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Detection of wave front set perturbations via correlation: Foundation for wave-equation tomography

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ABSTRACT

We discuss the mathematical aspects of wave field measurements used in travel-time inversion from seismograms. The primary information about the medium is assumed to be carried by the wave front set and its perturbation with respect to a hypothetical background medium is to be estimated. By a convincing heuristics a detection procedure for this perturbation was proposed based on optimization of wave field correlations. We investigate its theoretical foundation in simple mathematical case studies using the distribution theoretic definition of oscillatory integrals.

1 INTRODUCTION

In this paper, we investigate how to carry out tomography directly in terms of wavefield measurements. Tomography, in its original form, uses a ‘measured’ wavefront set as input in an inversion procedure which is solely (symplectic) geometric in nature, viz. based upon finding bicharacteristics that result through a canonical relation in matching the measurement. In ‘wave-equation’ tomography, one aims at replacing the geometric procedure by a wave-solution procedure, but keeping the wavefront set of the measurements as the primary source of information about the medium.

Following an embedding procedure to formulate the inverse problem, i.e. introducing a background medium and incident field and a medium contrast and scattered field, we then face the problem of detecting perturbations in the wavefront set associated with the scattered (perturbed – incident) field. An intuitive choice is based upon correlating the perturbed field with the incident field. We will show, by example, that such procedure should be carried out delicately. In fact, we conclude that the perturbation of the wavefront set can be derived from the singular support (of the derivative) of the proposed time correlation.

The outline of the paper is as follows. We briefly review the microlocal representation of solutions to the scalar wave equation (Section 2). In Section 3 we introduce the measuring process and its mathematical implementation; we describe how the wavefront set of the wavefield propagates through this measuring process. When we perturb the coefficient function in the wave equation (the wave speed) the solution representation will be perturbed. In particular, its wavefront set will

shift in the measurement-variables cotangent bundle. We formulate the process of correlating, within the measuring process, the perturbed representation with the original representation, and identify how such shift appears in the result. It is conjectured that the derivative of the (time) correlation at any given measurement position has its singular support precisely at the time shift associated with the perturbation of the wavefront set. In Section 4 we give examples to illustrate the conjecture. Special attention is paid how to define the product of distribution solutions within the correlation process. Finally, in Section 5, we discuss a method of detecting the singular support of the correlation in time at any measuring position by means of ‘localized’ Fourier transforms. The procedure defines a criterion to develop wave-equation tomography.

2 FOURIER INTEGRAL REPRESENTATION OF WAVE SOLUTIONS

The scalar wave equation for acoustic waves in a constant density medium is given by

$$Pu = f, \tag{1}$$

with

$$P = \partial_t^2 + \underbrace{D c(x)^2 D}_{A(x,D)}, \tag{2}$$

where $D = -i\partial_x$. The equation is considered on an open domain $\Omega \subset \mathbb{R}^n$ and in a time interval $]0, T[$.

We decouple the wave equation into its forward and backward components. To this end, we introduce the elliptic operator $A(x, D)$ and its square root $B(x, D) = \sqrt{A(x, D)}$. Decomposing the field according to

$$u_{\pm} = \frac{1}{2}u \pm \frac{1}{2}iB(x, D)^{-1}\partial_t u, \quad (3)$$

in combination with the source decomposition

$$f_{\pm} = \pm \frac{1}{2}iB(x, D)^{-1}f, \quad (4)$$

then results in the equivalent system of equations

$$[\partial_t \pm iB(x, D)]u_{\pm} = f_{\pm}. \quad (5)$$

Throughout, we assume that $c \in C^{\infty}(\Omega)$. We will construct operators G_{\pm} with distribution kernels $\mathcal{G}_{\pm}(x, x_0, t, t_0)$ that solve the initial value problem equivalent to (5) with $f_{\pm} = \pm\delta$.

Let $H = H(x, \xi, \tau) = \tau \pm B^{\text{prin}}(x, \xi)$ denote the Hamiltonian either for the forward or backward wave propagation. The Hamilton system of equations that generates the Hamiltonian flow or bicharacteristics is given by

$$\begin{aligned} \frac{\partial x}{\partial \lambda} &= \pm \frac{\partial}{\partial \xi} B^{\text{prin}} & , & \quad \frac{\partial t}{\partial \lambda} = 1, \\ \frac{\partial \xi}{\partial \lambda} &= \mp \frac{\partial}{\partial x} B^{\text{prin}} & , & \quad \frac{\partial \tau}{\partial \lambda} = 0. \end{aligned} \quad (6)$$

Observe that $H(x, \xi, \tau) = 0$ implies $\tau = \mp B^{\text{prin}}(x, \xi)$.

