CWP-041

KIRCHHOFF INVERSION FOR REFLECTOR IMAGING AND SOUNDSPEED AND DENSITY VARIATIONS

by

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Partially supported by the Consortium Project of the Center for Wave Phenomena and by the Selected Research Opportunities Program of the Office of Naval Research

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ABSTRACT

A brief description of an inversion formalism for acoustic data in a variable soundspeed and variable density medium is presented. An inversion operator is introduced and applied to Kirchhoff approximate data for a single reflector. By asymptotic analysis of the output, a number of conclusions are drawn. First, the operator produces a bandlimited singular function of the reflecting surface. The singular function constitutes a mathematical image of the reflector and its pictorial representation does indeed depict the reflector. Second, each singular function is scaled by a slowly varying function of spatial coordinates. At the peak of the bandlimited singular function, this scale becomes the geometrical optics reflection coefficient at some (as yet unknown) incidence angle. Since the peak value of the singular function is known, this reflection coefficient is determined, as well. By introducing a second inversion operator, differing from the first in only a minor way, I can determine that incidence angle. At this point, in a constant density medium, the soundspeed below the reflector can be determined from the known values of (i) the soundspeed above, (ii) the reflection coefficient, and (iii) the incidence angle. When the density varies, as well, a second experiment must be carried out. This second experiment must differ from the first sufficiently to provide the reflection coefficient at a different incidence angle. From the two experiments, sufficient information is available to determine the values of soundspeed and density below the reflector in terms of the values above the reflector.

This method applies to common source, common receiver or common offset experiments in which the soundspeed and density above the reflector are known functions of all three spatial variables. The density variations across the reflector need not be small. However, the "known" background must be close to the true values in order for proper location of the reflector and proper determination of the soundspeed and density below the reflector. Also, multiples from reflectors above the one being analyzed must be sufficiently negligible so as not to contaminate the response from the given reflector.

This inversion is partially based on the theory proposed by Beylkin [1985a]. That theory arises from a Born approximation of the upward scattered field. The results here demonstrate that his inversion, as well as the modification proposed here, apply much more broadly than their basis in the Born approximation would suggest.

GLOSSARY

$A(\underline{x},\underline{x}_s)$	Amplitude of ray-theoretic or WKBJ Green's function for background sound speed with source at $\underline{x}_s$ and observation point $\underline{x}$.
$c(\underline{x})$	Soundspeed above the reflector S.
$c_+(\underline{x})$	Soundspeed below the reflector S.
$D(\underline{\xi},\omega)$	Observed data, upward scattered wave from S.
$F(\omega)$	Filtered (smoothed and tapered) source function in the Fourier domain.
$\Phi(\underline{x},\underline{x}',\underline{x}_s,\underline{x}_r)$	Phase of inversion operator applied to Kirchhoff-approximate field data.
$[\Phi_{\xi\sigma}]$	Hessian matrix of the phase of the inversion operator.
g	First fundamental form of differential geometry evaluated on the reflector S.
$\gamma(\underline{x})$	Singular function of the surface S.
$\gamma_B(\underline{x})$	Bandlimited version of $\gamma(\underline{x})$.
$h(\underline{x},\underline{\xi})$	Determinant arising in the inversion operator.
$\underline{k}$	Approximate Fourier variable of the inversion theory.
$\underline{n}$	Upward normal vector on the reflector S.
$R(\underline{x}',\underline{x}_s)$	Geometrical optics reflection coefficient.

$\rho(\underline{x})$ — Density above the reflector S.

$\rho_+(\underline{x})$ — Density below the reflector S.

S — Reflector in Kirchhoff representation of upward propagating wave.

S_0 — Observation surface.

S_ξ — Domain of integration in ξ-variables.

$\underline{\sigma} = (\sigma_1, \sigma_2)$ — Parameters used to define a reflecting surface.

$\tau(\underline{x}, \underline{x}_s)$ — Ray-theoretic travel time between $\underline{x}$ and $\underline{x}_s$.

θ — Angle between the normal to a surface at the point $\underline{x}'$ and the ray from $\underline{x}_s$ or $\underline{x}_r$ to $\underline{x}'$, under the stationarity conditions. Opening angle between these rays and normal subject to Snell's law of reflection.

$\underline{x}$ — Point at which the output of inversion operator applied to $D(\underline{\xi}, \omega)$ is to be evaluated.

$\underline{x}' = \underline{x}'(\underline{\sigma})$ — Point on reflecting surface.

$\underline{x}_s$, $\underline{x}_r$ — Source and receiver coordinates, respectively.

$\underline{\xi} = (\xi_1, \xi_2)$ — Parameters labelling source and/or receiver points; i. e., $\underline{x}_s = \underline{x}_s(\underline{\xi})$ and/or $\underline{x}_r = \underline{x}_r(\underline{\xi})$.

INTRODUCTION

The Kirchhoff approximation applied to the Kirchhoff integral representation of a field on one side of a reflecting surface provides an accurate high frequency approximation to the fields reflected by that surface [Hilterman, 1970]. Thus, it would seem reasonable that one might attempt to use this representation of the upward propagating wave for the purposes of migration and inversion. Indeed, Schneider [1978] demonstrated the utility of this approach for migration. In a series of papers in which I was involved [Bleistein, 1976, Bleistein and Cohen, 1980, Cohen and Bleistein, 1979a], inversion based on the Kirchhoff approximation was employed. The results in those papers paralleled our work in Born inversion [Cohen and Bleistein 1977, 1979b, 1981, Bleistein and Cohen, 1979]; that is, the work was zero offset, constant background soundspeed and constant density inversion. However, all of the Kirchhoff based inversions except for Cohen and Bleistein [1981] also used the far field approximation.

More recently, our inversion research [Bleistein, 1986a, Bleistein and Gray, 1985, Bleistein, Cohen and Hagin, 1986, Cohen, Bleistein and Hagin, 1986, Cohen and Hagin, 1985], as well as the work by others, especially Beylkin and associates [Beylkin, 1984, 1985a, 1985b, Beylkin, Oristaglio and Miller, 1985], addresses the problem of two and three dimensional inversion for soundspeed in a variable background medium, depending on one, two or three spatial variables. Except for Bleistein [1986a], all of these results are based on the Born approximation of the upward scattered wave. The inversion in that reference is based on the Kirchhoff approximation.

