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SYMMETRIC PRE-STACK INVERSION OPERATOR FOR MULTI-SOURCE, MULTI-GEOPHONE DATA

by

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ABSTRACT

Multi-source, multi-geophone data provides a redundancy which can be exploited for image enhancement after pre-stack common shot or common geophone inversion. After common shot inversions are averaged over shots or common geophone data are averaged over geophones, the resulting inversions are both sums over all shots and geophones. We pose the question of how to weight each of inversions in the averaging process. In this paper, we answer that question by imposing the criterion that the resulting averages yield exactly the same inversion algorithm, except possibly for the order in which the summations are carried out. We show that this criterion is sufficient to determine the weighting factors and that the resulting inversion operator is symmetric in source and geophone coordinates, except in the extremely rare case when the background density at the source and geophone locations are different.

GLOSSARY

A_s	WKBJ amplitude of 3D Green's function for acoustic wave equation, constant density, source point at $\underset{\sim}{x}_s$ and obvservation point at $\underset{\sim}{x}$.
A_g	WKBJ amplitude of 3D Green's function for acoustic wave equation, constant density, source point at $\underset{\sim}{x}_g$ and obvservation point at $\underset{\sim}{x}$.
A_{s2}	WKBJ amplitude of 2D Green's function for acoustic wave equation, constant density, source point at $\underset{\sim}{x}_s$ and obvservation point at $\underset{\sim}{x}$.
A_{g2}	WKBJ amplitude of 2D Green's function for acoustic wave equation, constant density, source point at $\underset{\sim}{x}_g$ and obvservation point at $\underset{\sim}{x}$.
$\beta_\eta(\underset{\sim}{x})$	Inversion output from a single common geophone data set.
$\beta_\xi(\underset{\sim}{x})$	Inversion output from a single common shot data set.
$\beta_1(\underset{\sim}{x})$	Weighted inversion over shots (13).
$\beta_2(\underset{\sim}{x})$	Weighted inversion over geophones (15).
$\cos\theta$	$c^2(\underset{\sim}{x})\underset{\sim}{p}_s \cdot \underset{\sim}{p}_g$.
$c(\underset{\sim}{x})$	Background propagation speed.
$D(\underset{\sim}{\xi},\underset{\sim}{\eta},\omega)$	Observed data in the frequency domain.
$E(\underset{\sim}{\xi},\underset{\sim}{\eta},t)$	Inverse Fourier transform of filtered observed data (9) for 3D inversion.
$E_2(\underset{\sim}{\xi},\underset{\sim}{\eta},t)$	Inverse Fourier transform of filtered observed data (9) for 2.5D inversion.
$h_\xi(\underset{\sim}{x},\underset{\sim}{\eta})$	Determinant that arises in the inversion (4), defined by (9).
$h_\eta(\underset{\sim}{x},\underset{\sim}{\xi})$	Determinant that arises in the inversion (11), defined by (12).
$\Omega_\sigma(\underset{\sim}{x})$	Angular aperture of initial ray directions at $\underset{\sim}{x}$ to all source or geophone locations.
$\underset{\sim}{p}_s$	$\nabla\tau_s$.
$\underset{\sim}{p}_g$	$\nabla\tau_g$.
r_ξ, r_η	Distance from $\underset{\sim}{x}$ to source and receiver points, respectively.
$\rho(\underset{\sim}{x})$	Background density.
ρs, ρ_g	$\rho(\underset{\sim}{x}_s)$, $\rho(\underset{\sim}{x}_g)$.
τ_s	WKBJ travel time from $\underset{\sim}{x}_s$ to $\underset{\sim}{x}$ in background propagation speed.

INTRODUCTION

The inversion formalism that we presently use [Bleistein, Cohen and Hagin, 1986, Bleistein, 1986, 1987a, 1987b, Cohen, Bleistein and Hagin, 1987] provides a pre-stack inversion method for inverting either common shot or common receiver data. A typical seismic survey contains redundant data, allowing for either type of inversion. Therefore, one could contemplate averaging the inversion of the common shot data over all the shots, or averaging the common receiver data over all of the receivers. In either case, one obtains an inversion as a sum (integral, in a continuous model) over all sources and receivers.