Equation (5) can be solved, microlocally, in the form of a Fourier integral representation. The phase of the associated Fourier integral operator follows from the canonical relations

$$C_{\pm} = \{(x(x_0, \xi_0, \pm t), t, \xi(x_0, \xi_0, \pm t), \underbrace{\mp B^{\text{prin}}(x_0, \xi_0)}_{\tau}); x_0, -\xi_0\}.$$

Let

$$(x_I, x_0, \underbrace{\xi_J, \tau}_{\theta}) \quad \text{with} \quad I \cup J = \{1, \dots, n\}$$

denote coordinates on C_{\pm} . A function S will locally describe C_+ according to

$$\begin{aligned} x_J &= -\frac{\partial}{\partial \xi_J} S & , & \quad t = -\frac{\partial}{\partial \tau} S, \\ \xi_I &= \frac{\partial}{\partial x_I} S & , & \quad \xi_0 = -\frac{\partial}{\partial x_0} S, \end{aligned} \quad (7)$$

and generates the non-degenerate phase function

$$\phi_+(x, x_0, t, \xi_J, \tau) = S(x_I, x_0, \xi_J, \tau) + \langle \xi_J | x_J \rangle + \tau t. \quad (8)$$

In our notation, we will suppress the dependence on x_0 and collect ξ_J, τ in the phase variables θ . The canonical relation can then be written as

$$C_+ = \{((x, t, \partial_x \phi_+, \partial_t \phi_+); (x_0, \partial_{x_0} \phi_+)) \mid \partial_{\theta} \phi_+ = 0\}.$$

We synthesize the canonical relation $C_{\phi} = C_+ \cup C_-$ with associated (non-degenerate) phase function $\phi = \phi_-$ if $\tau > 0$, $\phi = \phi_+$ if $\tau < 0$. In accordance with (3) we obtain

$$G(x, x_0, t) = \frac{1}{2}i[G_+(x, x_0, t) - G_-(x, x_0, t)]B(x_0, D_{x_0})^{-1}. \quad (9)$$

With this fundamental solution, the solution of (5) and its dependence on the initial conditions can then be written in the form of a Fourier integral operator (FIO) with amplitude $a = a(x_I, x_0, \xi_J, \tau)$. In fact, a lives in the tensor product $M_{C_{\phi}} \otimes \Omega^{1/2}(C_{\phi})$ of the Keller-Maslov line bundle and the half-densities on C_M .

The kernel of the FIO admits an oscillatory integral (OI) representation. In the remainder of this paper we consider such OIs to represent ‘the wavefield’. Perturbation of this wavefield are induced by perturbation of the coefficient function $c(x)$.

3 DETECTION OF SINGULARITIES OF THE WAVE FIELD

As described above, each component of the wave field as well as the perturbed wave field can be represented by an OI,

$$u(x, t) = \int a(x, t, \theta) e^{i\phi(x, t, \theta)} d\theta \quad (10)$$

where ϕ is a non-degenerate phase function and a a symbol ((Hörmander, 1990), Sect. 7.8); note that the wave front set satisfies the inclusion ((Hörmander, 1990), Thm. 8.1.9)

$$\text{WF}(u) \subseteq \{(x, t; \partial_x \phi(x, t, \theta), \partial_t \phi(x, t, \theta)) \mid \partial_\theta \phi(x, t, \theta) = 0\}. \quad (11)$$

3.1 Measurements as restrictions to submanifolds

Measurements are recordings of the wave field u in stations at certain points x in the acquisition manifold over some time interval (t_0, t_1) ; mathematically, this corresponds to the restriction of the distribution u to the one-dimensional submanifolds $S_x = \{x\} \times \mathbb{R}$ followed by further restriction of the resulting one-dimensional distribution u_x of time to the open interval (t_0, t_1) .