Beylkin [1985a] is a particularly noteworthy paper, in that a method is presented for a general source/receiver configuration and arbitrary soundspeed variation, with the viability of the experimental set-up characterized by the nonvanishing of a certain Jacobi determinant. The method proposed is a high frequency inversion of seismic data which produces the discontinuity surfaces of the perturbation in the propagation speed (which might even be the mode converted speed from acoustic to shear or *vice versa*). The justification of the inversion is based on pseudo-differential operator theory. The relationship between the output of the operator and the location of soundspeed variations or their magnitude are not stated by Beylkin.

The major advantage of Beylkin's approach is that it treats a wide variety of source/receiver configurations of interest in inversion experiments in seismic exploration and in other fields. Before this paper apeared, we had developed in our own group an approach to these general problems which, though systematic, required a separate analysis for each source/receiver configuration. (See Cohen and Hagin, [1985], and Sullivan and Cohen, [1985].) The inversion formulas that were derived were expressed in terms of the determinant of a certain Hessian matrix of a phase function which is the sum of travel times from source to output point to receiver. Beylkin's method provides a general expression for these various Hessian's which subsumes all of our separate results. Equation (23) below is essentially a statement of the relationship between these two determinants expressions.

In Bleistein [1986a], I proposed a modification of Beylkin's inversion operator which produces a singular function for each discontinuity surface of the soundspeed. The singular function is a Dirac delta function whose argument is arclength measured on any family of curves normal to the surface. Consequently, the support of the singular function is the discontinuity surface or reflector. Thus, determination of the singular function constitutes mathematical imagipair is unknown. However, by a slight modification of the inversion operator, it is possible to determine this critical incidence angle at each point. Once the angle is determined, the reflection coefficient can be unraveled to produce the soundspeed below the reflector as a function of that incidence angle and the soundspeed above the reflector.

These results were derived by applying the inversion operator which I proposed to Kirchhoff-approximate data from a single reflector. Thus, although Beylkin's inversion operator was motivated by analysis of the Born approximation of the upward scattered wave, the analysis of my paper demonstrated that the increment in soundspeed across the reflector used for the test data need not be small, thus extending the validity of both Beylkin's inversion operator and mine. On the other hand, the soundspeed above the reflector must, of necessity, not deviate too much from the "true" soundspeed of the medium in order that both location of the reflector and computation of the soundspeed below the reflector be accurate. Furthermore, it is necessary that multiple reflections from other reflectors above the test reflector be sufficiently small so that the data for the particular reflector in question is not too badly contaminated by this multiple reflection data. Thus, this method does not completely dispense with the need for small variations in soundspeed.

On the other hand, given that the method allows for arbitrary soundspeed variations, one could contemplate recursive application of the algorithm. In this application, one would invert data down through the first major reflector, use the information gained to define a soundspeed deeper into the earth and then invert down to the next major reflector, and so on. This would tend to diminish the first problem of the previous paragraph, but not the second. While the inversion method proposed here could in theory be extended to deal with multiples, I do not believe that the subsurface structure would ever be known to sufficient accuracy to make that a viable technique.

In this paper, I will briefly show how to extend the method of Bleistein [1986a] to the case in which the density varies, as well as the soundspeed. In that paper, the inversion operator applied to Kirchhoff data was represented as a five fold integral over the surface of observations, the reflector surface, and frequency. The four fold spatial integration was approximated by the method of stationary phase. The remaining frequency domain integral was then recognized as the band limited singular function of the reflecting surface. The stationary phase analysis remains the same in this paper. There is a minor modification of the inversion operator which leads to a modification of the interpretation of the output to account for the density variations allowed here.

As described above for the constant density case, from one experiment, we obtain an estimate of the angularly dependent reflection coefficient and an estimate of that angle. That is not enough to determine both soundspeed and density variations. To do so, requires a second experiment for which the "dominant" source/receiver pair and opening angle associated with each subsurface point is different from what it was for the first experiment. The reflection coefficient at two different opening angles provides two equations in the two unknowns, the soundspeed and density below the test reflector. Of course, the equations become "stiff" when the two opening angles come too close together.

The typical seismic survey includes many common source experiments, each of which provides partial coverage of the subsurface. Two nearby experiments will provide the "double coverage" required for this method to work. Alternatively, the data can be rearranged as two (or many) common offset experiments. One narrow offset data set and another wide offset data set to provide a pair of experiments for an inversion in which the algebraic equations for the new soundspeed and density are dwell conditioned.

This method is applied to prestack data. Furthermore, the upper surface may be curved; it turns out that very little extra effort is required to allow for a curved -- slowly varying on the scale of wavelengths -- upper surface. Thus, the method dispenses with two preprocessing steps.

Clayton and Stolt [1981] proposed a Born-WKBJ inversion procedure. Their method required that the entire upper surface be covered with sources and receivers. The analogous theoretical model here requires only two common source experiments with full coverage of receivers, or two common receiver experiments with full coverage of sources, or common offset coverage with only two offsets. Furthermore, their method is based on a Born interpretation of the upward scattered wave. At reflectors, the Born approximation degrades with opening angle. (At critical reflection, the reflection coefficient is equal to unity and the upward scattered wave is not of the order of the perturbation!) Thus, the need for small opening angle to keep the Born approximation accurate and the need for a large opening angle for stable algebraic inversion of the equations for soundspeed and density variations are in conflict. Those authors dealt with this problem by applying least squares inversion at what is equivalent to "many" opening angles. I believe that the method proposed here has superior stability properties, although, in practice, I would expect that averaging over many source/receiver pairs or a least squares method would still be used to minimize the effects of noise.

