{1} s+i k_{3}\left(k_{1}\right)\left[n\left(x_{s}, z_{s}\right)+n\left(x_{g}, z_{g}\right)\right]\right\}  \tag{50}\\
& \cdot \exp \left\{-i k_{1}^{\prime \prime}\left[\xi_{s}-x_{s}\right]-i k_{3}^{*}\left(k_{1}^{\prime \prime}\right)\left[\zeta_{s}-z_{s}\right]\right\}
\end{align*}
$$

Here, $s$ and $n$ are given in (43).
If the integrations over $x_{s}$ and $x_{g}$ ranged from $-\infty$ to $\infty$, then the results of those integrations would be $2 \pi \delta\left(k_{1}^{\prime \prime}-k_{1} \cos \phi+k_{3}\left(k_{1}\right) \sin \phi\right)$ and $2 \pi \delta\left(-k_{1}^{\prime}+k_{1} \cos \phi+\right.$ $k_{3}\left(k_{1}\right) \sin \phi$ ), respectively. Here is where we make our approximation based on the assumption that $x_{0}$ is "very far" from the domain of interest. As a practical matter, for dipping and curved reflectors in the earth's subsurface, we rarely have to be concerned with their emergence at the upper surface. This is a technicality of this particularly simple problem. Thus, we proceed on the assumption that we can neglect the effects of the endpoint, $x_{0}$, in these two integrals.

We now carry out the integrals in $k_{1}^{\prime}$ and $k_{1}^{\prime \prime}$ by in these variables at the zeroes of the arguments of the delta functions; that is for

$$
\begin{equation*}
k_{1}^{\prime \prime}=k_{1} \cos \phi-k_{3}\left(k_{1}\right) \sin \phi \quad \text { and } \quad k_{1}^{\prime}=k_{1} \cos \phi+k_{3}\left(k_{1}\right) \sin \phi \tag{51}
\end{equation*}
$$

Associated with these values, we find that

$$
\begin{equation*}
k_{3}\left(k_{1}^{\prime \prime}\right)=k_{1} \sin \phi+k_{3}\left(k_{1}\right) \cos \phi \quad \text { and } \quad k_{3}\left(k_{1}^{\prime}\right)=-k_{1} \sin \phi+k_{3}\left(k_{1}\right) \cos \phi, \tag{52}
\end{equation*}
$$

The result of carrying out these integrations and evaluations is

$$
\begin{align*}
I= & \frac{1}{2 \pi} \int_{C} R\left(k_{1}, \omega\right) \frac{k\left(-k_{1}\right) \cdot \hat{\boldsymbol{n}}}{k_{3}\left(k_{1}\right)} d k_{1} \exp \{i \Phi\} \\
\Phi= & k_{1}\left[\left(\xi_{g}-\xi_{s}\right) \cos \phi+\left(\zeta_{g}-\zeta_{s}\right) \sin \phi\right]  \tag{53}\\
& +k_{3}\left(k_{1}\right)\left[\left(\xi_{g}+\xi_{s}-2 x_{0}\right) \sin \phi-\left(z_{g}+z_{s}\right) \cos \phi\right] \\
& -k_{3}^{*}\left(k_{1}\right)\left[\zeta_{g}-z_{g}+\zeta_{s}+z_{s}\right] \cos \phi
\end{align*}
$$

The evaluation of $\Phi$ becomes more meaningful if we write the result in separate forms for the propagating values of $k_{1}$ and the evanescent values of $k_{1}$. We find that

$$
\Phi=\left\{\begin{array}{l}
k_{1} \cdot s\left(\xi_{g}, \zeta_{g}, \xi_{s}, \zeta_{s}\right)+k_{3}\left(k_{1}\right) \cdot\left[n\left(\xi_{g}, \zeta_{g}\right)+n\left(\xi_{s}, \zeta_{s}\right)\right], \quad k_{1}^{2} \leq \omega^{2} / c_{1}^{2}  \tag{54}\\
k_{1} \cdot s\left(\xi_{g}, \zeta_{g}, \xi_{s}, \zeta_{s}\right)+k_{3}\left(k_{1}\right) \cdot\left[n\left(\xi_{g}, \zeta_{g}\right)+n\left(\xi_{s}, \zeta_{s}\right)+2 d\right], \quad k_{1}^{2}>\omega^{2} / c_{1}^{2}
\end{array}\right.
$$

In this equation, $d$ plays the same role as it did in the horizontal reflector case; it is the sum of the normal distances from the actual source and receiver positions to the downward continued source and receiver positions:

$$
d=\left[\zeta_{g}-z_{g}+\zeta_{s}-z_{s}\right] \cos \phi
$$

Since $n\left(\xi_{g}, \zeta_{g}\right)+n\left(\xi_{s}, \zeta_{s}\right)+2 d$ is always positive, and $\Im k_{3}$ is positive in the evanescent region, $i k_{3}[n+2 d]$ provides exponential decay for the evanescent values of $k_{1}$ and

$$
\begin{align*}
I=\frac{1}{2 \pi} & \int_{\omega^{2} / c_{1}^{2} \geq k_{1}^{2}} d k_{1} R\left(k_{1}, \omega\right) \frac{k\left(-k_{1}\right) \cdot \hat{n}}{k_{3}\left(k_{1}\right)} \\
& \cdot \exp \left\{i k_{1} s+i k_{3}\left(k_{1}\right)\left[n\left(\xi_{g}, \zeta_{g}\right)+n\left(\xi_{\Omega}, \zeta_{\Omega}\right)\right]\right\}  \tag{55}\\
+ & \frac{1}{2 \pi} \int_{\omega^{2} / c_{1}^{2}<k_{1}^{2}} d k_{1} R\left(k_{1}, \omega\right) \frac{k\left(-k_{1}\right) \cdot \hat{n}}{k_{3}\left(k_{1}\right)} \\
& \cdot \exp \left\{i k_{1} s+2 i k_{3}\left(k_{1}\right)\left[n\left(\xi_{g}, \zeta_{g}\right)+n\left(\xi_{\Omega}, \zeta_{\Omega}\right)+2 d\right]\right\}
\end{align*}
$$

When we compare this result with the predicted reflectivity matrix, (45), we see that they agree for the propagating range of values of $k_{1}$ and disagree for the evanescent range of values of $k_{1}$. This is exactly as it was in the case of the horizontal reflector.

We have shown here that the reflectivity matrix is approximated by our adjoint operators in exactly the same was as it was for the case of the horizontal reflector. Now we must turn the questions of imaging and inversion using this reflectivity that we have created.

## Imaging

Previously, we proposed that imaging is achieved with this method by evaluating the inversion output, (55), for coincident source and receiver. As in the previous section, in this limit, $s=0$, and $I$ is a function of $n$ and $d$. That is, for the purpose of imaging, we should evaluate

$$
I=\frac{1}{2 \pi} \int_{\omega^{2} / c_{1}^{2} \geq k_{1}^{2}} d k_{1} R\left(k_{1}, \omega\right) \frac{k\left(-k_{1}\right) \cdot \hat{\boldsymbol{n}}}{k_{3}\left(k_{1}\right)} \exp \left\{2 i k_{3}\left(k_{1}\right) n\left(\xi_{s}, \zeta_{s}\right)\right\}
$$

$$
\begin{equation*}
+\frac{1}{2 \pi} \int_{\omega^{2} / c_{1}^{2}<k_{1}^{2}} d k_{1} R\left(k_{1}, \omega\right) \frac{k\left(-k_{1}\right) \cdot \hat{\boldsymbol{n}}}{k_{3}\left(k_{1}\right)} \exp \left\{4 i k_{3}\left(k_{1}\right)\left[n\left(\xi_{s}, \zeta_{s}\right)+d\right]\right\} \tag{56}
\end{equation*}
$$

Figure 13 is a Mathematica rendition of the output of (56) for the case of a plane at



## COINCIDENT SOURCE \& RECEIVER

Fig. 13. Imaging for a dipping reflector
$30^{\circ} \mathrm{dip}$. The reflector is clearly visible here. Again, I used a frequency of 20 Hz and propagation speed of $5000 \mathrm{~m} / \mathrm{sec}$. The integration was truncated at the evanescent boundary, justified by our experience with the horizontal reflector.

## Inversion

Here we will show how to extract the reflection coefficient from the result (55). First, note that $I$ is initially a function of four spatial variables, $\xi_{s}, \zeta_{s}, \xi_{g}, \zeta_{g}$ as well as frequency. In the Berkhout approach, the downward continuation is carried out with the two depth variables being the same; that is, $\zeta_{g}=\zeta_{s}$, thus, making $I$ a function of three spatial variables and frequency. Now let us consider the Fourier transform of this function with respect to $\xi_{g}$ with the phase shift suggested when we first introduced this type of Fourier transform above equation (9):

$$
\begin{equation*}
\tilde{I}=\int I \exp \left\{-i \kappa_{1}\left(\xi_{g}-\xi_{s}\right)\right\} d \xi_{g} \tag{57}
\end{equation*}
$$

Actually, we have already carried out a similar integration in determining $\tilde{R}_{i j}$ in (46). The main difference here is that the phase in (55) is different for the propagating
and evanescent ranges of the variable $k_{1}$. However, this does not change the support of the delta function or its evaluation. Consequently, we find that

$$
\begin{equation*}
\tilde{I}=\frac{\kappa_{1} \sin 2 \phi+k_{3}\left(\kappa_{1}\right) \cos 2 \phi}{k_{3}\left(\kappa_{1}\right)} R\left(k_{1}\left(\kappa_{1}\right), \omega\right) \exp i \Psi \tag{58}
\end{equation*}
$$

where, just as in the analysis of (46),

$$
\begin{align*}
\Psi & =2\left[-\kappa_{1} \sin \phi+k_{3}\left(\kappa_{1}\right) \cos \phi\right]\left[\left(\xi_{s}-x_{0}\right) \sin \phi-\zeta_{s} \cos \phi\right] \\
& =2 k_{3}\left(k_{1}\left(\kappa_{1}\right)\right) \cdot n\left(\xi_{s}, \zeta_{s}\right) \tag{59}
\end{align*}
$$

Here, $k_{1}\left(\kappa_{1}\right)$ and $k_{3}\left(k_{1}\left(\kappa_{1}\right)\right)$ are defined by (48).
As a result of our previous imaging, we know how to choose $\xi_{s}$ and $\zeta_{s}$ to make $n=0$; it is a matter of placing the output point on the reflector. In that case, we find

$$
\begin{equation*}
\tilde{I}=\frac{\kappa_{1} \sin 2 \phi+k_{3}\left(\kappa_{1}\right) \cos 2 \phi}{k_{3}\left(\kappa_{1}\right)} R\left(k_{1}\left(\kappa_{1}\right), \omega\right), \quad n=0 \tag{60}
\end{equation*}
$$

That is, by evaluating the Fourier transform of the downward continued field, $W_{g}^{*} u W_{s}^{*}$, on the reflector, we obtain the exactly the result predicted in (49). If we introduce the incidence angle $\gamma$ as in (10), but now for $\kappa_{1}$,

$$
\begin{equation*}
\kappa_{1}=\frac{\omega}{c_{1}} \sin \gamma \tag{61}
\end{equation*}
$$

then from (48),

$$
\begin{equation*}
k_{1}\left(\kappa_{1}\right)=\frac{\omega}{c_{1}} \sin (\gamma-\phi) \tag{62}
\end{equation*}
$$

and

$$
\begin{equation*}
R\left(\left(\omega / c_{1}\right) \sin (\gamma-\phi), \omega\right)=\frac{\cos (\gamma-\phi) / c_{1}-\sqrt{1 / c_{2}^{2}-\sin ^{2}(\gamma-\phi) / c_{1}^{2}}}{\cos (\gamma-\phi) / c_{1}+\sqrt{1 / c_{2}^{2}-\sin ^{2}(\gamma-\phi) / c_{1}^{2}}} \tag{63}
\end{equation*}
$$

At the moment, we have no way of determining $\phi$ except from the graphical output, such as Figure 13. However, given an analytical formula, we should anticipate a modified asymptotic downward continuation operator whose output will differ from the one here by $\cos \phi$, similar to the Kirchhoff inversion results in the Bleistein, (1987), et al, approach Also, the range of the angle $\gamma$ for which we actually have reliable output will be a function of the completeness of the angular aperture of the original experiment; here, we have assumed that we have sources and receivers from $x_{0}$ to $\infty$, which is clearly not the case, in practice. Here, again, asymptotic analysis, most likely by the method of stationary phase, will reveal the extent of the aperture. (I expect results similar to the ones obtained in the analysis of Kirchhoff inversion.)

## Summary

We have examined an exact foward model of the upward propagating wave from a dipping planar reflector. We have applied the inversion formalism that was motivated by the previous study of the horizontal reflector. Imaging was achieved for a fixed frequency, suggesting that dispersion could be accommodated by processing data at different frequencies with different background velocities. Estimation of the reflection coefficient is a little more obscure, here, because it is masked by a factor that depends on the dip angle. Furthmore, the transverse wave number at which the reflection coefficient is to be evaluated is also a function of the dip angle. However, we believe that the basic objective of exhibiting a continuum analog of Berkhout inversion for the case of a dipping planar reflector has been accomplished.

## EXTENSION OF WRW BEYOND THE PLANAR REFLECTOR

This section is based on a thesis by von Vroonhoven (1993), in which a derivation is given of the extension of the form $W R W$ to curved reflectors. The result here differs from von Vroonhoven's: her $W$-functions depend on the normal to the reflector and therefore are not pure propagators, while the ones given here do not depend on properties of the reflector. However, the price we pay for this is a slightly more complicated, still fairly simple, representation.

The derivation starts from a Kirchhoff integral representation of the upward scattered field from a single reflector. Von Vroonhoven gives a detailed derivation, but it can also be found elsewhere, including Bleistein (1984). The most interesting derivation is probably in Baker and Copson (1939).

The upward scattered field can be represented in terms of its values on a reflecting surface by

$$
\begin{equation*}
u\left(x, x_{s}\right)=\int_{x^{\prime} \epsilon \mathcal{S}}\left[u\left(x^{\prime}, x_{s}\right) \frac{\partial G\left(x^{\prime}, x\right)}{\partial n^{\prime}}-G\left(x^{\prime}, x\right) \frac{\partial u\left(x^{\prime}, x_{s}\right)}{\partial n^{\prime}}\right] d A\left(x^{\prime}\right) \tag{64}
\end{equation*}
$$

In this equation, $G$ is the free space Green's function; $u$ is the upward scattered field and $\mathcal{S}$ is the reflecting surface.

By reciprocity, ${ }^{3}$ we can interchange $x$ and $x_{s}$. In fact, we could interchange these variables under the integral sign, only, leaving them unchanged on the left side of the equation. That is,

$$
\begin{equation*}
u\left(x, x_{s}\right)=\int_{x^{\prime} \epsilon \mathcal{S}}\left[u\left(x^{\prime}, x\right) \frac{\partial G\left(x^{\prime}, x_{s}\right)}{\partial n^{\prime}}-G\left(x^{\prime}, x_{s}\right) \frac{\partial u\left(x^{\prime}, x\right)}{\partial n^{\prime}}\right] d A\left(x^{\prime}\right) \tag{65}
\end{equation*}
$$

[^2]The point of doing this is that the scattered field under the integral sign no longer depends on the upper surface source point, $x_{s}$, while the Green's function (propagator) now does. The function, $u\left(x^{\prime}, x\right)$ is the field at $x^{\prime}$ due to a fictitious source at $x$ of the same type as the true source at $x_{s}$. The field at $x^{\prime}$ is to be interpreted as the limit of the upward propagating field for an observation point above the reflector moved down onto the reflector.

The point, $x$ is not to be our ultimate observation point. That will be $x_{g}$. The representation (64), used again, allows to write

$$
\begin{equation*}
u\left(x_{g}, x_{s}\right)=\int_{x \in \mathcal{S}}\left[u\left(x, x_{s}\right) \frac{\partial G\left(x_{g}, x\right)}{\partial n}-G\left(x_{g}, x\right) \frac{\partial u\left(x, x_{s}\right)}{\partial n}\right] d A(x) \tag{66}
\end{equation*}
$$

Now, for $u$ and $\partial u / \partial n$, use (65). That means we move the source point in $u\left(\boldsymbol{x}^{\prime}, \boldsymbol{x}\right)$ onto the reflecting surface, as well. The result is

$$
\begin{align*}
u\left(x_{g}, x_{s}\right)= & \int_{x \in \mathcal{S}} \int_{x^{\prime} \epsilon \mathcal{S}}\left\{\left[u\left(x^{\prime}, x\right) \frac{\partial G\left(x^{\prime}, x_{s}\right)}{\partial n^{\prime}}-G\left(x^{\prime}, x_{s}\right) \frac{\partial u\left(x^{\prime}, x\right)}{\partial n^{\prime}}\right] \frac{\partial G\left(x_{g}, x\right)}{\partial n}\right. \\
& \left.\quad-G\left(x_{g}, x\right) \frac{\partial}{\partial n}\left[u\left(x^{\prime}, x\right) \frac{\partial G\left(x^{\prime}, x_{s}\right)}{\partial n^{\prime}}-G\left(x^{\prime}, x_{s}\right) \frac{\partial u\left(x^{\prime}, x\right)}{\partial n^{\prime}}\right]\right\} d A(x) d A\left(x^{\prime}\right) \tag{67}
\end{align*}
$$

We expand the second line and rewrite this result as

$$
\begin{align*}
& u\left(x_{g}, x_{s}\right)=\int_{x \in \mathcal{S}} \int_{x^{\prime} \mathcal{S}} d A(x) d A\left(x^{\prime}\right) \quad\left\{u\left(x^{\prime}, x\right) \frac{\partial G\left(x^{\prime}, x_{s}\right)}{\partial n^{\prime}} \frac{\partial G\left(x_{g}, x\right)}{\partial n}\right. \\
&+G\left(x^{\prime}, x_{s}\right) G\left(x_{g}, x\right) \frac{\partial^{2} u\left(x^{\prime}, x\right)}{\partial n^{\prime} \partial n} \\
&-\frac{\partial G\left(x_{g}, x\right)}{\partial n} G\left(x^{\prime}, x_{s}\right) \frac{\partial u\left(x^{\prime}, x\right)}{\partial n^{\prime}}  \tag{68}\\
&\left.-G\left(x_{g}, x\right) \frac{\partial G\left(x^{\prime}, x_{s}\right)}{\partial n^{\prime}} \frac{\partial u\left(x^{\prime}, x\right)}{\partial n}\right\}
\end{align*}
$$

It only remains to write this result in terms of propagators and a reflection operator. To do so, it is necessary to separate the normal derivatives (surface effects) from the Greens' functions (propagators). Thus, we introduce the horizontal 4-tuple,

$$
\begin{equation*}
\underline{G}\left(x_{g}, x\right)=\left(G\left(x_{g}, x\right), \nabla G\left(x_{g}, x\right)\right) \tag{69}
\end{equation*}
$$

and the $4 \times 4$ reflectivity dyad

$$
\Re\left(x^{\prime}, x\right)=\left[\begin{array}{cc}
\frac{\partial^{2} u\left(x^{\prime}, x\right)}{\partial n^{\prime} \partial n} & -\frac{\partial u\left(x^{\prime}, x\right)}{\partial n^{\prime}} \hat{n}(x)  \tag{70}\\
-\frac{\partial u\left(x^{\prime}, x\right)}{\partial n} \hat{n}^{T}\left(x^{\prime}\right) & u\left(x^{\prime}, x\right) \hat{n}^{T}\left(x^{\prime}\right) \hat{n}(x)
\end{array}\right]
$$

In this equation, $T$ denotes transpose, so that the first line here connotes a scalar followed by a three component horizontal vector and the second line connotes a three component vertical vector followed by a $3 \times 3$ dyadic.

By using these representations in (68) we find that we can rewrite that integral as

$$
\begin{equation*}
u\left(x_{g}, x_{s}\right)=\int_{x \in \mathcal{S}} \int_{x^{\prime} \epsilon \mathcal{S}} d A(x) d A\left(x^{\prime}\right) \underline{G}\left(x_{g}, x^{\prime}\right) \Re\left(x^{\prime}, x\right) \underline{G}^{T}\left(x, x_{s}\right) . \tag{71}
\end{equation*}
$$

This result clearly has the form

$$
\begin{equation*}
u\left(x_{g}, x_{s}\right)=\mathcal{W}_{g} \mathcal{R} \mathcal{W}_{s} \tag{72}
\end{equation*}
$$

where the operators $\mathcal{W}_{g}$ and $\mathcal{W}_{s}$ are propagators from the reflector to the geophone and from the source to the reflector, respectively, and $\mathcal{R}$ is a reflectivity operator which carries the surface information through the reflectivity dyad in (70). The operations performed are integrations over the reflector. $\mathcal{R} \mathcal{W}_{s}$ creates the upward scattered field at the reflector through integration over the variable $x$ and $\mathcal{W}_{g}$ operating on this result propagates this surface field back to the geophone $x_{g}$ through integration over the variable $\boldsymbol{x}^{\prime}$.

For the case of the dipping planar reflector in two dimensions, through calculations much like the ones that were carried out earlier with the Fourier representations of all of these functions, one finds that each of the terms in (68) is identical and equal to one fourth of the exact solution presented earlier for this problem. The calculation of one of these terms will be carried out here. The others follow in a quite similar matter. (For completeness of this short note, I will include equations that were stated in the earlier notes on the Berkhout approach to inversion.)

The previously derived result for the upward propagating wave from a dipping planar reflector is (eqs 41, 42)

$$
\begin{align*}
u\left(x, x^{\prime}, \omega\right) & =\frac{1}{2 \pi} \int_{C} R\left(k_{1}, \omega\right) \frac{k \cdot \hat{n}}{k_{3}\left(k_{1}\right)} d k_{1} \exp \{i \Phi\} \\
z^{\prime}<\left(x^{\prime}-x_{0}\right) \tan \phi, & \quad z<\left(x-x_{0}\right) \tan \phi \tag{73}
\end{align*}
$$

In this equation, $\hat{\boldsymbol{n}}=(-\sin \phi, \cos \phi)$ is the downward pointing unit normal to the reflector; the vector, $k$ is given by $k=\left(k_{1}, k_{3}\left(k_{1}\right)\right)$; and

$$
\begin{equation*}
R\left(k_{1}, \omega\right)=\frac{k_{3}\left(k_{1}, \omega\right)-k_{4}\left(k_{1}, \omega\right)}{k_{3}\left(k_{1}, \omega\right)+k_{4}\left(k_{1}, \omega\right)} \tag{74}
\end{equation*}
$$

Finally,

$$
\begin{aligned}
& \Phi=k_{1}\left[\left(x^{\prime}-x\right) \cos \phi+\left(z^{\prime}-z\right) \sin \phi\right] \\
& \\
& \quad+k_{3}\left(k_{1}\right)\left[\left(x^{\prime}+x-2 x_{0}\right) \sin \phi-\left(z^{\prime}+z\right) \cos \phi\right]
\end{aligned}
$$

$$
\begin{align*}
& =k_{1} s\left(x^{\prime}, x\right)+k_{3}\left(k_{1}\right)\left[n\left(x^{\prime}\right)+n(x)\right]  \tag{75}\\
s\left(x^{\prime}, x\right) & =\left(x^{\prime}-x\right) \cos \phi+\left(z^{\prime}-z\right) \sin \phi \\
n(x) & \left.=\left(x-x_{0}\right) \sin \phi\right)-z \cos \phi
\end{align*}
$$

For the Green's functions we have the results,

$$
\begin{equation*}
G\left(x_{g}, x\right)=-\frac{1}{4 \pi i} \int \frac{d k_{1}^{\prime}}{k_{3}\left(k_{1}^{\prime}\right)} \exp i\left\{k_{1}^{\prime}\left(x_{g}-x\right)+k_{3}\left(k_{1}^{\prime}\right)\left(z-z_{g}\right)\right\}, z>z_{g} \tag{76}
\end{equation*}
$$

and

$$
\begin{equation*}
G\left(x^{\prime}, x_{s}\right)=-\frac{1}{4 \pi i} \int \frac{d k_{1}^{\prime \prime}}{k_{3}\left(k_{1}^{\prime \prime}\right)} \exp i\left\{k_{1}^{\prime \prime}\left(x^{\prime}-x_{s}\right)+k_{3}\left(k_{1}^{\prime \prime}\right)\left(z-z_{s}\right)\right\}, z>z_{s} \tag{77}
\end{equation*}
$$

In (68), let us consider the first integral,

$$
\begin{equation*}
I_{1}=\int_{x_{\epsilon} \mathcal{S}} \int_{x^{\prime} \epsilon \mathcal{S}} d s d s^{\prime} u\left(x^{\prime}, x\right) \frac{\partial G\left(x_{s}, x^{\prime}\right)}{\partial n^{\prime}} \frac{\partial G\left(x_{g}, x\right)}{\partial n} \tag{78}
\end{equation*}
$$

Here, $d s$ and $d s^{\prime}$ are differential arclengths along the reflector and

$$
\begin{align*}
x & =x_{0}+s \cos \phi, \quad z=s \sin \phi \\
x^{\prime} & =x_{0}+s^{\prime} \cos \phi, \quad z=s^{\prime} \sin \phi \tag{79}
\end{align*}
$$

Substitution of the above three representations for the functions appearing in this equation yields the equation

$$
\begin{equation*}
I_{1}=\frac{1}{32 \pi^{3}} \int \frac{\boldsymbol{k} \cdot \hat{\boldsymbol{n}}}{k_{3}\left(k_{1}\right)} d k_{1} \frac{\tilde{\boldsymbol{k}}^{\prime} \cdot \hat{\boldsymbol{n}}}{k_{3}\left(k_{1}^{\prime}\right)} d k_{1}^{\prime} \frac{\tilde{\boldsymbol{k}}^{\prime \prime} \cdot \hat{n}}{k_{3}\left(k_{1}^{\prime \prime}\right)} d k_{1}^{\prime \prime} R\left(k_{1}, \omega\right) \exp \{i \Psi\} \tag{80}
\end{equation*}
$$

In this equation,

$$
\begin{equation*}
\tilde{\boldsymbol{k}}=\left(-k_{1}, k_{3}\left(k_{1}\right)\right) \tag{81}
\end{equation*}
$$

and

$$
\begin{align*}
\Psi=k_{1}\left(s^{\prime}-s\right) & +k_{1}^{\prime}\left(x_{g}-x_{0}-s^{\prime} \cos \phi\right)+k_{3}\left(k_{1}^{\prime}\right)\left(s^{\prime} \sin \phi-z_{g}\right) \\
& +k_{1}^{\prime \prime}\left(x_{s}-x_{0}-s^{\prime \prime} \cos \phi\right)+k_{3}\left(k_{1}^{\prime \prime}\right)\left(s \sin \phi-z_{s}\right) \tag{82}
\end{align*}
$$

The integrations in $s$ and $s^{\prime}$ now yield a pair of delta functions $\delta\left(k_{1}-k_{1}^{\prime} \cos \phi+\right.$ $k_{3}\left(k_{1}^{\prime}\right) \sin \phi$ ) and $\delta\left(k_{1}-k_{1}^{\prime \prime} \cos \phi+k_{3}\left(k_{1}^{\prime \prime}\right) \sin \phi\right.$ ) which allow us to carry out those
integrations. The derivatives of these delta function arguments with respect $k_{1}^{\prime}$ and $k_{1}^{\prime \prime}$, respectively, are exactly the functions, $\tilde{k}^{\prime} \cdot \hat{n} / k_{3}\left(k_{1}^{\prime}\right)$ and $\tilde{k}^{\prime \prime} \cdot \hat{n} / k_{3}\left(k_{1}^{\prime \prime}\right)$. In evaluating the delta functions, it is necessary to divide by these derivatives. The solutions for $k_{1}^{\prime}$ and $k_{1}^{\prime \prime}$ are given in equations (51) and (52) of the earlier notes, namely,

$$
\begin{align*}
k_{1}^{\prime} & =k_{1} \cos \phi+k_{3}\left(k_{1}\right) \sin \phi \\
k_{3}\left(k_{1}^{\prime}\right) & =-k_{1} \sin \phi+k_{3}\left(k_{1}\right) \cos \phi \\
k_{1}^{\prime \prime} & =k_{1} \cos \phi-k_{3}\left(k_{1}\right) \sin \phi  \tag{83}\\
k_{3}\left(k_{1}^{\prime \prime}\right) & =k_{1} \sin \phi+k_{3}\left(k_{1}\right) \cos \phi
\end{align*}
$$

When these results are used in (80) we obtain

$$
\begin{equation*}
u\left(x_{g}, x_{s}, \omega\right)=\frac{1}{8 \pi} \int_{C} R\left(k_{1}, \omega\right) \frac{k \cdot \hat{\boldsymbol{n}}}{k_{3}\left(k_{1}\right)} d k_{1} \exp \{i \Phi\} \tag{84}
\end{equation*}
$$

where, now, in the definition of $\Phi$ in (43) above, $x$ is replaced by $x_{g}$ and $x^{\prime}$ is replaced by $x_{s}$. This is one fourth of the previously derived upward scattered field for this problem, as stated earlier. In each of the other integrals in (68), the derivatives of the delta functions produce exactly the right factor to "cancel" the corresponding multiplier in $\tilde{k}$, just as occurred here.

Having the form (71) is not apparently of any particular use, except that it makes literal the conceptual structure $W R W$ for downward propagation, reflection, and upward propagation. As was demonstrated earlier, the application of the scalar adjoints. $W^{*} u W^{*}$, produces an approximate inverse for both the horizontal and the dipping planar reflector. It remains to analyze the curved reflector.

## INVERSION OF DATA FOR A CURVED REFLECTOR

We now consider processing of reflecton data for an arbitrary curved reflector. Clearly, in this case, we cannot write down an exact solution, so we must content ourselves with an asymptotic solution, namely, a Kirchhoff-approximate solution. We will apply the same inversion operator, $W^{*}$, as given by equation (23) and show that we can image the reflector by calculating $W_{s}^{*} u W_{g}^{*}$ for coincident downward propagated source and receiver. This analysis will be carried out in two dimensions, as were the earlier discussions of inversion. However, the asymptotic analysis of the amplitude for extraction of the reflectivity by this method has not been accomplished yet.

Application of the Kirchhhoff approximation requires certain constraints on the curvature of the reflector, namely, that it satisfy the inequality,

$$
\Lambda=2 \pi f L / c \gg 1
$$

Here, $f$ is the frequency in $\mathrm{Hz}, L$ is the radius of curvature of the reflector at the point(s) where the Kirchhoff approximation is to be applied and $c$ is the propagation speed, as in previous sections.

Thus, we are abandoning analysis of the exact solution for this discussion. However, it should be pointed out that the very concept of a reflection coefficient is an asymptotic-geometrical optics-attribute of the scattering process for curved reflectors. Hence, we see no inconsistency here in using an asymptotic approximation. Quite the contrary, if we seek reflectivity, it would be questionable to use anything more than asymptotic analysis for a curved reflector.

For this discussion, we introduce the notation, $u_{I}\left(x, x_{s}, \omega\right)$ for the downward propagating dipole response, (15). We assume that this field gives rise to an upward propagating wave $u_{R}\left(x, x_{s}, \omega\right)$ at a reflecting curve, $\mathcal{C}$. Starting from (65) and using the Sommerfeld radiation condition, one can derive the following representation for $u_{R}$ in terms of $u_{I}$ :

$$
\begin{align*}
u_{R}\left(x_{g}, x_{s}, \omega\right)=-\int_{x_{\varepsilon} \mathcal{C}}[ & u_{R}\left(x(\sigma), x_{s}, \omega\right) \frac{\partial G\left(x_{g}, x(\sigma)\right)}{\partial n} \\
& \left.-G\left(x_{g}, x(\sigma)\right) \frac{\partial u_{R}\left(x(\sigma), x_{s}, \omega\right)}{\partial n}\right] d \sigma \tag{85}
\end{align*}
$$

See Baker and Copson (1939), Bleistein (1984) In this equation, $\sigma$ is arclength on the reflector, $\mathcal{C}, G$ is the free-space Green's function, and $\partial / \partial n$ is the downward directed normal derivative on the reflector.