The question arises, then, how this averaging is to take place, that is, how the outputs of the separate common shot inversions are to be weighted in the sum, or how the separate common receiver inversions are to be weighted in that sum. We propose that the weightings ought to be determined by the requirement that the two "averaged inversions" be identical, since they both represent an inversion as a sum over all shots and receivers. We show here that this criterion is sufficient to determine the weights. This results in an inversion operator that is symmetric in source and receiver coordinates except for a ratio of background densities at the source and receiver locations. In all but the rarest of cases, this ratio would be equal to unity. That is, the background densities at the source and receiver points will be equal. Thus, we view this operator as a symmetric operator for all practical applications.

We believe that averaging in this manner will enhance the image of the output as compared to the separate pre-stack inversions. However, we find that something is lost in this averaging process. The peak amplitude of the

DERIVATION OF WEIGHTING FACTORS

Let us suppose that sources and receivers are arrayed on a surface, S_σ, whose equation is

$$\underset{\sim}{x} = \underset{\sim}{f}(\underset{\sim}{\sigma}) \ , \ \underset{\sim}{\sigma} = (\sigma_1, \sigma_2) \ \varepsilon \ S_\sigma \quad . \tag{1}$$

That is, as the two parameters (σ_1,σ_2) vary in a domain S_σ, $\underset{\sim}{x}$ ranges over the source/receiver surface S_σ. In the totality of experiments, both the sources and receivers cover S_σ. In a common shot experiment, the source location, $\underset{\sim}{x}_s$, is fixed while the geophone locations vary over some subset of S_σ. This subset is different for each shot. Therefore, we characterize a common shot experiment by

$$\underset{\sim}{x}_s(\underset{\sim}{\xi}) = \underset{\sim}{f}(\underset{\sim}{\xi}) \ , \ \underset{\sim}{\xi} \ \varepsilon \ S_\sigma \ ; \ \underset{\sim}{x}_g(\underset{\sim}{\eta}) = \underset{\sim}{f}(\underset{\sim}{\eta}) \ , \ \underset{\sim}{\eta} \ \varepsilon \ S_\eta(\underset{\sim}{\xi}) \quad . \tag{2}$$

If we re-order the data as common geophone data, then for each geophone location, the source locations vary over some subset of S_σ. This subset also differs for each geophone. Therefore, one characterizes a common geophone experiment by

$$\underset{\sim}{x}_g(\underset{\sim}{\eta}) = \underset{\sim}{f}(\underset{\sim}{\eta}) \ , \ \underset{\sim}{\eta} \ \varepsilon \ S_\sigma \ ; \ \underset{\sim}{x}_s(\underset{\sim}{\xi}) = \underset{\sim}{f}(\underset{\sim}{\xi}) \ , \ \underset{\sim}{\xi} \ \varepsilon \ S_\xi(\underset{\sim}{\eta}) \quad . \tag{3}$$

We denote by $\beta_\xi(\underset{\sim}{x})$ the acoustic reflectivity function obtained by inverting a single common shot data set. From Bleistein [1987b], equation (4),

$$E(\underset{\sim}{\xi},\underset{\sim}{\eta},\underset{\sim}{x}) = \int_{-\infty}^{\infty} i\omega\, F(\omega)\, d\omega\, D(\underset{\sim}{\xi},\underset{\sim}{\eta},\omega)\, \exp\left[-i\omega\left[\tau_s + \tau_g\right]\right] \tag{9}$$

is the filtered $[i\omega\, F(\omega)]$ inverse Fourier transform of the frequency domain data set $D(\underset{\sim}{\xi},\underset{\sim}{\eta},\omega)$, evaluated at the sum of the traveltimes, $\tau_s + \tau_g$. Finally,

$$h_{\xi}(\underset{\sim}{x},\underset{\sim}{\eta}) = \det \begin{vmatrix} \underset{\sim}{p}_s + \underset{\sim}{p}_g \\ \dfrac{\partial \underset{\sim}{p}_g}{\partial \eta_1} \\ \dfrac{\partial \underset{\sim}{p}_g}{\partial \eta_2} \end{vmatrix} . \tag{10}$$