Computationally, the method described here follows the pattern described in Bleistein, Cohen and Hagin [1985] and Bleistein and Gray [1985]. Fourier transform (by FFT) is applied to the data; it is filtered, inverse transformed and evaluated at a time equal to the travel time from source to subsurface output point plus the travel time from the latter point to the receiver point. The inverse Fourier transform is also done by FFT and the evaluation at the specific times is done by three point interpolation. For each output point, a weighted spatial sum is computed, taking account of such features as causality and anti-aliasing. For complex background structure, a table of travel times and weights is created in advance. The more complex the background -- z dependent; x,z dependent; x,y,z dependent -- the more CPU intensive this table construction can be. The size of this table and the attendant CPU time can be dramatically reduced by calculating the elements relatively sparsely and interpolating intermediary values. In practice, the background chosen is only approximate. An interpolated background is only a slightly different approximation of reality, thus, just as good. As an example, Docherty [1985] reported a CPU reduction via interpolation by more than a factor of 70 with imperceptible change in output. Indeed, in our group, it was Docherty who proposed this method as a result of a collaboration with S. H. Gray at AMOCO Research.

Thus, the CPU is dominated by the last weighted spatial sum over the traces. The CPU time is linear in the number of output points and linear in the average number of traces over which one must integrate for each output point.

In his talk at the Workshop, Beylkin described this type of inversion as a statially varying frequency domain filter, which is a terminology used in the electrical engineering literature. Alternatively, the weighted spatial sum is exactly of the form of a Kirchhoff migration, except for the details of the weighting factors. Indeed, this type of inversion may be thought of as a Kirchhoff migration with careful attention to amplitude (an observation made by Ken Larner after my presentation at the Workshop, but also suggested by discussions in Bleistein and Cohen, [1982], and Bleistein, Cohen and Hagin, [1985]).

Thus, there are two reasons to call this a Kirchhoff inversion. First, the inversion really looks like Schneider's Kirchhoff migration. Second, the argument for the validity of this approach is based on application of the inversion formula to Kirchhoff approximate data from a single reflector.

My own personal view is that this type of inversion and migration differ more in philosophy than in detail of implementation. I think of the migrator as viewing the problem as one of downward (or backward) continuing the ensemble of observations subject to an imaging principle. On the other hand, the inverter defines an unknown (or unknowns) to be determined, writes down a governing equation (or equations) relating that unknown (unknowns) to the observations and proceeds to solve the equation(s), at least approximately, with perhaps some imaging type filtering thrown in.

This paper proceeds as follows. In the next section, I introduce the main ideas underlying the singular function theory. After that, I introduce the first inversion operator. Then I describe the results of applying that operator to Kirchhoff data and introduce the second inversion operator. I then show how the use of both operators provides two equations for the two unknowns, $c_+(\underline{x})$ and $\rho_+(\underline{x})$.

THE SINGULAR FUNCTION OF A SURFACE

In many inversion problems, the solution is determined only through aperture limited information about its Fourier transform. The aperture has the property that the angles of the transform variable, $\underline{k}$, are limited as well as its magnitude, $k = |\underline{k}|$. Typically, k is proportional to the temporal frequency variable, ω, in such a manner that the data can be characterized as high frequency data; i. e., that the length scales of the spatial variables are at least three wave lengths.

Information about trend or slow variation of a function is contained in the low frequency part of the spectrum, while information about discontinuities of a function is contained in the high frequency portion of the spectrum. Fortunately, reflectors in the earth can be characterized as surfaces where the earth parameters -- soundspeed, shear speed, density -- are discontinuous.

In its simplest form, a discontinuity may be thought of as a step function. It is well known that step functions are not easily detected from high frequency data. The singular function provides a means to facilitate the detection and analysis of steps or discontinuities from high frequency data.

The singular function of a surface is a Dirac delta function whose support lies on the surface. Thus, determination of the singular function constitutes mathematical imaging of a surface. (See Fig. 1.) In contrast to bandlimited step functions, delta functions are relatively easily recognized from bandlimited data. For example, in the simplest case in one dimension in which the band limited data is just two "boxes" symmetrically placed around the axis, the inverse transform is a difference of sinc functions whose height is just the area under the boxes divided by some constant which depends on the normalization of the Fourier transform.

Let us suppose now that $\alpha(\underline{x})$ is a piecewise smooth function with a discontinuity surface S across which it has a discontinuity $[\alpha]$. Denote by $\gamma(\underline{x})$ the singular function of the surface S. Then asymptotically for large k, the Fourier transforms, $\tilde{\alpha}(\underline{k})$ and $\tilde{\gamma}(\underline{k})$, are related as follows:

$$[\alpha]\tilde{\gamma}(\underline{k}) \sim ik \operatorname{sgn} k_3 \tilde{\alpha}(\underline{k}) \; ;$$

$$\tilde{\alpha}(\underline{k}) = \int \alpha(\underline{x})\, e^{-i\underline{k}\cdot\underline{x}}\, d^3x \, , \tag{1}$$

[Cohen and Bleistein, 1979a, Bleistein, 1984]. I can now say more precisely that $[\alpha]$ is the difference between the value at higher z minus the value at lower z across S.

In applications,

$$k = \text{const}\, |\omega|/c \, , \quad \operatorname{sgn} k_3 = \operatorname{sgn} \omega \, ;$$

$$k \operatorname{sgn} k_3 = \text{const}\, \omega/c \, . \tag{2}$$

By constant, I mean a function of anything except ω or k.

This multiplication in the Fourier domain is very close to a standard image processing technique in which $|\nabla\alpha(\underline{x})|$ is plotted rather than $\alpha(\underline{x})$, itself. To do this, one would compute three transforms, with multipliers of ik_1, ik_2 and ik_3, on $\tilde{\alpha}(\underline{k})$, then take the magnitude of the sum. The experience in my own group is that the single multiplication discussed above is adequate and, as will be seen below, has certain advantages as regards relevance of amplitude.

The surface is imaged by its bandlimited delta function as in Fig 1.

The peak amplitude of the output is proportional to the area under the filter in the k-domain. By dividing out the known proportionality factors, one is left with an estimate of $[\alpha]$. Thus, while the singular function of the surface contains no more data than was available in the original Fourier transform $\tilde{\alpha}(\underline{k})$, it certainly facilitates the detection of discontinuity surfaces and estimation of the size of the discontinuity.

When $\alpha(\underline{x})$ has many discontinuities, the inversion of $ik\,\mathrm{sgn}\,k_3\tilde{a}(\underline{k})$ yields a sum of singular functions, each appropriately scaled.