The Kirchhoff approximation amounts to replacing the upward reflected field, $u_{R}$, and its normal derivative on $\mathcal{C}$ by their geometrical optics approximations, namely,

$$
\begin{gather*}
u_{R}\left(x(\sigma), x_{s}, \omega\right)=R\left(x(\sigma), x_{s}\right) u_{I}\left(x(\sigma), x_{s}, \omega\right) \\
\frac{\partial u_{R}\left(x(\sigma), x_{s}, \omega\right)}{\partial n}=-R\left(x(\sigma), x_{s}\right) \frac{\partial u_{I}\left(x(\sigma), x_{s}, \omega\right)}{\partial n} \tag{86}
\end{gather*}
$$

Here, $R$ is the geometrical optics reflection coefficient, calculated for the incidence angle between the geometrical optics ray from $x$ to $x(\sigma)$ and the upward normal $\hat{n}(\sigma)$ on the reflector. We prefer not to write down $R$ in this spatial form, because we will immediately modify this result by using a wavenumber representation for both $u_{I}$ and $R$, below.

To this end, we return to the representation of the dipole response, (15), which we use for $u_{I}$ and write down a corresponding representation for its normal derviative,
as well:

$$
\begin{align*}
u_{I}\left(x, x_{s}, \omega\right) & =\frac{1}{2 \pi} \int_{C} \exp \left\{i \Phi\left(k_{1}, x, x_{s}\right)\right\} d k_{1} \\
\frac{\left.\partial u_{I}\left(x, x_{s}\right), \omega\right)}{\partial n} & =\frac{1}{2 \pi} \int_{C} i k\left(k_{1}\right) \cdot \hat{n} \exp \left\{i \Phi\left(k_{1}, x, x_{s}\right)\right\} d k_{1} \tag{87}
\end{align*}
$$

Here, $k_{3}\left(k_{1}\right)$ is given by the first expression in (4), and

$$
\begin{align*}
x=(x, z), \quad x_{s} & =\left(x_{s}, z_{s}\right), \quad k\left(k_{1}\right)=\left(k_{1}, k_{3}\left(k_{1}\right)\right)  \tag{88}\\
\Phi\left(k_{1}, x, x_{s}\right) & =k_{1}\left[x(\sigma)-x_{s}\right]+k_{3}\left(k_{1}\right)\left(z(\sigma)-z_{s}\right), z>z_{s}
\end{align*}
$$

We introduce corresponding representations for $G$ and its normal derivative:

$$
\begin{align*}
G\left(x_{g}, x, \omega\right) & =-\frac{1}{4 \pi i} \int_{C} \frac{1}{k_{3}\left(k_{1}^{\prime}\right)} \exp \left\{i \Phi\left(-k_{1}^{\prime}, x, x_{g}\right)\right\} d k_{1}^{\prime} \\
\frac{\partial G\left(x_{g}, x, \omega\right)}{\partial n} & =-\frac{1}{4 \pi} \int_{C} \frac{k\left(-k_{1}^{\prime}\right) \cdot \hat{n}}{k_{3}\left(k_{1}^{\prime}\right)} \exp \left\{i \Phi\left(k_{1}^{\prime}, x, x_{g}\right)\right\} d k_{1}^{\prime}  \tag{89}\\
\boldsymbol{k}\left(-k_{1}^{\prime}\right) & =\left(-k_{1}^{\prime}, k_{3}\left(k_{1}^{\prime}\right)\right) \quad x_{g}=\left(x_{g}, z_{g}\right), z>z_{g}
\end{align*}
$$

As a new feature, here, we will introduce a representation for the reflection coefficient under the integral sign in (87). Since the representation in the Fourier domain is just a plane wave decomposition, we need only use the plane wave representation of the reflection coefficient for a wave incident on a reflector with normal direction, $\hat{n}$, namely,

$$
\begin{equation*}
R\left(k_{1}, \omega, \sigma\right)=\frac{\hat{\boldsymbol{n}} \cdot k-\operatorname{sign}[\omega \hat{\boldsymbol{n}} \cdot k] \sqrt{\omega^{2} / c_{+}^{2}-\omega^{2} / c^{2}+(\hat{\boldsymbol{n}} \cdot k)^{2}}}{\hat{\boldsymbol{n}} \cdot k+\operatorname{sign}[\omega \hat{\boldsymbol{n}} \cdot k] \sqrt{\omega^{2} / c_{+}^{2}-\omega^{2} / c^{2}+(\hat{\boldsymbol{n}} \cdot k)^{2}}}, k=k\left(k_{1}\right) \tag{90}
\end{equation*}
$$

Remark: $R$ is really a function of $k_{1} / \omega$ and $\sigma$. To see this, one needs only to divide the numerator and the denominator here by $\omega$ and to realize that

$$
k_{3}\left(k_{1}\right)=\omega \sqrt{1-\left(k_{1} / \omega\right)^{2}}
$$

because for real values of $k_{3}, \operatorname{sign}\left(k_{3}\right)=\operatorname{sign}(\omega)$; see (4).
We substitute the results (86) - (89) in (85) to obtain the result,

$$
\begin{gather*}
u_{R}\left(x_{g}, x_{s}, \omega\right)=\frac{1}{2(2 \pi)^{2}} \int_{x_{\varepsilon \mathcal{C}}} d \sigma \int d k_{1} d k_{1}^{\prime} \frac{k\left(k_{1}\right) \cdot \hat{\boldsymbol{n}}+k\left(-k_{1}^{\prime}\right) \cdot \hat{\boldsymbol{n}}}{k_{3}\left(k_{1}^{\prime}\right)} R\left(k_{1}, \omega, \sigma\right) \\
\cdot \cdot \exp \left\{i \Phi\left(k, x, x_{s}\right)+i \Phi\left(k^{\prime}, x_{g}, x\right)\right\} \tag{91}
\end{gather*}
$$

This equation should be compared to (41), which is the result for the dipping plane. We see here two additional integrations for this solution. However, if we specialize to the case of a dipping plane, this leading order asympotitic solution actually yields the exact result, (41). This is a somewhat trick calculation to carry out and I will only outline it here.

First, note that for a linear reflector, the phase is linear in $\sigma$ and $R$ is actually independent of $\sigma$. Thus, the $\sigma$-integration yields a Dirac delta function with argument,

$$
\left(k_{1}-k_{1}^{\prime}\right) \cos \phi+\left(k_{3}\left(k_{1}\right)+k_{3}\left(k_{1}^{\prime}\right) \sin \phi\right.
$$

with $\phi$ being the dip angle as in the earlier sections.
We use this delta function to evaluate the $k_{1}^{\prime}$-integral when this argument is zero, or when

$$
k_{1} \cos \phi+\left(k_{3}\left(k_{1}\right) \sin \phi=\lambda=k_{1}^{\prime} \cos \phi-k_{3}\left(k_{1}^{\prime}\right) \sin \varphi .\right.
$$

Here, we show the "auxiliary" variable, $\lambda$, because the representation (91) reduces to the result, (41) when the remaining $k_{1}$-integral is rewritten as an integral in $\lambda$.

We propose to apply the inversion operators, $W^{*}$, in source and receiver, fashioned from the basic definition (23) as follows. Set

$$
\begin{equation*}
W^{*}\left(x_{s}, \xi_{s}\right)=\frac{\operatorname{sign}\left(\zeta_{s}-z_{s}\right)}{2 \pi} \int d \kappa_{1} \exp \left\{-i \Phi\left(k_{1}, \xi_{s}, x_{s}\right)\right\} \tag{92}
\end{equation*}
$$

and

$$
\begin{equation*}
W^{*}\left(x_{g}, \xi_{g}\right)=\frac{\operatorname{sign}\left(z_{g}-\zeta_{g}\right)}{2 \pi} \int d \kappa_{1}^{\prime} \exp \left\{-i \Phi\left(-\kappa_{1}^{\prime}, \xi_{g}, x_{g}\right)\right\} \tag{93}
\end{equation*}
$$

We do not need to be concerned with complex conjugates in this discussion. In considering only the leading order asymptotic solution, we are neglecting evanescent waves-imaginary $k_{3}$ values. It is our point of view that the evidence of the analysis of the previous sections is overwhelming that the evanescent energy contributes little to the inversion process in this method. Thus, we do not consider this approximation as a serious loss of accuracy in our analysis.

We now multiply the upward propagating wave representation (91) by these two functions and integrate over $x_{s}$ and $x_{g}$ in order to carry out the approximate inversion, $W^{*} u W^{*}$, as in (29). This equation takes the form,

$$
\begin{align*}
W^{*} u W^{*}=-\frac{1}{4(2 \pi)^{4}} & \int d x_{s} d x_{g} d k_{1} d k_{1}^{\prime} d \kappa_{1} d \kappa_{1}^{\prime} d \sigma \\
& \cdot \frac{k\left(k_{1}\right) \cdot \hat{\boldsymbol{n}}+k\left(-k_{1}^{\prime}\right) \cdot \hat{\boldsymbol{n}}}{k_{3}\left(k_{1}^{\prime}\right)} R\left(k_{1}, \omega, \sigma\right)  \tag{94}\\
& \cdot \exp \left\{i \Psi\left(k_{1}, k_{1}^{\prime}, \kappa_{1}, \kappa_{1}^{\prime}, \boldsymbol{x}, \boldsymbol{x}_{s}, \boldsymbol{x}_{g}\right)\right\},
\end{align*}
$$

where,

$$
\Psi\left(k_{1}, k_{1}^{\prime}, \kappa_{1}, \kappa_{1}^{\prime}, x, x_{s}, x_{g}\right)=\Phi\left(k_{1}, x, x_{s}\right)+\Phi\left(-k_{1}^{\prime}, x, x_{g}\right)
$$

$$
\begin{equation*}
-\Phi\left(\kappa_{1}, \xi_{s}, x_{s}\right)-\Phi\left(-\kappa_{1}^{\prime}, \xi_{g}, x_{g}\right) \tag{95}
\end{equation*}
$$

or,

$$
\begin{aligned}
& \Psi\left(k_{1}, k_{1}^{\prime}, \kappa_{1}, \kappa_{1}^{\prime}, x, x_{s}, x_{g}\right)=k_{1}\left(x-x_{s}\right)+k_{3}\left(k_{1}\right)\left(z-z_{s}\right) \\
& -k_{1}^{\prime}\left(x-x_{g}\right)+k_{3}\left(k_{1}^{\prime}\right)\left(z-z_{g}\right) \\
& -\kappa_{1}\left(\xi_{s}-x_{s}\right)-k_{3}\left(\kappa_{1}\right)\left(\zeta_{s}-z_{s}\right) \\
& +\kappa_{1}^{\prime}\left(\xi_{g}-x_{g}\right)-k_{3}\left(\kappa_{1}^{\prime}\right)\left(\zeta_{g}-z_{g}\right)
\end{aligned}
$$

As in previous discussions of this process, the integrals in $x_{s}$ and $x_{g}$ produce Dirac delta functions with arguments, $k_{1}-\kappa_{1}$ and $k_{1}^{\prime}-\kappa_{1}^{\prime}$, respectively. Hence, we can then carry out those two integrations, as well, yielding the result for (94),

$$
\begin{align*}
& W^{*} u W^{*}=-\frac{1}{4(2 \pi)^{2}} \int d x_{s} d x_{g} d k_{1} d k_{1}^{\prime} d \sigma \\
& \cdot \frac{k\left(k_{1}\right) \cdot \hat{n}+k\left(-k_{1}^{\prime}\right) \cdot \hat{n}}{k_{3}\left(k_{1}^{\prime}\right)} R\left(k_{1}, \omega, \sigma\right) \\
& \cdot \exp \left\{i \Phi\left(k_{1}, x, \xi_{s}\right)+i \Phi\left(k_{1}^{\prime}, x, \xi_{g}\right)+\right\}  \tag{96}\\
& \xi_{s}=\left(\xi_{s}, \zeta_{s}\right), \quad \xi_{g}=\left(\xi_{g}, \zeta_{g}\right)
\end{align*}
$$

At first glance, it is not clear that such an integral could provide an image of the reflector and information about the reflection coefficient. In fact, this integral will peak on the reflector. To make this more plausible, we propose to introduce alternative integration variables in place of $k_{1}$ and $k_{1}^{\prime}$. For each point on the reflectorthat is, for each $\sigma$-we use the dip angle of the tangent to define this new pair of wave number variables, as follows:

$$
\begin{equation*}
\lambda_{1}=k_{1} \cos \phi+k_{3}\left(k_{1}\right) \sin \phi, \quad \lambda_{1}^{\prime}=-k_{1}^{\prime} \cos \phi+k_{3}\left(k_{1}^{\prime}\right) \sin \phi \tag{97}
\end{equation*}
$$

This makes the new wave numbers functions of $\sigma$ as well as functions of the old wave number variables. For these variables, one can verify the auxiliary relationships,

$$
\begin{equation*}
k_{3}\left(\lambda_{1}\right)=-k_{1} \sin \phi+k_{3}\left(k_{1}\right) \sin \phi, \quad \text { and } \quad k_{3}\left(\lambda_{1}^{\prime}\right)=k_{1} \sin \phi+k_{3}\left(k_{1}\right) \sin \phi \tag{98}
\end{equation*}
$$

In terms of these new variables, the representation (96) becomes

$$
\begin{align*}
W^{*} u W^{*}=-\frac{1}{4(2 \pi)^{2}} & \int_{x_{\varepsilon} \mathcal{C}} d \sigma \int d \lambda_{1} d \lambda_{1}^{\prime} \\
& \frac{k_{3}\left(\lambda_{1}\right)+\lambda_{1}^{\prime} \sin 2 \phi+k_{3}\left(\lambda_{1}^{\prime} \cos 2 \phi\right.}{k_{3}\left(\lambda_{1}^{\prime}\right)} \cdot \frac{\lambda_{1} \sin \phi+k_{3}\left(\lambda_{1}\right) \cos \phi}{k_{3}\left(\lambda_{1}\right)}  \tag{99}\\
& R\left(k_{1}, \omega, \sigma\right) \exp \left\{i \Theta\left(\lambda_{1}, s_{s}, n_{s}\right)+i \Theta\left(\lambda_{1}^{\prime}, s_{g}, n_{g}\right)\right\} .
\end{align*}
$$

In this equation,

$$
\begin{align*}
& \Theta\left(\lambda_{1}, s_{s}, n_{s}\right)=\lambda_{1} s_{s}+k_{3}\left(\lambda_{1}\right) n_{s} \\
& \Theta\left(\lambda_{1}^{\prime}, s_{g}, n_{g}\right)=\lambda_{1}^{\prime} s_{g}+k_{3}\left(\lambda_{1}^{\prime}\right) n_{g} \tag{100}
\end{align*}
$$

with the signed distances as shown in Figure 14 and given by


Fig. 14. Curved reflector coordinates.

$$
\begin{array}{ll}
s_{s}=\left(x-\xi_{s}\right) \cos \phi+\left(z-\zeta_{s}\right) \sin \phi, & n_{s}=-\left(x-\xi_{s}\right) \sin \phi+\left(z-\zeta_{s}\right) \cos \phi \\
s_{g}=\left(x-\xi_{g}\right) \cos \phi+\left(z-\zeta_{g}\right) \sin \phi, & n_{g}=-\left(x-\xi_{g}\right) \sin \phi+\left(z-\zeta_{g}\right) \cos \phi \tag{101}
\end{array}
$$

Remark: For the planar reflector, integration in $\sigma$ now yields the delta function, $\delta\left(\lambda_{1}+\lambda_{1}^{\prime}\right)$. Evaluation of the $\lambda_{1}^{\prime}$ integral again leads to the result, (41).

## Imaging

This is as far as the analysis has progressed on this problem at this time. However, there is enough information here to predict imaging. To see why this is so, consider the final formulas for inversion, ( 96 or (99). Note that to obtain an image of the reflector in earlier sections, we proposed that the data be processed for coincident source and receiver, that is for

$$
\xi_{s}=\xi_{g}=\xi
$$

Let us consider that case now and, further, let us suppose that $x i$ is on the reflector, $\mathcal{C}$. In this case, there is one choice of $\sigma$, say $\sigma=\sigma_{0}$, for which $s_{s}=s_{g}=n_{s}=n_{g}=0$ and the oscillatory exponential no longer appears in the integrand. In this case, we expect that the result of the $\lambda_{1}$ and $\lambda_{1}^{\prime}$ integrals would be much larger than for any other choice of $\sigma$. That is, the pair of integrals in $\lambda_{1}$ and $\lambda_{1}^{\prime}$ behave something like a delta function, $\operatorname{delta}\left(\sigma-\sigma_{0}\right)$. This is mathematical imaging!

To go further with this type of argument, suppose that $\boldsymbol{\xi}$ is "near" $\mathcal{C}$ and we choose for $\sigma_{0}$ the value that identifies the point on $\mathcal{C}$ closest to $\xi$. We expect this value to be a critical value of the asymptotic analysis of the $\sigma$-integral. In this case, let us consider linearizing the exponent around $\sigma=\sigma_{0}$ and evaluating the amplitude at $\sigma=\sigma_{0}$ as a first order approximation of the integral.

We already know the result of this linearized analysis: $\mathcal{C}$ is replaced by a dipping plane with dip angle, $\phi=\phi\left(\sigma_{0}\right)$ ! In this case, we showed in this section, that the double integral in $\sigma$ and /lambdás effectively reduce to the processing for a dipping linear reflector of an earlier section, just as the forward Kirchhoff model reduces exactly to the model of a plane when $\mathcal{C}$ was specialized to this case. Analysis of the processing formalism for the dipping plane was carried out earlier and, indeed, we saw that the output of the formalism was an image of the reflector.

Thus, out intuition tells us that the result we seek is contained in the formulas (96) or (99); it only remains to be carried out in detail.

In summary, what we have done here is model propagating part of the energy from a reflector as by the Kirchhoff approximation and apply the $W^{*} u W^{*}$ formalism to that representation. We have proceeded far enough with the analysis to see that it is at least very likely that this output will produce an image of the curved reflector. The details of this latter analysis are a subject for further study.

## CONCLUSIONS

The objective of this paper was to expose research in progress on the development of a continuum analog of the Berkhout inversion formalism. Beyond the desire for a basic understanding, it is hoped that this study will lead to methods of adapting results based on stationary phase for integral inversion operators to the discrete Berkhout inversion. We have in mind here the development a 2.5 D Berkhout formalism as well as methods of processing for geometrical attributes, such as incidence
angle, travel time, etc.-with this discrete formalism. Furthermore, it is hoped that by asymptotic analysis of the continuum analog, the artefacts of limited aperture for this method can be better understood.

An important new idea that has been exposed here is that for full aperture data from a reflector, each fixed frequency leads to an image of the reflector. This suggests the possibility of processing with different velocities for different frequencies, thereby accounting for dispersion. It also suggests the possiblity of developing a velocity analysis scheme in which one examines residual moveout as a function of frequency, thereby developing a frequency dependent background velocity-essentially, deriving a dispersion relationship from residual moveout in frequency.

## ACKNOWLEDGEMENTS

This project is outgrowth of a discussion with Sam Gray at the time of Svenfest and continuing discussions with and encouragement from Sam thereafter, for which I am extremely grateful. Subsequently, Guus Berkhout, Kees Wapenaar and Jacob Fokkema have all provided stimulating discussion and encouragement, despite the fact that I am sometimes critical and often irreverent in my judgments of their approach to inversion. They take my lack of grace with good spirit and keep returning for more discussion of the science in apparent understanding that underlying my irreverence is a fundamental respect (awe!) for the quality of their work. I thank them for their patience, understanding and support.

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Seismic data compression: a tutorial

Tong Chen

# Seismic data compression: a tutorial 

Tong Chen


#### Abstract

A data compression system generally has three building blocks: the transformation, quantization and coding.

In this paper, I use the discrete wavelet packet transform (DWPT) as an example to introduce these building blocks. Specifically, I discuss issues as to why DWPT can be particularly helpful in compressing seismic data, how quantization and coding compresses data, and how to perform quantization and coding optimally, in the sense that might be appropriate for seismic data.


## INTRODUCTION

Seismic data volumes, these days, are huge and growing. With the emergence of 3D technology, the data volume is particularly large ( $>10^{12}$ bytes are common in 3D surveys). Simply archiving these data will require a vast amount of storage. Moreover, as more data are processed and interpreted on workstations, more data transfer among the workstations through local area networks is required. It is therefore desirable to compress the data, in order to reduce the costs of storage and transmission.

There are two categories of data compression techniques: lossless and lossy. Lossless compression means no information is lost during the cycle of compression and decompression, and the original signal can be perfectly reconstructed from the compressed one. Lossy compression, on the other hand, means some information is lost during compression. Cost aside, lossless compression is what every customer would like since it provides a flawless reproduction of the original. Unfortunately, because seismic data are generally represented by floating point numbers throughout processing, true lossless compression of seismic data is inefficient for that compact representation. On the other hand, when data are sampled and recorded, some amount of error is already introduced. So, instead of trying to perfectly reproduce the original signal, it is realistic to make compromises that yield reproductions that are satisfactory for our purposes. Therefore, I focus the discussion here on lossy compression and specifically on transform-based lossy compression.

A transform-based, lossy compression technique consists of three building blocks: transformation, quantization and coding. First, some transform is applied to the
signal. After an appropriate transform, the energy of the signal can be concentrated into a relatively smaller region in the transformed domain than in the original data domain. Then the transformed coefficients (real numbers) are converted into a finite set of integer numbers through quantization. After quantization, fewer bits are used to approximate the coefficients, though at the expense of introducing some errors. Finally, the integer numbers are encoded to further reduce the number of bits required to represent the data.

Much of the original work in lossy compression can be found in the area of speech and image processing (e.g. Bellamy, 1991; Wallace, 1991; Gall, 1991). For seismic data, Wood (1974) discussed compression by truncating the Walsh transform of each trace. Bordley (1983), on the other hand, used linear predictive coding (LPC) to compress marine seismic data. Spanias et al. (1991) compared LPC with some of the transform-based compression techniques, such as the Karhunen-Loeve transform (KLT), the Walsh-Hadamard transform (WHT) and the discrete cosine transform (DCT). More recently, Luo and Schuster (1992) applied the wavelet packet transform to the compression of seismic data by discarding the small coefficients of the transform. Bosman and Reiter (1993) studied how the errors in the wavelet transformbased compression propagate through some processing modules. Reiter and Heller (1994) compared the compression errors for NMO-corrected CDP gathers and stacked sections and found that stacking can actually reduce the compression errors.

Different from the previous publications, in this paper I focus on gaining an understanding of data compression, particularly as it relates to the peculiarities of typical reflection seismic data. I will use the discrete wavelet packet transform (DWPT) as an example to discuss the issues as to why DWPT can be particularly helpful in compressing seismic data, how quantization and coding compresses data, and how to perform quantization and coding optimally, in the sense that might be appropriate for seismic data.

## DISCRETE WAVELET PACKET TRANSFORM

Wavelet packets, introduced by Coifman and Wickerhauser (1992) to compress speech signals, are closely related to the theory of wavelet transformation (Daubechies, 1992). Here, instead of giving rigorous mathematical definitions, I describe discrete transforms from a signal processing point of view.

Both the discrete wavelet transform and discrete wavelet packet transform involve two important filters: a high-pass filter $D$ and a low-pass filter $A$. In the discrete wavelet transform, a signal $x(n)$ is first decomposed by applying the two filters, and then the filtered data are subsampled - retaining only one sample in two so that the total number of samples remains unchanged. For notation purpose, I call the operation of high-pass filtering followed by subsampling $G$ and the corresponding one for low-pass filtering $H$. After the first-stage decomposition, the output from $H$ is further decomposed, and the process goes on until only one sample is left for the $H$ operator, as illustrated by Figure 1. In the discrete wavelet packet transform, both


Fig. 1. The diagram for the discrete wavelet transform. The $G$ and $H$ operators correspond to first filtering by high-pass (for $G$ ) and low-pass (for $H$ ) filters and then subsampling the output by a factor of two.
the outputs from $H$ and $G$ are further decomposed, as shown in Figure 2. Consider


Fig. 2. The diagram for the discrete wavelet packet transform. The $G$ and $H$ operators correspond to first filtering by high-pass (for $G$ ) and low-pass (for $H$ ) filters and then subsampling the output by a factor of two.
the structure in this figure as a tree. For each node in the tree, there exists the choice of decomposing further or not. Therefore, there results a huge collection of possible valid decompositions, with the discrete wavelet transform being one of them. Here, for simplicity, I just decompose all the components to some fixed level.

Since the discrete wavelet packet transform (DWPT) involves iteratively applying the low- and high-pass filters, it achieves frequency partitioning as a result of the decomposition. Moreover, it is an unitary transform so that the root-mean-square (RMS) amplitude remains the same before and after the transform. Also, the process can be reversed to obtain the reconstruction or inverse transform using the same filters $D$ and $A$ (called conjugate quadrature filters in signal processing literature).

It is not difficult to extend DWPT to higher dimensions. A straightforward way to generate a 2D DWPT is to apply two 1D transforms separately along the two
dimensions, i.e., to cascade two 1D DWPTs. As can be imagined, 2D DWPT partitions the data into different frequency bands, along both dimensions. Figure 3 shows a stacked section, and Figure 4 shows the section after 2D DWPT. Here, I did three


Fig. 3. A stacked section.
levels of decomposition along both the time and space dimensions. The filters $D$ and $A$ used here correspond to a specific type of wavelets called fourth-order coiflets (Daubechies, 1992). As shown in the figure, the transformed section consists of small blocks. Each block represents one frequency-wavenumber partition. From the figure, the DWPT concentrates the energy of the data in Figure 3 in the lower-right corner, which corresponds to the partition of low frequency and small wavenumber. This is not surprising, because the original data contain mainly horizontal events resulting in the strong small-wavenumber components.

After DWPT, therefore, the coefficients are concentrated in a region of the new (frequency-wavenumber) domain. But how this action of DWPT help in compressing seismic data?

Figures 5 and 6 may help give understanding of this. Figure 5 shows a histogram obtained from the stacked section shown in Figure 3. Since the data are represented


Fig. 4. The stacked section after 2D DWPT. It is obtained by three levels of decomposition along both space and time dimensions using the filters corresponding to the fourth-order coiflet. Each block represents a frequency-wavenumber partition.
as floating point numbers, they are first converted into integers based on a chosen quantization, thus incurring some error. In generating this figure, I fixed the RMS amplitude of this error to be $1 \%$ of the RMS amplitude of the signal. After the conversion, the occurrence of each integer is then counted and normalized to generate the histogram. Figure 6 shows the histogram of the section after DWPT, as shown in Figure 4. Here again, $1 \%$ relative RMS error is allowed. Since the RMS amplitude of the data remains the same before and after DWPT, the absolute errors are the same in both cases as well. However, the number of integer levels for the data before and after the transform might differ. Here, for plotting purpose, the integer levels are truncated to the range of -128 to 127 . Clearly from the figures, the data after the transform have a much narrower distribution than do those before the transform. As I define later, the information entropy is lower for narrower distributions. Entropy is a quantity that determines the average bits per sample needed to represent a signal. Therefore, the lower the entropy, the fewer bits needed to represent the signal, and the more compression that can be obtained. It turns out that the entropy of the signal in Figure 3 is 6.7 bits while that of the signal in Figure 4 is 5.8 bits.


Fig. 5. Histogram for the stacked section, shown in Figure 3.

## QUANTIZATION

We will find that such reduction in the entropy of a signal is sufficient to yield cost-effective compression of seismic data. But no compression has occurred so far, because the number of coefficients is the same as the number of samples in the original signal. To achieve compression, we need to approximate the transform coefficients using fewer bits. In their approach, Luo and Schuster (1992) applied an approximation by discarding the small coefficients. In order to reconstruct the data, however, they also had to store the locations of the remaining coefficients. An alternative, more practical approach, however, is to approximate the coefficients by a set of integers; this is called quantization.

Since quantization is the only step where approximations are made in representing the signal, how one designs a quantizer - an algorithm performing the quantization - will determine how the error is distributed in the approximation. This, in turn, will have direct influence on how the approximated signal looks, how the waveform in the approximation differs from that in the original, and how the approximation error propagates through different processing modules. Therefore, a good quantizer is one that is tuned for a specific type of signal and processes that will be applied to it. For example, the quantizers used in speech signal compression are different from those used in image compression. Applying the quantizers (and the compression techniques using those quantizers) designed for one type of signal to another type is thus generally inappropriate.

In order to design quantizers that might be appropriate for seismic signals, it is necessary to understand the theory of quantization, a subject that is more complicated than it appears. In their book, Gersho and Gray (1992) discussed many different quantizers and therefore provided many options. However, after some transformation,


Fig. 6. Histogram for the stacked section after DWPT, shown in Figure 4.
scalar quantizers - where each sample is quantized independently - are often used for simplicity.

A scalar quantizer is an operator $Q$ that maps real numbers $x$ within a range $(a, b)$ into a finite set of output levels $y_{1}, y_{2}, \ldots, y_{N}$. After quantization, a real number $x$ can be uniquely approximated by the nearest output level $y_{i}$ and therefore represented by the integer $i$. Depending on how the $y_{i}$ 's are distributed, scalar quantizers are further categorized as uniform - where the $y_{i}$ 's are uniformly distributed, as shown in Figure 7 - and nonuniform, otherwise (Figure 8). The distance $\Delta_{i}=\left|y_{i}-y_{i-1}\right|$ is


Fig. 7. A uniform quantizer where the output levels $y_{i}$ are uniformly distributed.
called the stepsize. Therefore, the maximum error of a quantizer is just max $x_{i} \frac{\Delta_{i}}{2}$. The
mean-squared-error (MSE, which is the square of RMS) between the original signal $x$ and the approximation $Q(x)$ is also called the $L^{2}$-average distortion (I will simply call it average distortion throughout the rest of the paper.), which is given by

$$
\begin{equation*}
D=\int|x-Q(x)|^{2} f_{X}(x) d x \tag{1}
\end{equation*}
$$

where $f_{X}(x)$ is the probability density function of $x$.
Given these definitions, following are some results important in designing a quantizer. The proofs for most of the observations can be found in Gersho and Gray (1992).

1. For a given number of output levels $N$, the uniform quantizer minimizes the maximum error.
2. The average distortion of the uniform quantizer is

$$
\begin{equation*}
D=\frac{\Delta^{2}}{12} \tag{2}
\end{equation*}
$$

provided that $f_{X}(x)$ is smooth and the stepsize $\Delta$ (all the stepsizes are the same for the uniform quantizer) is small.
3. For a given number of output levels $N$, a nonuniform quantizer that matches the input probability density function $f_{X}(x)$ minimizes the average distortion.
The next two observations are related to entropy. The entropy of a discretealphabet random variable $f$ (i.e., random variable that can take on a discrete number of values) is defined as

$$
\begin{equation*}
H_{Q} \equiv-\sum_{i=1}^{N} P(i) \log _{2} P(i) \tag{3}
\end{equation*}
$$

where $P()$ is the probability mass function of $f$; less rigorously, $P(i)$ is the frequency of occurrence of the symbol $i$. There is a continuous-alphabet analog of $H_{Q}$ called the Shannon's differential entropy $h(X)$ of the random variable $X$.
4. For a fixed entropy, the uniform quantizer minimizes the average distortion. Equivalently, for a fixed average distortion, the uniform quantizer achieves the minimum entropy.
5. The minimum entropy for a fixed average distortion is given approximately by

$$
\begin{equation*}
H_{Q} \approx h(X)-\frac{1}{2} \log _{2} 12 D \tag{4}
\end{equation*}
$$

From these observations, there are different optimal quantizers for different purposes. For example, in digital telephone communication, it is desirable to have a fixed number of output levels $N$ (fixed rate codes). From Observation 3, a nonuniform quantizer is therefore required that matches the amplitude distribution of speech signals. Although the shape might be different, most of the natural (as opposite to synthetic) signals have an amplitude distribution similar to the one shown in Figure 5 ; i.e., there are more small-amplitude samples than large-amplitude samples. Intuitively, to have as small an average distortion as possible, the nonuniform quantizer will allocate smaller errors for the small amplitude values than for the large amplitude values (Figure 8), because there are more of them. A nonuniform quantizer such as this is what is used in the current North American standard for digital telephony (CCITT G.711, e.g. Bellamy, 1991). It happens that, besides minimizing the average


Fig. 8. Nonuniform quantizer. The output levels $y_{i}$ are nonuniformly distributed so that the stepsizes for smaller input values are smaller than those for larger input values.
distortion, the nonuniform quantizer fits the purpose of telephony as well. This is because the human auditory system is not very sensitive to the volume of the sound. For a range of large-amplitude events, the content is already known, and the volume does not make too much difference (it might make some difference in expressing emotions though). In contrast, for the small-amplitude events (the whispers) only small errors can be tolerated in order that the content be understandable.