While the second of those restrictions is always possible and straightforward, the first can be carried out as continuous map only on distributions satisfying the following condition ((Hörmander, 1990), Thm. 8.2.4 and Cor. 8.2.7)

$$\text{WF}(u) \cap \{(x, t; \xi, 0) \mid t \in \mathbb{R}, \xi \in \mathbb{R}^n\} = \emptyset. \quad (12)$$

Note that by (11) this condition is satisfied if and only if $\partial_t \phi(x, t, \theta) \neq 0$ whenever $\partial_\theta \phi(x, t, \theta) = 0$. If it holds, the restriction u_x can be defined as the pullback $\iota_x^* u$ of u under the embedding map $\iota_x : S_x \hookrightarrow \mathbb{R}^{n+1}$ and by (11) we have the wave front set relation

$$\text{WF}(u_x) \subseteq \{(t; \tau) \mid \exists \xi : (x, t; \xi, \tau) \in \text{WF}(u)\} \subseteq \{(t; \partial_t \phi(x, t, \theta)) \mid \partial_\theta \phi(x, t, \theta) = 0\}. \quad (13)$$

Let ψ be a phase function, b a symbol, both with the same domains and supports as ϕ , a , and v be the oscillatory integral defined by them; assume that v also satisfies (12) and set $v_x = \iota_x^* v$. In case we are interested only in a certain time window of measurement we may use further cut-offs and achieve that u_x and v_x are compactly supported.

3.2 The correlation function

For $a \in \mathbb{R}$ denote by T_a the translation by a on \mathbb{R} . If the distributional product $w_{x,t} = u_x \cdot \overline{T_t^* v_x}$ can be defined and yields an integrable distribution ((Horváth, 1966), Sect. 4.5) we define the value of the *correlation function* at t by

$$c[u_x, v_x](t) = \langle u_x \cdot \overline{T_t^* v_x}, 1 \rangle = \langle w_{x,t}, 1 \rangle. \quad (14)$$

The correlation is bilinear in $[\cdot, \cdot]$. Whenever there is no ambiguity about the distributions u and v and the point x under consideration we will denote the correlation briefly by $c(t)$.

Whenever u and v represent the unperturbed and perturbed solution, then typically $\partial_t \phi = \partial_t \psi$ (the frequencies coincide) and therefore for certain values of t we expect the cotangent components of the wave front sets of u_x and $T_t v_x$ to be identical on the overlap of singular supports. That means that, unless both cotangent parts are only half rays on the same side of 0, Hörmander's condition ((Hörmander, 1990), Thm. 8.2.1) for defining the product does not apply. But within the hierarchy of distributional products described by Oberguggenberger (1992), Ch. II, this condition, 'WF favorable', appears only as one out of a variety of consistent possibilities to give a distributional meaning to the product under consideration. We apply some of these to the analysis of the correlation function in some examples below to explore and illustrate whether and how the correlation, after restriction, can provide information about shifts in wave front set from u to v . It will become clear that the customary criterion of searching for the 'stationary point' of the correlation (Dahlen *et al.*, 2000), (Zhao *et al.*, 2000), and (Luo & Schuster, 1991) for detecting the shift in wave front sets is generally incorrect.

Here, we would like to point out that the appropriate mathematical framework to deal with the multiplication (and also the restrictability) in a uniform and systematic manner is Colombeau's theory of generalized functions (cf. (Colombeau, 1985; Oberguggenberger, 1992)). Such framework will enable us to cope with the integrability question (forming the correlation) at the same time (Hörmann, 1999).

Remark 1. Note that the projection $p_x : \mathbb{R}^{n+1} \rightarrow S_x$ onto S_x , the right inverse of ι_x , is surjective and hence

the pullback $p_x^*u_x$ is always well-defined. Estimating the wave front set of the true wave field $\text{WF}(u)$ from the measurements $\text{WF}(u_x)$ amounts to stating an upper bound in the form

$$\text{WF}(u) \subseteq \bigcup_x \text{WF}(p_x^*u_x)$$

where x ranges over \mathbb{R}^n in an ideal model situation.

Practically, we will have to consider regularizations or approximations to the formal expression $c(t) = \langle w_{x,t}, 1 \rangle$ of the correlation either to give a meaning to the product or to make the integration (i.e., distributional action on 1) well-defined. This amounts to the attempt of defining $c(t)$ as the pointwise (in t) limit of sequences

$$c_n(t) = \langle w_{x,t}^n, 1 \rangle$$

as $n \rightarrow \infty$ where $w_{x,t}^n$ is a suitable regularization or approximation of $w_{x,t}$.