Consider a seismic experiment as shown schematically in Fig. 2. Sources and receivers are arrayed on a surface, S_0. The source/receiver pairs on S_0 are parametrized by $\xi = (\xi_1, \xi_2)$. For example, a common source experiment would have $\underline{x}_s$ equal to a constant vector while $\underline{x}_r(\xi)$ would be a paramatric representation of the observation surface. Other source/receiver configurations are similarly defined. The parameter ξ ranges over its own two dimensional domain S_ξ to sweep out the array of source/receiver pairs.

A wave propagates into the subsurface and reflects from a surface S. The soundspeed and density above S are known. However, it is not known where these background values end -- that is, where S is -- and new values begin.

The wave propagation is governed by the acoustic wave equation,

$$\rho \nabla \cdot \left[\frac{1}{\rho} \nabla u \right] + \frac{\omega^2}{c^2} u = - \delta(\underline{x} - \underline{x}_s). \tag{3}$$

In this equation, ∇ is the gradient operator, ω is frequency, $u(\underline{x},\omega)$ is the pressure, $c(\underline{x})$ is the soundspeed and $\rho(\underline{x})$ is the density. For each source/receiver pair -- for each ξ -- the upward scattered data, $D(\xi,\omega)$, is observed. The values of the soundspeed and density below S will be denoted by $c_+(\underline{x})$ and $\rho_+(\underline{x})$. The objective is to find S and these values from the observed data.

Motivated by Beylkin [1985a], I propose the following inversion operator for the determination of these unknowns:

$$\beta(\underline{x}) \sim \frac{1}{8\pi^3} \int_{S_\xi}\!\!\int d^2\xi \sqrt{\frac{\rho(\underline{x}_s)}{\rho(\underline{x}_r)}} \frac{|h(\underline{x},\xi|}{A(\underline{x},\underline{x}_s)A(\underline{x},\underline{x}_r) \left|\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)\right|}$$

$$\cdot \int i\omega\, d\omega\, F(\omega) \exp\{-i\omega[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)]\}\, D(\xi,\omega)\ . \tag{4}$$

In this equation,

$$g(\underline{x},\underline{x}_s) = \sqrt{\frac{\rho(\underline{x})}{\rho(\underline{x}_s)}}\ A(\underline{x},\underline{x}_s)\ e^{i\omega\tau(\underline{x},\underline{x}_s)} \tag{5}$$

is the WKB Green's function for (3). The phase and amplitude of the Green's function satisfy the eikonal equation and the transport equation, respectively,

$$(\nabla\tau)^2 = 1/c^2(\underline{x})\ ;\quad 2\nabla\tau\cdot\nabla A + A\nabla^2\tau = 0, \tag{6}$$

subject to the conditions,

$$\tau(\underline{x}\ ,\underline{x}_s) = 0,\quad A(\underline{x},\underline{x}_s)\,|\underline{x} - \underline{x}_s| \to 1/4\pi,\ \text{as}\ |\underline{x} - \underline{x}_s| \to 0, \tag{7}$$

and similarly for the functions which depend on $\underline{x}_r$.

The determinant $h(\underline{x},\underline{\xi})$ is defined by

$$h(\underline{x},\underline{\xi}) = \det \begin{vmatrix} \nabla[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)] \\ \frac{\partial}{\partial \xi_1} \nabla[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)] \\ \frac{\partial}{\partial \xi_2} \nabla[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)] \end{vmatrix} . \qquad (8)$$

This determinant is of fundamendal importance to both Beylkin's theory and successful computer implementation of the inversion formula. It is assumed throughout that $h(\underline{x},\underline{\xi}) \neq 0$. The reader is referred to the cited papers by that author for interpretations of this assumption.

The function, $F(\omega)$, is a smoothed version of the wave shape of the original signal. Actually, the right side of (3) should have had some factor to represent the signal shape. However, that would require the introduction of a function which would be eliminated at this step. Instead, I simply introduce $F(\omega)$ at this final step to take account of both signal and smoothing. I assume that the inversion of this function is a bandlimited Delta function.

I arrived at this inversion operator in the following manner. First, I rewrote Beylkin [1985a], eq. 4 in terms of the data in the temporal Fourier domain. Note that Beylkin's formula provides an asymptotic inversion of a perturbation in a function characterizing the soundspeed (through the inverse square slowness function). Thus, motivated by the singular function theory, I introduced an extra factor of $i\omega$ in the integrand to image discontinuity surfaces of the slowness, instead. The spatial multiplier arises from identification of the relationship between frequency and the wave vector $\underline{k}$, defined by Beylkin,

$$\underline{k} = \omega[\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)] . \qquad (9)$$

APPLICATION TO KIRCHHOFF DATA

The inversion operator (4) is to be applied to the upward scattered response from a single reflector, S, as defined by the Kirchhoff approximation. This representation can be found in many sources, including Bleistein [1986b], eq. (74). In the notation used here the result is

$$D(\underline{\xi},\omega) \sim i\omega \sqrt{\frac{\rho(\underline{x}_r)}{\rho(\underline{x}_s)}} \int_S R(\underline{x}',\underline{x}_s)A(\underline{x}',\underline{x}_s)A(\underline{x}',\underline{x}_r)\ \tilde{n}\cdot[\nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r)]$$
$$\cdot \exp\{i\omega[\tau(\underline{x}',\underline{x}_s) + \tau(\underline{x}',\underline{x}_r)]\}\ dS' \ . \tag{10}$$

In this equation, ∇' denotes a gradient with respect to the $\underline{x}'$ variables and $R(\underline{x}',\underline{x}_s)$ is the geometrical optics reflection coefficient,

$$R(\underline{x}',\underline{x}_s) = \frac{\dfrac{1}{\rho(\underline{x}')}\left|\dfrac{\partial\tau(\underline{x}',\underline{x}_s)}{\partial n}\right| - \dfrac{1}{\rho_+(\underline{x}')}\sqrt{\dfrac{1}{c_+^2(\underline{x}')} - \dfrac{1}{c^2(\underline{x}')} + \left[\dfrac{\partial\tau(\underline{x}',\underline{x}_s)}{\partial n}\right]^2}}{\dfrac{1}{\rho(\underline{x}')}\left|\dfrac{\partial\tau(\underline{x}',\underline{x}_s)}{\partial n}\right| + \dfrac{1}{\rho_+(\underline{x}')}\sqrt{\dfrac{1}{c_+^2(\underline{x}')} - \dfrac{1}{c^2(\underline{x}')} + \left[\dfrac{\partial\tau(\underline{x}',\underline{x}_s)}{\partial n}\right]^2}} \ . \tag{11}$$