In (4-10), the cited equation (4), Bleistein [1987a] has been specialized, with notation appropriately modified to account for the fact that this is a common shot inversion - fixed $\underset{\sim}{x}_s$, hence fixed $\underset{\sim}{\xi}$ and integration over geophones, hence varying $\underset{\sim}{\eta}$.

Similarly, one can write down a common receiver inversion as an integration over the shot array. This is also deduced from Bleistein [1987b], equation (4). The result is

$$\beta_{\eta}(\underset{\sim}{x}) = \frac{1}{8\pi^3} \int_{S_{\xi}(\underset{\sim}{\eta})} d^2\xi \sqrt{\frac{\rho_s}{\rho_g}}\, \frac{\left|h_{\eta}(\underset{\sim}{x},\underset{\sim}{\xi})\right|\, E(\underset{\sim}{\xi},\underset{\sim}{\eta},\underset{\sim}{x})}{A_s A_g \left|\underset{\sim}{p}_s + \underset{\sim}{p}_g\right|} . \tag{11}$$

The only new function in this result is

$$\bar{W}_2(\underset{\sim}{x}) = \int_{S_\sigma} d^2\eta \; W_2(\underset{\sim}{\eta}, \underset{\sim}{x}) \quad . \tag{16}$$

Substitution of (4) into (13) and (11) into (15) yields the two results

$$\beta_1(\underset{\sim}{x}) = \frac{1}{8\pi^3 \, \bar{W}_1(\underset{\sim}{x})} \int_{S_\sigma} d^2\xi \; W_1(\underset{\sim}{\xi}, \underset{\sim}{x}) \int_{S_\eta(\underset{\sim}{\xi})} d^2\eta \sqrt{\frac{\rho_s}{\rho_g}} \; \frac{h_\xi(\underset{\sim}{x}, \underset{\sim}{\eta}) \; E(\underset{\sim}{\xi}, \underset{\sim}{\eta}, \underset{\sim}{x})}{A_s A_g \; | \, \underset{\sim}{p}_s + \underset{\sim}{p}_g \, |} \quad , \tag{17}$$

$$\beta_2(\underset{\sim}{x}) = \frac{1}{8\pi^3 \, \bar{W}_2(\underset{\sim}{x})} \int_{S_\sigma} d^2\eta \; W_2(\underset{\sim}{\eta}, \underset{\sim}{x}) \int_{S_\xi(\underset{\sim}{\eta})} d^2\xi \sqrt{\frac{\rho_s}{\rho_g}} \; \frac{h_\eta(\underset{\sim}{x}, \underset{\sim}{\xi}) \; E(\underset{\sim}{\xi}, \underset{\sim}{\eta}, \underset{\sim}{x})}{A_s A_g \; | \, \underset{\sim}{p}_s + \underset{\sim}{p}_g \, |} \quad . \tag{18}$$

Our objective is to make these two averages the same. First, we interchange the order of integration in the first of these equations. Since both lines represent integration over the identical source/receiver array, this interchange must result in the integrations having identical ranges of integration. Thus, to make the integrals agree, we need only make the integrands agree. This reduces to the requirement

$$\frac{W_1(\underset{\sim}{\xi}, \underset{\sim}{x}) \; h_\xi(\underset{\sim}{x}, \underset{\sim}{\eta})}{\bar{W}_1(\underset{\sim}{x})} = \frac{W_2(\underset{\sim}{\eta}, \underset{\sim}{x}) \; h_\eta(\underset{\sim}{x}, \underset{\sim}{\xi})}{\bar{W}_2(\underset{\sim}{x})} \quad . \tag{19}$$

This requirement can be simplified by using a result derived in Beylkin [1985] and, alternatively, in equations (45-49) in Cohen, Hagin and Bleistein [1987]. In the present notation,