This nonuniform quantizer, however, might not be appropriate for seismic signals. When a nonuniform quantizer is used, more error is allocated to large-amplitude events, because they occur less frequently, as shown in Figure 5. However, in seismic data large-amplitude events (the stand-outs) are what we are often most interested in. Those are the events from which we estimate various earth parameters. Keeping those
events in position and their amplitudes as accurate as possible, intuitively, will help alleviate the possible exaggeration of the quantization errors in further processing. On the other hand, the small-amplitude events have a good chance of being random noise. The nonuniform quantizer therefore might expend too much effort in approximating possible random noise.

Until a better quantizer is found, the uniform quantizer might be a safe choice. From Observation 1, the uniform quantizer minimizes the maximum error for a given number of output levels $N$. Therefore, the uniform quantizer is robust in that good performance can be maintained for a wide variety of input signals. With the error allocated uniformly, the targeting features (large-amplitude events) are approximated accurately, while the small-amplitude events are treated with some care as well. The above reasoning remains valid for the transformed domain in a transform-based compression technique, as well as in the original data domain.

The uniform quantizer might be a safe choice for allocating the error. But will it provide enough compression for a given amount of average distortion? From Observation 4, the uniform quantizer minimizes the entropy. Therefore, if the uniform quantizer is followed by an entropy coder, it will provide the most compression for a given average distortion. The entropy coder will be discussed in the next section, but before going to that, let me show an example.

The approach of discarding the small coefficients discussed in some of the literature can be considered as a special form of quantizer. In this quantizer, small amplitudes are set to zero while the large amplitudes are kept intact. Therefore, large-amplitude events are treated with extreme care (with no approximation at all) while smallamplitude events are totally ignored. Though it might be difficult to argue the possible disadvantages of this error allocation approach in terms of further processing, the compression ratios can be evaluated. For the stacked section shown in Figure 3, I tried the compression technique of quantization with coding, as well as the method of discarding small coefficients. The transformations used are identical for both cases, with five levels along the time direction and four levels along the space dimension of wavelet packets decomposition using the fourth-order coiflet. Under $1 \%$ relative RMS error (RMS amplitude of the error is $1 \%$ of the RMS amplitude of the signal), the quantization (with a uniform quantizer) and coding technique gives about 5.75:1 compression. To achieve this same amount of compression, the method of discarding small coefficients would have to throw away more than $80 \%$ of the smallest coefficients and also store the indices of the remaining coefficients. This however gives an RMS error as large as $20 \%$ even though the coefficients discarded are smaller than $2 \%$ of the largest coefficient. This result gives support to Observation 4: the uniform quantizer minimizes the entropy for a given average distortion.

## CODING

As suggested in the previous section, entropy coders are needed after using a uniform quantizer, in order to achieve good compression. Entropy coding is a lossless
compression step. It attempts to compress the data so that the average number of bits per symbol is close to the entropy of a sequence of symbols, defined by equation (3). The literature contains extensive study on entropy coding, and detailed accounts may be found in many books and papers, e.g., Gersho and Gray (1992).

Example of the many forms of entropy coders include Huffman coders (Huffman, 1952), arithmetic coders (Witten et al., 1987) and dictionary-based coders (Welch, 1984). Here, I use a simple example to show how Huffman coders compress data.

| $i$ | $P(i)$ | Natural Code | Huffman Code |
| :---: | :---: | :---: | :---: |
| 0 | $1 / 2$ | 000 | 0 |
| 1 | $1 / 4$ | 001 | 10 |
| 2 | $1 / 8$ | 010 | 110 |
| 3 | $1 / 16$ | 011 | 1110 |
| 4 | $1 / 32$ | 100 | 11110 |
| 5 | $1 / 64$ | 101 | 111110 |
| 6 | $1 / 128$ | 110 | 1111110 |
| 7 | $1 / 128$ | 111 | 1111111 |

Suppose there is a sequence of symbols. Each symbol belongs to the set of $\{0,1,2, \ldots, 7\}$. Their corresponding binary (natural) codes are shown in the above table. The binary code requires 3 bits per symbol, no matter what the distribution of the symbols in a sequence. Now suppose each symbol $i$ has a frequency of occurrence or probability $P(i)$ shown in the table. In Huffman coding, each symbol $i$ is assigned a code according to its probability $P(i)$. The Huffman code length for symbol $i$ approaches $-\log _{2} P(i)$. Therefore, symbols occurring frequently will have shorter code length, as shown by their Huffman codes in the above table. Huffman code therefore requires $1 \times \frac{1}{2}+2 \times \frac{1}{4}+3 \times \frac{1}{8}+4 \times \frac{1}{16}+5 \times \frac{1}{32}+6 \times \frac{1}{64}+7 \times \frac{1}{128}+7 \times \frac{1}{128} \approx 1.98$ bits per sample on average. Therefore, for this example Huffman coding compresses the sequence by a factor of more than $3: 2$ relative to binary coding.

After entropy coding, the average number of bits per sample will approach the entropy. Therefore, the lower the entropy of the data, the fewer bits required per sample of the representation and the more compression will be achievable. From the definition [equation (3)], it is not difficult to show that the more evenly distributed is $P(i)$, the higher the entropy. If in the previous example, all the symbols $i$ have the same probability $P(i)$, the entropy will be 3 bits, and no compression can be achieved. For the stacked section in Figure 3, we saw that DWPT helped reduce the entropy (from 6.7 to 5.8 bits). Therefore, applying DWPT in this case will result in more compression (for a given level of accuracy) than that achievable for the original data. (We can always compress data with quantization and coding, no matter whether a transform is applied or not.)

Quantization error is another factor that can change the entropy, as shown in Observation 5 in the previous section. From equation (4), it is easy to understand
the trade-off between the quantization error, or the average distortion $D$, and the compression ratio $r$, which is defined as the ratio of the average number of bits per sample before and after compression. For some given data, the number of bits per sample is a fixed quantity $b$ before compression, while it can ideally be the entropy $H_{Q}$ after compression. Therefore, the compression ratio

$$
r \equiv \frac{b}{H_{Q}}
$$

Since $H_{Q}$ is related to the average distortion $D$ according to equation (4), the compression ratio is therefore a function of $D$, as

$$
r(D) \equiv \frac{b}{H_{Q}}=\frac{b}{h(X)-\frac{1}{2} \log _{2} 12 D}
$$

Generally, it is difficult to estimate $h(X)$ and absolute $D$ from the data. On the other hand, $r(D)$ can be measured for some given initial value of $D_{0}\left(D_{0}\right.$ is related to the stepsize in the quantization according to equation (2).) to obtain $r_{0}=r\left(D_{0}\right)$. Since

$$
r_{0}=r\left(D_{0}\right)=\frac{b}{h(X)-\frac{1}{2} \log _{2} 12 D_{0}}
$$

$r(D)$ can be represented using $r_{0}=r\left(D_{0}\right)$, as

$$
r(D)=\frac{b}{\frac{b}{r_{0}}-\frac{1}{2} \log _{2} \frac{D}{D_{0}}}
$$

Defining the relative error $e$ as the MSE (which is just the average distortion $D$ ) divided by the mean squared amplitude of the data $E$,

$$
e \equiv \frac{D}{E}
$$

the compression ratio can then be represented as a function of $e$

$$
\begin{equation*}
r(e)=\frac{b}{\frac{b}{r_{0}}-\frac{1}{2} \log _{2} \frac{e}{e_{0}}} \tag{5}
\end{equation*}
$$

where $r_{0}=r\left(e_{0}\right)$.
Figure 9 shows how the compression ratio $r(e)$ changes with the relative error $e$ for two hypothetical data sets, one with $r_{0}=6$ and the other with $r_{0}=4$, when $e_{0}=.01 \%$ and $b=32$ assumed for both cases. It looks similar to the one shown in Reiter and Heller (1994), where they compared how the compression ratios change with the relative error for an NMO-corrected common-midpoint (CMP) gather and a stacked section. By comparison, they concluded that the error increases with compression ratio more rapidly for CMP gathers than for stacked sections. Figure 9 gives a possible explanation for this phenomenon. Compared to CMP gathers, stacked sections often


Fig. 9. Compression ratios changing with compression error.
have higher signal-to-noise ratios and therefore more coherency. After the transform, they will have less entropy. Since

$$
r_{0}=\frac{b}{h(X)-\frac{1}{2} \log _{2}\left(12 D_{0}\right)},
$$

the gathers with less entropy will have a larger compression ratio $r_{0}$. From equation (5), then, the error will increase more slowly for stacked sections, as illustrated by the example shown in Figure 9.

## CONCLUSION

Using DWPT based compression technique, I show that DWPT helps compressing seismic data because it reduces the entropy of the data. I reason that uniform quantizers might be more appropriate for seismic data, and they can achieve the same amount of compression with less error than simply throwing away the small transform coefficients. I also give a possible explanation for the phenomenon that the compression error grows more rapidly with compression ratio for CMP gathers than for stacked sections.

## ACKNOWLEDGEMENTS

Most of the work in this paper was done during my summer employment at Advance Geophysical Inc. and I thank Advance for allowing me to continue and publish this work after I came back to school. Thanks to Professor Ken Larner for his technical discussion and valuable suggestions as well as critical reviewing of the paper.

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# Converting expressions to Thomsen notation 

Jack K. Cohen

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#### Abstract

Leon Thomsen introduced a set of parameters that allow specialization to weakly transverse isotropic (TI) media without losing the capability of treating the general TI medium. For this reason, many studies of transverse isotropic media involve the conversion of expressions in the conventional notations to the corresponding expression in terms of Thomsen's parameters $\gamma, \delta$, and $\epsilon$. These conversions are awkward because one of the transformation equations is nonlinear in $\delta$. For example, in the Voight tensor notation, $$
C_{13}=-\rho c_{S}^{2}+\rho \sqrt{\left(c_{P}^{2}-c_{S}^{2}\right)^{2}+2 c_{P}^{2}\left(c_{P}^{2}-c_{S}^{2}\right) \delta} .
$$

By introducing a modified parameter, $\tilde{\delta}$, this relation can be made linear. Indeed, the entire transformation from Voight notation to Thomsen notation becomes linear. The same is true for the conversion from the other conventional notations. If an expression in the pure Thomsen parameters is desired, one can replace $\dot{\delta}$ by its definition in terms of $\delta$ as the last step in the calculation. In the limit of weak transverse isotropy, the parameter $\tilde{\delta}$ reduces to Thomsen's $\delta$. Thus, in this important special case, the conversion to pure Thomsen notation amounts to just replacing $\tilde{\delta}$ by $\delta$. The Mathematica package, Thomsen.m, containing functions to automate conversions between the various TI notations accompanies this article.


## INTRODUCTION

Leon Thomsen (1986) introduced a set of parameters that allow specialization to weakly transverse isotropic (TI) media without losing the capability of treating the general TI medium. For this reason, many studies of transverse isotropic media involve the conversion of expressions in the Love, Voight or Hookean tensor notations to the corresponding expression in terms of Thomsen's parameters $\gamma, \delta$, and $\epsilon$. These conversions are awkward because one of the transformation equations is nonlinear in $\delta$. For example, in the Voight notation,

$$
\begin{equation*}
C_{13}=-\rho c_{S}^{2}+\rho \sqrt{\left(c_{P}^{2}-c_{S}^{2}\right)^{2}+2 c_{P}^{2}\left(c_{P}^{2}-c_{S}^{2}\right) \delta}, \tag{1}
\end{equation*}
$$

where $\rho$ is the density and $c_{P}, c_{S}$ are the compressional and shear speeds. By introducing a modified parameter, $\tilde{\delta}$, this relation can be made linear. Indeed, the entire transformation from Voight notation to Thomsen notation becomes linear. It also turns out that many expressions are simpler in terms of a parameter $f$ introduced by Ilya Tsvankin (1994) The five parameters $c_{P}, f, \gamma, \epsilon$, and $\tilde{\delta}$ form a convenient canonical set of parameters. Once expressions have been cast in terms of these parameters, one can make subsequent substitutions that eliminate $f$ in favor of $c_{S}$ and/or $\tilde{\delta}$ in favor of $\delta$.

In this report, I describe a Mathematica package to convert between the various notations for the TI parameters. In particular, the user has the choice of using $c_{S}$ or $f$ and the choice of using $\tilde{\delta}$ or $\delta$.

## THE PARAMETERS $\tilde{\delta}$ AND $f$

First, simplify equation (1) by introducing Tsvankin's $f$ :

$$
\begin{equation*}
f=\frac{c_{P}^{2}-c_{S}^{2}}{c_{P}^{2}} \tag{2}
\end{equation*}
$$

yielding

$$
\begin{equation*}
C_{13}=-\rho c_{S}^{2}+\rho c_{P}^{2} \sqrt{f^{2}+2 f \delta} \tag{3}
\end{equation*}
$$

Next, eliminate the square root by introducing $\tilde{\delta}$ as

$$
\begin{equation*}
f^{2}+2 f \delta=(f+\tilde{\delta})^{2} \tag{4}
\end{equation*}
$$

obtaining the linear expression

$$
\begin{equation*}
C_{13}=-\rho c_{S}^{2}+\rho c_{P}^{2}(f+\tilde{\delta}) \tag{5}
\end{equation*}
$$

Equation (4) allows us to express $\tilde{\delta}$ in terms of $\delta$ or vice versa:

$$
\begin{equation*}
\tilde{\delta}=f\left(\sqrt{1+\frac{2 \delta}{f}}-1\right) \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
\delta=\tilde{\delta}\left(1+\frac{\tilde{\delta}}{2 f}\right) \tag{7}
\end{equation*}
$$

An application of the binomial theorem to equation (6) shows that in the limit of weak transverse isotropy, the parameter $\tilde{\delta}$ reduces to Thomsen's $\delta$. Thus, in this important special case, the conversion to the Thomsen parameter amounts to just replacing $\tilde{\delta}$ by $\delta$.

The quantity $\tilde{\delta}$ introduced in this section also plays a role in the basic TI wave equations. In a companion report (Thomsen Operators and Thomsen Matrices: this volume), I show that the TI wave equation operator has the exact form,

$$
\begin{equation*}
\mathcal{L}=\mathcal{L}^{(0)}+\gamma \mathcal{L}^{(\gamma)}+\tilde{\delta} \mathcal{L}^{(\delta)}+\epsilon \mathcal{L}^{(\epsilon)} \tag{8}
\end{equation*}
$$

Here, the operator $\mathcal{L}^{(0)}$ is the isotropic wave equation operator, while the "Thomsen operators," $\mathcal{L}^{(\gamma)}, \mathcal{L}^{(\delta)}$, and $\mathcal{L}^{(\epsilon)}$, characterize the anisotropic contributions. The isotropic operator and the three Thomsen operators are independent of the Thomsen parameters $\gamma, \delta$, and $\epsilon$, so that the TI operator is linear in $\gamma, \epsilon$, and the modified Thomsen parameter $\tilde{\delta}$.

## NOTATIONS FOR THE TI ELASTIC PARAMETERS

Using $f$ and $\tilde{\delta}$ as introduced above, the Voight parameters are expressed in terms of the Thomsen parameters, $\gamma, \delta, \epsilon$, and the material parameters by the equations (Thomsen, 1986):

$$
\begin{align*}
& C_{11}=\rho c_{P}^{2}(1+2 \epsilon) \\
& C_{33}=\rho c_{P}^{2} \\
& C_{13}=-\rho c_{S}^{2}+\rho c_{P}^{2}(f+\tilde{\delta})  \tag{9}\\
& C_{44}=\rho c_{S}^{2} \\
& C_{66}=\rho c_{S}^{2}(1+2 \gamma)
\end{align*}
$$

Note that here, " 3 " is used as the index corresponding to the axis of symmetry-many authors use " 1 " as the distinguished direction.

The Love parameters are simply aliases for the Voight parameters (Musgrave, 1970):

$$
\begin{align*}
A & =C_{11} \\
C & =C_{33} \\
F & =C_{13}  \tag{10}\\
L & =C_{44} \\
N & =C_{66}
\end{align*}
$$

In terms of the Voight notation, the fundamental Hookean parameters are given by:

$$
\begin{align*}
& C_{1111}=C_{33} \\
& C_{2222}=C_{3333}=C_{11} \\
& C_{2233}=C_{3322}=C_{11}-2 C_{66} \\
& C_{1133}=C_{3311}=C_{1122}=C_{2211}=C_{13}  \tag{11}\\
& C_{1212}=C_{2121}=C_{2112}=C_{1221}=C_{44} \\
& C_{1313}=C_{3131}=C_{3113}=C_{1331}=C_{44} \\
& C_{2323}=C_{3232}=C_{3223}=C_{2332}=C_{66}
\end{align*}
$$

Finally, mention a lesser used notation appearing in Kuprazde (1976):

$$
C_{1}=C_{11}
$$

$$
\begin{align*}
& C_{2}=C_{11}-C_{66} \\
& C_{3}=C_{13}  \tag{12}\\
& C_{4}=C_{33} \\
& C_{5}=C_{44}
\end{align*}
$$

## THE MATHEMATICA IMPLEMENTATION

The Mathematica implementation reserves the following variables for their meaning as notations for the elastic parameters:

```
rho, cp, cs, gamma, delta, deltaTilde, epsilon, eta, f,
C11, C33, C13, C44, C66,
A, C, F, L, N,
C1, C2, C3, C4, C5, C6,
C1111, C2222, C3333, C2233, C3322, C1133, C3311,
C1122, C2211, C1212, C2121, C2112, C1221,
C1313, C3131, C3113, C1331, C2323, C3232, C3223, C2332
```

Note well that users of this package cannot use these variables for any other purpose!
The conversion rules are implemented by the user functions:
ConvertCij converts an expression in the Voight parameters to an expression in the canonical variables $c_{P}, f, \gamma, \epsilon$, and $\tilde{\delta}$.

ConvertWeak assumes the weak TI limit and converts an expression in the Voight and/or Thomsen parameters to an expression in the variables $c_{p}, f, \gamma, \epsilon$ and $\delta$.

EliminateF replaces $f$ by its definition in terms of $c_{P}$ and $c_{S}$.
EliminateCs is a partial inverse of EliminateF; it replaces $c_{S}$ in favor of $c_{P}$ and $f$.
EliminateDeltaTilde replaces $\tilde{\delta}$ by its definition in terms of $\delta$ and $f$.
EliminateDelta is a partial inverse of EliminateDeltaTilde; it replaces $\delta$ in favor of $\tilde{\delta}$ and $f$.

EliminateEta replaces $\eta$ by its definition in terms of $\delta$ and $\epsilon$.
EliminateEpsilon is a partial inverse of EliminateEta; it replaces $\eta$ in favor of $\delta$ and $\epsilon$.

EliminateLove replaces the Love parameters by the Voight parameters.
EliminateKuprazde replaces the Kuprazde parameters by the Voight parameters.
EliminateHooke replaces the Hooke parameters by the Voight parameters.

ConvertThomsen converts an expression from Thomsen notation to Voight notation. Any of the variants, $\delta, \tilde{\delta}, \eta, c_{S}$, and $f$ may appear in the expression.

SineForm writes a trigonometric expression using powers of sine.
CosineForm writes a trigonometric expression using powers of cosine.
The user functions are implemented with the aid of "private" or "hidden" rules such as:

```
CsToFRule =
    cs^n_Integer?Positive ->
                        (cp-2(1 - f))`Quotient[n,2] cs^Mod[n,2]
DeltaRule = deltaTilde }->\textrm{f}(\mathrm{ Sqrt [1+2delta/f] - 1)
...
CijRules =
{
    C11 -> rho cp^2 (1 + 2 epsilon),
    C33 -> rho cp^2,
    C13 -> -rho cs^2 + rho cp^2 (f + deltaTilde),
    C44 -> rho cs^2,
    C66 ->> rho cs^2 (1 + 2 gamma)
}
```

These rules cannot be directly accessed by the user-the "public" functions described above provide the user interface. For example, the private DeltaRule rule is accessed by the public function EliminateDeltaTilde:

```
EliminateDeltaTilde[expression_] := expression /. DeltaRule//Simplify
```

The main conversion engines are the functions ConvertCij and ConvertWeak. The first merely provides public access to the appropriate private rules:

```
ConvertCij[expression_] := expression/. CijRules/. CsToFRule//Simplify
```

The implementation of ConvertWeak is a bit harder. While Mathematica has a facility for expanding functions in multiple power series, there is no built-in way to impose that terms like $\epsilon^{2}$ and $\epsilon \delta$ are equally to be neglected. A simple way to impose such conditions is to introduce a scaling parameter, here $x$, and expand in this single
parameter (this is analogous to the methodology used in the Calculus of Variations and other applications where functional derivatives are needed). Despite the programming maxim of having modules do a single job, for user convenience I decided to embed a call to ConvertCij within this code:

```
ConvertWeak[expression_] :=
Module[{x, tmp},
    tmp = ConvertCij[expression];
    tmp = tmp /. EpsilonRule /. DeltaTildeRule /.
        {
            epsilon -> epsilon x,
            deltaTilde }->\mathrm{ d delta }x\mathrm{ ,
            gamma ->> gamma x
        };
Normal@Series[tmp, {x,0,1}] /. x->1 /. CsToFRule//
Simplify//PowerExpand
]
```


## USAGE EXAMPLES

## Voight to Thomsen Examples

The cracks expression defined below is used as an example to illustrate use of the functions in the Thomsen.m package. The corresponding equation,

$$
\begin{equation*}
C_{11} C_{33}-C_{13}^{2}=2 C_{66}\left(C_{13}+C_{33}\right) \tag{13}
\end{equation*}
$$

represents the relation between $C_{i j}$ 's for TI media formed by a system of thin parallel horizontal cracks in a purely isotropic matrix. It comes from the fact that in this case there are only four independent parameters instead of five for general TI media. See the more detailed discussion in the paper by Schoenberg and Sayers (1995) (the only difference is that their cracks are vertical). First, convert the expression to the canonical parameters and store the result in the variable generalCracks:

```
cracks = C11 C33 - C13^2 - 2 C66 (C13 + C33);
generalCracks = ConvertCij[cracks]
    cp (-deltaTilde }\mp@subsup{}{}{4}+2\mathrm{ epsilon - 2 deltaTilde f -
        4 deltaTilde gamma - 8 f gamma +
    2 2
        4 deltaTilde f gamma + 8 f gamma) rho
```

Next, eliminate $f$ in favor of the speed $c_{S}$ :

Cohen

```
EliminateF[%]
```



```
    2
    cp
    2 epsilon - 8 cs gamma 8 cs gamma
    4 cs deltaTilde gamma 2
```

Just as an illustration, switch back to the canonical representation by eliminating $c_{S}$ in favor of $f$ :

## EliminateCs [\%]

```
    4 2
    cp (-deltaTilde + 2 epsilon - 2 deltaTilde f -
        4 deltaTilde gamma - 8 f gamma +
        2 2
        4 deltaTilde f gamma + 8 f gamma) rho
```


## Voight to Weak TI Thomsen Examples

Do the canonical conversion in the limit of weak TI. Recall that in this limit, $\delta$ and $\delta$ are equal.

```
weakCracks = ConvertWeak[cracks]
            4 2
    2cp (epsilon - delta f-4 f gamma + 4f gamma)
        2
    rho
```

Use Mathematica to solve for $\delta$ :

```
delta /. Flatten@Solve[weakCracks \(=0\), delta]//Simplify
```

```
epsilon
------- - 4 gamma + 4f gamma
    f
```

Use a typical value $c_{P} / c_{S}=2$ which is equivalent to the value $f=3 / 4$ to get a feel for this result:
$\% / . f$-> $3 / 4$

```
4 epsilon
    3
```

Introduce $c_{S}$ in favor of $f$ :

```
EliminateF [weakCracks]
    \(2\left(-\left(c p^{4}\right.\right.\) delta \()+c p^{2} c s^{2}\) delta \(+c p^{4}\) epsilon -
\(4 \mathrm{cp}^{2} \mathrm{cs}^{2}\) gamma \(+4 \mathrm{cs}^{4}\) gamma) \(\mathrm{rho}^{2}\)
```

Solve for $\delta$ in the new variables:

```
delta /. Flatten@Solve[% == 0, delta] //Simplify
    cp epsilon-4 cp cm cs gamma + 4 cs gamma
```

Use a standard Mathematica function to again get the result in our typical case:
Limit[\%, cp $\rightarrow 2$ cs]//Simplify
4 epsilon
3

ConvertWeak can also handle expressions involving the standard Thomsen parameters, as well as $f$ and $\tilde{\delta}$ :

```
ConvertWeak[C33(1 + 2 epsilon)^3 - C44(1 + deltaTilde + delta)^3]
    2
    cp (-6 delta + 6 epsilon + f + 6 delta f) rho
    Collect[%, {cp, rho, delta}]
        2
    cp (6 epsilon + f + delta (-6 + 6 f)) rho
```


## Eliminating $\tilde{\delta}$

When $\tilde{\delta}$ is eliminated, the underlying square root becomes explicit (unless we make the weak TI assumption):

## EliminateDeltaTilde[generalCracks]

```
            4 2
    2 cp (epsilon - deltaf-2fgamma \(2 f\) gamma -
                2 delta +f
            2 f Sqrt[-----------] gamma +
                f
```



And if we want to use the pure Thomsen parameters, continue by eliminating $f$ :

## EliminateF [\%]

$p^{4}$ delta) $+\mathrm{cp}^{2} \mathrm{cs}^{2}$ delta $+\mathrm{cp}^{4}$ epsilon -



Now go the weak TI limit and recover our typical case for the third time:
ConvertWeak[\%]
$\mathrm{cp}^{4}$ (epsilon - deltaf-4fgamma $+4 \mathrm{f}^{2}$ gamma)
2
rho
$\% / . f$-> $3 / 4$


Solve[\% == 0, delta]//Flatten
4 epsilon - 3 gamma
\{delta $\rightarrow$---------------------

## Using the $\boldsymbol{\eta}$ Parameter

Alkhalifah and Tsvankin (1994) introduced the parameter

$$
\eta=\frac{\epsilon-\delta}{1+2 \delta}
$$

to facilitate time processing in transversely isotropic media. In the package, we offer routines to trade $\epsilon$ for $\eta$ and vice versa:

```
EliminateEpsilon[epsilon]
    delta + eta + 2 delta eta
```

EliminateEta [\%]
epsilon
EliminateEpsilon[(epsilon-delta)/(1 + 2 delta)]
eta

## Love Notation Example

You can convert Love notation expressions to Voight notation and then proceed as above:

EliminateLove[F + L]
C13 + C44

## ConvertCij [\%]

2
cp (deltaTilde + f) rho
EliminateDeltaTilde[\%]


EliminateF[\%]


## Kuprazde Notation Example

Again, you can convert Kuprazde notation expressions to Cij notation and then proceed as above:

```
EliminateKuprazde [C2]
    C11 - C66
ConvertCij[%]
        2
    cp (2 epsilon + f - 2 gamma + 2 f gamma) rho
EliminateF[%]
    (cp 2 - cs }\mp@subsup{}{}{2}+2c\mp@subsup{c}{}{2}\mathrm{ epsilon - 2 cs (gamma) rho
Collect[%, {cp, cs, rho}]
    2 2
    cp (1 + 2 epsilon) rho + cs (-1 - 2 gamma) rho
```


## Hooke Notation Example

EliminateHooke[C3322 + 2 C3223]
C11

## Converting from Thomsen to Love or Voight Notation

Problem: Verify that the expression
$(1+F)(-1+F+2 L) /(2-2 L)$
is equal to Thomsen's $\delta$ when the Love parameter $C=1$ :
ConvertThomsen[delta] /. C $\rightarrow 1$
$(1+F)(-1+F+2 L)$
$2(1-L)$
If we want the result in terms of the $C_{i j}$ :
EliminateLove[\%]

$$
\frac{(1+C 13)(-1+C 13+2 C 44)}{2-2 C 44}
$$

## A Final Realistic Example

Despite the general utility of Mathematica, and the specific utility of the package introduced here, real examples often require the skillful intervention of the scientist. Consider the expression in equation (6) of Tsvankin (1994):

```
expr = ((C11+C44) Sin[theta] }~2+(C33 + C44) Cos[theta] -2 +
    Sqrt[((C11 - C44) Sin[theta] 2 -
        (C33 - C44)Cos[theta] -2) -2 +
        4(C13 + C44) ^2 Sin[theta] ^2 Cos[theta] `2
    ])/(2 C33);
```

Converting to Thomsen parameters gives:

```
exprThom = EliminateDeltaTilde@ConvertCij[expr]
    2 2
    (cp (2 - f) rho Cos[theta] +
        2
        cp (2 + 2 epsilon - f) rho Sin[theta] +
            4 2
    Sqrt[cp rho (epsilon - epsilon Cos[2 theta] -
```

```
                        2
    f Cos[2 theta]) +
        4 2 2 2
    4 cp f (2 delta + f) rho Cos[theta] Sin[theta]]
        2
) / (2 cp rho)
```

We can see the common factor of $\rho c_{p}^{2}$ even if Mathematica cannot. Rather than wasting time trying to get Mathematica to recognize this factor, it is better to force it out directly. Here is one way:

```
ratiosq = Limit[exprThom, rho -> 1/cp^2]
        2 2
    ((2 - f) Cos[theta] + (2 + 2 epsilon - f) Sin[theta] +
    Sqrt[(epsilon - epsilon Cos[2 theta] -
        2
        f Cos[2 theta]) +
        2 2
    4f(2 delta + f) Cos[theta] Sin[theta] ]) / 2
```

The routines in the Thomsen package call on Mathematica's Simplify routine; this routine has a canonical form for the trigonometric functions that does not always give what you may want. Thus, the package contains the functions SineForm and CosineForm to respectively emphasize powers of sine and powers of cosine. Sometimes none of the three forms will give exactly what you want, but often you will be able to "optimize" yourself after viewing the three possibilities.

```
ratiosq1 = SineForm[ratiosq]
            f 2
    1 - + epsilon Sin[theta] +
        2
            2
                                2
    Sqrt[f + 8 delta f Sin[theta] -
        2 2 4
        4 epsilon f Sin[theta] + 4 epsilon Sin[theta] -
        4
            8 delta f Sin[theta] + 8 epsilon f Sin[theta] ] / 2
```

To simplify further, one has to isolate the square root term:
$a=1-f / 2+\operatorname{Sin}[t h e t a]^{-} 2$ epsilon;
b $=(\text { ratiosq1 }-\mathrm{a})^{\wedge} 2 / /$ Simplify
222
(f +8 delta $f \operatorname{Sin}[$ theta] -4 epsilon $f \operatorname{Sin}[$ theta] +
4 epsilon ${ }^{2} \operatorname{Sin}\left[\right.$ theta] ${ }^{4}-8$ delta $f \operatorname{Sin}\left[\right.$ theta] ${ }^{4}+$
4
8 epsilon $f$ Sin[theta] ) / 4
b1 = Collect[b, \{epsilon, delta, f\}]
2

4

2 4
delta $f(2 \operatorname{Sin}[$ theta] - $2 \operatorname{Sin}[$ theta] $)+$

24
epsilon $f$ (-Sin[theta] $+2 \operatorname{Sin}$ [theta])

Finally, look at the weak limit:

```
weakratiosq = ConvertWeak[ratiosq1]//SineForm
                            2 4
    1 + 2 delta Sin[theta] - 2 delta Sin[theta] +
                            4
        2 epsilon Sin[theta]
Collect[weakratiosq, {epsilon, delta}]
    4
    1 + 2 epsilon Sin[theta] +
```

        24
        delta ( \(2 \operatorname{Sin}[\) theta] - \(2 \operatorname{Sin}[\) theta] )
    \%//Simplify

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Thomsen operators and Thomsen matrices

Jack K. Cohen

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#### Abstract

Leon Thomsen introduced a set of parameters that allow specialization to weakly transverse isotropic (TI) media without losing the capability of treating the general TI medium. The Thomsen parameters have proven useful in a variety of transverse isotropic media studies-it turns out that they also lead to an elegant formulation of the TI wave equations. Indeed, the TI wave equation operator takes the form, $$
\mathcal{L}=\mathcal{L}^{(0)}+\gamma \mathcal{L}^{(\gamma)}+\tilde{\delta} \mathcal{L}^{(\delta)}+\epsilon \mathcal{L}^{(\epsilon)}
$$

The operator $\mathcal{L}^{(0)}$ is the isotropic wave equation operator, while the "Thomsen operators," $\mathcal{L}^{(\gamma)}, \mathcal{L}^{(\delta)}$, and $\mathcal{L}^{(\epsilon)}$, characterize the anisotropic contributions. The isotropic operator and the three Thomsen operators are independent of the Thomsen parameters $\gamma, \delta$, and $\epsilon$, so that the TI operator is linear in $\gamma, \epsilon$, and the modified Thomsen parameter $\tilde{\delta}$. The parameter $\tilde{\delta}$ reduces to Thomsen's $\delta$ in the limit of weak transverse anisotropy. The three Thomsen operators are spatial differential operators and the "Thomsen matrices," $M^{(\gamma)}, M^{(\delta)}$, and $M^{(\epsilon)}$ are their respective spatial Fourier transforms. The matrices $M^{(\gamma)}$ and $M^{(\epsilon)}$ have rank one, while the matrix $\mathrm{M}^{(\delta)}$ is of rank two.