The unit normal $\tilde{n}$ points upward and $\partial/\partial n = \tilde{n}\cdot\nabla'$. This result is substituted into (4) to obtain the following multi-fold integral representation of the output $\beta(\underline{x})$ when applied to this synthetic data:

$$\beta(\underline{x}) \sim -\frac{1}{8\pi^3} \iint_{S_\xi} d^2\xi \frac{|h(\underline{x},\xi)|}{A(\underline{x},\underline{x}_s)A(\underline{x},\underline{x}_r)\,|\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)|} \int \omega^2\, d\omega\, F(\omega)$$

$$\cdot \int_S R(\underline{x}',\underline{x}_s)\, A(\underline{x}',\underline{x}_s)\, A(\underline{x}',\underline{x}_r) \exp\{i\omega\, \Phi(\underline{x},\underline{x}',\underline{x}_s,\underline{x}_r)\}$$

$$\tilde{n}\cdot[\nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r)]\, dS' \quad . \tag{12}$$

In this equation

$$\Phi(\underline{x},\underline{x}',\underline{x}_s,\underline{x}_r) = \tau(\underline{x}',\underline{x}_s) + \tau(\underline{x}',\underline{x}_r) - [\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)] \tag{13}$$

is the difference of travel times, source point to input point to receiver point minus source point to output point to receiver point. The surface S is described parametrically in terms of two parameters, (σ_1,σ_2), by an equation of the form

$$\underline{x}' = \underline{x}'(\underline{\sigma}),\ \underline{\sigma} = (\sigma_1,\sigma_2). \tag{14}$$

In terms of these parameters,

$$dS' = \sqrt{g}\, d\sigma_1 d\sigma_2, \tag{15}$$

with g the first fundamental form of differential geometry for S,

$$g = \left| \frac{d\underline{x}'}{d\sigma_1} \times \frac{d\underline{x}'}{d\sigma_2} \right|^2 = \left| \det \left[\frac{d\underline{x}'}{d\sigma_k} \cdot \frac{d\underline{x}'}{d\sigma_m} \right] \right| , \quad k,m = 1,2 . \tag{16}$$

Here $\times$ denotes the vector cross product.

The method of stationary phase is applied to (12) in the four variables $(\xi,\underline{\sigma})$. The phase Φ is a function of these variables through the dependence of $\underline{x}'$ on $\underline{\sigma}$ and the dependence of $\underline{x}_s$ and $\underline{x}_r$ on ξ. The conditions that the phase be staionary are given by

$$\begin{aligned} &\nabla_s\left[\tau(\underline{x}',\underline{x}_s) - \tau(\underline{x},\underline{x}_s) \right] \cdot \frac{d\underline{x}_s}{d\xi} + \nabla_r\left[\tau(\underline{x}',\underline{x}_r) - \tau(\underline{x},\underline{x}_r) \right] \cdot \frac{d\underline{x}_r}{d\xi} = 0 ; \\ &\nabla'\left[\tau(\underline{x}',\underline{x}_s) + \tau(\underline{x}',\underline{x}_r) \right] \cdot \frac{d\underline{x}'}{d\sigma} , \quad m = 1,2. \end{aligned} \tag{17}$$

In this equation, $\nabla_s(\nabla_r)$ is a gradient with respect to the variables $\underline{x}_s(\underline{x}_r)$.

In Bleistein [1986a], I discuss the conditions under which Φ is stationary. I show that, for $\underline{x}$ on the surface S and $h(\underline{x},\xi) \neq 0$, there is a unique stationary triple, $\underline{x}'$, $\underline{x}_s$ and $\underline{x}_r$, with $\underline{x}' = \underline{x}$. An important feature of this stationary point for $\underline{x}$ on S is that the value of ξ determined by (17) is the one for which the geometrical optics rays associated with the traveltimes from $\underline{x}_s$ and $\underline{x}_r$ to $\underline{x}$ satisfy Snell's law at $\underline{x}$. That is, they make equal angles with the normal to S at $\underline{x}$. This is shown for the following source/receiver configurations of practical interest: common source, common receiver and common offset. Although I have only considered here the fully three dimensional problem, this analysis specializes to the cases of 2.5D inversion. (In 2.5D, it is assumed that a line of data is gathered over a medium with no out-of-plane variation. Thus, 2.5D connotes three dimensional propagation over a medium with two dimensional parameter variations.)

I will proceed below by focusing attention on this stationary triple when $\underline{x}$ is in the neighborhood of S. That is, this is the stationary point which has limit $\underline{x}' = \underline{x}$ and $\underline{x}_s$ and $\underline{x}_r$ as described above, as $\underline{x}$ approaches S. If there were no source/receiver pair in the seismic survey under consideration which included the particular $\underline{x}_s$ and $\underline{x}_r$ needed to complete the stationary triple, then the asymptotic contribution for that point $\underline{x}$ would be of lower order in ω and almost always of smaller magnitude after the ω integration than the result I state below. Thus, I proceed under the assumption that such a stationary triple has been determined and that the corresponding values of $\underline{\sigma}$ and $\underline{\xi}$ are interior points of their respective domains of integration.