$\underset{\sim}{x}$, independent of $\underset{\sim}{\xi}$. Thus, a necessary condition for equality here is that

$$W_1(\underset{\sim}{\xi},\underset{\sim}{x}) = \frac{f_1(\underset{\sim}{x})}{|p_{3s}|} \left| \frac{\partial(p_{1s},p_{2s})}{\partial(\xi_1,\xi_2)} \right| \tag{24}$$

and

$$W_2(\underset{\sim}{\eta},\underset{\sim}{x}) = \frac{f_2(\underset{\sim}{x})}{|p_{3g}|} \left| \frac{\partial(p_{1s},p_{2s})}{\partial(\eta_1,\eta_2)} \right| , \tag{25}$$

with $f_1(\underset{\sim}{x})$ and $f_2(\underset{\sim}{x})$ arbitrary functions of $\underset{\sim}{x}$. However, the choice of f_1 and f_2 affects neither (23) nor the averages, (17) and (18). Hence, we choose both of these functions to be equal to one.

We use these results to further simplify the criterion, (23) to

$$\bar{W}_1(\underset{\sim}{x}) = \int_{S_\sigma} \left| \frac{\partial(p_{1s},p_{2s})}{\partial(\xi_1,\xi_2)} \right| \frac{d\xi_1 d\xi_2}{|p_{3s}|} = \bar{W}_2(\underset{\sim}{x}) = \int_{S_\sigma} \left| \frac{\partial(p_{1g},p_{2g})}{\partial(\eta_1,\eta_2)} \right| \frac{d\eta_1 d\eta_2}{|p_{3g}|} . \tag{26}$$

These integrals are identical, since $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ simply represent different dummy variables of integration on the same surface S_σ and $\underset{\sim}{p}_s$ is the same function of $\underset{\sim}{\xi}$ as $\underset{\sim}{p}_g$ is of $\underset{\sim}{\eta}$. Before stating our conclusion about W_1 and W_2, we derive an interesting interpretation of this integral.

Let us consider the left side of (26), which can be rewritten as

$$\bar{W}_1(\underset{\sim}{x}) = \int_{P_\sigma(\underset{\sim}{x})} \frac{dp_{1s}dp_{2s}}{p_{3s}} . \tag{27}$$

$$\beta_1(\underset{\sim}{x}) = \frac{c(\underset{\sim}{x})}{4\pi^3 \Omega_\sigma(\underset{\sim}{x})} \int_{S_\sigma} d^2\xi \int_{S_\eta(\underset{\sim}{\xi})} d^2\eta \sqrt{\frac{\rho_s}{\rho_g}} \frac{|h_\xi(\underset{\sim}{x},\underset{\sim}{\eta}) h_\eta(\underset{\sim}{x},\underset{\sim}{\xi})| E(\underset{\sim}{\xi},\underset{\sim}{\eta},\underset{\sim}{x})}{A_s A_g |\underset{\sim}{p}_s + \underset{\sim}{p}_g|^3} , \qquad (31)$$

and

$$\beta_2(\underset{\sim}{x}) = \frac{c(\underset{\sim}{x})}{4\pi^3 \Omega_\sigma(\underset{\sim}{x})} \int_{S_\sigma} d^2\eta \int_{S_\xi(\underset{\sim}{\eta})} d^2\xi \sqrt{\frac{\rho_s}{\rho_g}} \frac{|h_\xi(\underset{\sim}{x},\underset{\sim}{\eta}) h_\eta(\underset{\sim}{x},\underset{\sim}{\xi})| E(\underset{\sim}{\xi},\underset{\sim}{\eta},\underset{\sim}{x})}{A_s A_g |\underset{\sim}{p}_s + \underset{\sim}{p}_g|^3} . \qquad (32)$$

It is now apparent that the averaged reflectivities differ only in the order of integration.

$$\frac{ds}{d\sigma_\xi} = \frac{1}{c(\underset{\sim}{x})} , \quad \frac{d\tau}{d\sigma_\xi} = \frac{1}{c^2(\underset{\sim}{x})} . \tag{35}$$