Two simple applications are presented to illustrate the utility of the formulating the TI wave equations in terms of the Thomsen operators/matrices. The first is a direct derivation of the phase speeds in the limit of weak TI by application of the standard matrix perturbation theory for the eigenvalue-eigenvector problem. The second application is a derivation of the exact TI Green's tensor in the special case $\delta=\epsilon=0$.


## THE TI WAVE EQUATIONS IN THOMSEN NOTATION

Leon Thomsen (1986) introduced a set of parameters that allow specialization to weakly transverse isotropic (TI) media without losing the capability of treating the general TI medium. The Thomsen parameters have proven useful in a variety of transversely isotropic media studies-the following considerations show that their use also leads to an elegant formulation of the TI wave equations.

The Love parameters are expressed in terms of the Thomsen parameters, $\gamma, \delta, \epsilon$, and the material parameters by the equations:

$$
\begin{align*}
A & =\rho c_{P}^{2}(1+2 \epsilon) \\
C & =\rho c_{P}^{2} \\
F & =-\rho c_{S}^{2}+\rho c_{P}^{2} \sqrt{f(f+2 \delta)}=-\rho c_{S}^{2}+\rho c_{P}^{2} f \sqrt{1+2 \delta / f}  \tag{1}\\
L & =\rho c_{S}^{2} \\
N & =\rho c_{S}^{2}(1+2 \gamma)
\end{align*}
$$

Here, the quantity $f$ is

$$
\begin{equation*}
f=\frac{c_{P}^{2}-c_{S}^{2}}{c_{P}^{2}} \tag{2}
\end{equation*}
$$

as introduced by Ilya Tsvankin (1994) In TI calculations, the following consequences of this definition are used repeatedly:

$$
\begin{equation*}
c_{P}^{2}-c_{S}^{2}=f c_{P}^{2}, \quad \text { and } \quad 1-f=\frac{c_{S}^{2}}{c_{P}^{2}} \tag{3}
\end{equation*}
$$

The material parameters are denoted by $\rho$ for the density, and $c_{P}$ and $c_{S}$ for the speeds. These speeds represent the phase velocities along the distinguished axis (here the vertical or " 3 " axis)-alternately, they can be construed as the speeds that would prevail if the medium were isotropic. With this convention, the TI wave operators in Love notation are:

$$
\begin{align*}
& \mathcal{L}_{1} u=\left(A \partial_{1}^{2}+N \partial_{2}^{2}+L \partial_{3}^{2}\right) u_{1}+(A-N) \partial_{1} \partial_{2} u_{2}+(F+L) \partial_{1} \partial_{3} u_{3}-\rho \partial_{t}^{2} u_{1} \\
& \mathcal{L}_{2} u=\left(N \partial_{1}^{2}+A \partial_{2}^{2}+L \partial_{3}^{2}\right) u_{2}+(A-N) \partial_{1} \partial_{2} u_{1}+(F+L) \partial_{2} \partial_{3} u_{3}-\rho \partial_{t}^{2} u_{2}  \tag{4}\\
& \mathcal{L}_{3} u=\left(L \partial_{1}^{2}+L \partial_{2}^{2}+C \partial_{3}^{2}\right) u_{3}+\quad(F+L) \partial_{3}\left(\partial_{1} u_{1}+\partial_{2} u_{2}\right)-\rho \partial_{t}^{2} u_{3}
\end{align*}
$$

Observe that $\delta$ enters the wave equations only through $F$, and, in turn, $F$ enters the wave equations only in the combination $F+L$. Write this combination as

$$
\begin{equation*}
F+L=\rho c_{P}^{2} f \sqrt{1+2 \delta / f}=\rho c_{P}^{2}(f+\tilde{\delta}) \tag{5}
\end{equation*}
$$

where we have introduced the modified Thomsen parameter,

$$
\begin{equation*}
\tilde{\delta}=f(\sqrt{1+2 \delta / f}-1) \tag{6}
\end{equation*}
$$

Notice that for small $\delta, \tilde{\delta}=\delta$ to first order. Thus, $\delta$ and $\tilde{\delta}$ are equally valid parameters for passing to the weak TI limit. The advantage of $\tilde{\delta}$ in the present study is that the TI wave equations are linear in this variable. Indeed, since the TI wave equations are already linear in $\gamma$ and $\epsilon$, we may write the TI operators in the form,

$$
\begin{equation*}
\mathcal{L}=\mathcal{L}^{(0)}+\gamma \mathcal{L}^{(\gamma)}+\tilde{\delta} \mathcal{L}^{(\delta)}+\epsilon \mathcal{L}^{(\epsilon)} \tag{7}
\end{equation*}
$$

where the matrix operators, $\mathcal{L}^{(0)}, \mathcal{L}^{(\gamma)}, \mathcal{L}^{(6)}$, and $\mathcal{L}^{(\epsilon)}$, are independent of the Thomsen parameters. Notice that this decomposition of the TI wave operator is valid for strong as well as for weak transverse anisotropy. Observe that a decomposition for weak TI using Thomsen's $\delta$ only requires expanding the scalar quantity $\tilde{\delta}$ to the desired order in $\delta$-in the usual first order case, this amounts to merely replacing $\tilde{\delta}$ by $\delta$.
Remark: For other purposes, $\tilde{\delta}$ is less useful that Thomsen's $\delta$ or other combinations of the Thomsen parameters, so I am not proposing a replacement of $\delta$ by $\tilde{\delta}$ in general TI studies.

Explicitly, we have for the components of $\mathcal{L}^{(0)}$ :

$$
\begin{align*}
\mathcal{L}_{1}^{(0)} u & =\left(\rho c_{P}^{2} \partial_{1}^{2}+\rho c_{S}^{2} \partial_{2}^{2}+\rho c_{S}^{2} \partial_{3}^{2}\right) u_{1}+\rho c_{P}^{2} f \partial_{1} \partial_{2} u_{2}+\rho c_{P}^{2} f \partial_{1} \partial_{3} u_{3}-\rho \partial_{t}^{2} u_{1} \\
\mathcal{L}_{2}^{(0)} u & =\left(\rho c_{S}^{2} \partial_{1}^{2}+\rho c_{P}^{2} \partial_{2}^{2}+\rho c_{S}^{2} \partial_{3}^{2}\right) u_{2}+\rho c_{P}^{2} f \partial_{1} \partial_{2} u_{1}+\rho c_{P}^{2} f \partial_{2} \partial_{3} u_{3}-\rho \partial_{t}^{2} u_{2}  \tag{8}\\
\mathcal{L}_{3}^{(0)} u & =\left(\rho c_{S}^{2} \partial_{1}^{2}+\rho c_{S}^{2} \partial_{2}^{2}+\rho c_{P}^{2} \partial_{3}^{2}\right) u_{3}+\rho c_{P}^{2} f \partial_{3}\left(\partial_{1} u_{1}+\partial_{2} u_{2}\right)-\rho \partial_{t}^{2} u_{3}
\end{align*}
$$

After writing the speeds in terms of the Lamé parameters as

$$
\begin{align*}
c_{S}^{2} & =\frac{\mu}{\rho}  \tag{9}\\
c_{P}^{2} & =\frac{\lambda+2 \mu}{\rho} \tag{10}
\end{align*}
$$

it is straightforward to show that these equations are just the ordinary isotropic elastic wave equations for a homogeneous medium.

The new operators, which characterize the anisotropic contributions, have the components,

$$
\begin{align*}
\mathcal{L}_{1}^{(\gamma)} u & =2 \rho c_{S}^{2} \partial_{2}\left(\partial_{2} u_{1}-\partial_{1} u_{2}\right) \\
\mathcal{L}_{2}^{(\gamma)} u & =2 \rho c_{S}^{2} \partial_{1}\left(\partial_{1} u_{2}-\partial_{2} u_{1}\right)  \tag{11}\\
\mathcal{L}_{3}^{(\gamma)} u & =0
\end{align*}
$$

$$
\begin{align*}
\mathcal{L}_{1}^{(\delta)} u & =\rho c_{P}^{2} \partial_{1} \partial_{3} u_{3} \\
\mathcal{L}_{2}^{(\delta)} u & =\rho c_{P}^{2} \partial_{2} \partial_{3} u_{3}  \tag{12}\\
\mathcal{L}_{3}^{(\delta)} u & =\rho c_{P}^{2} \partial_{3}\left(\partial_{1} u_{1}+\partial_{2} u_{2}\right)
\end{align*}
$$

and,

$$
\begin{align*}
\mathcal{L}_{1}^{(\epsilon)} u & =2 \rho c_{P}^{2} \partial_{1}\left(\partial_{1} u_{1}+\partial_{2} u_{2}\right) \\
\mathcal{L}_{2}^{(\epsilon)} u & =2 \rho c_{P}^{2} \partial_{2}\left(\partial_{1} u_{1}+\partial_{2} u_{2}\right)  \tag{13}\\
\mathcal{L}_{3}^{(\epsilon)} u & =0
\end{align*}
$$

The explicit matrix form of the "Thomsen operators" introduced in equation (7) are:

$$
\begin{align*}
\mathcal{L}^{(\gamma)} & =2 \rho c_{S}^{2}\left(\begin{array}{ccc}
\partial_{2}^{2} & -\partial_{1} \partial_{2} & 0 \\
-\partial_{1} \partial_{2} & \partial_{1}^{2} & 0 \\
0 & 0 & 0
\end{array}\right),  \tag{14}\\
\mathcal{L}^{(\delta)} & =\rho c_{P}^{2}\left(\begin{array}{ccc}
0 & 0 & \partial_{1} \partial_{3} \\
0 & 0 & \partial_{2} \partial_{3} \\
\partial_{1} \partial_{3} & \partial_{2} \partial_{3} & 0
\end{array}\right), \tag{15}
\end{align*}
$$

and,

$$
\mathcal{L}^{(\epsilon)}=2 \rho c_{P}^{2}\left(\begin{array}{ccc}
\partial_{1}^{2} & \partial_{1} \partial_{2} & 0  \tag{16}\\
\partial_{1} \partial_{2} & \partial_{2}^{2} & 0 \\
0 & 0 & 0
\end{array}\right)
$$

The Thomsen operators can also be written in terms of dyadic differential operators. Indeed, $\mathcal{L}^{(\gamma)}$ is the rank one operator:

$$
\mathcal{L}^{(\gamma)}=2 \rho c_{S}^{2}\left(\begin{array}{c}
-\partial_{2}  \tag{17}\\
\partial_{1} \\
0
\end{array}\right)\left(\begin{array}{lll}
-\partial_{2} & \partial_{1} & 0
\end{array}\right)
$$

Thus

$$
\mathcal{L}^{(\gamma)}=2 \rho c_{S}^{2} D^{\perp} D^{\perp}, \quad \text { where } \quad D^{\perp}=\left(\begin{array}{c}
-\partial_{2}  \tag{18}\\
\partial_{1} \\
0
\end{array}\right)
$$

Similarly, the rank one representation of $\mathcal{L}^{(\epsilon)}$ is

$$
\mathcal{L}^{(\epsilon)}=2 \rho c_{P}^{2} D D, \quad \text { where } \quad D=\left(\begin{array}{c}
\partial_{1}  \tag{19}\\
\partial_{2} \\
0
\end{array}\right)
$$

and a rank two representation of $\mathcal{L}^{(\delta)}$ is

$$
\mathcal{L}^{(\delta)}=\rho c_{P}^{2}\left(D D_{3}+D_{3} D\right), \quad \text { where } \quad D_{3}=\left(\begin{array}{c}
0  \tag{20}\\
0 \\
\partial_{3}
\end{array}\right)
$$

## THE THOMSEN MATRICES

Now study the Thomsen form of the TI wave equations in Fourier domain. Apply the four-fold Fourier transform,

$$
\begin{equation*}
U(k, t)=\int d r \int d t e^{i(\omega t-k \cdot r)} u(r, t) \tag{21}
\end{equation*}
$$

and define $M, M^{(0)}, M^{(\gamma)}, M^{(6)}$, and $M^{(\epsilon)}$ as the negatives of the transforms of the corresponding differential operators in equation (7). From equations (14), (15), and (16),
the negatives of the transformed Thomsen operators can be written down at once in matrix form as

$$
\begin{align*}
M^{(\gamma)} & =2 \rho c_{S}^{2}\left(\begin{array}{ccc}
k_{2}^{2} & -k_{1} k_{2} & 0 \\
-k_{1} k_{2} & k_{1}^{2} & 0 \\
0 & 0 & 0
\end{array}\right),  \tag{22}\\
M^{(\delta)} & =\rho c_{P}^{2}\left(\begin{array}{ccc}
0 & 0 & k_{1} k_{3} \\
0 & 0 & k_{2} k_{3} \\
k_{1} k_{3} & k_{2} k_{3} & 0
\end{array}\right), \tag{23}
\end{align*}
$$

and,

$$
M^{(\epsilon)}=2 \rho c_{P}^{2}\left(\begin{array}{ccc}
k_{1}^{2} & k_{1} k_{2} & 0  \tag{24}\\
k_{1} k_{2} & k_{2}^{2} & 0 \\
0 & 0 & 0
\end{array}\right)
$$

Similarly, the transforms of equations (18), (19), and, (20) yield the dyadic representations of the Thomsen matrices as

$$
\begin{gather*}
M^{(\gamma)}=2 \rho c_{S}^{2} \kappa^{\perp} \kappa^{\perp}, \quad \text { where } \quad \kappa^{\perp}=\left(\begin{array}{c}
-k_{2} \\
k_{1} \\
0
\end{array}\right),  \tag{25}\\
M^{(\epsilon)}=2 \rho c_{P}^{2} \kappa \kappa, \quad \text { where } \quad \kappa=\left(\begin{array}{c}
k_{1} \\
k_{2} \\
0
\end{array}\right), \tag{26}
\end{gather*}
$$

and,

$$
\mathrm{M}^{(\delta)}=\rho c_{P}^{2}\left(\kappa k_{3}+k_{3} \kappa\right), \quad \text { where } \quad k_{3}=\left(\begin{array}{c}
0  \tag{27}\\
0 \\
k_{3}
\end{array}\right)
$$

The vector system $\kappa, \kappa^{\perp}$, and $\boldsymbol{k}_{3}$ is closely related to the ordinary cylindrical unit vector basis, here denoted by $\hat{\kappa}, \hat{\phi}$, and $\hat{k}_{3}$. Indeed,

$$
\begin{gather*}
\kappa=\kappa \hat{\kappa}, \quad \hat{\kappa}=\left(\begin{array}{c}
\cos \phi \\
\sin \phi \\
0
\end{array}\right)  \tag{28}\\
\kappa^{\perp}=\kappa \hat{\phi}, \quad \hat{\phi}=\left(\begin{array}{c}
-\sin \phi \\
\cos \phi \\
0
\end{array}\right), \tag{29}
\end{gather*}
$$

and,

$$
k_{3}=k_{3} \hat{k}_{3}, \quad \hat{k}_{3}=\left(\begin{array}{l}
0  \tag{30}\\
0 \\
1
\end{array}\right)
$$

In cylindrical notation, the Thomsen matrices are

$$
\begin{equation*}
M^{(\gamma)}=2 \rho c_{S}^{2} \kappa^{2} \hat{\phi} \hat{\phi} \tag{31}
\end{equation*}
$$

$$
\begin{equation*}
M^{(\epsilon)}=2 \rho c_{P}^{2} \kappa^{2} \hat{\boldsymbol{\kappa}} \hat{\kappa}, \tag{32}
\end{equation*}
$$

and,

$$
\begin{equation*}
\mathrm{M}^{(\delta)}=\rho c_{P}^{2} \kappa k_{3}\left(\hat{\kappa} \hat{k}_{3}+\hat{k_{3}} \hat{\kappa}\right) \tag{33}
\end{equation*}
$$

To get the Thomsen matrices in terms of the spherical coordinate unit vectors, here denoted by $\hat{\boldsymbol{k}}, \hat{\boldsymbol{\theta}}$, and the aforementioned $\hat{\boldsymbol{\phi}}$, observe the relations,

$$
\begin{align*}
& \hat{k}=\left(\begin{array}{c}
\cos \phi \sin \theta \\
\sin \phi \sin \theta \\
\cos \theta
\end{array}\right)=\sin \theta \hat{\kappa}+\cos \theta \hat{k}_{3} \\
& \hat{\boldsymbol{\theta}}=\left(\begin{array}{c}
\cos \phi \cos \theta \\
\sin \phi \cos \theta \\
-\sin \theta
\end{array}\right)=\cos \theta \hat{\kappa}-\sin \theta \hat{k}_{3} \tag{34}
\end{align*}
$$

or,

$$
\begin{array}{r}
\hat{\boldsymbol{\kappa}}=\sin \theta \hat{\boldsymbol{k}}+\cos \theta \hat{\boldsymbol{\theta}} \\
\hat{\boldsymbol{k}}_{3}=\cos \theta \hat{\boldsymbol{k}}-\sin \theta \hat{\boldsymbol{\theta}} \tag{35}
\end{array}
$$

In terms of spherical coordinates, the Thomsen matrices are

$$
\begin{gather*}
M^{(\gamma)}=2 \rho c_{S}^{2} k^{2} \sin ^{2} \theta \hat{\phi} \hat{\phi}  \tag{36}\\
M^{(\epsilon)}=2 \rho c_{P}^{2} k^{2} \sin ^{2} \theta\left(\hat{k} \hat{k} \sin ^{2} \theta+(\hat{k} \hat{\theta}+\hat{\theta} \hat{k}) \sin \theta \cos \theta+\hat{\theta} \hat{\theta} \cos ^{2} \theta\right) \tag{37}
\end{gather*}
$$

and,

$$
\begin{equation*}
\mathrm{M}^{(\delta)}=\rho c_{P}^{2} k^{2} \sin \theta \cos \theta(\hat{k} \hat{k} \sin 2 \theta+(\hat{\boldsymbol{k}} \hat{\boldsymbol{\theta}}+\hat{\boldsymbol{\theta}} \hat{\boldsymbol{k}}) \cos 2 \theta-\hat{\theta} \hat{\theta} \sin 2 \theta) \tag{38}
\end{equation*}
$$

For the application given in the next section, it is convenient to also have explicit representations for the isotropic matrix, $M^{(0)}$ :

$$
\begin{align*}
M^{(0)} & =\rho\left(c_{S}^{2} k^{2}-\omega^{2}\right) \mathbb{I}+\rho\left(c_{P}^{2}-c_{S}^{2}\right) k k  \tag{39}\\
& =\lambda_{S} \mathbb{I}+\left(\lambda_{P}-\lambda_{S}\right) \hat{k} \hat{k} \\
& =\lambda_{S}(\mathbb{I}-\hat{k} \hat{k})+\lambda_{P} \hat{k} \hat{k} \tag{40}
\end{align*}
$$

Here II denotes the identity matrix.
Remark: The representation of $M^{(0)}$ in terms of the Lamé parameters is

$$
\begin{equation*}
\mathrm{M}^{(0)}=\left(\mu k^{2}-\rho \omega^{2}\right) \mathbb{I}+(\lambda+\mu) k k \tag{41}
\end{equation*}
$$

Before proceeding, review the theory of the "spectral representation" for a real symmetric matrix A (say 3 by 3 , for simplicity). The eigenvectors of such a matrix
can be taken as an orthonormal basis of $\mathcal{R}^{3}$, say $e_{1}, e_{2}$, and $e_{3}$. Denoting the corresponding eigenvalues by $\lambda_{1}, \lambda_{2}$, and $\lambda_{3}$, the spectral representation of A is

$$
\begin{equation*}
\mathrm{A}=\lambda_{1} e_{1} e_{1}+\lambda_{2} e_{2} e_{2}+\lambda_{3} e_{3} e_{3} \tag{42}
\end{equation*}
$$

Then, for any function $f(z)$ defined on the $\lambda_{j}$, we have

$$
\begin{equation*}
f(\mathrm{~A})=f\left(\lambda_{1}\right) e_{1} e_{1}+f\left(\lambda_{2}\right) e_{2} e_{2}+f\left(\lambda_{3}\right) e_{3} e_{3} \tag{43}
\end{equation*}
$$

The form of the matrix $M^{(0)}$ in equation (40) is its spectral representation with the eigenvalues being $\lambda_{S}$ (double eigenvalue) and $\lambda_{P}$. The rank two tensor $\mathbb{I I}-\hat{k} \hat{k}$ may be replaced by a sum of dyadics based on any pair of orthonormal vectors that are orthogonal to $\hat{k}$, but we have no immediate need to introduce a specific pair, so we allow this mild generalization of the spectral representation and write the matrix functions of $M^{(0)}$ as

$$
\begin{equation*}
f\left(M^{(0)}\right)=f\left(\lambda_{S}\right)(\mathbb{I}-\hat{k} \hat{k})+f\left(\lambda_{P}\right) \hat{k} \hat{k} \tag{44}
\end{equation*}
$$

A principal application of the spectral form of $M^{(0)}$ is in finding the isotropic Green's tensor $G^{(0)}$ satisfying the differential system

$$
\begin{equation*}
\mathcal{L}^{(0)} \mathbf{g}^{(0)}=-\mathbb{I} \delta(r) \delta(t) \tag{45}
\end{equation*}
$$

In transform domain, this is

$$
\begin{equation*}
M^{(0)} \cdot G^{(0)}=I I \tag{46}
\end{equation*}
$$

Thus, $G^{(0)}$ is just the matrix inverse of $M^{(0)}$ and we can apply the spectral theory with $f(z)=1 / z$ to obtain

$$
\begin{equation*}
G^{(0)}=\frac{1}{\lambda_{S}}(\mathbb{I I}-\hat{k} \hat{k})+\frac{1}{\lambda_{P}} \hat{k} \hat{k} \tag{47}
\end{equation*}
$$

## APPLICATION: PHASE SPEEDS IN THE WEAK TI LIMIT

As an application of the Thomsen matrices, seek the plane wave solutions in the weak TI limit. These are the solutions $u$ of the homogeneous equation,

$$
\begin{equation*}
\mathcal{L} u=\mathbf{o} \tag{48}
\end{equation*}
$$

that have the form

$$
\begin{equation*}
u=v e^{i(\omega t-k \cdot x)} \tag{49}
\end{equation*}
$$

where $v$ is a constant amplitude vector. Insert this plane wave ansätz into the TI wave equation to obtain

$$
\begin{equation*}
M \cdot v=0 . \tag{50}
\end{equation*}
$$

Here, to first order in the Thomsen parameters, $M$ is given by

$$
\begin{equation*}
M \approx M^{(0)}+\gamma M^{(\gamma)}+\delta M^{(\delta)}+\epsilon M^{(\epsilon)} \tag{51}
\end{equation*}
$$

Again, $M^{(0)}$ is the transform of the isotropic elastic wave operator and the remaining matrices on the right are the Thomsen matrices defined earlier. From equation (39), write

$$
\begin{equation*}
\mathrm{M}^{(0)}=\rho\left(c_{P}^{2}-c_{S}^{2}\right) k k+\rho c_{S}^{2} k^{2} \mathbb{I}-\rho \omega^{2} \mathbb{I} . \tag{52}
\end{equation*}
$$

Thus, the plane wave problem is related to the eigenvalue-eigenvector problem by equating $\rho \omega^{2}$ to the eigenvalue $\lambda$ in the zero order matrix. Indeed, we consider both the unperturbed eigen-problem,

$$
\begin{equation*}
N^{(0)} v^{(0)}=\lambda^{(0)} v^{(0)}, \quad N^{(0)}=\rho\left(c_{P}^{2}-c_{S}^{2}\right) k k+\rho c_{S}^{2} k^{2} I I \tag{53}
\end{equation*}
$$

and the perturbed problem with perturbation specified by the Thomsen matrices,

$$
\begin{equation*}
N v=\lambda v, \quad N=N^{(0)}+\gamma \mathrm{M}^{(\gamma)}+\delta \mathrm{M}^{(\delta)}+\epsilon \mathrm{M}^{(\epsilon)} \tag{54}
\end{equation*}
$$

Then, we obtain the plane wave solutions by using the eigenvectors as the amplitudes of the plane waves with dispersion relations obtained by setting the eigenvalues equal to $\rho \omega^{2}$. Notice that $N^{(0)}$ and $N$ are respectively the same as $M^{(0)}$ and $M$ except for the omitted $\rho \omega^{2} I I$ term.

The results cited below for the perturbed eigen-problem rely on the theory expounded in the classic Courant-Hilbert text (Courant \& Hilbert, 1953). Computing the perturbation corrections in the present application has two complications over the simplest case:

1. The eigenvalues in the unperturbed (isotropic) case are degenerate.
2. There are three small parameters instead of just one.

The second problem isn't serious: the corrections corresponding to each Thomsen parameter can be computed separately and the total correction is just the sum of the individual ones. We overcome the associated notational problem by first stating the results for a generic small parameter $\beta$ and then applying the generic result for each of the perturbations $\gamma, \delta$, and $\epsilon$. In particular, denote the expansions to first order of the generic eigenvalue and eigenvector by

$$
\begin{equation*}
\lambda=\lambda^{(0)}+\beta \lambda^{(\beta)} \tag{55}
\end{equation*}
$$

and

$$
\begin{equation*}
v=v^{(0)}+\beta v^{(\beta)} \tag{56}
\end{equation*}
$$

For the unperturbed eigen-problem in equation (53), the eigenvectors and associated eigenvalues are:

$$
v^{(0)}=\left\{\begin{array}{l}
\hat{\boldsymbol{k}}  \tag{57}\\
c_{11} \hat{\boldsymbol{\theta}}+c_{12} \hat{\phi} \equiv \hat{\boldsymbol{l}} \\
c_{21} \hat{\boldsymbol{\theta}}+c_{22} \hat{\phi} \equiv \hat{\boldsymbol{m}}
\end{array}, \quad \lambda^{(0)}=\left\{\begin{array}{l}
\rho c_{P}^{2} k^{2} \\
\rho c_{S}^{2} k^{2} \\
\rho c_{S}^{2} k^{2}
\end{array}\right.\right.
$$

As expected, the second eigenvalue is repeated. Putting the eigenvalues equal to $\rho \omega^{2}$, obtain the plane wave dispersion relations $\omega^{2}=c_{P}^{2} k^{2}$ and $\omega^{2}=c_{S}^{2} k^{2}$ with the associated phase speeds $\omega / k$ being $c_{P}$ and $c_{S}$, also as expected.