The result of applying the method of stationary phase to (12) is the following:

$$\beta(\underline{x}) = H(\underline{x})\gamma_B(\underline{x}) \ . \tag{18}$$

In this equation, $\gamma_B(\underline{x})$ is the bandlimited singular function of the reflector, S and $H(\underline{x})$ is a slowly varying function (on the length scale of the wave lengths in $\tilde{\gamma}_B(\underline{k})$ or on the scale of the length of the main lobe of $\gamma_B(\underline{x})$). The function $H(\underline{x})$ is given by

$$H(\underline{x}) = -R(\underline{x}',\underline{x}_s)\ \frac{A(\underline{x}',\underline{x}_s)\ A(\underline{x}',\underline{x}_r)}{A(\underline{x},\underline{x}_s)\ A(\underline{x},\underline{x}_r)}\ \frac{|h(\underline{x},\underline{\xi})|}{\left|\det\left[\Phi_{\xi\sigma}\right]\right|^{1/2} |\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)|^2} \cdot \tilde{n}\cdot[\nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r)]\ \sqrt{g}\ , \tag{19}$$

with $[\Phi_{\xi\sigma}]$ the 4X4 matrix

$$\left[\Phi_{\xi\sigma}\right] = \begin{bmatrix} \dfrac{\partial^2\Phi}{\partial\xi_i\partial\xi_j} & \dfrac{\partial^2\Phi}{\partial\xi_i\partial\sigma_j} \\ \\ \dfrac{\partial^2\Phi}{\partial\xi_i\partial\sigma_j} & \dfrac{\partial^2\Phi}{\partial\sigma_i\partial\sigma_j} \end{bmatrix}, \quad i,j = 1,2: \tag{20}$$

The vectors $\underline{x}'$, $\underline{x}_s$ and $\underline{x}_r$ are determined here as functions of $\underline{x}$ by the stationarity conditions (17), so that the entire result is a function of $\underline{x}$. The function $\beta(\underline{x})$ images the reflector through its dependence on the function $\gamma_B(\underline{x})$. It only remains to determine the peak amplitude of this result when $\underline{x}$ is on the reflector.

The analysis of the peak value is facilitated by introducing certain intermediary results established in Bleistein [1986a]. I first introduce the acute angle θ between the upward normal to the surface and the incident and reflected rays on the surface. Note that the downward gradients $\nabla'\tau(\underline{x}',\underline{x}_s)$ and $\nabla'\tau(\underline{x}',\underline{x}_r)$ make angles of $\pi - \theta$ with this normal and make an angle of 2θ with one another. Therefore,

$$\tilde{n}\cdot[\nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r)] = -\frac{2\cos\theta}{c(\underline{x}')}, \tag{21}$$

$$\left|\nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r)\right|^2 = \frac{2}{c^2(\underline{x}')}[1 + \cos 2\theta] = \left[\frac{2\cos\theta}{c(\underline{x}')}\right]^2. \tag{22}$$

Finally, in Appendix D to Bleistein [1986a], I show that

$$\frac{|h(\underline{x},\xi)|}{\left|\det\left[\Phi_{\xi\sigma}\right]\right|^{1/2}}\sqrt{g} = |\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)| = \frac{2\cos\theta}{c(\underline{x})}, \quad \underline{x} \text{ on } S. \tag{23}$$

By inserting the results (21-23) into (19), one obtains the following result for $\beta(\underline{x})$ at its peak; that is for $\underline{x}$ on S:

$$\beta(\underline{x}) \sim R(\underline{x}',\underline{x}_s)\, \gamma_B(\underline{x}), \quad \underline{x} \text{ on } S. \tag{24}$$

This confirms my original claim about the inversion operator defined by (4). That is, when applied to Kirchhoff approximate data and evaluated asymptotically, the operator produces a bandlimited singular function of the reflecting surface multiplied by the geometrical optics reflection coefficient evaluated for some particular choice of incident angle (through its dependence on $\partial\tau/\partial n$).

As noted in the introduction, this incidence angle is not known *a priori* and we require some means for determining this angle from the processed output. This is discussed below.

DETERMINATION OF θ, c_+ and ρ_+

In order to determine the values of θ, c_+ and ρ_+, I require one other intermediary result from Bleistein [1986a], namely,

$$\gamma_B(\underline{x}) \sim \frac{2\cos\theta}{c(\underline{x})} \frac{1}{2\pi} \int F(\omega)\, d\omega \;, \; \underline{x} \text{ on } S. \tag{25}$$

Substitution of this result into (24) yields

$$\beta(\underline{x}) \sim \frac{2\cos\theta}{c(\underline{x})} R(\underline{x},\underline{x}_s) \frac{1}{2\pi} \int F(\omega)\, d\omega \;, \; \underline{x} \text{ on } S. \tag{26}$$

That is, the actual numerical value at the peak depends on the area under the filter in the frequency domain, the opening angle θ between the normal and each of the rays from $\underline{x}_s$ and $\underline{x}_r$ to $\underline{x}$ on S, and the reflection coefficient at that opening angle. We know the filter and, hence, the area under the filter, but the separate elements, θ, c_+ and ρ_+ remain coupled in this equation.

As a first step, I address the determination of θ. From (23), it can be seen that the first fraction in (26) arises from the evaluation of $|\nabla\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r|$ at the stationary point. This factor appears in the denominator in the inversion operator defined by (4). By changing the power of this factor in that inversion operator, it is possible to change the power of the multiplicative factor, $2\cos\theta/c(\underline{x})$, at the peak of the output of the inversion operator. Therefore, in addition to processing the data with the inversion operator (4), I propose that the data be process with the operator

$$\beta_1(\underline{x}) \sim \frac{1}{8\pi^3} \int_{S_\xi}\int d^2\xi \sqrt{\frac{\rho(\underline{x}_s)}{\rho(\underline{x}_r)}} \frac{|h(\underline{x},\xi|}{A(\underline{x},\underline{x}_s)A(\underline{x},\underline{x}_r) \left|\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)\right|^2}$$

$$\cdot \int i\omega \, d\omega \, F(\omega) \exp\{-i\omega[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)]\} D(\xi,\omega) \; . \tag{27}$$

Since it is necessary to calculate $|\nabla\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r|$ anyway, simultaneous computation of this second inversion operator imposes no severe burden on computer time. The asymptotic analysis of the output $\beta_1(\underline{x})$ applied to Kirchhoff data is readily determined from the results for $\beta(\underline{x})$. This function also produces the band limited singular function, $\gamma_B(\underline{x})$, scaled by a different factor. At the peak, that scale factor differs from the scale for $\beta(\underline{x})$ by $|\nabla\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r|^{-1}$ evaluated at the stationary point on S, given by (23). That is,

$$\beta_1(\underline{x}) \sim R(\underline{x},\underline{x}_s) \frac{1}{2\pi} \int F(\omega) \, d\omega \; , \; \underline{x} \text{ on } S, \tag{28}$$

and

$$\frac{\beta(\underline{x})}{\beta_1(\underline{x})} \sim \frac{2\cos\theta}{c(\underline{x})} \; , \; \underline{x} \text{ on } S. \tag{29}$$

Thus, when both inversion operators are applied to the data, the locations of the peaks of either of them determine the reflector and then the ratio of the peak values determine $\cos\theta$. With θ determined, either peak amplitude provides a single equation for the two unknowns, $c_+(\underline{x})$ and $\rho_+(\underline{x})$.