A similar definition obtains for A_{g2} and τ_g and σ_η. The data is defined by

$$E_2(\xi,\eta,\underset{\sim}{x}) = \int \sqrt{|\omega|}\, F(\omega)\, d\omega\, D(\xi,\eta,\omega)\, \exp\left[-i\omega\, [\tau_s+\tau_g] + i\pi/4\ \mathrm{sgn}\omega\right] . \tag{36}$$

The 2X2 determinant $H_\xi(\underset{\sim}{x},\underset{\sim}{\eta})$ replaces the 3X3 determinant defined by (10). It is given by

$$H_\xi(\underset{\sim}{x},\eta) = \det \begin{vmatrix} \underset{\sim}{p}_s + \underset{\sim}{p}_g \\ \frac{\partial}{\partial_\eta}\ \underset{\sim}{p}_g \end{vmatrix} . \tag{37}$$

The analysis proceeds as in the previous section and results in the symmetric inversion

$$\beta_1(\underset{\sim}{x}) = \frac{2}{(2\pi)^{3/2}\Omega_\sigma(\underset{\sim}{x})} \int_{S_\sigma} d\xi \int_{S_\eta(\xi)} d\eta \sqrt{\frac{\rho_s}{\rho_g}}\ \frac{|H_\xi(\underset{\sim}{x},\eta) H_\eta(\underset{\sim}{x},\xi)|\, E_2(\xi,\eta,\underset{\sim}{x}) \sqrt{\sigma_\xi+\sigma_\eta}}{A_{s2}A_{g2}\ |\underset{\sim}{p}_s+\underset{\sim}{p}_g\ |^3} . \tag{38}$$

In this equation, $H_\eta(\underset{\sim}{x},\xi)$ is defined by (32) when we interchange ξ and η and interchange s and g. Also,

IMPULSE RESPONSES

We consider here the impulse response in a constant background medium for a common shot inversion and for the symmetric inversion operator over all shots in 2.5D. The point of this is to see the assymetry of the former and confirm the symmetry of the latter. It will possible to demonstrate this both analytically and numerically.

We consider first the impulse response for the common shot inversion defined by equation (33). Therefore, let us suppose that there is a single shot at $\xi = -h_0$ in the form of an impulse,

$$D(-h_0,\eta,t) = \delta(\eta-h_0)\delta(t-t_0). \tag{40}$$

Thus, for a source at $-h_0$, the only nonzero data occurs at the receiver located at h_0. If the output is to be symmetric, then it should be possible to interchange these two points without change the result. Alternatively, the output ought to be a symmetric function of x. That is, replacing x by -x should not change the output, since x = 0 should be the axis of symmetry.

For this data, the Fourier transform is given by

$$D(-h_0,\eta,\omega) = \delta(\eta-h_0)\exp\{i\omega t_0\}. \tag{41}$$

We use (36) to define E_2. However, we make a minor modification in that definition. The filter, $\sqrt{|\omega|}\cdot\exp\{i\pi/4\,\mathrm{sgn}\,\omega\}$ is designed to produce bandlimited delta functions as output when the input is an array of connected events spread over a number of traces. For impulse data, this filter will produce a different distribution (in fact, $1/t^{3/2}$, peaking on the reflector). To produce a bandlimited delta function for this data, we

$$\tau_s = r_\xi / c \; , \; \tau_g = r_\eta / c \; . \tag{47}$$

When the specific function E_2 defined by (42) is substituted into (44), the delta function, $\delta(\eta - h_0)$, can be exploited to carry out the η integration. The result is

$$\beta_\xi(\underset{\sim}{x}) = \frac{4z\sqrt{2\pi}}{c^{3/2}} \; \frac{\cos\theta \sqrt{r_\xi + r_\eta} \; \sqrt{r_\xi r_\eta}}{r_\eta^2} \; \delta_B(t_0 - \tau_s - \tau_g), \tag{48}$$