As far as the unperturbed problem is concerned, the matrix $C=\left(c_{i j}\right)$ can be any orthogonal matrix. It is perhaps surprising that a consistent perturbation theory for the degenerate case puts the following constraint on C :

Theorem 1 C must be chosen such that

$$
\begin{equation*}
\hat{l} \cdot \mathrm{M}^{(\beta)} \cdot \hat{m}=\mathrm{D} \tag{58}
\end{equation*}
$$

where D is a diagonal matrix,

$$
\mathrm{D}=\left(\begin{array}{cc}
d_{11} & 0  \tag{59}\\
0 & d_{22}
\end{array}\right) .
$$

Observe that equations ( $36-38$ ) imply that this condition is satisfied simultaneously for all three Thomsen matrices with the choices $\hat{\boldsymbol{l}}=\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{m}}=\hat{\boldsymbol{\phi}}$. Thus, we can dispense with the matrix C and simplify the result for the unperturbed eigen-problem to:

$$
v^{(0)}=\left\{\begin{array}{l}
\hat{k}  \tag{60}\\
\hat{\theta} \\
\hat{\phi}
\end{array} \quad \quad \lambda^{(0)}=\left\{\begin{array}{l}
\lambda_{k}^{(0)} \\
\lambda_{\theta}^{(0)} \\
\lambda_{\phi}^{(0)}
\end{array}=\left\{\begin{array}{l}
\rho c_{P}^{2} k^{2} \\
\rho c_{S}^{2} k^{2} \\
\rho c_{S}^{2} k^{2}
\end{array}\right.\right.\right.
$$

The theory dictates expanding the first order perturbation in the eigenvectors in terms of the zeroth order eigenvectors (chosen consistent with Theorem 1). In generic notation, specialized to the case of perturbations from our unperturbed results in equation (60):

Theorem 2 The perturbations in the eigenvalues are

$$
\begin{align*}
\lambda_{k}^{(\beta)} & =\hat{\boldsymbol{k}} \cdot M^{(\beta)} \cdot \hat{\boldsymbol{k}} \\
\lambda_{\theta}^{(\beta)} & =\hat{\boldsymbol{\theta}} \cdot \mathbf{M}^{(\beta)} \cdot \hat{\boldsymbol{\theta}}  \tag{61}\\
\lambda_{\phi}^{(\beta)} & =\hat{\boldsymbol{\phi}} \cdot M^{(\beta)} \cdot \hat{\boldsymbol{\phi}}
\end{align*}
$$

Moreover, the perturbations in the eigenvectors have the anti-symmetric form

$$
\begin{align*}
& \hat{k}^{(\beta)}=b_{k \boldsymbol{\theta}}^{(\beta)} \hat{\boldsymbol{\theta}}+b_{k \phi}^{(\beta)} \hat{\boldsymbol{\phi}} \\
& \hat{\boldsymbol{\theta}}^{(\beta)}=-b_{k \theta}^{(\beta)} \hat{\boldsymbol{k}}+b_{\theta \phi}^{(\beta)} \hat{\boldsymbol{\phi}}  \tag{62}\\
& \hat{\boldsymbol{\phi}}^{(\beta)}=-b_{k \phi}^{(\beta)} \hat{k}-b_{\theta \phi}^{(\beta)} \hat{\boldsymbol{\theta}}
\end{align*}
$$

with

$$
\begin{align*}
b_{k \theta}^{(\beta)} & =\frac{\hat{\boldsymbol{k}} \cdot M^{(\beta)} \cdot \hat{\boldsymbol{\theta}}}{\lambda_{k}-\lambda_{\theta}} \\
b_{k \phi}^{(\beta)} & =\frac{\hat{\boldsymbol{k}} \cdot \mathrm{M}^{(\beta)} \cdot \hat{\boldsymbol{\phi}}}{\lambda_{k}-\lambda_{\phi}}  \tag{63}\\
b_{\theta \phi}^{(\beta)} & =\frac{\left(\hat{\boldsymbol{k}} \cdot \mathrm{M}^{(\beta)} \cdot \hat{\boldsymbol{\theta}}\right)\left(\hat{\boldsymbol{\phi}} \cdot M^{(\beta)} \cdot \hat{\boldsymbol{\theta}}\right)}{\left(\lambda_{\theta}^{(\beta)}-\lambda_{\phi}^{(\beta)}\right)\left(\lambda_{\theta}-\lambda_{k}\right)} .
\end{align*}
$$

Apply this generic form of the result to each of the Thomsen perturbations given in equations (36-38) to obtain for the $\gamma$ corrections:

$$
\begin{equation*}
\lambda_{k}^{(\gamma)}=0, \quad \lambda_{\theta}^{(\gamma)}=0, \quad \lambda_{\phi}^{(\gamma)}=2 \rho c_{S}^{2} k^{2} \sin ^{2} \theta \tag{64}
\end{equation*}
$$

and

$$
\begin{equation*}
b_{k \theta}^{(\gamma)}=0, \quad b_{k \phi}^{(\gamma)}=0, \quad b_{\theta \phi}^{(\gamma)}=0, \tag{65}
\end{equation*}
$$

for the $\delta$ corrections:

$$
\begin{equation*}
\lambda_{k}^{(\delta)}=2 \rho c_{P}^{2} k^{2} \sin ^{2} \theta \cos ^{2} \theta, \quad \lambda_{\theta}^{(\delta)}=-2 \rho c_{P}^{2} k^{2} \sin ^{2} \theta \cos ^{2} \theta, \quad \lambda_{\phi}^{(\delta)}=0 \tag{66}
\end{equation*}
$$

and

$$
\begin{equation*}
b_{k \theta}^{(\delta)}=\frac{1}{f} \sin \theta \cos \theta\left(1-2 \sin ^{2} \theta\right), \quad b_{k \phi}^{(\delta)}=0, \quad b_{\theta \phi}^{(\delta)}=0, \tag{67}
\end{equation*}
$$

and for the $\epsilon$ corrections:

$$
\begin{equation*}
\lambda_{k}^{(\epsilon)}=2 \rho c_{P}^{2} k^{2} \sin ^{4} \theta, \quad \lambda_{\theta}^{(\epsilon)}=2 \rho c_{P}^{2} k^{2} \sin ^{2} \theta \cos ^{2} \theta, \quad \lambda_{\phi}^{(\epsilon)}=0 \tag{68}
\end{equation*}
$$

and

$$
\begin{equation*}
b_{k \theta}^{(\epsilon)}=\frac{2}{f} \sin ^{3} \theta \cos \theta, \quad b_{k \phi}^{(\epsilon)}=0, \quad b_{\theta \phi}^{(\epsilon)}=0 \tag{69}
\end{equation*}
$$

In these equations, $f$ is the quantity defined in equation (2).
Adding these results gives the following first order solution to the eigen-problem:

$$
\begin{align*}
& \lambda_{k} \approx \rho c_{P}^{2} k^{2}\left(1+2 \sin ^{2} \theta\left[\delta+(\epsilon-\delta) \sin ^{2} \theta\right]\right)  \tag{70}\\
& v_{k} \approx \hat{k}+q \hat{\theta}  \tag{71}\\
& \lambda_{\theta} \approx \rho k^{2}\left(c_{S}^{2}+2 c_{P}^{2} \sin ^{2} \theta \cos ^{2} \theta(\epsilon-\delta)\right)  \tag{72}\\
& v_{\theta} \approx \hat{\theta}-q \hat{k}  \tag{73}\\
& \lambda_{\phi} \approx \rho c_{S}^{2} k^{2}\left(1+2 \gamma \sin ^{2} \theta\right)  \tag{74}\\
& v_{\phi} \approx \hat{\phi} \tag{75}
\end{align*}
$$

Here, the shorthand notation

$$
\begin{equation*}
q=\frac{\sin \theta \cos \theta}{f}\left[\delta+2(\epsilon-\delta) \sin ^{2} \theta\right] \tag{76}
\end{equation*}
$$

has been introduced.
Equating the eigenvalues to $\rho \omega^{2}$ yields the phase velocities:

$$
\begin{align*}
& V_{k}=V_{Q P} \approx c_{P}\left\{1+\left[\delta+(\epsilon-\delta) \sin ^{2} \theta\right] \sin ^{2} \theta\right\}  \tag{77}\\
& V_{\theta}=V_{Q S} \approx c_{S}\left\{1+\frac{c_{P}^{2}}{c_{S}^{2}}(\epsilon-\delta) \sin ^{2} \theta \cos ^{2} \theta\right\}  \tag{78}\\
& V_{\phi}=V_{S P} \approx c_{S}\left\{1+\gamma \sin ^{2} \theta\right\} \tag{79}
\end{align*}
$$

Remark: These results could also be obtained by power series expansion of the eigenvalues of the full TI wave equation (Thomsen, 1986).

## APPLICATION: THE GREEN'S TENSOR FOR $\delta=\epsilon=0$

As another example of using the Thomsen matrices, derive the Green's tensor in the special case when $\delta=\epsilon=0$. Note that this case is distinctly easier than the general case; the calculations here should be regarded as only a "warm-up" to obtaining fuller results. There is a substantial literature on the TI Green's function, however, most of it assumes the far field (or high frequency or ray) approximation. Some excellent papers on this topic are (Ben-Menahem \& Sena, 1990; Ben-Menahem et al.. 1991; Buchwald, 1959; Kazi-Aoual et al., 1988; Tverdokhlebov \& Rose, 1988). Here. although only a special case of the Thomsen parameters is treated, the near field or low frequency terms are included.

Begin with the defining equation for the Green's tensor g,

$$
\begin{equation*}
\mathcal{L} \mathrm{g}=-\mathbb{I} \delta(r) \delta(t) \tag{80}
\end{equation*}
$$

where II denotes the three by three identity matrix. After the Fourier transform defined in equation (21), this becomes

$$
\begin{equation*}
M \cdot G=I I \tag{81}
\end{equation*}
$$

Here the matrix $M$ is given by

$$
\begin{equation*}
M=M^{(0)}+\gamma M^{(\gamma)}+\tilde{\delta} M^{(\delta)}+\epsilon M^{(\epsilon)} \tag{82}
\end{equation*}
$$

where $M^{(0)}$ is the transform of the isotropic elastic wave operator and the remaining matrices on the right are the Thomsen matrices defined earlier.

Observe that

$$
\begin{equation*}
M^{(0)} \cdot G^{(0)}=\mathbb{I} \tag{83}
\end{equation*}
$$

where $G^{(0)}$ represents the isotropic Green's tensor in transform domain-given explicitly in equation (47).

Equation (83) implies that $M^{(0)}$ and $G^{(0)}$ are inverses, thus equation (81) can be written as

$$
\begin{equation*}
\left(\mathbb{I I}+\mathrm{G}^{(0)} \cdot\left(\gamma \mathrm{M}^{(\gamma)}+\tilde{\delta} \mathrm{M}^{(\delta)}+\epsilon \mathrm{M}^{(\epsilon)}\right)\right) \cdot \mathrm{G}=\mathrm{G}^{(0)} \tag{84}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\mathrm{G}=\mathrm{C}^{-1} \cdot \mathrm{G}^{(0)} \tag{85}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathrm{C}=\mathbb{I}+\mathrm{G}^{(0)} \cdot\left(\gamma \mathrm{M}^{(\gamma)}+\tilde{\delta} \mathrm{M}^{(\delta)}+\epsilon \mathrm{M}^{(\epsilon)}\right) \tag{86}
\end{equation*}
$$

At this point, make the simplifying assumption that $\delta$ and $\epsilon$ vanish. From equations (47) and (25), and the relation $\boldsymbol{k} \cdot \kappa^{\perp}=0$, conclude

$$
\begin{equation*}
\mathrm{G}^{(0)} \cdot \mathrm{M}^{(\gamma)}=\frac{2 \rho c_{S}^{2}}{\lambda_{S}} \kappa^{\perp} \kappa^{\perp} \tag{87}
\end{equation*}
$$

Thus, in our special case,

$$
\mathrm{C}=\mathbb{I}+\frac{2 \gamma \rho c_{S}^{2}}{\lambda_{S}} \kappa^{\perp} \kappa^{\perp}=\left(\begin{array}{ccc}
1+\nu k_{2}^{2} & -\nu k_{1} k_{2} & 0  \tag{88}\\
-\nu k_{1} k_{2} & 1+\nu k_{2}^{2} & 0 \\
0 & 0 & 1
\end{array}\right)
$$

where

$$
\begin{equation*}
\nu=\frac{2 \gamma \rho c_{S}^{2}}{\lambda_{S}} \tag{89}
\end{equation*}
$$

Using the block structure of $C$, obtain

$$
C^{-1}=\left(\begin{array}{ccc}
\frac{1+\nu k_{1}^{2}}{\Delta} & \frac{\nu k_{1} k_{2}}{\Delta} & 0  \tag{90}\\
\frac{\nu k_{1} k_{2}}{\Delta} & \frac{1+\nu k_{2}^{2}}{\Delta} & 0 \\
0 & 0 & 1
\end{array}\right)
$$

where

$$
\begin{equation*}
\Delta=\left(1+\nu k_{2}^{2}\right)\left(1+\nu k_{1}^{2}\right)-\nu^{2} k_{1}^{2} k_{2}^{2}=1+\nu \kappa^{2} \tag{91}
\end{equation*}
$$

On using this result in equation (85), after some calculations, find that the formula for $G$ can be written

$$
\begin{equation*}
G=G^{(0)}+G^{(1)} \tag{92}
\end{equation*}
$$

where the exact "correction" for anisotropy in the special case $\delta=\epsilon=0$ is given by

$$
\begin{equation*}
G^{(1)}=-\frac{\nu}{\lambda_{S} \Delta} \kappa^{\perp} \kappa^{\perp} \tag{93}
\end{equation*}
$$

The factor $\frac{\nu}{\lambda_{s} \Delta}$ expands to

$$
\begin{equation*}
\frac{\nu}{\lambda_{S} \Delta}=\frac{2 \gamma c_{S}^{2}}{\left(\omega^{2}-\omega_{0}^{2}\right)\left(\omega^{2}-\omega_{1}^{2}\right)} \tag{94}
\end{equation*}
$$

where

$$
\begin{equation*}
\omega_{0}^{2}=c_{S}^{2} k^{2}, \text { and } \omega_{1}^{2}=c_{S}^{2}\left(k_{3}^{2}+(1+2 \gamma) \kappa^{2}\right) \tag{95}
\end{equation*}
$$

It is convenient to express these quantities as

$$
\begin{equation*}
\omega_{n}^{2}=c_{S}^{2}\left(k_{3}^{2}+a_{n}^{2} \kappa^{2}\right) \tag{96}
\end{equation*}
$$

with

$$
\begin{equation*}
a_{0}^{2}=1, \text { and } a_{1}^{2}=1+2 \gamma \tag{97}
\end{equation*}
$$

The above results lead to the Fourier inversion of $G^{(1)}$ as the integral

$$
\begin{equation*}
\mathbf{g}^{(1)}=\frac{-2 \gamma c_{S}^{2}}{(2 \pi)^{4} \rho} \int d k \kappa^{\perp} \kappa^{\perp} \int d \omega \frac{e^{-i(\omega t-k \cdot r)}}{\left(\omega^{2}-\omega_{0}^{2}\right)\left(\omega^{2}-\omega_{1}^{2}\right)} \tag{98}
\end{equation*}
$$

or by use of partial fractions as

$$
\begin{equation*}
\mathbf{g}^{(1)}=\frac{1}{(2 \pi)^{4} \rho} \int d k \hat{\kappa}^{\perp} \hat{\kappa}^{\perp} \int d \omega e^{-i(\omega t-k \cdot r)}\left(\frac{1}{\omega^{2}-\omega_{0}^{2}}-\frac{1}{\omega^{2}-\omega_{1}^{2}}\right) \tag{99}
\end{equation*}
$$

where $\hat{\kappa}^{\perp}$ denotes the unit vector $\kappa^{\perp} / \kappa$. The $\omega$ integrals are done by residue integration, yielding

$$
\begin{equation*}
\int d \omega \frac{e^{-i \omega t}}{\omega^{2}-\omega_{n}^{2}}=-2 \pi H(t) \frac{\sin \omega_{n} t}{\omega_{n}}, \quad n=0,1 \tag{100}
\end{equation*}
$$

The integral over $k_{3}$ is done using a cosine transform result as

$$
\begin{equation*}
\int d k_{3} e^{i k_{3} z} \frac{\sin \omega_{n} t}{\omega_{n}}=2 \int_{0}^{\infty} d k_{3} \cos k_{3}|z| \frac{\sin \omega_{n} t}{\omega_{n}}=H\left(c_{S} t-z\right) \frac{\pi}{c_{S}} J_{0}\left(a_{n} \kappa \sqrt{c_{S} t-z}\right) \tag{101}
\end{equation*}
$$

To accomplish the remaining integrations over $k_{1}, k_{2}$, introduce the plane polar coordinates $\kappa$ and $\phi$. Since $\hat{\kappa}^{\perp}$ is just $\hat{\phi}$ (see equation 29), the $\phi$ integration can be written

$$
\begin{align*}
& \int_{-\pi}^{\pi} d \phi \\
& \quad\left(\begin{array}{ccc}
\sin ^{2} \phi & -\sin \phi \cos \phi & 0 \\
-\sin \phi \cos \phi & \cos ^{2} \phi & 0 \\
0 & 0 & 0
\end{array}\right) e^{i \kappa R \cos \phi}  \tag{102}\\
&=2 \int_{0}^{\pi} d \phi\left(\begin{array}{ccc}
1-\cos ^{2} \phi & 0 & 0 \\
0 & \cos ^{2} \phi & 0 \\
0 & 0 & 0
\end{array}\right) e^{i \kappa R \cos \phi},
\end{align*}
$$

where the reduction on the right follows from elementary properties of the trigonometric functions. Introducing the unit vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ along the transverse axes, and recalling the integral representation of the Bessel function, this integral reduces to

$$
\begin{equation*}
2 \pi J_{0}(\kappa R) \hat{x} \hat{x}+2(\hat{y} \hat{y}-\hat{x} \hat{x}) \int_{0}^{\pi} d \phi \cos ^{2} \phi e^{i \kappa R \cos \phi} \tag{103}
\end{equation*}
$$

The remaining angular integral can also be reduced to an explicit Bessel function as follows:

$$
\begin{equation*}
\int_{0}^{\pi} d \phi \cos ^{2} \phi e^{i \kappa R \cos \phi}=-\frac{1}{\kappa^{2}} \frac{d^{2}}{d R^{2}} \int_{0}^{\pi} d \phi e^{i \kappa R \cos \phi}=-\frac{1}{\kappa^{2}} \frac{d^{2}}{d R^{2}} J_{0}(\kappa R) \tag{104}
\end{equation*}
$$

Using the defining differential equation for $J_{0}$ and the relation $J_{0}^{\prime}(z)=J_{1}(z)$, derive

$$
\begin{equation*}
\frac{1}{\kappa^{2}} \frac{d^{2}}{d R^{2}} J_{0}(\kappa R)=\frac{1}{\kappa R} J_{1}(\kappa R)-J_{0}(\kappa R) \tag{105}
\end{equation*}
$$

Assembling our results, we have

$$
\begin{align*}
& \mathbf{g}^{(1)}=\frac{H(t) H\left(c_{S} t-z\right)}{4 \pi \rho c_{S}} \int_{0}^{\infty} d \kappa \kappa\left(J_{0}\left(a \kappa \sqrt{c_{S}^{2} t^{2}-z^{2}}\right)-J_{0}\left(\kappa \sqrt{c_{S}^{2} t^{2}-z^{2}}\right)\right) \\
&\left(J_{0}(\kappa R) \hat{y} \hat{y}+\frac{1}{\kappa R} J_{1}(\kappa R)(\hat{x} \hat{x}-\hat{y} \hat{y})\right) \tag{106}
\end{align*}
$$

where $a=a_{1}=\sqrt{1+2 \gamma}$.
The final integration over $\kappa$ is accomplished with the aid of the identities

$$
\begin{equation*}
\int_{0}^{\infty} d \kappa \kappa J_{0}(\kappa R) J_{0}(\kappa S)=\frac{\delta(R-S)}{\sqrt{R S}}=\frac{\delta(R-S)}{R} \tag{107}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{0}^{\infty} d \kappa J_{0}(\kappa R) J_{1}(\kappa S)=\frac{H(R-S)}{R} \tag{108}
\end{equation*}
$$

These lead to the closed form result

$$
\begin{align*}
\mathbf{g}^{(1)}=\frac{H(t)}{4 \pi \rho c_{S}} & \left(\left[\delta\left(R-a \sqrt{c_{S}^{2} t^{2}-z^{2}}\right)-\delta\left(R-\sqrt{c_{S}^{2} t^{2}-z^{2}}\right)\right] \frac{\hat{y} \hat{y}}{R}\right. \\
+ & {\left.\left[H\left(R-a \sqrt{c_{S}^{2} t^{2}-z^{2}}\right)-H\left(R-\sqrt{c_{S}^{2} t^{2}-z^{2}}\right)\right] \frac{\dot{x} \hat{x}-\hat{y} \hat{y}}{R^{2}}\right) } \tag{109}
\end{align*}
$$

which, using standard properties of the Dirac and Heaviside functions, may be cast as

$$
\begin{align*}
& \mathbf{g}^{(1)}=\frac{H(t)}{4 \pi \rho}\left(\left[\frac{\delta\left(t-\frac{1}{c_{S}} \sqrt{\frac{R^{2}}{a^{2}}+z^{2}}\right)}{c_{S}^{2} a^{2} \sqrt{\frac{R^{2}}{a^{2}}+z^{2}}}-\frac{\delta\left(t-\frac{r}{c_{S}}\right)}{c_{S}^{2} r}\right] \frac{\hat{y} \hat{y}}{R}\right. \\
&\left.+\left[H\left(t-\frac{1}{c_{S}} \sqrt{\frac{R^{2}}{a^{2}}+z^{2}}\right)-H\left(t-\frac{r}{c_{S}}\right)\right] \frac{\hat{\boldsymbol{x}} \hat{x}-\hat{y} \hat{y}}{c_{S} R^{2}}\right) \tag{110}
\end{align*}
$$

To complete the determination of $g=g^{(0)}+g^{(1)}$, note that the Fourier inversion of $G^{(0)}$ is the isotropic Green's function which may be written as

$$
\begin{align*}
\mathbf{g}^{(0)}=\frac{H(t)}{4 \pi r \rho} & {\left[\left(\frac{1}{c_{P}^{2}} \delta\left(t-t_{P}\right)+\frac{2 t}{r^{2}} \mathcal{X}_{\left[t_{P}, t_{S}\right]}(t)\right) \hat{r} \hat{r}\right.} \\
& \left.+\left(\frac{1}{c_{S}^{2}} \delta\left(t-t_{S}\right)-\frac{t}{r^{2}} \mathcal{X}_{\left[t_{P}, t_{s}\right]}(t)\right)(\mathbb{I}-\hat{r} \hat{r})\right] \tag{111}
\end{align*}
$$

Here the indicator function $\mathcal{X}_{[a, b]}$ is defined as

$$
\mathcal{X}_{[a, b]}(x)= \begin{cases}1 & a<x<b  \tag{112}\\ 0 & \text { otherwise }\end{cases}
$$

and the $P$-wave and $S$-wave arrival times at $r$ are defined as

$$
\begin{equation*}
t_{P}=\frac{r}{c_{P}}, \quad \text { and } \quad t_{S}=\frac{r}{c_{S}} \tag{113}
\end{equation*}
$$

## Discussion of results

It should be emphasized again that the purpose of these Green's tensor calculations is only to suggest of the utility of using Thomsen notation as a starting point in TI wave equation studies. The full result would entail inverting the entire $C$ matrix which would "scramble" the terms considered above. However, the result is sufficient to indicate that, in contrast to the phase velocity application, a perturbation expansion of the Green's tensor in the Thomsen parameters is only valid under restricted circumstances-for example, the difference of the Dirac functions can only be replaced by a derivative of the Dirac function for small $\gamma$ under the restriction of small $R$ (or in frequency domain under the restriction of low frequency).

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# Using multi-resolution analysis to study the complexity of inverse calculations 

H. Lydia Deng

# Using multi-resolution analysis to study the complexity of inverse calculations 

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#### Abstract

Optimization is a tool for many inverse calculations. However, in practice it is found that functions we wish to optimize are often highly non-convex (multimodal). An example of optimizations with such difficulties is in seismic waveform inversion. The waveform misfit functions are generally multi-modal, partly because of the oscillatory nature of seismic waves.

In this paper, I show how multiresolution analysis can be used to deal with the multi-modal nature of objective functions arising in seismic inversion problems. Using residual statics estimation as an example, this paper shows that the waveform misfit function is multi-modal even for the simplest case, and, indeed, MRA simplifies the waveform misfit function. By studying the complexity of objective functions on MRA decomposed data, increased understanding of the complexity of objective functions, such as those arising in inverse problems, can be gained.


## REALITY IN OPTIMIZATIONS

Optimization is a tool for many inverse calculations. However, in practice it is found that functions we wish to optimize, objective functions, are often highly nonconvex (multi-modal). Many optimization algorithms have been developed to handle different inverse problems; each of them usually works well for some situations but fails for the others. Therefore, it is important to understand what makes some inverse problems difficult, while others are not.

The multi-modalities in optimizations are especially serious for seismic waveform misfit functions. This is partly caused by the oscillatory nature of seismic waves. When there are many local minima, the gradient-based searching methods have little chance of finding the correct global minimum.

Global search methods are used by many researchers when dealing with multimodal misfit functions. Rothman $(1985 ; 1986)$ solved a residual statics problem by simulated annealing (SA). Scales et al. (1992), Smith et al. (1992), Sen and Stoffa (1991a; 1991b; 1992) and Gouveia (1993) studied SA and genetic algorithms (GAs)
on a variety of multi-modal optimization problems, received satisfying results for problems they studied.

Although it can be proved that SA and GAs converge to global extrema asymptotically, it is not guaranteed that they would find the global extrema in a finite amount of computational time. Gouveia (1994) studied some hybrid methods of the traditional gradient-based searching and distributed-parallel GAs. They show that these hybrid methods are more efficient for the residual statics estimation than a distributed GA.

Many researchers found other alternative ways to overcome problems of local minima, while keeping the computation effort relatively low. Here I describe two strategies.

The first approach is to choose initial models that are close to the global extrema by integrating into the solution formalism a priori knowledge other than data itself. Although applying a priori information to inversions does not reduce the complexity of the objective function in global sense, it helps to confine the searches to a smaller range, which increases the chance of convergence to global minima (Tarantola, 1987). In a seismic waveform inversion, Chapman (1985; 1988) suggested using travel-time information to infer a smooth velocity model, which is used as the initial guess to the waveform inversion. Scales and Tarantola (1994) conducted statistical analysis on geologic and well-log information in order to obtain a priori information for waveform inversion.

The second approach is to simplify the objective function of optimizations so as to reduce the total number of local minima. Shaw and Orcutt (1985) suggested using the envelope of seismic data as the fitting target for simplifying the objective function of waveform inversions. Unfortunately, they found that the "envelope" in seismic data was sensitive to noise.

Despite the failure of "envelope inversion", there has been significant interest in simplifying the waveform misfit function while not having to extract indirect information from the data. In waveform inversions, the success of the differential semblance optimization (DSO) method (Symes and Carazzone, 1991; Symes, 1993) demonstrates that the complexity of objective functions in inversions is significantly affected by the parameterization of the target models. Chevent (1994) and Symes (1994) proved theoretically that DSO produces almost convex objective functions in some waveform inversions.

Multi-scale ideas are also used to deal with the multi-modality of objective functions in inverse calculations. Seismic waveform data can be decomposed into several data sets by low-pass filters, each of which contains progressively higher frequency data. Optimizations are applied to these data sets iteratively in order to increase the chance of finding the global minima (Saleck et al., 1993; Chen, 1994). This paper describes a similar approach by means of the multiresolution analysis (MRA). As the first step of studying the complexity of inverse problems, this paper studies a simple residual statics problem.

## MULTIRESOLUTION ANALYSIS

## What is a Multiresolution Analysis?

Multiresolution analysis (MRA) was formulated based on the study of orthonormal, compactly supported wavelet bases. Wavelets theory and its applications are rapidly developing fields in applied mathematics and signal analysis. Wavelet basis representation of certain signals show advantages over the traditional Fourier basis representation both theoretically and practically. The MRA concept was initiated by Meyer (1992) and Mallat (1989), which provides a natural framework for the understanding of wavelet bases. Here, I give a brief description of orthonormal, compactly supported wavelet bases; detailed information can be found, for example, in Daubechies (1992) and Jawerth and Sweldens (1994).

An orthonormal, compactly supported wavelet basis of $L^{2}(\mathbf{R})$ is formed by the dilation and translation of a single function $\psi(x)$, called the wavelet function:

$$
\begin{equation*}
\psi_{j, k}(x)=2^{-j / 2} \psi\left(2^{-j} x-k\right) ; \quad j, k \in \mathbf{Z} \tag{1}
\end{equation*}
$$

where Z is the set of integers. In equation (1), the function $\psi$ has $M$ vanishing moments up to order $M-1$, and it satisfies the following "two-scale" difference equation,

$$
\begin{equation*}
\psi(x)=\sqrt{2} \sum_{k=0}^{L-1} g_{k} \psi(2 x-k) \tag{2}
\end{equation*}
$$

The wavelet function $\psi(x)$ has a companion, the scaling function $\phi(x)$, which also forms a set of orthonormal bases of $\mathbf{L}^{2}(\mathbf{R})$,

$$
\begin{equation*}
\phi_{j, k}(x)=2^{-j / 2} \phi\left(2^{-j} x-k\right) ; \quad j, k \in \mathbf{Z} \tag{3}
\end{equation*}
$$

The scaling function $\phi(x)$ satisfies,

$$
\int_{-\infty}^{+\infty} \phi(x) d x=1
$$

and the "two-scale difference" equation,

$$
\begin{equation*}
\phi(x)=\sqrt{2} \sum_{k=0}^{L-1} h_{k} \phi(2 x-k) \tag{4}
\end{equation*}
$$

In equations (2) and (4), two coefficient sets $\left\{g_{k}\right\}$ and $\left\{h_{k}\right\}$ have the same finite length $L$ for a certain basis, where $L$ is related to the number of vanishing moments $M$ in $\psi(x)$. For example, $L$ equals $2 M$ in the Daubechies wavelets. In the wavelet representation of signals, $\left\{h_{k}\right\}_{k=0, \ldots, L-1}$ behaves as a low-pass filter and $\left\{g_{k}\right\}_{k=0, \ldots, L-1}$ behaves as a high-pass filter to signals. These two filters are related by

$$
\begin{equation*}
g_{k}=(-1)^{k} h_{L-k} ; \quad k=0, \ldots, L-1 \tag{5}
\end{equation*}
$$



Fig. 1. Illustration of the sequence of multiresolution analysis subspaces $V_{j} . W_{j}$ is the orthogonal complement of $V_{j}$ in $V_{j-1}$. Space $V_{0}$ represents the space that contains the finest resolution data, and $V_{0}=V_{3} \oplus W_{3} \oplus W_{2} \oplus W_{1}$.
and are called quadrature mirror filters (QMF). An extensive study of the QMF can be found in (Monzon, 1994).

The MRA of $\mathbf{L}^{2}(\mathbf{R})$ is a set of nested, closed subspaces $\left\{V_{j} ; j \in \mathbf{Z}\right\}$, such that

$$
\begin{equation*}
\ldots V_{3} \subset V_{2} \subset V_{1} \subset V_{0} \ldots \tag{6}
\end{equation*}
$$

where the basis for the subspace $V_{j}$ is a set of orthonormal, translated functions, and each of these functions sets is a fixed dilation of the scaling function, $\left\{\phi_{j, k} ; k \in \mathbf{Z}\right\}$. Therefore, these subspaces have the property

$$
\begin{equation*}
f(x) \in V_{0} \Longleftrightarrow f\left(2^{-j} x\right) \in V_{j} ; \quad \forall j \in \mathbf{Z} \tag{7}
\end{equation*}
$$

Defining $W_{j}$ to be the orthogonal complement of $V_{j}$ in $V_{j-1}$, they are related by

$$
\begin{equation*}
V_{j-1}=V_{j} \oplus W_{j} \tag{8}
\end{equation*}
$$

The wavelet basis $\left\{\psi_{j, k} ; k \in \mathbf{Z}\right\}$, as in equations (1) and (2), forms the orthonormal basis of the subspace $W_{j}$. Therefore, for $j<n_{0}$, we can have

$$
\begin{equation*}
V_{j}=V_{n_{0}} \oplus W_{n_{0}} \oplus W_{n_{0}-1} \ldots \oplus W_{j+1} \tag{9}
\end{equation*}
$$

Figure 1 illustrates the nesting of subspaces $V_{j}$ and their orthogonal complements $W_{j}$. In Figure $1, V_{0}$ contains the original data which has the finest resolution; the projection of the data on $\left\{V_{j} ; j=1,2,3\right\}$ has increasingly coarser resolution. In this paper, the data projected onto the subspace $V_{j}$ is referred as the decomposition of data at resolution level $j$.

We define the projection of a function $f \in V_{0}$ on $V_{j}$ to be $f^{j}(x)$. Then the $j$ th resolution level of the function has the form

$$
\begin{equation*}
f^{j}(x)=\sum_{k} s_{j, k} \phi_{j, k}(x) \tag{10}
\end{equation*}
$$

where $s_{j, k}$ is the projection of the function $f(x)$ on the basis $\phi_{j, k}$; that is,

$$
s_{j, k}=\int f(x) \phi_{j, k}(x) d x
$$

Next, define the projection of $f(x)$ on the subspace $W_{j}$ to be

$$
\begin{equation*}
d f^{j}(x)=\sum_{k} d_{j, k} \psi_{j, k}(x) \tag{11}
\end{equation*}
$$

where $d_{j, k}$ is the projection of function $f(x)$ on the basis $\left({ }^{\prime} \jmath, k\right.$

$$
d_{j, k}=\int f(x) \psi_{j, k}(x) d x
$$

Then, equation (9) implies that the original function $f(x) \in l_{i}$ can be represented by

$$
\begin{align*}
f(x) & =f^{n_{0}}(x)+\sum_{j=n_{0}}^{1} d f^{j}(x) \\
& =\sum_{k} s_{n_{0}, k} \phi_{n_{0}, k}(x)+\sum_{j=n_{0}}^{1} \sum_{k} d_{j, k} \iota_{j, k}(x) . \tag{12}
\end{align*}
$$

Figure 2 shows the decomposition of a simple synthetic scismic trace at various resolution levels for two different wavelet functions. The original trace is a Ricker wavelet, i.e. a normalized second-order derivative of a Gaussian function, with a peak frequency of 30 Hz . The left figure shows the decomposition by a Daubechies orthonormal basis with 2 vanishing moments, while the right figure shows the same decomposition with 3 vanishing moments. The Ricker wavelet $(f)$ is the left most trace in each box, while the remaining traces correspond to $f^{j}$ of equation (10), where $j=1,2,3$, respectively. From Figure 2, it can be seen that the decomposed traces contains progressively lower frequencies with the increase of decomposition levels while the major features of the original signal are preserved. Comparing the two plots in Figure 2, we also observe that the increasing the number of vanishing moments increases the smoothness of the decomposed signal.

## A Symmetric and Shift-Invariant Wavelet Basis

In many applications, it is required that the processes applied to the obtained signals be shift-invariant. For example, in examining the multi-scale property in residual statics correction problems, it is important that the error-fitting function


Fig. 2. Decomposition of a Ricker wavelet at increasingly coarser resolution levels. The bases of the decompositions are Daubechies wavelets with 2 and 3 vanishing moments for the left and right figure respectively. The first traces represents the signal at the finest level, which is the original signal.
at each scale have a common - or at least close to common - global minimum. Therefore, we expect that the relative time-shifts among traces at each scale to be almost the same as it was in the original data, and that the waveforms not be deformed from one trace to another. However, the orthonormal wavelet bases representations are generally not shift-invariant. This shift-variance can be seen directly from the construction of their bases, equations (2) and (4), because of the change of step sizes among different scales in these definitions. Therefore, the Daubechies wavelet bases are not suitable for our purpose. Figure 3 shows ten copies of randomly shifted Ricker-wavelet traces, and their projections onto the subspace $V_{3}$ in the Daubechies bases with 2 vanishing moments. The decomposed waveforms on the right of Figure 3 are deformed to different shapes among traces with different time-shifts, and they do not have the same relative time shifts of those shown on the left of Figure 3.