Often, density variations are sufficiently small that they may be neglected; that is, $\rho_+(\underline{x}) = \rho(\underline{x})$. In this case, the peak amplitude, with $\cos\theta$ known, provides a single equation for the unknown, $c_+(\underline{x})$. Solution of this equation provides an estimate of $c_+(\underline{x})$ consistent with the geometrical optics reflection coefficient and not constrained to small values of the difference, $c_+(\underline{x}) - c(\underline{x})$.

When both parameters vary across the reflector, more information is necessary if both are to be determined. I propose two experiments. That is, if the experiment in question is a common source experiment with an array of geophones, then a second common source experiment is carried out with the source moved to another location. In this case, it is certain that the stationary value of θ is different for the second experiment. The reason is that θ must be the angle between the ray from $\underline{x}_s$ to the output $\underline{x}$ and the upward normal to S at $\underline{x}$. Moving $\underline{x}_s$ changes the ray and changes this angle. A similar argument holds for two different common receiver experiments. If the data were generated by an array of common offset experiments, then a second array of common offset experiments at different offset distance will produce a different stationary value of θ. Thus, in each case, two data sets provide two equations for $c_+(\underline{x})$ and $\rho_+(\underline{x})$ through determination of the reflection coefficient at $\underline{x}$ at two different (known) values of θ.

To see how this works out in detail, first rewrite the reflection coefficient in (11) in terms of θ and $\underline{x}' = \underline{x}$, on S. Note first that from the stationarity conditions

$$\frac{\partial\tau(\underline{x},\underline{x}_s)}{\partial n} = \frac{\cos\theta}{c(\underline{x})} . \tag{30}$$

Now, with a slight abuse of notation, that is, replacing $R(\underline{x},\underline{x}_s)$ by $R(\underline{x},\theta)$,

(11) can be rewritten as follows:

$$R(\underline{x},\theta) = \frac{\dfrac{\cos\theta}{\rho(\underline{x})\ c(\underline{x})} - \dfrac{1}{\rho_+(\underline{x})}\sqrt{\dfrac{1}{c_+^2(\underline{x})} - \dfrac{\sin^2\theta}{c^2(\underline{x})}}}{\dfrac{\cos\theta}{\rho(\underline{x})\ c(\underline{x})} + \dfrac{1}{\rho_+(\underline{x})}\sqrt{\dfrac{1}{c_+^2(\underline{x})} - \dfrac{\sin^2\theta}{c^2(\underline{x})}}} \ . \tag{31}$$

Let us assume that two sets of experiments have been carried out. For each, both inversion operators, (4) and (27) were applied to the data, so that the two values of θ, say, θ_1 and θ_2, were determined, along with the reflection coefficents, $R_1 = R(\underline{x},\theta_1)$ and $R_2 = R(\underline{x},\theta_2)$. These values are then used in (31) to determine $c_+(\underline{x})$ and $\rho_+(\underline{x})$. The results of that calculation are

$$\frac{1}{c_+^2} = \frac{1}{c^2}\,\frac{P_2^2\cos^2\theta_2\sin^2\theta_1 - P_1^2\cos^2\theta_1\sin^2\theta_2}{P_2^2\cos^2\theta_2 - P_1^2\cos^2\theta_1}\ , \quad \rho_+^2 = \rho^2\,\frac{\sin^2\theta_1 - \sin^2\theta_2}{P_2^2\cos^2\theta_2 - P_1^2\cos^2\theta_1} . \tag{32}$$

In these results, P_1 and P_2 are given in terms of the observed reflection coefficients by

$$P_j = \frac{1 - R_j}{1 + R_j}\ , \ j = 1,2. \tag{33}$$

This completes the determination of c_+ and ρ_+.

CONCLUSIONNS

I have introduced here an inversion operator for acoustic data in a variable soundspeed and density medium. By applying the operator to Kirchhoff approximate data from a single reflector, I drew certain conclusions about the output of this operator. First, the operator produces a (an array of) bandlimited singular function(s) of the reflecting surface(s) in the interior of the earth. The singular function constitutes a mathematical image of the reflector and its pictorial representation does indeed depict the reflector. Second, each singular function is scaled by a slowly varying function of spatial coordinates. At the peak of the bandlimited singular function, this scale becomes the geometrical optics reflection coefficient at some (as yet unknown) incidence angle. Since the peak value of the singular function is known, this reflection coefficient is determined, as well. By introducing a second inversion operator, differing from the first in only a minor way, I can determine that incidence angle. At this point, in a constant density medium, the soundspeed below the reflector can be determined from (i) the known values of the soundspeed above, (ii) the reflection coefficient, and (iii) the incidence angle. When the density varies, as well, a second experiment must be carried out. This second experiment must differ from the first sufficiently to provide the reflection coefficient at a different incidence angle. From the two experiments, sufficient information is available to determine the values of soundspeed and density below the reflector in terms of the values above the reflector.

This method applies to common source, common receiver or common offset experiments in which the soundspeed and density above the reflector are known functions of all three spatial variables. The density variations across the reflector need not be small. However, the "known" background must be close to the true values in order for proper location of the reflector and proper determination of the soundspeed and density below the reflector. Also, multiples from reflectors above the one being analyzed must be sufficiently negligible so as not to contaminate the response from the given reflector.

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FIGURE CAPTIONS

Figure 1: The singular function of a surface.

Figure 2: Schematic of the seismic experiment.