except that now η is evaluated at h_0, so that

$$r_\xi = \sqrt{(x + h_0)^2 + z^2} \; , \quad r_\eta = \sqrt{(x - h_0)^2 + z^2} \; , \tag{49}$$

$$\cos\theta = \frac{(x + h_0)(x - h_0) + z^2}{r_\xi r_\eta} \; . \tag{50}$$

The impulse response, (48), is a scaled bandlimited delta function which peaks on the ellipse,

$$ct_0 = \tau_s + \tau_g = \sqrt{(x + h_0)^2 + z^2} + \sqrt{(x - h_0)^2 + z^2} \; . \tag{51}$$

This is the expected location of the peak of the output. The scaling on the delta function in (48) can be seen to be symmetric about the origin except for the factor $1/r_\eta^2$. This factor characterizes the entire assymmetry of the impulse response. Recall that r_η measure the distance from the output point, (x,z) to the geophone location, $(h_0,0)$. Thus, the effect of this

$$\beta_1(\underset{\sim}{x}) = \frac{8}{c\sqrt{2\pi c}} \; \frac{z^2\cos\theta\sqrt{r_\xi + r_\eta}}{\left[r_\xi r_\eta\right]^{3/2}} \; \delta_B(t_0 - \tau_s - \tau_g), \; \xi = -h_0, \; \eta = h_0. \qquad (56)$$

Again, the output peaks on the appropriate ellipse. However, the symmetry of the amplitude is now self evident. It should be noted however, that the peak amplitude increases with depth, approximately like O(z). The reason for this is that the input was taken to have the same amplitude now matter how deep the reflector is, that is, no matter how large t_0 is. In order for the input amplitude to remain constant, the reflector strength would have to increase with depth. This is what is seen in this output.

Figures 1-3 demonstrate these results numerically. Figure 1 is the impulse response for common shot inversion and should be compared to the result (48). Figure 2 is the common geophone impulse response. This is merely a reflection of the previous result through the line, $x = h_0$. Finally, Figure 3 is the impulse response to the averaged inversion and should be compared to the result (56). The predicted symmetry is evident in the figure.