Saito and Beylkin (1993) suggested using the shell of an orthonormal basis when shift-invariant is required. Without loss of generality, let us assume that the signal we consider having finite length $N=2^{J}$. Consider a family of functions

$$
\left\{\tilde{\psi}_{j, k}(x)\right\}_{1 \leq j \leq J, 0 \leq k \leq N-1}
$$

and

$$
\left\{\tilde{\phi}_{j, k}(x)\right\}_{1 \leq j \leq J, 0 \leq k \leq N-1}
$$

where

$$
\begin{align*}
& \tilde{\psi}_{j, k}(x)=2^{-j / 2} \psi\left(2^{-j}(x-k)\right),  \tag{13}\\
& \tilde{\phi}_{j, k}(x)=2^{-j / 2} \phi\left(2^{-j}(x-k)\right), \tag{14}
\end{align*}
$$



Fig. 3. Ten traces of randomly shifted Ricker-wavelet traces (l.ff) and their decomposition at resolution level 3 in the Daubechies wavelet bases with tanishing moments of 2 (right).



Level-4 Shifted Traces

Fig. 4. The decomposition of ten copies of randomly shifted Ricker-wavelet traces, in the shell of the Daubechies basis with 2 vanishing moments, at resolution levels 3 and 4. The original traces are shown on the left of Figure 3.


Fig. 5. The decomposition of ten copies of randomly shifted Ricker-wavelet traces, in the auto-correlation shell of Daubechies basis with 2 vanishing moments, at resolution levels 3 and 4. The original traces are shown on the left of Figure 3.
where the functions $\psi(x)$ and $\phi(x)$ are a wavelet and scaling function, respectively. The new family of functions defined by equations ( 13 and ( 14 can also serve as bases for subspaces $V_{j}$ and $W_{j}$ in MRA. They are complete, but they are redundant and not orthonormal (Saito, 1994). Therefore, the decomposition of a function in these bases is not unique. However, by forcing an additional constraint to the projection, a function $f \in V_{0}$ may still be decomposed in the shell of an orthonormal basis much the same way as it was in an orthonormal wavelet basis itself. In this case, the basis functions in equations (10) and (11) are replaced by $\tilde{\psi}_{j, k}(x)$ and $\tilde{\phi}_{j, k}(x)$.

The representation of signals using this family of bases are shift-invariant among different scales. Figure 4 shows the same numerical experiment as that in Figure 3, except using the shell of orthonormal bases expansion at resolution levels 3 and 4 . The relative time-shifts among traces are preserved while the waveforms are deformed to the same amount. However, the original symmetric waveforms are deformed to asymmetric waveforms. This deformation of the waveforms is not desirable, and may cause problems for some applications.

To overcome this problem, a family of symmetric, shift-invariant bases are introduced (Saito and Beylkin, 1993). Let $\Phi(x)$ and $\Psi(x)$ be auto-correlation functions of scaling function and wavelet function respectively,

$$
\begin{align*}
\Phi(x) & =\int \phi(y) \phi(y-x) d y  \tag{15}\\
\Psi(x) & =\int \psi(y) \psi(y-x) d y \tag{16}
\end{align*}
$$

where $\psi$ and $\phi$ satisfy equations (2) and (4) respectively. Construct a family of bases

$$
\left\{\Psi_{j, k}(x)\right\}_{l \leq j \leq J, 0 \leq k \leq N-1}
$$

and

$$
\left\{\Phi_{j, k}(x)\right\}_{l \leq j \leq J, 0 \leq k \leq N-1},
$$

where

$$
\begin{align*}
& \Phi_{j, k}(x)=2^{-j / 2} \Phi\left(2^{-j}(x-k)\right),  \tag{17}\\
& \Psi_{j, k}(x)=2^{-j / 2} \Psi\left(2^{-j}(x-k)\right) \tag{18}
\end{align*}
$$

Now, we have an auto-correlation shell of an orthonormal basis that is both symmetric and shift-invariant. Figure 5 shows the expansion of shifted Ricker-wavelet traces in the auto-correlation shell of Daubechies basis. It can be seen that both the symmetry of the waveforms and the relative time-shifts are preserved at resolution levels 3 and 4.

There exists a fast algorithm for expanding a function $f \in V_{0}$ using the autocorrelation shell of orthonormal basis (Saito and Beylkin, 1993). I only give the formulas of the discrete expansion; detailed derivation can be found in (Saito and Beylkin, 1993).

Suppose that $S_{k}^{j}$ and $D_{k}^{j}$ are the projected signal onto the subspaces $V_{j}$ and $W_{j}$ at the sampled positions respectively, that is

$$
S_{k}^{j}=f^{j}(k \Delta), \quad D_{k}^{j}=D f^{j}(k \Delta)
$$

where $\Delta$ is the sampling interval. Then, two symmetric filters, $P=\left\{p_{k}\right\}_{-L+1 \leq k \leq L-1}$ and $Q=\left\{q_{k}\right\}_{-L+1 \leq k \leq L-1}$ are applied recursively to the signal we wish to decompose,

$$
\begin{align*}
S_{k}^{j} & =\sum_{l=-L+1}^{L-1} p_{l} S_{k+2^{j-1} l}^{j-1} \\
D_{k}^{j} & =\sum_{l=-L+1}^{L-1} q_{l} S_{k+2^{j-1} l}^{j-1} \tag{19}
\end{align*}
$$

where $0 \leq k<N, 1 \leq j \leq J$, and $L$ is the filter length in the "two-scale difference" equations of wavelet and scaling functions as in equations (2) and (4). In equation (20), $N=2^{J}$ is the number of samples of the signal and the filter coefficients $p_{k}$ and $q_{k}$ are,

$$
p_{k}= \begin{cases}2^{-1 / 2}, & \text { for } k=0  \tag{21}\\ 2^{-3 / 2} a_{|k|}, & \text { otherwise }\end{cases}
$$

and

$$
q_{k}= \begin{cases}2^{-1 / 2}, & \text { for } k=0  \tag{22}\\ -p_{k}, & \text { otherwise }\end{cases}
$$

In equations (21) and (22), coefficients $\left\{a_{k}\right\}_{k=1, \ldots, L-1}$ are the correlation of the lowpass filter $\left\{h_{l}\right\}_{l=0, \ldots, L-1}$ in equation (4),

$$
a_{k}= \begin{cases}2 \sum_{l=0}^{L-1-k} h_{l} h_{l+k}, & \text { for } k \text { is odd }  \tag{23}\\ 0, & \text { for } k \text { is even } .\end{cases}
$$

## MULTIRESOLUTION ANALYSIS FOR INVERSE CALCULATIONS

Many inverse problems are solved by optimization methods. Mathematically, gradient-search optimization methods work well when the objective function is convex (e.g., a "basin") in the searching range; and the wider the basin of attraction leading to the bottom, the more likely that the optimizations converge to the optimum point. Optimizations have difficulties when there exists more than one point with zero gradient (e.g. local minima, flat area) in the searching range. Unfortunately, this is usually the case in many realistic inverse problems. The complexity of objective functions can be affected by many factors, such as noise, frequency bandwidth, and features of the information in the observed data.

As studied in the above section, an MRA can decompose signals into various resolution levels. The data with coarse resolutions contain less detailed information and lower frequencies, while keeping major features of the original signal consistent with the low frequency information. These less-information data can serve as a relaxation to optimizations. Therefore, by using data at coarser resolution levels, complexity of objective functions may be reduced, which increases the performance of optimizations.

## A Simple Residual Statics Problem

Let us first consider a simple residual statics problem. Consider a trace containing one Ricker wavelet; duplicate the trace with an unknown shift. Figure 6 shows two traces as described above. Now, we look for the time-shift between the two traces by applying an optimization, that is, searching for the time-shift which maximally aligns the two traces. This is a simple residual statics estimation problem using the stacking power method; there is only one unknown in the optimization. The objective function is formulated as a least-squared error,

$$
\begin{equation*}
E(\delta)=\sum_{i=0}^{N-1}\left(P_{0}(i-\delta)-P_{1}(i)\right)^{2} \tag{24}
\end{equation*}
$$

where $P_{0}(t)$ and $P_{1}(t)$ are the two data traces, $N$ is the number of samples per trace, and $\delta$ is the unknown time-shift. The goal is to find the time-shift $\delta$ that minimizes the error function $E(\delta)$. Figure 7 shows the error function as in equation (24) for the fitting of these two traces. In addition to possible problems caused by the localminima, the basin of attraction leading to the global-minimum is "steep" and narrow, while the two areas to the sides are "flat". The global structure of this objective function suggests that the global minimum point may be hard to find by traditional


Fig. 6. The observed data in the first example. Two traces contains identical waveforms of Ricker wavelet with a 30 Hz peak frequency. The relative time-shift is the unknown we are seeking.


Fig. 7. The error-fitting function with respect to the relative time-shift between two traces. The goal is to find the optimal point where the mean-squared error is minimum.


Fig. 8. The histogram of the obtained time-shifts of 50 conjugate-gradient optimization experiments starting from uniformly distributed random initial models between $[-0.2,0.2] \mathrm{s}$. The horizontal axis is the number of shift-samples, where the sample interval is 0.01 s , and the grid size of the histogram is 4 samples. The number of times that found the true global minimum is 8 out of 50 .
gradient-based searching methods. Assuming that we know a priori the time-shift between the traces lies in the range of $[-0.2,0.2] \mathrm{s}$, the searching range is restricted to this interval. Figure 8 shows the histogram of the obtained time-shift for 50 optimizations by using the Conjugate-Gradient and Cubic-Line-Search tools provided in the CWP Object-Oriented Optimization Library (Deng et al., 1995); initial models are randomly chosen between $[-0.2,0.2] \mathrm{s}$. As expected, the chances of finding the correct global minimum is small. In the case of this test, there are 8 out of 50 experiments that the correct time-shift was found.

## Using the MRA for Optimizations

Let us decompose the observed data into various resolution levels by representing them with wavelet bases. For the above example, the traces $\left\{P_{i}(x) ; i=0,1\right\}$ of lengths $N=2^{J}$ can be represented in the form of equation (12),

$$
\begin{equation*}
P_{i}(x)=P_{i}^{n_{0}}(x)+\sum_{j=n_{0}}^{1} \sum_{k=0}^{N-1} d_{j, k} \psi_{j, k}(x) \tag{25}
\end{equation*}
$$

where $1 \leq n_{0} \leq J$ and $P_{i}^{n_{0}}(x)$ is the projection of the original data onto the subspace $V_{n_{0}}$. Therefore, equation (24) can be rewritten as,

$$
\begin{equation*}
E^{n_{0}}(\delta)=\sum_{i=0}^{N-1}\left(P_{0}^{n_{0}}(i-\delta)-P_{1}^{n_{0}}(i)\right)^{2}+R^{n_{0}}(\delta) \tag{26}
\end{equation*}
$$

where $R^{n_{0}}(\delta)$ is the residual error term which is related to the detailed information being projected onto subspaces $\left\{W_{j} ; j=1, \ldots, n_{0}\right\}$.


Fig. 9. The mean-squared error functions for two seismic traces at various resolution levels. The traces are decomposed in the Daubechies basis with 2 vanishing moments.

Ignoring certain levels of fine-resolution information, i.e., ignoring the residual term in equation (26), the resolution level $n_{0}$ representation of the seismic traces can be used for optimization. Figure 9 shows the objective function $E^{n_{0}}(\delta)$ at various resolution levels, $n_{0}=1,2,3,4$. It shows that the global complexity of the objective function is reduced with the increasingly coarser level of resolution, and there are wider basins of attraction leading to the global minimum. However, Figure 9(d) shows that the global structure of the objective function is severely distorted when detailed information is ignored.

This phenomenon is caused by the shift-variance nature of compactly supported, orthonormal wavelet bases. In the inverse problem discussed here, it is required that the bases used to represent the signals be shift-invariant. According to the discussion of the previous section, two families of bases are shift-invariant. Because of the symmetric feature of the auto-correlation shell of orthonormal bases, we choose this family of bases for this study. From this point on. the paper uses only the auto-correlation shell of orthonormal bases to decompose the signal. unless otherwise indicated. Figure 10 shows the objective function at resolurion levels $n_{0}=2,3,4,5$ in an auto-correlation shell of the Daubechies wavelet basis with? waishing moments. The global structure of the objective function also shows the desired simplification as that in Figure 9, such as a wider basin of attraction leading to the global minimum, less oscillations and smaller "flat" area in the searching rauge. Moreover, the global minimum is not shifted at any decomposed resolution level. Finure 10(d) show that the whole searching range transformed to one wide basin of attraction, which would lead all initial models to the global minimum.

Figure 11 shows the same histograms as that shown in Figure 8, except the data used for optimizations are decomposed at various resolution levels. These results confirm our prediction that there are increasing chances for local-search optimizations to find the global minimum when coarse-resolution data are used. For Figure 11(d), all searches converge to the global minimum when data are decomposed to resolution level five.

For the simple problem discussed above, five levels of decomposition are needed to reduce the objective function to a convex function in the searching range. In addition, the global minimum of this simplified objective function coincides with that of the original objective function. Therefore, the correct solution is reached when only coarse resolution data are used in this example. In the next example, an optimization applied to the coarse-resolution data will not suffice.

## More Examples

For more complex optimization problems, further reduction of resolution may be needed to make objective functions convex. The severe loss of information may cause an erroneous global minimum of the objective function.

Here, I show another example of residual statics correction problem for a trace with complex waveforms and unknown noise. Figure 12 shows a trace taken from a


Fig. 10. The mean-squared error functions for two seismic traces at various resolution levels. The traces are decomposed in the auto-correlation shell of the Daubechies basis with 2 vanishing moments.


Fig. 11. Histograms of the obtained time-shifts of 50 conjugate-gradient optimization experiments for data at various resolution levels. Initial models are chosen randomly between $[-0.2,0.2] \mathrm{s}$. The horizontal axis is the number of shift-samples, where the sample interval is 0.01 s , and the grid size of the histograms is 4 samples. All 50 experiments found the true solution when the data are decomposed to resolution level 5.


Fig. 12. A real seismic trace and its duplication with an unknown shift.


Fig. 13. The mean-squared error functions for two real seismic traces shown in Figure 12.
field seismic record, and its duplication with an unknown shift. We repeat the process discussed in the previous section on these two traces. Figure 13 shows the objective function for this optimization. Due the oscillatory nature of the seismic field data and unknown noise, the objective function shows complicated local and global structure. The basin of attraction leading to the global minimum point is extremely narrow and steep, which makes it almost impossible for any gradient searching methods to find the correct solution.

Again, the auto-correlation shell of the Daubechies basis is used to decompose the traces to coarse resolution levels. Figure 14 shows objective functions when applying various level of decomposition to traces in Figure 12. As expected, the complexity of the objective function is greatly reduced after the data being decomposed to coarse levels.

However, it is worth noticing the global minimum point are slightly shifted in Figure $14(\mathrm{~d})$, though the objective function shows a nice, convexity shape. This problem may be caused by the loss of information when too much resolution was discarded from the data. In this case, an iterative process similar to a multi-grid iteration can be used to enhance the resolution progressively; i.e. the solution of a coarse-level optimization is used as the initial model to the following optimization at a finer level (e.g., Chen 1994).

## DISCUSSION

The MRA can be used for analyzing signals at various scales. One of these first studies in the field of wave propagation was conducted by Morlet et al. (1982a; 1982b). In recent years, many researchers have been applying the technique successfully to data compression and processing. Cohen and Chen (1993) gave some intuitive insight as well as suggestions on possible applications in seismic imaging. This paper shows that wavelet theory can also be used to study optimization as applied to inverse theory.

Taking advantage of MRA in wavelet theory, seismic data can be decomposed to coarse resolution levels, while keeping major features in the original signal. This paper has described the first step of the study on the complexity of inverse calculations; the influence of MRA on objective functions has significant effects on the performance of optimizations. The objective functions can be simplified and they approach convexity when data are decomposed to a low resolution. This initial study demonstrates that MRA may be a useful tool for characterizing complexity of objective functions in the optimization approach to inverse problems.

Using the residual statics correction as an example, I have shown in this report that the waveform misfit function is multi-modal even for a simple problem, and MRA indeed can simplify the complexity waveform misfit function. Comparing Figure 7 and Figure 10, Figures 13 and 14, the objective functions for the wavelet decomposed data show a wider basin of attraction leading to the global minimum, a reduced


Fig. 14. The mean-squared error functions for two real seismic traces shown in Figure 12 at various resolution levels. The traces are decomposed with the auto-correlation shell of the Daubechies basis with 2 vanishing moments.
number of local minima, but non-distorted global feature.
As discussed in this paper, the choice of bases used to decompose the data is critical for this application. In addition to the time-invariance and the symmetry issues of the bases, the influence of vanishing moments to objective functions remains to be investigated, especially its tradeoff with computational intensity.

More tests will be done for more realistic optimization problems. For example, inverse problems we encounter usually have many unknown parameters to be recovered; the observed data may also be contaminated by noise. Therefore, it is important to study the influence of the MRA on these realistic problems. Results of these studies can be used to characterize complexities of certain inverse problems. As all the other inverse algorithms, the computation cost is also an issue that needs to be studied.

## ACKNOWLEDGMENT

I appreciate very much the encouragement of Dr. John Scales during the progress of this project, his continuous support on the optimization library - COOOL, and sharing his insights in inversions. I also would like to thank Drs. Bin Wang and Keh Pann of Mobil MEPTEC for initially suggesting that I look at the wavelet transform literature. I benefited a great deal from the "wavelets" class I took from Dr. Gregory Beylkin this spring. I am sincerely grateful to him for offering this wonderful opportunity and his suggestion of looking at the symmetric, shift-invariant wavelet bases. I also benefited from the brief discussion with Dr. Stew Levin of Mobil MEPTEC during SEG, Los Angeles.

Finally, I want to thank my colleague and friend, Wences Gouveia, who helped with the optimization algorithm part of this project.

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# A study of model covariances in amplitude seismic inversion 

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# A study of model covariances in amplitude seismic inversion 

Wences Gouveia


#### Abstract

Bayesian inversion (Tarantola, 1987) provides a concise mathematical framework that formally allows the incorporation of a priori information into geophysical data inversion. In this methodology, the general solution of an inverse problem can be regarded as a probability density $\sigma(\mathrm{m})$ over the space of models, that consists of the product of two probability density functions. One, known as the likelihood function $L(\mathrm{~m})$, measures the extent that the observed data are fit by model data. The other, $\rho(\mathbf{m})$ quantifies the a priori knowledge that is possibly available about the inverse problem. This information, derived for instance from regional geology considerations, well-log data and other types of geophysical data, can be incorporated into the inversion problem, via model covariance matrices of $\rho(\mathbf{m})$. The construction of these matrices from such sources of information is a complicated problem, and in the large majority of cases, ad hoc simplifying assumptions are made. As a consequence the significance of the model covariance matrices is lost.

In this work I study the effect of model covariance matrices in a linear, iterative amplitude-inversion algorithm. I illustrate in a simple example some advantages of building covariance matrices from statistical considerations about the underlying model, as opposed to using the Tikhonov regularization method (Tikhonov and Arsenin, 1977). This method builds covariance matrices generally under the assumption of model smoothness, providing little flexibility to incorporate more realistic information about the inverse problem.

The linear, iterative amplitude-inversion algorithm is proposed in Jin et al. (1992). In their work the inversion problem is formulated under the small scatterer, or Born, approximation. The resulting linear system of equations is solved by a minimization of a weighted least-squares norm, with weights derived from ray theory. The solution, i.e., perturbations to a given background velocity model, is obtained by a quasi-Newton optimization method, possibly an expensive approach since it makes use of an approximation to the second derivatives of the objective function. However, Jin et al. (1992) showed that the Hessian matrix can be approximated by a diagonal matrix with good results.


## INTRODUCTION

Seismic amplitude or travel time inversion methods are a major topic of geophysical research due to their potential capability of extracting detailed lithologic information about the subsurface. Several inversion methodologies are described in the technical literature. Although the procedures differ, it is acknowledged in all of them that the data alone do not constrain all the model features that one aims to estimate. To reduce the ambiguity of the inverse problem it is necessary to incorporate a priori information about the underlying model. The Bayesian approach for geophysical data inversion (Tarantola, 1987) paves the way for the incorporation of such knowledge. In this methodology, the general solution of an inverse problem is defined as a probability density $\sigma(\mathbf{m})$ over the space of models, that consists of the product of two probability density functions. One, known as the likelihood function $L(\mathbf{m})$, measures the extent that the observed data are fit by model data. This function accounts for uncertainties in the data, i.e., data features that were not taken into account in the forward modeling step. Examples are noise in the data, multiples in the situation where the forward modeling procedure just generates primaries, and so on. The other probability density function, $\rho(\mathbf{m})$, quantifies the a priori knowledge that is possibly available about the inverse problem. In this work I will assume that $\rho(\mathbf{m})$ and $L(\mathbf{m})$ are Gaussian probability distributions, defined by the following expressions

$$
\begin{align*}
& \rho(\mathbf{m})=\left((2 \pi)^{M} \operatorname{det} C_{M}\right)^{-\frac{1}{2}} \exp \left[-\frac{1}{2}\left(\mathbf{m}-\mathbf{m}_{\mathbf{0}}\right)^{T} C_{M}^{-1}\left(\mathbf{m}-\mathbf{m}_{\mathbf{0}}\right)\right] \\
& L(\mathbf{m})=\left((2 \pi)^{N} \operatorname{det} C_{D}\right)^{-\frac{1}{2}} \exp \left[-\frac{1}{2}\left(g(\mathbf{m})-\mathbf{d}_{\mathbf{o b s}}\right)^{T} C_{D}^{-1}\left(g(\mathbf{m})-\mathbf{d}_{\mathbf{o b s}}\right)\right] . \tag{1}
\end{align*}
$$

Here, $M$ is the number of model parameters; $N$ is the number of observations; $C_{M}$ and $C_{D}$ are the model and data covariance matrices, respectively; $\mathbf{d}_{\mathrm{obs}}$ is the observed data vector; $g(\mathbf{m})$ represents the modeled (synthetic) data for the model $m$, and $m_{0}$ is the mean or most likely model.

In this situation the probability density $\sigma(\mathbf{m})$, also known as a posteriori probability density function is also Gaussian and given by

$$
\begin{equation*}
\sigma(\mathbf{m}) \propto \exp \left[-\frac{1}{2}\left(g(\mathbf{m})-\mathbf{d}_{\mathbf{o b s}}\right)^{T} C_{D}^{-1}\left(g(\mathbf{m})-\mathbf{d}_{\mathbf{o b s}}\right)+\left(\mathbf{m}-\mathbf{m}_{\mathbf{0}}\right)^{T} C_{M}^{-1}\left(\mathbf{m}-\mathbf{m}_{\mathbf{0}}\right)\right] . \tag{2}
\end{equation*}
$$

The covariance matrix $C_{M}$ of the probability density function $\rho(\mathrm{m})$ is a possible connection between the a priori information and the inverse problem. This information can be derived from regional geological considerations, well-logs, interpretative work and so on. To build covariance matrices from those sources is not trivial and is seldom attempted, at least in the published inversion literature. A specific case where model covariance matrices, and higher order statistical moments are derived from well-logs is described in Scales and Tarantola (1994). In view of this difficulty,
$a d$ hoc techniques are commonly used to build the model covariance matrices. Consequently, the significance of these matrices (and also of $\rho(\mathbf{m})$ ) is lost.

The objective of this work is to illustrate with a simple example how a specific seismic amplitude inversion algorithm can benefit from a model covariance matrix built using statistical considerations about the model one seeks, derived from some source of information (for instance, well logs). I compare the result obtained with this approach with the one obtained when I used the Tikhonov regularization method (Tikhonov and Arsenin, 1977) to construct the model covariance matrix, based on model smoothness assumptions. As will be shown later, both results are equivalent for the case considered here. However, two advantages can be pointed out in favor of the statistical construction of the model covariance matrices. First, the absence of a weighting factor, required by the Tikhonov approach, to incorporate the a priori information into the inverse problem. Second, the assessment of the uncertainties of the inversion procedure is probably more accurate when the model covariances are constructed honoring, at least to some extent, the statistics of the model parameters.

The amplitude seismic-inversion algorithm discussed here is based on the work of Jin et al. (1992). They linearize the isotropic elastic inversion problem with the Born approximation (Cohen and Bleistein, 1979) yielding a system of equations that is weighted according to ray-theoretic considerations and solved by a quasi-Newton method. They derived a diagonal approximation to the second-derivative matrix, which is a direct consequence of the weighting applied to the system.

This paper is structured as follows. I begin with a brief exposition of the inversion algorithm proposed by Jin et al. (1992). Here I restrict this outline to the acoustic approximation. Following that, I review the theoretical aspects of the regularization theory and present an example to illustrate its utility in the situation of inversion of noisy data. I also point out the connection between this theory and the more general Bayesian approach. Next I carry out the comparison, for a given inverse problem, between the results of the asymptotic linear iterative inversion when model covariances derived from the model statistics are used as opposed to Tikhonov regularization matrices. Finally, I present conclusions and future research directions for this work.

## ITERATIVE ASYMPTOTIC AMPLITUDE INVERSION

Jin et al. (1992) proposed a linearized asymptotic inversion method where the seismic inversion is formulated as the optimization of a data misfit objective function for elastic parameter estimation. Essentially the method consists of solving an overdetermined system of equations obtained from the linearization of the integral solution of the wave equation via the Born approximation. In their work Jin et al. (1992) solved this system using a weighted least-squares criterion. This weighting is derived from ray theory considerations. Following is a brief description of the algorithm for the acoustic inverse problem.

## Linearization

As in many inversion procedures the velocity $c(\mathbf{r})$ of the medium is characterized by a long-wavelength velocity profile $c_{0}(\mathbf{r})$ plus small deviations $\delta(\mathbf{r})$ (scatterers) from this background velocity, where $r$ is the position vector. The ultimate objective of the type of inversion algorithm discussed here is to estimate such deviations, given the background velocity. This approach, derived from perturbation theory, is relatively common in the inversion literature (e.g. Beylkin, 1985, Cohen and Bleistein, 1979, and others).

Leaving the algebraic details to the references (Bleistein et al., 1994), it is possible to show that the recorded (scattered) wave field $u_{s}\left(\mathrm{r}_{\mathrm{g}}, \mathrm{r}_{\mathrm{s}}, t\right)$ and the source (incident) wave field $u_{i}\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{s}}, t\right)$ satisfy the following integral relationship, here expressed in the frequency domain ${ }^{1}$ :

$$
\begin{equation*}
u_{s}\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{s}}, \omega\right)=\int_{D} d \mathbf{r} g\left(\mathbf{r}, \mathbf{r}_{\mathbf{g}}, \omega\right) m(\mathbf{r})\left(u_{i}\left(\mathbf{r}, \mathbf{r}_{\mathbf{s}}, \omega\right)+u_{s}\left(\mathbf{r}, \mathbf{r}_{\mathbf{s}}, \omega\right)\right) \omega^{2} \tag{3}
\end{equation*}
$$

where $D$ is the domain of integration over the diffraction points; $m(r)=\frac{-2 \delta c(\mathbf{r})}{c_{0}(\mathbf{r})^{3}}$, the unknown perturbation scaled by the background velocity, is the parameter sought in the inversion, and $g\left(\mathbf{r}_{1}, \mathbf{r}_{2}, \omega\right)$ is the Green's function for an impulsive source at $\mathbf{r}_{2}$ recorded at $\mathbf{r}_{1}$, computed in the present work from ray theory. For a 2D medium this Green's function is given by

$$
\begin{equation*}
g\left(\mathbf{r}_{1}, \mathbf{r}_{2}, \omega\right)=A\left(\mathbf{r}_{1}, \mathbf{r}_{2}\right) e^{i \omega \tau\left(\mathbf{r}_{1}, \mathbf{r}_{2}\right)} \frac{1}{\sqrt{-i \omega}} \tag{4}
\end{equation*}
$$

Here, $\tau\left(\mathbf{r}_{1}, \mathbf{r}_{2}\right)$ and $A\left(\mathbf{r}_{1}, \mathbf{r}_{2}\right)$ are the ray theoretical traveltime and amplitude that satisfy the eikonal and transport equations, respectively. Equation (3) is nonlinear with respect to $m(\mathbf{r})$ because it contains a product of this unknown quantity and the scattered (observed) field $u_{s}\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{s}}, \omega\right)$. The Born approximation, which basically neglects the scattered wave field in comparison with the incident wavefield under the "small scatterer" assumption, is used in the linearization of Equation (3). Noticing that $u_{i}\left(\mathbf{r}, \mathbf{r}_{\mathbf{s}}, \omega\right)=g\left(\mathbf{r}, \mathbf{r}_{\mathbf{s}}, \omega\right)$, this linearization results in the following equation:

$$
\begin{equation*}
u_{s}\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{s}}, \omega\right)=\int_{D} d \mathbf{r} g\left(\mathbf{r}, \mathbf{r}_{\mathbf{g}}, \omega\right) m(\mathbf{r}) g\left(\mathbf{r}, \mathbf{r}_{\mathbf{s}}, \omega\right) \omega^{2} \tag{5}
\end{equation*}
$$

or, in matrix form:

$$
\begin{equation*}
G \mathbf{m}=\mathbf{u}_{\mathbf{s}} \tag{6}
\end{equation*}
$$

Here, $G$ is the Born operator matrix; $\mathbf{m}$ is the unknown normalized scattering vector, and $u_{s}$ is the recorded field.

In the inversion algorithm, the linear system of equations (6) is solved for the model m by a quasi-Newton technique described next.

[^3]
## Solution of the linear system

The linear system of equations (6) is usually over-determined, since the number of observations (data points) is larger than the number of unknowns. Therefore it is necessary to define what it is meant by the solution of the system. In their work Jin et al. (1992) used the following weighted least-squares criterion for this definition:

$$
\begin{equation*}
\min _{\mathbf{m}} S\left(\mathbf{m}, \mathbf{r}_{\mathbf{0}}\right)=\min _{\mathbf{m}} \frac{1}{2} \int d \xi \int d \psi \int d \omega\left(\mathbf{u}_{\mathbf{s}}-G \mathbf{m}\right)^{T} Q\left(\mathbf{u}_{\mathbf{s}}-G \mathbf{m}\right) \tag{7}
\end{equation*}
$$

Where $r_{0}$ is the output point, i.e., the coordinate of the scatter point to be estimated; $Q$ is a diagonal matrix that implements the weighting and will be described later. In matrix form Equation (7) can be written as:

$$
\begin{equation*}
\min _{\mathbf{m}} S\left(\mathbf{m}, \mathbf{r}_{\mathbf{0}}\right)=\min _{\mathbf{m}}\left(\mathbf{u}_{\mathbf{s}}-G \mathbf{m}\right)^{T} Q\left(\mathbf{u}_{\mathbf{s}}-G \mathbf{m}\right) \tag{8}
\end{equation*}
$$

Notice that the sum in Equation (7) is carried out over the angles $\psi$ and $\xi$ defined in Figure 1. Ideally, for the best resolution, it would be desirable to sample the scattering point from all angles, which is not the case for seismic experiments. Moreover, instead of summing over the angles $\psi$ and $\xi$, a more suitable coordinate system is defined by the source and receiver locations. Considering this coordinate system, Equation (7) can be rewritten as:

$$
\begin{equation*}
\min _{\mathbf{m}} S\left(\mathbf{m}, \mathbf{r}_{\mathbf{0}}\right)=\min _{\mathbf{m}} \frac{1}{2} \int d \mathbf{r}_{\mathbf{s}} \int d \mathbf{r}_{\mathbf{g}} \int d \omega\left(\mathbf{u}_{\mathbf{s}}-G \mathbf{m}\right)^{T} Q\left(\mathbf{u}_{\mathbf{s}}-G \mathbf{m}\right) J\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{s}}, \xi, \psi\right) \tag{9}
\end{equation*}
$$

Here, $J\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{s}}, \xi, \psi\right)$ is the Jacobian of the transformation, that should account for the discretization of the data acquisition. One of the most interesting aspects of the work of Jin et al. (1992) is the weighting diagonal matrix $Q$. The $i i^{\text {th }}$ element of the $Q$ matrix relates to a given source-receiver pair and is defined as follows:

$$
\begin{equation*}
Q\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{s}}, \omega, \mathbf{r}_{\mathbf{0}}\right)=\frac{\left\|\mathbf{p}\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{0}}, \mathbf{r}_{\mathbf{s}}\right)\right\|^{2}}{4 \pi^{2} \omega A^{2}\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{0}}, \mathbf{r}_{\mathbf{s}}\right)} \tag{10}
\end{equation*}
$$

where (see Figure 1) $\left\|p\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{0}}, \mathbf{r}_{\mathbf{g}}\right)\right\|^{2}$ is the square modulus of the total slowness vector at the scattering point, defined as $\mathbf{p}\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{0}}, \mathbf{r}_{\mathbf{s}}\right)=\nabla \tau\left(\mathbf{r}_{\mathbf{0}}, \mathbf{r}_{\mathbf{s}}\right)+\nabla \tau\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{0}}\right)=$ $\mathbf{p}_{\mathbf{s}}\left(\mathbf{r}_{\mathbf{0}}, \mathbf{r}_{\mathbf{s}}\right)+\mathbf{p}_{\mathbf{g}}\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{0}}\right)$, and $A\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{0}}, \mathbf{r}_{\mathbf{s}}\right)$ is an amplitude factor defined as $A\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{0}}, \mathbf{r}_{\mathbf{g}}\right)=$ $A\left(\mathbf{r}_{\mathbf{0}}, \mathbf{r}_{\mathbf{s}}\right) A\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{0}}\right)$.