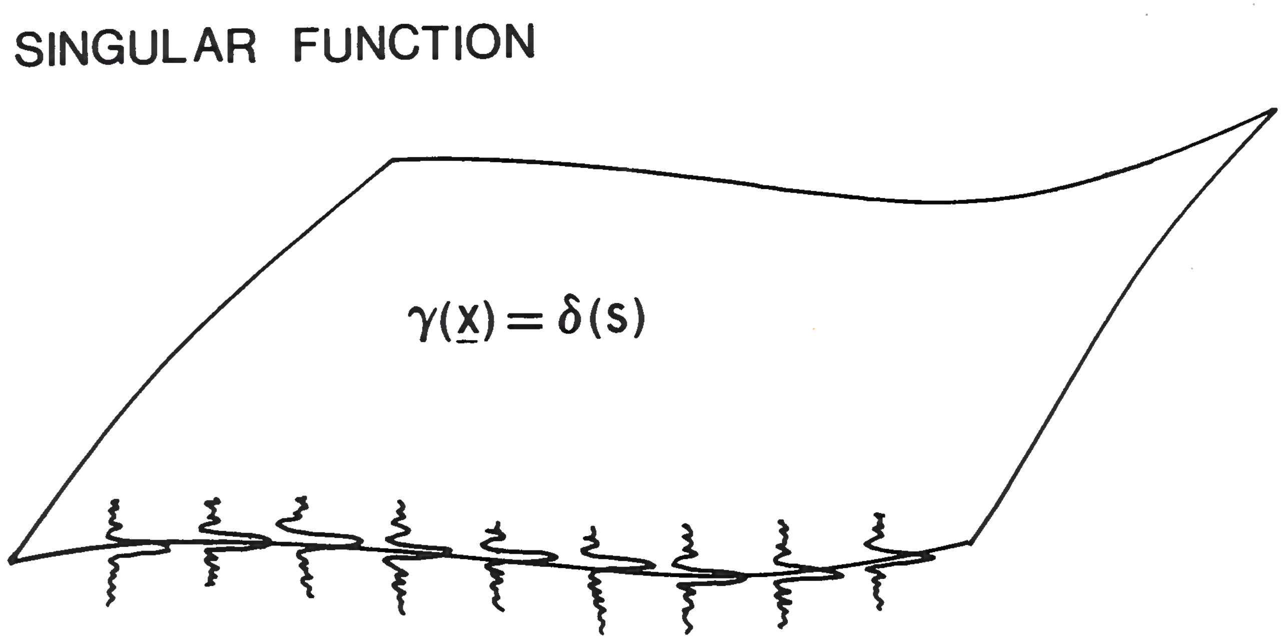

FIGURE 1

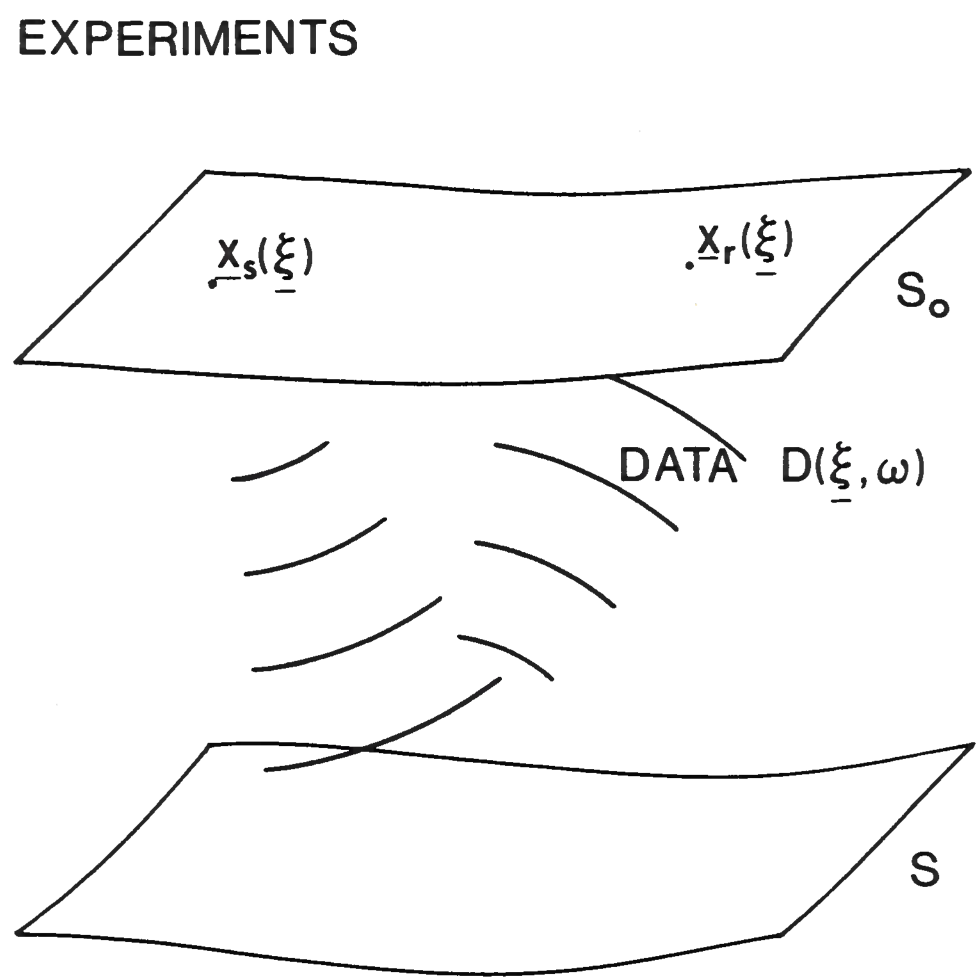
EXPERIMENTS
$\underline{x}_s(\underline{\xi})$
$\underline{x}_r(\underline{\xi})$
S_o
DATA $D(\underline{\xi},\omega)$
S

FIGURE 2

ACKNOWLEDGEMENT

The author gratefully acknowledges the support of the Office of Naval Research, Mathematics Division, through its Selected Research Opportunities Program, and the Consortium Project on Seismic Inverse Methods for Complex Structures at the Center for Wave Phenomena, Colorado School of Mines. Consortium members are Amoco Production Company, Conoco, Inc., Geophysical Exploration Company of Norway A/S, Marathon Oil Company, Mobil Research and Development Corp., Phillips Petroleum Company, Sun Exploration and Research, Texaco USA, Union Oil Company of California, and Western Geophysical.