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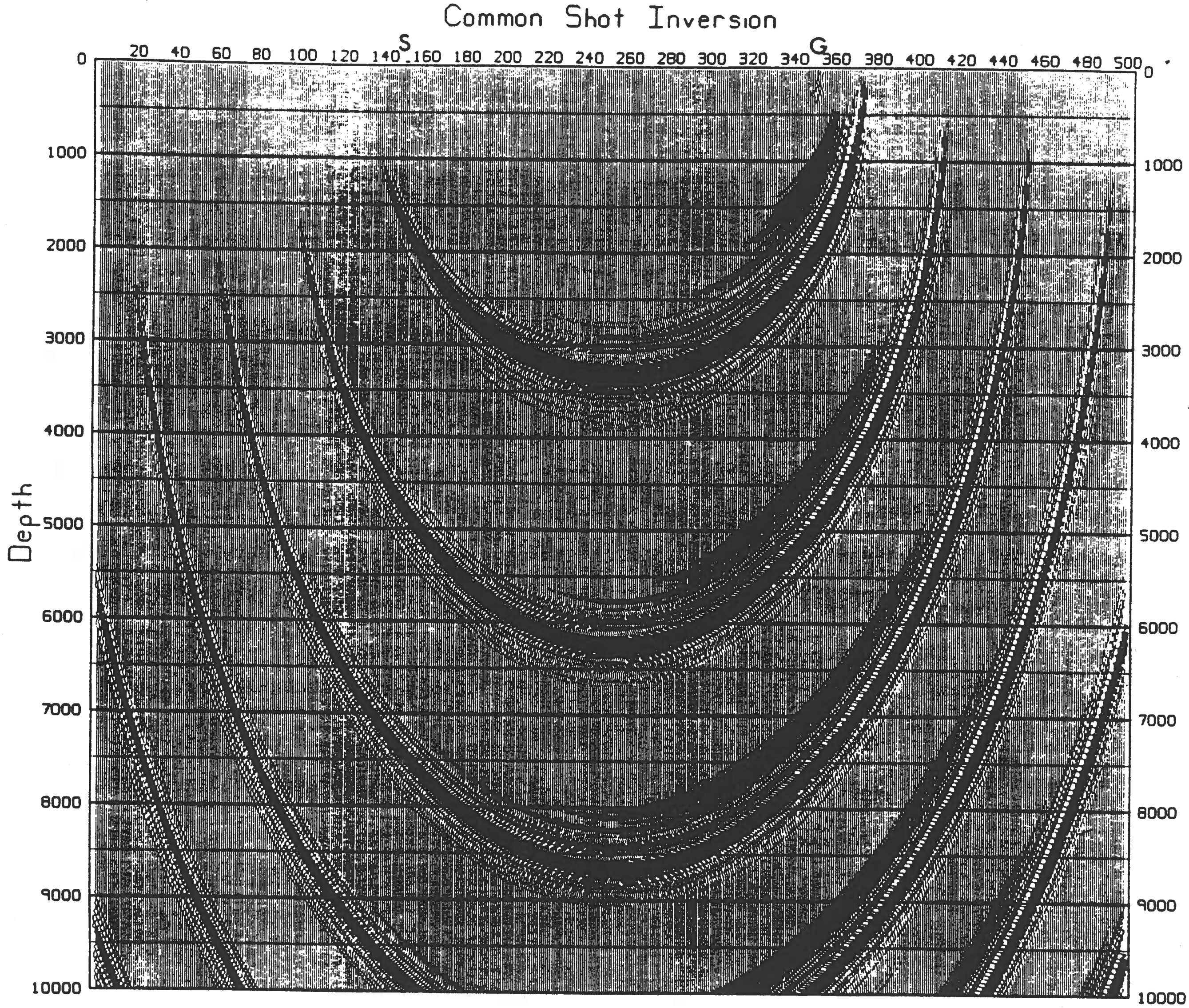

Fig. 1: **Common Shot Inversion Operator. Source and geophone are at the locations marked. Trace spacing is 50 ft. Depth spacing is 20 ft. Velocity is a constant 10,000 ft/sec. The amplitudes in these figures have been compressed by taking the square root.**