The justification for the weighting matrix $Q$ comes from ray-theory. This matrix compensates for geometrical spreading losses and tends to eliminate wide angle reflections from the inversion (small values of $\|\mathbf{p}\|$ ). Gray (1994) also avoided wide angle reflections in his migration procedure, under the justification that those events might be spatially aliased. Notice that the $Q$ matrix depends on the coordinate $r_{0}$. A drawback of this formulation of $Q$ is the presence of the temporal frequency $\omega$ in the denominator, weighting down the higher-frequency information. However, as will be shown later, the main benefit of using such weighting is accomplished by a


Fig. 1. Geometry for one scattering point.
diagonal approximation of the Hessian matrix ( $2^{\text {nd }}$ derivative matrix of the objective function with respect to the model parameters) allowing an efficient implementation of a quasi-Newton algorithm for minimizing the objective function defined in Equation (9).

The minimizer of the objective function defined in Equation (9) satisfies the familiar normal system of equations

$$
\begin{equation*}
G^{H} Q G \mathrm{~m}=G^{H} Q \mathbf{u}_{\mathbf{s}} . \tag{11}
\end{equation*}
$$

Where $G^{H}$ is the Hermitian adjoint of $G$. It is well known that direct or iterative techniques are available for solving linear system of equations. Jin et al. (1992) opted for a quasi-Newton iteration method given by

$$
\begin{equation*}
\mathbf{m}_{\mathbf{n + 1}}=\mathbf{m}_{\mathbf{n}}-H_{a}^{-1} \gamma\left(\mathbf{m}_{\mathbf{n}}\right) . \tag{12}
\end{equation*}
$$

Here, $\mathbf{m}_{\mathbf{n}+\mathbf{1}}$ is the updated model; $\mathbf{m}_{\mathbf{n}}$ is the current model; $H_{a}^{-1}$ is an approximation for the inverse of the Hessian matrix evaluated at model $\mathbf{m}_{\mathbf{n}}$, and $\gamma\left(\mathbf{m}_{\mathbf{n}}\right)$ is the gradient of the objective function evaluated at model $\mathbf{m}_{\mathbf{n}}$. Each iteration performed in Equation (12) has a computational cost equivalent to a pre-stack migration algorithm.

The analytic computation of the derivatives of the objective function in Equation (9) is facilitated by the fact that WKBJ Green's functions and the Born approximation have been used. Therefore it is not complicated to show that the gradient of the objective function with respect to the model parameters is given by
$\gamma\left(\mathbf{r}, \mathbf{r}_{\mathbf{0}}\right)=\frac{1}{2 \pi} \int d \mathbf{r}_{\mathbf{s}} \int d \mathbf{r}_{\mathbf{g}} \frac{A\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{,} \mathbf{r}_{\mathbf{s}}\right)}{A^{2}\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{0}}, \mathbf{r}_{\mathbf{g}}\right)}\|\mathbf{p}\|^{2} J\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{s}}, \xi, \psi\right) \mathcal{H}\left[\delta \mathbf{u}_{\mathbf{s}}\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{s}}, t=\tau\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}, \mathbf{r}_{\mathbf{s}}\right)\right]\right.$.
$\mathcal{H}\left[\delta \mathbf{u}_{\mathrm{s}}\left(\mathrm{r}_{\mathrm{g}}, \mathrm{r}_{\mathrm{s}}, t=\tau\left(\mathrm{r}_{\mathrm{g}}, \mathbf{r}, \mathrm{r}_{\mathrm{s}}\right)\right]\right.$ is the Hilbert transform of the data residual evaluated at the total travel time $\tau\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}, \mathbf{r}_{\mathbf{s}}\right)=\tau\left(\mathbf{r}, \mathbf{r}_{\mathbf{s}}\right)+\tau\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}\right)$.

Along the same lines it is possible to show that the $i j^{\text {th }}$ element of the Hessian matrix $H=G^{H} Q G$ is given by

$$
\begin{equation*}
H\left[\mathbf{r}_{\mathbf{i}}, \mathbf{r}_{\mathbf{j}}\right]=\frac{1}{4 \pi^{2}} \int d \mathbf{r}_{\mathbf{g}} \int d \mathbf{r}_{\mathbf{g}} \int d \omega \omega\|\mathbf{p}\|^{2} \frac{A^{2}\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{j}}, \mathbf{r}_{\mathbf{s}}\right)}{A^{2}\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{i}}, \mathbf{r}_{\mathbf{g}}\right)} J\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{g}}, \xi, \psi\right) e^{-i \omega\left[\tau\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{j}}, \mathbf{r}_{\mathbf{s}}\right)-\tau\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{i}}, \mathbf{r}_{\mathbf{s}}\right)\right]} \tag{14}
\end{equation*}
$$

The above equation can be considerably simplified with a sequence of approximations. First notice that in the process of the quasi-Newton iterations the $i^{\text {th }}$ row of the Hessian matrix will be dotted with the gradient vector to yield the updated model parameter at the coordinate $\mathbf{r}_{\mathbf{i}}$. Therefore $\mathbf{r}_{\mathbf{i}}$ is the output point $\mathbf{r}_{\mathbf{0}}$ mentioned previously. So, by using the same methodology as in Bleistein et al. (1994) and Beylkin (1985) the traveltime $\tau\left(\mathbf{r}_{\mathbf{s}}, \mathbf{r}, \mathbf{r}_{\mathbf{g}}\right)$ and amplitude $A\left(\mathbf{r}_{\mathbf{s}}, \mathbf{r}, \mathbf{r}_{\mathbf{g}}\right)$ are expanded about the output point $\mathbf{r}_{\mathbf{i}}$. This procedures results in the following expression, after keeping terms to first order for the travel time and just the zero-th order term for the amplitudes:

$$
\begin{equation*}
H_{a}\left[\mathbf{r}_{\mathbf{i}}, \mathbf{r}_{\mathbf{j}}\right]=\frac{1}{4 \pi^{2}} \int d \mathbf{r}_{\mathbf{s}} \int d \mathbf{r}_{\mathbf{g}} \int d \omega \omega\|\mathbf{p}\|^{2} J\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{s}}, \xi, \psi\right) e^{-i \omega \mathbf{p}\left(\mathbf{r}_{\mathbf{j}}-\mathbf{r}_{\mathbf{i}}\right)} \tag{15}
\end{equation*}
$$

Furthermore assuming that $J \approx 1$, and going to the $[\psi, \omega]$ domain we get:

$$
\begin{equation*}
H_{a}\left[\mathbf{r}_{\mathbf{i}}, \mathbf{r}_{\mathbf{j}}\right]=\frac{1}{4 \pi^{2}} \int d \mathbf{r}_{\mathbf{s}} \int d \psi \int d \omega \omega\|\mathbf{p}\|^{2} e^{-i \omega \mathbf{p}\left(\mathbf{r}_{\mathbf{j}}-\mathbf{r}_{\mathbf{i}}\right)} \tag{16}
\end{equation*}
$$

Introducing the variable $\mathbf{K}=\omega \mathbf{p}$ we obtain:

$$
\begin{equation*}
H_{a}\left[\mathbf{r}_{\mathbf{i}}, \mathbf{r}_{\mathbf{j}}\right]=\frac{1}{4 \pi^{2}} \int d \mathbf{r}_{\mathbf{s}} \int d \psi \int d\|\mathbf{K}\|\|\mathbf{K}\| e^{-i \mathbf{K} .\left(\mathbf{r}_{\mathbf{j}}-\mathbf{r}_{\mathbf{i}}\right)} \tag{17}
\end{equation*}
$$

Notice that the integrals over the $[K, \psi]$ domain represent the integration in cylindrical coordinates of a constant. If $\|\mathbf{K}\|$ ranged from 0 to $+\infty$, and $\psi$ from 0 to $2 \pi$, this integral would result in a delta function. This is obviously not the case; nonetheless Jin et al. (1992) make this assumption, and the final expression for the approximation of the Hessian is given by:

$$
\begin{equation*}
H_{a}\left[\mathbf{r}_{\mathbf{i}}, \mathbf{r}_{\mathbf{j}}\right]=\frac{1}{2 \Delta \mathbf{r}_{\mathbf{g}}} \delta\left(\mathbf{r}_{\mathbf{j}}-\mathbf{r}_{\mathbf{i}}\right) \int d \mathbf{r}_{\mathbf{s}} \tag{18}
\end{equation*}
$$

Here, $\Delta \mathrm{r}_{\mathrm{g}}$ is the receiver spacing. Equation (18) is the final approximation to the Hessian that will be used in the quasi-Newton iterations described in Equation (12). Notice that it is invariant from iteration to iteration. Although this approximation worked fine with the examples shown in this work and also in Jin et al. (1992) it remains to be seen how it behaves for the situation of more complex data sets.

## An Example

Consider the simple velocity model in Figure 2, which consists of just one horizontal interface. I generated the five shot gathers for this model shown in Figure 3.


Fig. 2. One-layer velocity model. The length of a shot gather is illustrated by the horizontal line in the figure.

The forward modeling procedure used here is, as the inversion method, based on the Born approximation. To illustrate the action of regularization in amplitude inversion of noisy data, this is acceptable.

The objective of the inversion in this problem is, given a background velocity of $1.5 \mathrm{~km} / \mathrm{s}$, to estimate the magnitude of the velocity contrast to the second layer, i.e., $.2 \mathrm{~km} / \mathrm{s}$. In this noise-free situation the result, obtained in three iterations, is perfect, and is illustrated in Figure 4. This plot illustrates the perturbations to the background for several receiver positions as a function of depth. The spikes peak at the depth of 1.0 km with magnitude equal $.2 \mathrm{~km} / \mathrm{s}$ as expected.


Fig. 3. Five shot gathers generated for Figure 2.

In the next section I illustrate the behavior of this inversion algorithm in the presence of noisy data. I also review how to regularize this procedure using Tikhonov's approach, and present numerical examples that show the lower sensitivity of the regularized inversion with respect to noise, as compared to non-regularized inversion.


Fig. 4. Result of the inversion for the data in Figure 3.

## EFFECTS OF REGULARIZATION ON AMPLITUDE INVERSION

## Motivation

Consider the data set shown in Figure 5. It is the same data set illustrated in Figure 3, but with the addition of band-limited random noise such that the signal-tonoise ratio is now 2. Repeating the inversion procedure for this data set, I obtained the result illustrated in Figure 6, which is a considerably degraded version of the noise-free inversion result. The magnitude of the spikes contain errors larger than $50 \%$.


Fig. 5. Data of Figure 3 with band-limited noise. The signal to noise ratio is 2.

If just one shot gather of Figure 5 is used in the inversion, the final image, illustrated in Figure 7, is even a worse one. This is an expected result since the noise is attenuated when five shot gathers are used due to the larger data redundancy.

I applied Tikhonov regularization in the inversion algorithm discussed here, aiming at reducing its sensitivity with respect to perturbations (noise) in the data. I show the results later in this section, but first I briefly discuss this procedure and illustrate, with examples, the behavior of the regularized inversion algorithm for noisy data.


Fig. 6. Result of the inversion for the data in Figure 5.


Fig. 7. Result of the inversion for only the first gather in Figure 5.

## Basics of regularization

Consider the linear system

$$
\begin{equation*}
A \mathbf{x}=\mathbf{b} \tag{19}
\end{equation*}
$$

where $A$ is an operator of the forward problem that computes the data $\mathbf{b}$ for $\mathbf{a}$ given model $\mathbf{x}$. The solution for $\mathbf{x}$ in Equation (19) is said to be ill-conditioned if it is non-unique and/or if a small perturbation on the data $\mathbf{b}$ corresponds to a large perturbation in the model $\mathbf{x}$. The fundamental idea of Tikhonov's regularization method (Tikhonov and Arsenin, 1977) is to replace the operator $A$ by a family of approximate operators, functions of the so-called regularization parameter $\alpha$, such that the solution $x_{\alpha}{ }^{*}$ for each one of those parameters is well-conditioned, but, in some sense tends to $\mathbf{x}$ as $\alpha$ goes to zero. The approximated solution $\mathbf{x}_{\alpha}{ }^{*}$ can be defined as the minimizer of the quadratic functional:

$$
\begin{equation*}
\|A \mathbf{x}-\mathbf{b}\|^{2}+\alpha\|R \mathbf{x}\|^{2} \tag{20}
\end{equation*}
$$

in the domain of $R$. The matrix $R$ defines the correlation between different elements of model space according to some criterion, usually related to model smoothness, As it will be shown latter, the regularization matrix $R$ and the model covariance of the probability density function $\rho(\mathbf{m})$ are closely related.

Similarly it is possible to introduce $\mathbf{x}^{*}{ }_{\alpha}$ as the solution to the regularized normal equations

$$
\begin{equation*}
\left(A^{T} A+\alpha R^{T} R\right) \mathbf{x}=A^{T} \mathbf{b} \tag{21}
\end{equation*}
$$

Note that $R^{T} R$ is positive semidefinite, so a direct consequence of Tikhonov's method is to shift the spectrum of singular values of $A^{T} A$ in the positive direction. Generally
this implies that the solution of the regularized normal system of equations should be less susceptible to perturbations in the data vector b (Karlsruhe and Lyngby, 1993). However, that is not always the case. As shown in Scales et al. (1990), a singular value decomposition of the matrix $A$ is needed to understand what is actually being accomplished with regularization.

The regularization parameter $\alpha$ controls the influence of the penalty term on the optimization problem described in Equation (20). If it is chosen too small, Equation (20) is close to the original ill-posed problem, and the regularization would be of no or little effect. It $\alpha$ is too large, the problem solved would have little connection with the original Equation (19). Choosing the "optimum" value of $\alpha$ is a complicated matter in practice. Algorithms do exist with this intent (e.g., Hansen, 1992), and one of them is described later in this paper. A drawback is that the computational cost of those procedures is sometimes too high to make this method reasonable in problems such as amplitude seismic inversion.

## Regularization of Amplitude Inversion

In the specific problem of seismic inversion, a possible approach would be to use $R$ to add a priori knowledge about the model one seeks. For example, if lateral velocity variations are negligible, $R$ can be constructed such that the scatterers in the horizontal direction are correlated with (or imposed to be similar to) each other. In this case just for illustrative purposes, assume that we have five scatterers per layer. The matrix $R$ that correlates those scatterers is:

$$
R=\left[\begin{array}{rrrrr}
1 & -1 & 0 & 0 & 0  \tag{22}\\
1 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & -1
\end{array}\right] .
$$

Other regularization schemes are available. It is also possible to regularize the inverse problem by attenuating the roughness of the final solution. This is accomplished by imposing small variations on the first or second derivatives of the model parameters. Schematically those two regularization procedures are represented for the situation of five scatterers by the matrix $R$ in Equation (23) and (24), respectively.

$$
R=\left[\begin{array}{rrrrr}
1 & -1 & 0 & 0 & 0  \tag{23}\\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

$$
R=\left[\begin{array}{rrrrr}
-2 & 1 & 0 & 0 & 0  \tag{24}\\
1 & -2 & 1 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 \\
0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 1 & -2
\end{array}\right]
$$

To incorporate Tikhonov's regularization (for example the one described in Equation (22)) in the iterative asymptotic inversion presented in the last section, the following steps have to be undertaken. The regularization term should be added to the objective function in Equation (9). This results in the following expression:

$$
\begin{align*}
\min _{\mathbf{m}} S\left(\mathbf{m}, \mathbf{r}_{\mathbf{0}}\right)= & \min _{\mathbf{m}} \frac{1}{2} \int d \mathbf{r}_{\mathbf{s}} \int d \mathbf{r}_{\mathbf{g}} \int d \omega\left(\mathbf{u}_{\mathbf{s}}-G \mathbf{m}\right)^{T} Q\left(\mathbf{u}_{\mathbf{s}}-G \mathbf{m}\right) J\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{s}}, \xi, \psi\right) \\
& +\alpha \sum_{i} \sum_{j} \lambda\left(\mathbf{r}_{\mathbf{i}}, \mathbf{r}_{\mathbf{j}}\right)\left(m\left(\mathbf{r}_{\mathbf{i}}\right)-m\left(\mathbf{r}_{\mathbf{j}}\right)\right)^{2} \tag{25}
\end{align*}
$$

Here, $\alpha$ is the regularization parameter. The matrix $R$ is constructed with the parameter $\lambda\left(\mathbf{r}_{\mathbf{i}}, \mathbf{r}_{\mathbf{j}}\right)$, which has the value of 1 if the scatterers $\mathbf{r}_{\mathbf{j}}$ and $\mathbf{r}_{\mathbf{i}}$ are to be correlated, or 0 otherwise. $m(\mathbf{r})$ is the magnitude of the scatterer at $\mathbf{r}_{\mathbf{i}}$. As in Equation (8) the corresponding matrix form of Equation (25) is:

$$
\begin{equation*}
\min _{\mathbf{m}} S\left(\mathbf{m}, \mathbf{r}_{\mathbf{0}}\right)=\min _{\mathbf{m}}\left(\mathbf{u}_{\mathbf{s}}-G \mathbf{m}\right)^{T} Q\left(\mathbf{u}_{\mathbf{s}}-G \mathbf{m}\right)+\alpha \mathbf{m}^{T} R^{T} R \mathbf{m} \tag{26}
\end{equation*}
$$

where $R$ is the regularization operator that couples the model parameters according to $\lambda\left(\mathbf{r}_{\mathbf{i}}, \mathbf{r}_{\mathbf{j}}\right)$.

The differentiation of the regularization term with respect to the model parameters is incorporated in the gradient (13), yielding the following expression:

$$
\begin{align*}
\gamma\left(\mathbf{r}, \mathbf{r}_{\mathbf{0}}\right)= & \frac{1}{2 \pi} \int d \mathbf{r}_{\mathbf{s}} \int d \mathbf{r}_{\mathbf{g}} \frac{A\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}, \mathbf{r}_{\mathbf{s}}\right)}{A^{2}\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{0}}, \mathbf{r}_{\mathbf{s}}\right)}\|\underline{p}\|^{2} J\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{s}}, \xi, \psi\right) \mathcal{H}\left[\delta \mathbf{u}_{\mathbf{s}}\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}_{\mathbf{s}}, t=\tau\left(\mathbf{r}_{\mathbf{g}}, \mathbf{r}, \mathbf{r}_{\mathbf{s}}\right)\right]\right. \\
& +\alpha \sum_{i} \sum_{j} \lambda\left(\mathbf{r}_{\mathbf{i}}, \mathbf{r}_{\mathbf{j}}\right)\left(m\left(\mathbf{r}_{\mathbf{i}}\right)-m\left(\mathbf{r}_{\mathbf{j}}\right)\right) \tag{27}
\end{align*}
$$

The approximation used in Equation (18) is used in the regularized version of the algorithm.

Finally the search direction in the Newton's iterations (Equation (12)) should take into account the regularization operator $R$, resulting in the following update scheme:

$$
\begin{equation*}
\mathbf{m}_{\mathbf{n}+\mathbf{1}}=\mathbf{m}_{\mathbf{n}}-\left[H_{a}+R^{T} R\right]^{-1} \gamma\left(\mathbf{m}_{\mathbf{n}}\right) \tag{28}
\end{equation*}
$$

In the next section I assess the performance of the regularized asymptotic inversion with the data set of Figure 5.

## Choosing the Regularization Parameter

Equation (20) defines an objective functional formed by two terms. The first one relates to the data misfit while the second incorporates the regularization scheme. An "optimum" regularization parameter $\alpha$ would provide an ideal balance between those two components, minimizing the regularization error and the perturbation error in the solution $\mathbf{x}_{\alpha}{ }^{*}$. Several algorithms are available in the literature (Karlsruhe and Lyngby, 1993) for this purpose. Those methods are usually subdivided into two main categories, according to the assumption as to whether the magnitude of the perturbation on the data vector is known or not.

In this section I present the algorithm described in Scales et al. (1990) and used in this work. This procedure relies on the definition of a data misfit threshold as a stopping criterion for the iterative inversion. Here, this stopping criterion is satisfied when the root-mean-square (RMS) of the residual is less than the R.MS amplitude of the noise for a given time window. In this case convergence is assumed.

To find an "optimum" $\alpha$, the method proceeds as follows. The inverse problem is solved for several values of the regularization parameter. The data misfit for each one of the solutions $\mathbf{x}_{\alpha}{ }^{*}$ is plotted as a function of $\alpha$, as shown schematically in Figure 8.


Fig. 8. Data misfit as a function of the regularization parameter $\alpha$.

The "optimum" value of $\alpha, \alpha^{\text {opt }}$ in Figure 8, is postulated as the largest regularization parameter for which the pre-specified data fit is achieved. Therefore the solution $\mathbf{x}_{\alpha}^{\mathbf{o p t} *}$ would have the desired data fitness and also be the most consistent with the a priori information used to build the operator $R$ in Equation (20).

This can be expensive procedure to carry on in practice, since the full inverse problem is solved several times for different regularization parameters. Nonetheless, this procedure has been successfully applied in some situations as described in Scales et al. (1990) and Pratt et al. (1993).

## An Example

Carrying out the inversion procedure for the data set shown in Figure 5 under the assumption that the medium is laterally homogeneous, the regularization described in Equation (22), leads to the result shown in Figure 9. This should be compared with Figures 4 and 6 . The regularization was effective in reducing the sensitivity of the inversion procedure to the noise in the data, yielding scatterers shown in Figure 9 with the correct magnitude.

Figure 10 shows the convergence of a particular scattering point as a function of the number of iterations for the noise-free inversion and the regularized inversion. The smaller number of iterations needed to obtain convergence in the latter case is an indication that the condition number of the problem was reduced in comparison with the non-regularized approach. Note that convergence to $0 \%$ error in the presence of noise is not possible since the amplitudes of the data are corrupted.


Fig. 9. Result of the inversion for in Figure 5 using regularization.


FIG. 10. Convergence plots for the noise-free and regularized inversions.
Using the first- and second-order derivative schemes described in Equations (23) and (24) to invert the data set of Figure 5, I obtained the results shown in Figures 11 and 12 , respectively. The first-order derivative scheme provided comparable results
to the one shown in Figure 9. Such is not the case for the second-order derivative regularization. This scheme yields a smooth result, as illustrated in Figure 13. However, due to the limited lateral extent of the velocity model, and the fact that the smoothing operator $R$ in Equation (24) does not allow rapid changes in the model parameters, this procedure does not produce satisfactory results.

Figure 14 shows a plot of the regularization parameter as a function of data misfit for the three regularization schemes. Notice that since the model is indeed laterally homogeneous, once the data misfit of the optimum solution is below the threshold, it becomes independent of the regularization parameter. According to this curve, a value for $\alpha$ of 5 for the regularization scheme described in Equation (22) and of 25 for the first-order derivative scheme were chosen. In the case of the second-order derivative regularization the value was arbitrary, since the curve never drops below the data misfit threshold.


Fig. 11. Result with first-order derivative regularization.


Fig. 12. Result with second-order derivative regularization.

## RELATING BAYESIAN INVERSION AND REGULARIZATION

In the inversion procedure proposed by Jin et al. (1992) the solution of the inversion problem was defined as the minimizer of the weighted least-squares norm defined by Equation (7), or by Equation (26) when regularization is used. The simple statement of these equations implies important assumptions about the statistical nature of the noise in the observed data and on the correlation between model parameters. The purpose of this section is to describe what those assumptions are.


Fig. 13. Velocity at 1.0 km depth obtained from second-order regularization.


Fig. 14. Regularization curves for the data of Figure 5.

As mentioned before, for Gaussian statistics the a posteriori probability density function $\sigma(\mathbf{m})$ is given by Equation (2), repeated here for convenience.

$$
\begin{equation*}
\sigma(\mathbf{m}) \propto \exp \left[-\frac{1}{2}\left(g(\mathbf{m})-\mathbf{d}_{\mathbf{o b s}}\right)^{T} C_{D}^{-1}\left(g(\mathbf{m})-\mathbf{d}_{\mathbf{o b s}}\right)+\left(\mathbf{m}-\mathbf{m}_{\mathbf{0}}\right)^{T} C_{M}^{-1}\left(\mathbf{m}-\mathbf{m}_{\mathbf{0}}\right)\right] \tag{29}
\end{equation*}
$$

When approaching an inverse problem using the Bayesian framework, we intend to determine which models, if any, are associated with large values of $\sigma(\mathbf{m})$. In other words we are interested in finding the maximizers of Equation (29). Under the Gaussian hypothesis this corresponds to minimizing the following quantity:

$$
\begin{equation*}
\min _{\mathbf{m}} S(\mathbf{m})=\min _{\mathbf{m}}\left(g(\mathbf{m})-\mathbf{d}_{\mathbf{o b s}}\right)^{T} C_{D}^{-1}\left(g(\mathbf{m})-\mathbf{d}_{\mathbf{o b s}}\right)+\left(\mathbf{m}-\mathbf{m}_{\mathbf{0}}\right)^{T} C_{M}^{-1}\left(\mathbf{m}-\mathbf{m}_{\mathbf{0}}\right) \tag{30}
\end{equation*}
$$

If $m_{0}$ is a null vector, Equations (30) and (26) are completely equivalent. The inverse of the weighting matrix $Q^{-1}$ plays the role of a data covariance matrix $C_{D}$, and the product $\left[\alpha R^{T} R\right]^{-1}$ implements the model covariance matrix $C_{M}$. Therefore, the leastsquares formulation of the inverse problem expressed in Equation (26) implies the assumption of uncorrelated Gaussian noise in the data and that the model parameters are also described by a Gaussian probability density function with zero mean. The model covariance matrix is defined by the regularization scheme.

In the next section, I consider the situation where the model covariance matrix is built by assuming a given correlation length (Tarantola, 1987) between the scatterers that form the interface between two acoustic layers. The results will be compared with those obtained by the regularized inversion procedure.

## MODEL COVARIANCE ESTIMATION

The random sequence illustrated in Figure 15 was generated by filtering a random white process with a filter of a given correlation length. Figure 16 illustrates the autocorrelation of this series and a possible exponential fit that will be used later in building the covariance matrix. A portion of this sequence was used to construct the lateral variation in velocity of the second layer as illustrated in the model shown in Figure 17.

Five shot gathers generated for the velocity model of Figure 17, are illustrated in Figure 18. Notice the change in polarity of the reflection from the interface, caused by the lateral inhomogeneity of the second layer.

As described in Tarantola (1987), a sequence such as that in Figure 15 can be approximately modeled as a Gaussian process with covariance function given by:

$$
\begin{equation*}
C[i, j]=\sigma^{2} e^{\left(\frac{i-j \mid}{\Delta}\right)} \tag{31}
\end{equation*}
$$

where $\Delta$ is the correlation length of the sequence, and $\sigma$ its standard deviation. A plot of this covariance matrix for a small problem consisting of 27 scatterers is shown


Fig. 15. Sequence with a given correlation length.


Fig. 16. Correlation and exponential fit for the sequence of Figure 15.


Fig. 17. Laterally inhomogeneous velocity model.


Fig. 18. Five shot gathers generated for the model of Figure 17.
in Figure 19. Here, the correlation length is determined by fitting an exponential of the form $e^{\frac{x}{\Delta}}$ to the autocorrelation of the scatterers (Figure 16). Although the fitting is not very accurate, it provides an initial estimation of the correlation matrix that is likely to be superior to the assumption that the model parameters are not correlated.

Again, the objective of the inversion is to determine the scatterer distribution given the velocity of the first layer. In this example, I compare the results obtained for the inversion of the model shown in Figure 17 for the target depth of 1 km obtained for the following cases: 1) non-regularized inversion; 2) regularized inversion using the operator $R$ described in Equations (22), (23), and (24), and 3) Bayesian inversion with the covariance matrix is given by Equation (31).

Figure 20 shows the result obtained with the non-regularized inversion. The inverse result is reasonably close to the true scatterer distribution, represented by the dashed curve in the Figure. The residual of this final solution is illustrated in Figure 21 .

Figures 22,23 and 24 show the results of the inversion using the three regularization schemes described in Equations (22), (23) and (24), respectively. As expected, a poor result was obtained with the regularization implemented by Equation (22) (Figure 22), since the assumption of lateral homogeneity is a bad one for this situation. The other two approaches smoothed the final solution to a some degree, defined by the parameter $\alpha$ described earlier. I experimented several values for $\alpha$, and the best results are illustrated by Figures 23 and 24, obtained with $\alpha=1$ in both cases. The first and second-derivative regularizations provided superior solutions than the one obtained with the non-regularized inversion.

Finally, Figure 24 shows the inverse result when I used the model covariance


Fig. 19. Model covariance matrix as define in Equation (31).


Fig. 20. Result of the non-regularized inversion.


Fig. 21. Residual of the non-regularized inversion result.


Fig. 22. Result of the regularized inversion using Equation (22).


Fig. 23. Result of the regularized inversion using Equation (23).


Fig. 24. Result of the regularized inversion using Equation (24).
shown in Figure 19. This solution is comparable to the ones obtained by the first and second-order regularization schemes. An appealing advantage of this approach is the absence of the parameter $\alpha$, since the covariance matrix is built based on statistical considerations.

The fact that the results obtained with the derivative-based regularizations (Figures 23 and 24) and with the model covariance defined by Equation (31) are equivalent, should not be surprising since the true scatterer distribution is smooth. Examples dealing with more complex models are required to carry out a more thorough comparison between the two procedures.


Fig. 25. Result of the exponential model covariance inversion.
Another important aspect of the Bayesian methodology is the possibility of computing the a posteriori model covariance, which provides insights on the resolution of the inverse problem solution. The a posteriori covariance $C_{M^{\prime}}$ (Tarantola, 1987) is given by:

$$
\begin{equation*}
C_{M^{\prime}}=\left[G^{H} Q G+C_{M}^{-1}\right]^{-1} \tag{32}
\end{equation*}
$$

Notice that, since the a posteriori covariance includes the model covariance $C_{M}$, this analysis would be of little significance if this matrix is built without resorting to the statistics of the model parameters, as it is done in the Tikhonov regularization.

The a posteriori and a priori standard deviations (square-root of the main diagonal of the a posteriori and a priori covariance matrices, respectively) are plotted in Figure 26. As expected, the a posteriori are smaller than the a priori standard deviations indicating that the inversion succeeded in reducing the uncertainties of the model parameters. Also, the deviations reduce towards the center of the model, which corresponds to a larger data redundancy available at this location.