3.3 The shift of singular supports

We compare the singular supports, or rather the wave front sets, of u_x and v_x . This expresses the amount of time shift of the wave fronts / singularities at location x by the perturbation.

First we observe that under a natural time evolution condition on the phase function a restrictable OI is representable as an OI in one dimension.

Lemma 2. If $u = \int a(\cdot, \theta) e^{i\phi(\cdot, \theta)} d\theta \in \mathcal{D}'(\mathbb{R}^{n+1})$ satisfies condition (12) at x and $\partial_t \phi(x, t, \theta) \neq 0$ for all t and $\theta \neq 0$ such that $(x, t, \theta) \in \text{supp}(a)$ then the restriction u_x to S_x is the OI on \mathbb{R} (i.e., in the time variable) where x is considered as a parameter in the phase and amplitude. Therefore

$$u_x = \int a(x, \cdot, \theta) e^{i\phi(x, \cdot, \theta)} d\theta. \quad (15)$$

Proof. By assumption $\phi_x(t, \theta) = \phi(x, t, \theta)$ defines a phase function on $\mathbb{R} \times \mathbb{R}^N$. We have $u_x = \iota_x^*(u)$ and ι_x^* is continuous on the subspace of restrictable distributions. Therefore we may use any standard OI regularization $u = \lim_{\epsilon \rightarrow 0} u_\epsilon$ and obtain $u_x = \lim_{\epsilon \rightarrow 0} \iota_x^*(u_\epsilon)$. Since the latter is an OI regularization in one dimension with phase function ϕ_x and symbol $a(x, \cdot, \cdot)$ the assertion is proved. \square

Note that the usual stationary phase argument applied to this one-dimensional OI gives the same upper bound for the wave front set as established above in (13). Assuming that the perturbed solution v is given as an OI with phase function ψ and amplitude b we can compare the wave front sets of their measurements at x (restrictions to S_x).

As pointed out above, the perturbation will affect the phase function only in its x - and θ -gradient, i.e., we may assume that $\partial_t \phi = \partial_t \psi$. If $(t_0, \tau_0) \in \text{WF}(u_x)$ then $\tau_0 = \partial_t \phi(x, t_0, \theta_0)$ for some θ_0 with $\partial_\theta \phi(x, t_0, \theta_0) = 0$; similarly if $(t_1, \tau_1) \in \text{WF}(v_x)$ then $\tau_1 = \partial_t \phi(x, t_1, \theta_1)$ for some θ_1 with $\partial_\theta \phi(x, t_1, \theta_1) = 0$. In any microlocal representation of the solution to the wave equation, in the absence of attenuation, the phase contains t only in the form $t\theta_N$ say.

As was shown in Section 1, typical phase functions are of the special form $\phi(x, t, \eta, \rho) = \phi_0(x, \eta, \rho) - t\rho$ and $\psi(x, t, \eta, \rho) = \psi_0(x, \eta, \rho) - t\rho$. In this case the stationary phase conditions in the wave front sets read

$$t_0 = \partial_\rho \phi_0(x, \eta_0, \rho_0), \quad \partial_\eta \phi_0(x, \eta_0, \rho_0) = 0 \quad (16)$$

$$t_1 = \partial_\rho \psi_0(x, \eta_1, \rho_1), \quad \partial_\eta \psi_0(x, \eta_1, \rho_1) = 0 \quad (17)$$

and the respective t -derivatives of the phases yield cotangent components $\tau_0 = -\rho_0$ and $\tau_1 = -\rho_1$. By the (positive) homogeneity of ϕ and ψ w.r.t. (η, ρ) their first order derivatives w.r.t. those variables are (positively) homogeneous of degree 0.