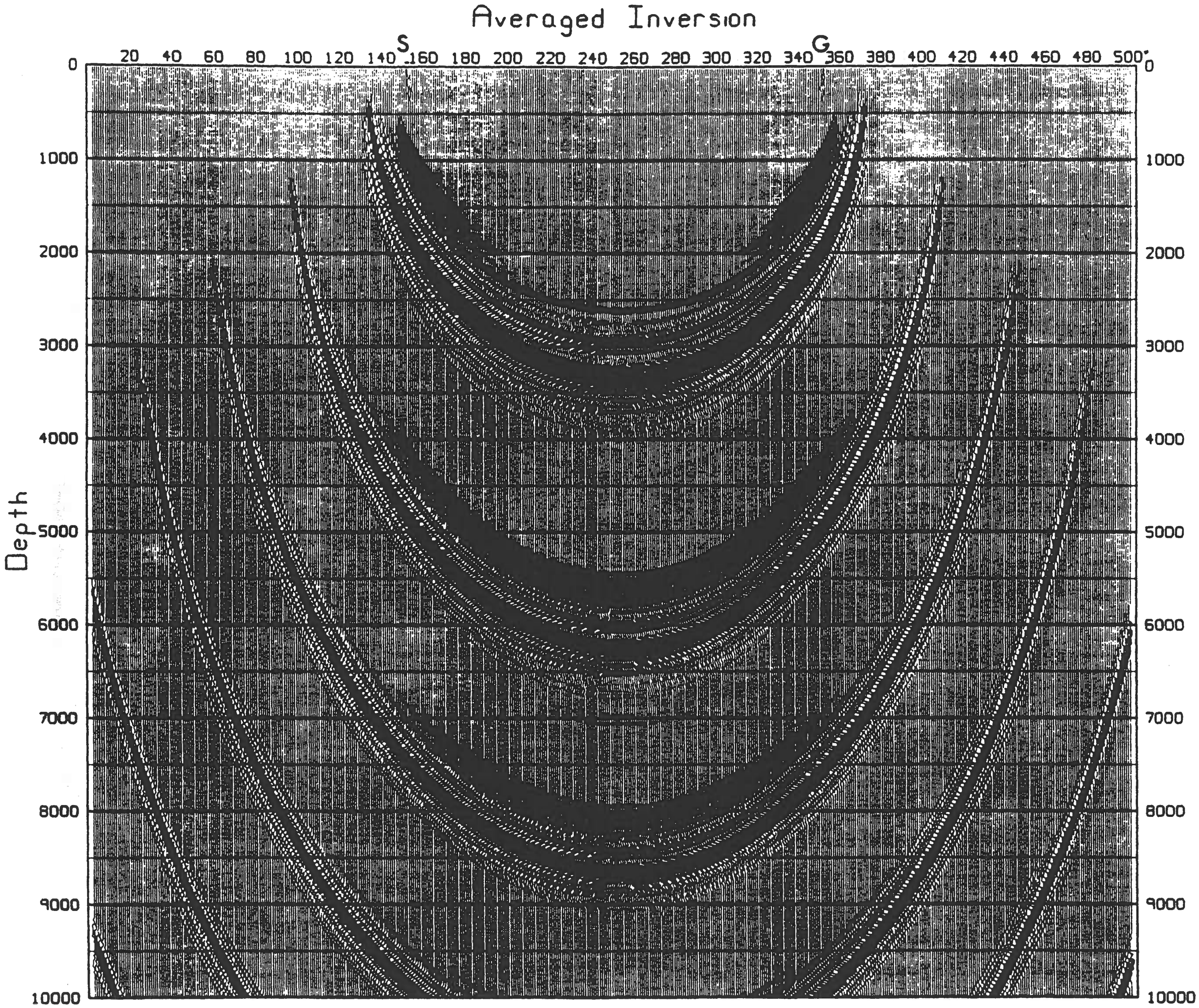

Fig. 3: Averaged Inversion Operator. Source and geophone are at the locations marked. Trace spacing is 50 ft. Depth spacing is 20 ft. Velocity is a constant 10,000 ft/sec.

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APPENDIX

In this appendix, we derive equations (44) and (52). That is, we specialize equations (33) and (38), respectively, and assume that the background speed and density are constants. In this case, all of the geometrical optics variables can be found in a number of places, including, Bleistein [1986]. The geometrical optics rays are straight lines, the travels times are just distance divided by propagation speed, and σ is just distance times sound speed. In summary then,

$$r_\xi = \sqrt{(x-\xi)^2 + z^2}\ , \quad r_\eta = \sqrt{(x-\eta)^2 + z^2}\ ,$$
$$\tau_\xi = r_\xi/c\ , \quad \tau_\eta = r_\eta/c\ , \quad \sigma_\xi = cr_\xi\ , \quad \sigma_\eta = cr_\eta\ . \tag{A1}$$

The amplitudes of the the WKBJ Green's functions are given by

$$A_{s2} = \frac{1}{2}\sqrt{\frac{c}{2\pi r_\xi}}\ , \quad A_{g2} = \frac{1}{2}\sqrt{\frac{c}{2\pi r_\eta}}\ . \tag{A2}$$

We use (37) to compute $H_\xi(\underline{x},\eta)$ and the definition below (38) to compute $H_\eta(\underline{x},\xi)$ for this case, with $\underline{p}_s$ and $\underline{p}_g$ the two dimensional gradietns of the coresponding travel times. The results are

$$H_\xi(\underline{x},\eta) = \frac{2z\cos^2\theta}{c^2 r_\eta^2}\ , \quad H_\eta(\underline{x},\xi) = \frac{2z\cos^2\theta}{c^2 r_\xi^2}\ , \tag{A3}$$

with