The results presented in this work, including the solution of the inverse problem shown in Figure 25 are still preliminary and require further research. The likely next


FIg. 26. Comparison between the a priori and a posteriori standard deviations.
step is to estimate the model covariance matrix directly from the data (Figure 15), without resorting to exponential models.

## CONCLUSIONS

Here, I presented a study on model covariances under the framework of amplitude seismic inversion. For simple examples I compared inversion results when the model covariance was built taking into account statistical considerations about the underlying model with the case in which model covariances were derived from Tikhonov regularization. As indicated in this paper, the advantages of the former approach is the absence of a regularization parameter ( $\alpha$ ) and a more reliable a posteriori uncertainty analysis of the inverse problem solution.

Tikhonov regularization, although an effective procedure for reducing the sensitivity of the inversion method to perturbations in the data (noise), might not provide the necessary flexibility for incorporating more general information about the inverse problem. In a more complex situation, not only in terms of the difficulty posed by the inverse problem, but also in the presence of different levels of information one would like to consider in the inverse problem, Tikhonov regularization is probably a limited approach for constructing covariance matrices.

The simple results discussed in this paper motivate the use of realistic model covariance matrices in geophysical parameter estimation. For instance, one of the objectives to be accomplished in future work is to estimate model covariances directly from some source of information (for instance well logs), probably without resorting to exponential models as done in this paper.

Closely related to the estimation of model covariance matrices is the quantification of the uncertainties of the inverse problem solution. This is an very important subject
that will be addressed in this research.

## ACKNOWLEDGMENTS

I thank John Scales, Norman Bleistein and Ken Larner for their critical review.

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# Prestack migration error in a transversely isotropic media 

Herman Jaramillo and Ken Larner

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#### Abstract

Previous studies have shown the dependence of migration error on reflector dip when poststack migration is done with an algorithm that ignores the presence of anisotropy. Here we do a numerical study of the offset dependence of migration error that can be expected when common-offset data from factorized transversely isotropic media are imaged by an isotropic prestack migration algorithm. Anisotropic ray tracing, velocity analysis and prestack migration in the common-offset domain are the basic tools for this analysis, which we apply to models with constant vertical gradient in velocity that are characterized by a particular combination of Thomsen's anisotropy parameters: $\eta \equiv(\epsilon-\delta) /(1+2 \delta)$. The results show that the offset dependence of error in imaged position, and therefore the quality of stacked, imaged data, depends largely, but not completely, on the anisotropy parameter $\eta$. Generally, the larger the value of $\eta$, the larger the problem of mis-stacking. Over a wide range of reflector dip, time-misalignment of imaged features on common-reflection-point gathers is considerably less than the error in imaged position on the zero-offset data. For all the model parameters studied, we expect stacking quality to be worst for reflector dip around 50 degrees. Reflections from horizontal reflectors and those with dip close to 90 degrees should stack well in all cases, and mistacking is not severe for overturned reflectors.


## INTRODUCTION

Previous studies have estimated the errors that can be expected in the results of seismic data processing as a consequence of the assumption that earth is isotropic where it is not. Poststack migration (Larner and Cohen, 1993; Alkhalifah and Larner, 1994) and dip moveout (DMO) (Larner, 1993; Tsvankin, 1995), for example, have been found to introduce mis-positioning of events when anisotropy is ignored. Errors in poststack migration and shortcomings of DMO when anisotropy is ignored bracket the issue of errors in prestack migration, but may not fully characterize the problems that might be expected when anisotropy is ignored in prestack migration.

Jaramillo and Larner (1994) did a qualitative demonstration of the positioning error in prestack migration based on a common-offset synthetic data set for a transversely isotropic medium with a vertical axis of symmetry (VTI medium) and constant velocity gradient in depth. Although several algorithms are presently available for doing anisotropic migration, it is of interest to gain a more thorough understanding of the perceived need to take anisotropy into account in migration.

For these reasons, here we do a systematic numerical study along the lines of the analysis for poststack migration error done by Larner and Cohen (1993) and by Alkhalifah and Larner (1994). Since offset is the new dimension, we focus our attention on the offset dependence of error. Taking the zero-offset error as a reference, we study the spread of migration-position errors over the range of offsets typically encountered in the seismic exploration. This spread of position errors gives information about the quality of stacking to be expected when data are prestack migrated with an isotropic algorithm, as is done in routine practice.

To perform the error analysis we create a common-offset diffractor response using a ray-trace modeling technique described in Larner (1993), Larner and Cohen (1993), and Alkhalifah (1995), then simulate the stacking velocities that would be obtained in practice, and finally apply prestack isotropic, depth migration using the velocity information. Position errors are computed in terms of the time error described in Larner and Cohen (1993). Once errors are computed for the different offsets, their differences from the zero-offset error are evaluated. These differences give an estimation of the offset- and dip-dependence of error and, by implication, stacking quality.

For the tests, the TI media are characterized by the anisotropy parameters of Thomsen (1986). In particular, following conclusions of Alkhalifah and Larner (1994) and Tsvankin (1995), we study media having a range of realistic values of Thomsen's parameters $\epsilon$ and $\delta$. Also, we limit consideration to media with vertical axis of symmetry (i.e., VTI media) and constant gradient in depth of the $P$-wave velocity along the vertical symmetry axis. (For all of our tests, the gradient is $k=0.6 \mathrm{~s}^{-1}$.) Moreover, the media are factorized anisotropic inhomogeneous (FAI), which means that $\epsilon$ and $\delta$ are constants, independent of depth.

Following the numerical study of error as a function of dip and offset, we generate synthetic multi-offset sections for a VTI model with reflecting segments having dip from 0 to 90 degrees in intervals of 15 degrees. Results of migrating these data support the results of the numerical study.

## MODELING

To perform the error analysis, for each offset we initially create diffraction curves in a vertical plane. Shearer and Chapman (1988) have developed an efficient method for ray tracing in the type of media considered here - FAI media with constant velocity gradient in depth. (For FAI media, all ratios among the various elasticity parameters are independent of position.) For transversely isotropic media, the core


Fig. 1. Illustration of Snell's law. Here $\mathbf{s}$ and $\mathbf{r}$ are the source and receiver positions respectively, $\mathbf{d}$ is the diffractor position, $\mathbf{u}$ is the unit vector pointing along the reflector, $\mathrm{p}_{\mathrm{s}}$ and $\mathrm{pr}_{\mathrm{r}}$ are the source and receiver slowness vectors at the scatterer, and $\theta$ is the dip of the reflector.
of their result is the property that raypaths simply are scaled, rotated versions of the slowness curve, the curve that relates horizontal and vertical slowness at any point in the medium. Also, for such media, two-point ray tracing can be done by solving a quadratic equation. The details are explained in Larner (1993).

To find the reflector dip associated with a given point on the diffraction curve requires use of Snell's law of reflection for zero offset. Larner and Cohen (1993) solved for $p_{1}$ and $p_{3}$ as a by-product of the ray-tracing routine. Here $p_{1}$ is the ray parameter or horizontal component of the slowness vector and $p_{3}$ is the vertical component of the slowness vector. Snell's law requires continuity of the slowness vectors along an interface. For nonzero offset, let us define $p_{s}=\left(p_{1 s}, p_{3 s}\right)$ as the slowness vector for the incident ray, $\mathrm{p}_{\mathbf{r}}=\left(p_{1 r}, p_{3 r}\right)$ as the slowness vector for the reflected ray, and $\mathbf{u}=(\cos \theta, \sin \theta)$, a unit vector pointing along reflector (Figure 1). Snell's law says

$$
\mathbf{p}_{\mathbf{s}} \cdot \mathbf{u}=\mathbf{p}_{\mathbf{r}} \cdot \mathbf{u}
$$

or

$$
p_{1 s} \cos \theta+p_{3 s} \sin \theta=p_{1 r} \cos \theta+p_{3 r} \sin \theta
$$

Solving for $\theta$, we find

$$
\theta=\arctan \left(\frac{p_{1 r}-p_{1 s}}{p_{3 s}-p_{3 r}}\right)
$$

Two simple cases exemplify this formula. If the reflector is horizontal $p_{1 r}=p_{1 s}$ and $p_{3 s}=-p_{3 r}$, so $\theta=0$ as expected. The zero-offset case ( $\mathbf{p}_{\mathrm{s}}=-\mathbf{p}_{\mathrm{r}}$ ) gives
$\theta=\arctan \left(p_{1} / p_{3}\right)$, where $p_{1}=p_{1 s}=p_{1 r}$ and $p_{3}=p_{3 s}=p_{3 r}$. This formula was derived by Larner and Cohen (1993).

For all of our tests, we consider just the case where the velocity gradient in the vertical direction is $0.6 s^{-1}$, which we judge to be a sufficiently representative value for the subsurface, considering that the constant gradient is itself a simplification. Also, for all tests, the depth $D$ of the scatterer is 1500 m .

For the error studies, we consider media with the following range of Thomsen's parameters: $0 \leq \epsilon \leq .3$ and $-0.2 \leq \delta \leq 0.3$. Alkhalifah and Larner (1994) and Tsvankin (1995) have shown that the two parameters, $\delta$ and $\epsilon$, are sufficient to characterize the error behavior of $P$-waves in VTI media. Tsvankin (1995), in particular, found that DMO behavior was governed primarily by the difference of the two parameters, $\epsilon-\delta$. More recently, Alkhalifah and Tsvankin (1995) have shown that a particular combination of $\epsilon$ and $\delta$, given by $\eta=(\epsilon-\delta) /(1+2 \delta)$, fully characterizes all time-related processing of $P$-wave data (e.g., moveout, dip moveout, and poststack and prestack migration) in homogeneous VTI media. We shall collect most of our results in terms of the anisotropy parameter $\eta$.

Viewed in terms of Kirchhoff migration, errors arise in migration when data characterized by one set of diffraction curves are migrated with medium parameters for which the diffraction curves are different. Figure 2 shows a set of five diffraction curves for a scatterer depth of 1500 m and offsets $0,1000,2000,3000$, and 4000 m , in a medium with Thomsen's parameters $\epsilon=0.2, \delta=0.1$. The curves show the familiar increased flattening near the apex with increasing offset, and an inflection point at the midpoint distance from the scatterer corresponding to the raypath that is horizontal at the scatterer. Note that, for midpoints greater than about 4 km , here, the curves for all offsets seem to overlap, perhaps suggesting that when data are migrated, errors in imaged positions will be independent of offset for larger dips. We shall find that such is not the case. While the curves are close, they do not overlap. The small differences in the diffraction curves will account for finite offset dependence of migration error.

## ISOTROPIC PRESTACK MIGRATION

## Velocity Analysis

Certainly, migration results depend on the velocity function used in the migration. Here, we wish to mimic a common way in which migration velocity is obtained in practice. We start by approximating the stacking velocity that would be found for reflections from horizontal reflectors.

First, we generate a zero-offset diffraction curve for a scatterer depth of 1500 m , i.e., the solid black curve in Figure 2. Least-square fitting a straight line to $t^{2}-x^{2}$ gives the stacking velocity $V_{s t k}$ at the given depth. To find the migration velocity at any depth from this stacking velocity, we assume that the vertical gradient in interval velocity, $d v / d z$, is constant and has been computed from velocity analysis

## Midpoint (km)



Fig. 2. Diffraction curves for different offsets. The offsets are 0-black solid, 1000 m gray solid, 2000 m -black dot, 3000 m-black dash and 4000 m -thin black. Thomsen's parameters $\epsilon=0.2 \delta=0.1$ are used here. The vertical velocity function is given by $v(z)=v_{0}+k z$, where, for this model, surface velocity $v_{0}=2000 \mathrm{~m} / \mathrm{s}$ and velocity gradient $k=0.6$. The diffractor depth is $D=1500 \mathrm{~m}$.
for reflections from within the overburden. We then solve a transcendental equation to find the migration velocity at the surface $v_{m}(0)$. The resulting migration velocity, $v_{m}=v_{m}(0)+(d v / d z) z$, is not the true vertical velocity of the TI medium.

## Prestack Migration

To simulate migration for a given offset, we apply common-offset isotropic migration to the diffraction curve that would be observed for that offset in the modeled VTI medium (e.g., one of the curves in Figure 2). Any of several techniques can be used to compute analytically a diffractor position given the slope and traveltime along the diffraction curve in a common-offset section. We illustrate the one used here by starting with the simplest situation of a zero-offset diffraction response in a homogenous isotropic media. Consider the function

$$
\begin{equation*}
\left.f(x, z, t) \equiv v^{2} t^{2}-4\left[\left(x-x_{d}\right)^{2}+\left(z-z_{d}\right)^{2}\right)\right]=0 \tag{1}
\end{equation*}
$$

which is the surface of a cone for any given diffractor position. Here $v$ is the medium velocity, $\left(x_{d}, z_{d}\right)$ is the diffractor position and $(x, z)$ is the source (also receiver) position. For a fixed $z$, for example $z=0$ representing the earth's surface, $f$ is a hyperbola (the diffraction curve) in the time-midpoint $(t-x)$ domain. (For zero offset, midpoint, source point, and receiver point are the same). For a fixed time $t$, and midpoint $x$ (let us assume again that $z=0$ ), $f$ is a circle with center at $(x, z)=(x, 0)$ and radius
$r=v t / 2$. This circle is the aplanatic curve associated with midpoint $(x, z)$ and traveltime $t$, the locus of all possible reflection points given the reflection time. Note that the diffractor point $\left(x_{d}, z_{d}\right)$ is a point of the aplanatic curve. The inverse problem consists of finding the diffractor point $\left(x_{d}, z_{d}\right)$ given the traveltime $t$, velocity $v$, and slope $p=d t / d x$ at a given midpoint $(x, z)$. To find the two unknowns $x_{d}$ and $z_{d}$, we need one more equation. Given that we can measure $p=d t / d x$, it seems natural to take the implicit derivative of $f$ in equation (1) with respect to $x$. This gives the second equation,

$$
\begin{equation*}
g\left(x_{d}, z_{d}\right) \equiv v^{2} t p-4\left(x-x_{d}\right)=0 \tag{2}
\end{equation*}
$$

To see the explicit dependence of $g$ on $z_{d}$ we recognize that $z_{d}=(v t / 2) \cos \theta$, where $\theta$ is the angle that the raypath from the source to the diffractor makes with the vertical. By recognizing also that $p=2 \sin \theta / v$, we find

$$
\begin{equation*}
g\left(x_{d}, z_{d}\right) \equiv z \tan \theta-\left(x-x_{d}\right)=0 \tag{3}
\end{equation*}
$$

the equation of a straight line from source to diffractor, with angle $\theta$ respect to the vertical, i.e., the raypath from the source to the scatterer. This simple interpretation will not hold for the case of nonzero-offset geometry, as we will show later.

Figure 3 shows the full geometrical interpretation of this problem. Equation (1) serves the double purpose of creating the diffraction response and the aplanatic curve by fixing the appropriate parameters as explained above. After constructing the diffraction response, we select a midpoint $(x, 0)$ and traveltime $t$. Given that time, we construct the aplanatic curve and then trace a ray with ray-parameter $p$ from the source to the aplanatic curve. The point $\left(x_{d}, z_{d}\right)$ where the ray intersects the curve is the solution of the inverse problem - the diffractor position given the traveltime curve. The algebraic problem is as simple as the solution of a quadratic equation (1) and a linear equation (3), with unknowns $x_{d}$ and $z_{d}$.

Nonzero-offset homogeneous migration.-We start with the double squareroot equation

$$
\begin{equation*}
f(x, z, t) \equiv \sqrt{\left(x-h-x_{d}\right)^{2}+\left(z-z_{d}\right)^{2}}+\sqrt{\left(x+h-x_{d}\right)^{2}+\left(z-z_{d}\right)^{2}}-v t=0 \tag{4}
\end{equation*}
$$

The parameters are the same as before, with the addition of the new parameter $h$, half the distance between source and receiver. Equation (4) is a conical surface with elliptical cross section. As before, the diffraction response is generated by fixing $z=0$. The function $f$ describes a hyperbola only for $h=0$ (previous case). As $h / z_{d}$ increases, the curve flattens near its apex (Cheops's Pyramid; Claerbout 1982). For a fixed time $t$, fixed midpoint $x$, and $z=0, \mathrm{f}$ represents an ellipse - the aplanatic curve associated with the midpoint $(x, z)=(x, 0)$ and time $t$. Again we seek $\left(x_{d}, z_{d}\right)$, given time and slope on the diffraction curve. Following the ideas above, we take the


Fig. 3. Zero-offset migration for a homogenous, isotropic medium. (a) A hyperbola is generated from the function $f$, fixing the diffractor point $\left(x_{d}, z_{d}\right)=(0 \mathrm{~m},-500 \mathrm{~m})$ and velocity $v=2000 \mathrm{~m} / \mathrm{s}$. A midpoint $x_{m}=1500 \mathrm{~m}$ is selected and the traveltime $t$ and slope $p$ of a point along the diffraction curve are computed. (b) Given $x_{m}$ and $t$ the circular aplanatic curve is computed from the same function $f$. A ray with rayparameter $p$ and source location ( $x_{m}, 0$ ), intersects the aplanatic curve at the point $\left(x_{d}, z_{d}\right)$. This is the solution of the inverse problem.
implicit derivative of $f$ with respect to midpoint $x$, and replace $d t / d x$ by $p$, yielding

$$
\begin{equation*}
g\left(x_{d}, z_{d}\right) \equiv \frac{\left(x-h-x_{d}\right)}{p v \sqrt{\left(z-z_{d}\right)^{2}+\left(x-h-x_{d}\right)^{2}}}+\frac{\left(x+h-x_{d}\right)}{p v \sqrt{\left(z-z_{d}\right)^{2}+\left(h-x_{d}+x\right)^{2}}}-p=0 \tag{5}
\end{equation*}
$$

Since this equation is nonlinear, the implicit derivative of $f$ no longer represents the equation of a ray. As we shall see, this equation corresponds to an ellipse. Figure 4 shows the geometrical interpretation of this problem. Equation (4) serves the double purpose of creating the diffraction response and the aplanatic curve by fixing the appropriate parameters. The algebraic problem is simply the solution of the ellipse (4) and its implicit derivative (5) for the unknown position ( $x_{d}, z_{d}$ ).

An alternative way to solve this problem, which will be more convenient for the inhomogeneous case, is to reduce $f$ to the explicit form of an ellipse equation (Claerbout, 1982). The new equation for $f$ is

$$
\begin{equation*}
f(x, y, t) \equiv 1-\frac{\left(x-x_{d}\right)^{2}}{(v t / 2)^{2}}-\frac{\left(z-z_{d}\right)^{2}}{(v t / 2)^{2}-h^{2}}=0 \tag{6}
\end{equation*}
$$

where the distance $z-z_{d}$ is the diffractor depth below the surface $z=0$, and $x-x_{d}$ is the lateral migration distance by which a midpoint $x$ is displaced from the position directly above the diffractor. For a data point in the midpoint-time domain, this is equivalent to the amount of time and lateral distance required to move a given point on the flank of a hyperbola to the apex of the hyperbola. Let us shift coordinates, letting $x-x_{d} \rightarrow x$ and $z-z_{d} \rightarrow z$ and solve directly for the new $x$ and $z$ in the equation

$$
\begin{equation*}
f(x, y, t) \equiv 1-\frac{x^{2}}{(v t / 2)^{2}}-\frac{z^{2}}{(v t / 2)^{2}-h^{2}}=0 \tag{7}
\end{equation*}
$$

Here we identify $x$ and $z$ as the lateral and vertical distances from a given data point (in space) to the diffractor position. The new function $f$ is not only simpler than equation (6) but is also written in terms of the needed parameters. Regarding $t$ as a function of $x$, we take the implicit derivative of $f$ with respect to $x$ and find, after simplification,

$$
\begin{equation*}
g=-t\left(t^{2} v^{2}-4 h^{2}\right)^{2} x+p\left(t^{2} v^{2}-4 h^{2}\right)^{2} x^{2}+p t^{4} v^{4} z^{2} \tag{8}
\end{equation*}
$$

Note that $p=d t / d x$ is the same in the shifted coordinate system since the new $x$ is a constant shift of the old one. By letting $a \equiv v t / 2$ and $b \equiv \sqrt{v^{2} t^{2} / 4-h^{2}}$, we can write equations (7) and (8) as

$$
\begin{aligned}
& f \equiv 1-\frac{x^{2}}{a^{2}}-\frac{z^{2}}{b^{2}}=0 \\
& g \equiv 1-\frac{(x-A)^{2}}{A^{2}}-\frac{z^{2}}{B^{2}}=0
\end{aligned}
$$



FIG. 4. Common-offset migration for a homogeneous isotropic medium. (a) A diffraction response is generated by using the function $f$ and fixing the diffractor point $\left(x_{d}, z_{d}\right)=(0 \mathrm{~m},-500 \mathrm{~m})$ for a medium velocity $v=2000 \mathrm{~m} / \mathrm{s}$ and half-offset $h=500$ m . A midpoint $x_{m}=1500 \mathrm{~m}$ is selected, and its corresponding traveltime $t$ and slope $p$ are computed. With $x_{m}$ and $t$, the aplanatic ellipse (b) is computed using the same function $f$. The intersection of the curve $g$ with the aplanatic curve is the solution to the inverse problem, the diffractor position $\left(x_{d}, z_{d}\right)$.
where $A \equiv t / 2 p$ and $B \equiv b t / 2 a^{2} p . f$ is an ellipse with center at $(0,0)$ and semi-axes $a$ (horizontal) and $b$ (vertical), and $g$ is an ellipse with center at ( $A, 0$ ) and semi-axes $A$ (horizontal) and $B$ (vertical). Note that $a, b, A, B$ are functions of the parameters $v, t, p, h$, and $(0,0)$ is a point of $g$. The solution to the inverse problem is the lower intersection of these two ellipses. Figure 5 describes the geometrical interpretation of this problem. The algebraic solution is found to be

$$
\begin{aligned}
& x=\frac{t}{8 h^{2} p}\left(4 h^{2}-t^{2} v^{2}+\sqrt{16 h^{4}-8 h^{2} t^{2} v^{2}+4 h^{2} p^{2} t^{2} v^{4}+t^{4} v^{4}}\right) \\
& z=\frac{1}{32 h^{4} p^{2} v^{2}}\left(-4 h^{2}+t^{2} v^{2}\right)^{2}\left(4 h^{2}-2 h^{2} p^{2} v^{2}-t^{2} v^{2}+\sqrt{16 h^{4}-8 h^{2} t^{2} v^{2}+4 h^{2} p^{2} t^{2} v^{4}+t^{4} v^{4}}\right)
\end{aligned}
$$

Conversion from depth to migrated time is easily achieved by using

$$
\begin{equation*}
t_{m}=2 z / v \tag{9}
\end{equation*}
$$

The nonzero-offset migration for a constant velocity gradient in depth.Slotnick (1986) shows that for an isotropic medium with a constant velocity gradient in depth, the raypaths are arcs of circles, and the traveltime is given by

$$
\tau=\frac{1}{k} \cosh ^{-1} \frac{x^{2}+\hat{z}^{2}+z_{0}^{2}}{2 \hat{z} z_{0}} .
$$

Here, $k$ is the velocity gradient, $v(z)=v_{0}+k z, \hat{z}=v(z) / k$, and $z_{0}=v_{0} / k . x$ is the horizontal distance from the diffractor to the source, and $z$ is the depth of the diffractor. The total traveltime in a common-offset section is computed as $t=\tau_{s}+\tau_{r}$, where $\tau_{s}$ is the time from the source to the scatterer, and $\tau_{r}$ is the time from the receiver to the scatterer. That is,

$$
t=\frac{1}{k}\left[\cosh ^{-1} \frac{(x-h)^{2}+z^{2}+z_{0}^{2}}{2 z z_{0}}+\cosh ^{-1} \frac{(x+h)^{2}+z^{2}+z_{0}^{2}}{2 z z_{0}}\right] .
$$

Here, $h$ is the half-offset between source and receiver, and $x$ is now the horizontal position of the midpoint. We obtain reflection times in a common-offset gather by fixing $h$ in this equation; in the same way, we obtain reflection times in a commonmidpoint gather by fixing $x$, and we obtain an aplanatic curve by fixing $t$. That is, for fixed $t$, the desired diffractor location $(x, z)$ relative to the midpoint must be a point of the aplanatic curve. The inverse problem consists of finding the coordinates $(x, z)$ given the traveltime $t$, velocity $v$, and slope $p=d t / d x$ at midpoint relative to the diffractor location. Dietrich and Cohen (1993) reduced the above equation to

$$
\begin{align*}
f(x, z, t) & \equiv\left(x^{2}+y^{2}+\hat{z}^{2}+{\hat{z_{0}}}^{2}+h^{2}\right)^{2}-4 x^{2} h^{2} \operatorname{coth}^{2}(k t / 2)-4 \hat{z}^{2} \hat{z}_{0}^{2} \cosh ^{2}(k t / 2) \\
& =0 \tag{10}
\end{align*}
$$

Proceeding the same as before, we obtain a second equation by taking the implicit derivative of $f$ with respect to midpoint $x$. This gives

$$
\begin{array}{r}
g(x, z) \equiv 8 h^{2} x \operatorname{coth}^{2}(k t / 2)+4 h^{2} k x^{2} \operatorname{coth}(k t / 2) \operatorname{csch}^{2}(k t / 2) p- \\
4 k \hat{z}^{2} \hat{z}_{0}^{2} \cosh (k t / 2) \sinh (k t / 2) p=0 . \tag{11}
\end{array}
$$




Fig. 5. Alernative method for common-offset migration for a homogeneous isotropic medium. Given the input parameters $v, t, p, h$ we construct two ellipses. One, $f$, is associated with the aplanatic curve and the other, $g$, with is the implicit derivative respect to $x$. The intersection of the two is the solution to the inverse problem, the relative distance from the midpoint to the scatterer. The solution $(x, z)=(1500 \mathrm{~m},-500 \mathrm{~m})$ agrees with the location of the scatterer in the specified model.
where, again, $p=d t / d x$. Figure 6 shows the traveltime curve as well as the curves corresponding to equations (10) and (11). The solution $-x=1500 \mathrm{~m}, z=-500 \mathrm{~m}$ - is consistent with the scatterer location in the given model.

We solve equation (10) for $\hat{z}^{2}$ and insert this into equation (11). The result is the fourth-degree polynomial equation in $x$

$$
\begin{equation*}
a x^{4}+b x^{3}+c x^{2}+d x+e=0 \tag{12}
\end{equation*}
$$

The coefficients and solution of this equation are presented in Appendix A.
The explicit solution for $z$ in terms of the appropriate solution $x$ and the original parameters $v_{0}, t, p, h, k$ is given by

$$
\begin{aligned}
& z= \sqrt{-\frac{h^{2} x+x^{3}+x v_{0}^{2} / k^{2}-2 h^{2} x \operatorname{coth}^{2}(k t / 2)+h^{2} k p x^{2} \operatorname{coth}(k t / 2) \operatorname{csch}^{2}(k t / 2)}{x-p v_{0}^{2} \cosh (k t / 2) \sinh (k t / 2) / k}} \\
& \quad-v_{0} / k .
\end{aligned}
$$

For this constant-gradient model, the conversion from depth to time is then given by

$$
\begin{equation*}
t_{m}=\frac{2}{k} \log \left(\frac{v(D)}{v_{0}}\right) \tag{13}
\end{equation*}
$$

The homogeneous case can be seen as a limiting case of the inhomogeneous one, obtained by taking $k \rightarrow 0$. However, it is easier to solve the homogeneous problem as above than to take the mathematical limit of the inhomogeneous case.

## COMPUTATION OF ERROR AND ANALYSIS OF RESULTS

In seismic data processing and interpretation we do not consider the positions of isolated points, but rather of reflections that are collection of points; that is, we interpret on a macro-scale. Larner and Cohen (1993) devised a way to estimate errors due to mis-positioning of reflections. We will use the same technique described there, but for a different reason. The error estimate is the difference between the times of the true and the incorrect migrated reflections at a given output position [i.e., at a given common-midpoint gather (CMP) location]. This time difference will vary with offset, and we use the variation of the time difference across all live offsets as our measure of expected stack quality. Specifically, if the range of time differences is smaller than half a period (here measured at a reference frequency of 30 Hz ) we might say that the stacking quality is good. Thus, we focus our attention on the range in error over offset. In essence, we use the zero-offset error studied by Larner and Cohen (1993) and by Alkhalifah and Larner (1994) as the reference time error, and consider offset-dependent departures from this value, for all dips.

For the tests, we use offsets of $1000 \mathrm{~m}, 2000 \mathrm{~m}, 3000 \mathrm{~m}$ and $4000 \mathrm{~m}, 4000 \mathrm{~m}$ being a typical length of streamer today. Also, we simulate muting of wide-angle data by



Fig. 6. Common-offset migration for an isotropic medium with constant velocity gradient in depth. Given the input parameters $v, t, p, h, k$, we construct two curves. One, $f$, is associated with the aplanatic curve and the other, $g$, with is the implicit derivative respect to $x$. The intersection of the two is the solution to the inverse problem, the relative distance from the midpoint to the scatterer. The solution $(x, z)=(1500 \mathrm{~m},-500 \mathrm{~m})$ agrees with the location of the scatterer in the specified model.


Fig. 7. Time error for four different offsets, for an isotropic medium (i.e., Thomsen parameters $\epsilon=0.0 \delta=0.0$ ). The offsets are 1000 -black solid, 2000 m -black dash 3000 m - gray solid, and 4000 m -gray dash. The migration velocity coincides with the true vertical velocity for this isotropic case, so the migration is accurate for all offsets.
ignoring portions of traces for which $X>d$, where $X$ is offset, and $d$ is the distance between the midpoint and the diffractor.

As mentioned above, the velocity that we use for migration is based on the stacking velocity for horizontal reflectors. For reference, we first consider a test in which the data are for an isotropic medium and are thus migrated with the appropriate algorithm. The expected result should be zero position error for any dip, and any offset. Given that the ray tracing and migration algorithms are independent programs, we can trust their functionality if the resulting error is zero. Figure 7 shows the (zero) error computed in this case.

In our tests with models of TI media, we characterize the media in terms of the Thomsen parameters $\epsilon$ and $\delta$. As mentioned above, Tsvankin (1995) observed that, when anisotropy is ignored for VTI media, DMO error depends primarily on one particular combination of Thomsen parameters, the difference $\epsilon-\delta$. Subsequently, Alkhalifah and Tsvankin (1995) showed that all time-related $P$-wave processing in homogeneous VTI media are fully described by the combination of Thomsen parameters given by $\eta=(\epsilon-\delta) /(1+2 \delta)$, plus the stacking velocity for horizontal reflectors. Patterned on their results, we therefore group our test results, in which $\delta$ and $\epsilon$ vary, by constant value of $\eta$.

Figure 8 shows results for four sets of $(\epsilon, \delta)$, all with $\eta=0.2$. (Media for which $\eta=0$ are elliptically anisotropic. Therefore, the magnitude of $\eta$ is a measure of the departure of a medium from elliptical anisotropy.) The difference error is the


[^0]:    ${ }^{1}$ The Fourier transform here is multiplication by $\exp \left\{-i \kappa_{1}\left(x_{g}-x_{s}\right)\right\}$ and integration over $x_{g}$, or, equivalently, multiplication by $\exp \left\{-i \kappa_{1} s\right\}$ and integration over $s$.

[^1]:    ${ }^{2}$ Note, however, that we have redefiined the variable $n$ here to write the total normal distance as the sum of the two distances from source to reflector and receiver to reflector.

[^2]:    ${ }^{3}$ When the acoustic wave equation is written in non-self-adjoint form, minor adjustments must be made to account for non-symmetric reciprocity. In fact, von Vroonhoven starts from self-adjoint coupled equations for pressure and particle velocity, so that this derviation applies to her equations. This is only appropriate, since this derivation follows hers.

[^3]:    ${ }^{1}$ Throughout this paper $\mathrm{rg}_{\mathrm{g}}$ and $\mathrm{r}_{\mathrm{s}}$ will represent spatial coordinates of receiver and source respectively, $t$ the traveltime and $\omega$ the temporal frequency.