Hence, if we are detecting time-like forward (resp. backward) cotangent directions, i.e., $\rho > 0$ (resp. $\rho < 0$), we may rescale the arguments in the phase and obtain the time shifts

$$t_1 - t_0 = \partial_\rho \psi_0(x, \eta'_1, \pm 1) - \partial_\rho \phi_0(x, \eta'_0, \pm 1) \quad (18)$$

for the corresponding (slowness co-vector) projections η'_0, η'_1 satisfying the conditions

$$\partial_\eta \phi_0(x, \eta'_0, \pm 1) = 0, \quad \partial_\eta \psi_0(x, \eta'_1, \pm 1) = 0. \quad (19)$$

3.4 Correlation optimization

In Luo & Schuster (1991) a traveltime inversion method is described that uses optimal fitting of traveltimes from synthetic seismograms according to wave equation solutions of velocity model perturbations. The fitting criterion is

based upon a crosscorrelation function of the observed (v) and the synthetic (u) seismic data. This crosscorrelation of Luo & Schuster (1991) corresponds to the correlation function defined in (14) above.

We give a brief schematic description of this interesting fitting strategy and test its theoretical validity in three simple examples below. Assume that v represents the observed (or perturbed) wave field and $u = u[\gamma]$ is the solution of a velocity model which is parametrized by the variable velocity $\gamma(x)$. We assume that γ is a real-valued smooth function. Therefore the correlation function is actually dependent on time t and the velocity γ which we indicate in the notation

$$c(t)[\gamma] = \langle u_x[\gamma] \cdot \overline{T_t^* v_x}, 1 \rangle,$$

where (\cdot) denotes the scalar and $[\cdot]$ the functional argument of c . An intuitive expectation would then be that at the exact travelttime shift induced by the perturbation, we find optimum match (overlap) of the corresponding seismograms and therefore the crosscorrelation should be maximal. Leaving possible maxima at time interval boundaries aside, we search for a (γ, t) relation that gives stationarity of the crosscorrelation, i.e.,

$$F(t)[\gamma] = \partial_t c(t)[\gamma] \equiv 0. \quad (20)$$

Naively speaking we can consider this to be an implicit definition of a functional relationship between γ and t^* . Under the condition that $\partial_t F = \partial_t^2 c \neq 0$ we would therefore try to solve equation (20) locally for t as a function of γ and find a quasi-explicit representation by

$$\partial_\gamma t = -\frac{\partial_\gamma F}{\partial_t F}.$$

4 CASE STUDIES

4.1 Two propagating delta waves

Consider $u = \delta_0(x - s)$ and $v = \delta_0(x - \gamma s)$, two Dirac deltas travelling along the lines $x = s$ and $x = \gamma s$ respectively. (These are distributional pullbacks of $\delta_0 \in \mathcal{D}'(\mathbb{R})$, the Dirac measure located at 0, via the maps $(x, s) \mapsto x - s$ and $(x, s) \mapsto x - \gamma s$.) Assume that $x > 0$; the opposite sign case is completely symmetric. We clearly have $u_x = \delta_x$ and $\overline{T_t^* v_x} = \frac{1}{\gamma} \delta_{\frac{x}{\gamma} - t}$, and therefore

$$\text{singsupp}(u_x) = \{x\}, \quad \text{singsupp}(v_x) = \left\{\frac{x}{\gamma}\right\}$$

yielding a singularity shift of $t_1 - t_0 = -x(1 - 1/\gamma)$.

Observe that u_x and $\overline{T_t^* v_x}$ have disjoint singular supports unless $t = -x(1 - 1/\gamma)$ in which case their product would require to multiply δ_x with itself. This cannot be done consistently within the hierarchy of distributional products (cf. (Oberguggenberger, 1992)) and calls for a systematic treatment in the framework of algebras of generalized functions. However, here we touch upon those aspects only in terms of regularizations.

Choose a rapidly decaying smooth function ρ on \mathbb{R} such that $\int \rho = 1$, in other words ρ is a mollifier, and set $\rho_\varepsilon(s) = \rho(s/\varepsilon)/\varepsilon$. Denote by u_x^ε and v_x^ε the convolutions of u_x and v_x with ρ_ε . Then we have

$$u_x^\varepsilon(s) = \rho_\varepsilon(s - x), \quad \overline{T_t^* v_x^\varepsilon}(s) = \frac{1}{\gamma} \overline{\rho_\varepsilon(s + t - x/\gamma)}$$

and upon integration of $u_x^\varepsilon(s) \overline{T_t^* v_x^\varepsilon}(s)$ w.r.t. s with a change of the variable $y = (s - x)/\varepsilon$ we obtain for the regularized correlation function

$$c_\varepsilon(t) = \frac{1}{\gamma \varepsilon} \int \rho(y) \overline{\rho\left(y + \frac{t + x(1 - 1/\gamma)}{\varepsilon}\right)} dy. \quad (21)$$

If we let $\varepsilon \rightarrow 0$ we observe that $c_\varepsilon(t) \rightarrow 0$ pointwise for $t \neq -x(1 - 1/\gamma)$ and $|c_\varepsilon(-x(1 - 1/\gamma))| \rightarrow \infty$. Hence, in an approximative sense, the singular support of the correlation $c(t)$ contains the time shift information. To be more precise, it is not difficult to show that in the sense of distributions

$$c_\varepsilon \rightarrow \frac{1}{\gamma} \delta_{-x(1 - 1/\gamma)}. \quad (22)$$

*Observe that here γ is an infinite-dimensional variable and therefore more attention is to be paid to the exact meaning of applying an 'implicit function theorem'.

For this, we just note that for arbitrary $\varphi \in \mathcal{D}(\mathbb{R})$ one may change the variable in $\int \varphi(t)c_\varepsilon(t) dt$ to $r = (t+x(1-1/\gamma))/\varepsilon$ and use the fact that $\int f * g = \int f \cdot \int g$ for rapidly decreasing functions f and g .

In particular, this result shows that in this case the correlation is stable under changes within this class of representations since the limit does not depend on ρ .

Curiously enough, the regularization approach also gives the correct answer when using the procedure of Luo & Schuster (1991). Define the short-hand notation $k_\varepsilon(x, t, \gamma) = (t + x(1 - 1/\gamma))/\varepsilon$ and consider

$$c'_\varepsilon(t) = \frac{1}{\gamma\varepsilon^2} \int \rho(s)\overline{\rho'(s + k_\varepsilon(x, t, \gamma))} ds$$

and set $F_\varepsilon(x, t, \gamma) = \gamma\varepsilon^2 c'_\varepsilon(t)$. We see that

$$\partial_{t\varepsilon} F_\varepsilon(x, t, \gamma) = \int \rho(s)\overline{\rho''(s + k_\varepsilon(x, t, \gamma))} ds,$$

which is proportional to $\|\rho'\|_{L^1}$ at $t = -x(1-1/\gamma)$ and hence nonzero near this point. Therefore, in that neighborhood, we can solve the implicit equation $F_\varepsilon(x, t, \gamma) = 0$ for $t = t(x, \gamma)$ and find locally

$$\partial_\gamma t(x, \gamma) = -\frac{\partial_\gamma F_\varepsilon(x, t, \gamma)}{\partial_t F_\varepsilon(x, t, \gamma)} = -\frac{\frac{x}{\gamma^2\varepsilon} \int \rho(s)\overline{\rho''(s + k_\varepsilon(x, t, \gamma))} ds}{\frac{1}{\varepsilon} \int \rho(s)\overline{\rho''(s + k_\varepsilon(x, t, \gamma))} ds} = -\frac{x}{\gamma^2}.$$

Assuming the initial condition $t(x, 1) = 0$ (no perturbation \rightarrow no singularity shift) we find from this by integration over γ that

$$t(x, \gamma) = \frac{x}{\gamma} - x = -x(1 - 1/\gamma)$$

which is the correct shift of the singular support.

4.2 A delta wave interacting with a shock

We set $u = \delta_0(x - t)$ and $v = H(x - \gamma t)$ (where H is the Heaviside function) yielding exactly the same configuration of wave front sets as in the previous case. In this case, restricting our attention again to $x > 0$, $u_x = \delta_x$ and $\overline{T_t^* v_x}(s) = H(x - \gamma(s + t))$. The only critical product appears if $t = \bar{t} = -x(1 - 1/\gamma)$: At this point we have to deal with $\delta_x(s) \cdot H(x - s)$ which exists as a so-called ‘strict product (7.4)’ in the notion of Oberguggenberger (1992), Ch. II, assigning the value $\frac{1}{2}\delta_x$ to it. For $t < \bar{t}$ we obtain $u_x \cdot \overline{T_t^* v_x} = \delta_x$ and for $t > \bar{t}$ we have $u_x \cdot \overline{T_t^* v_x} = 0$ because the Heaviside contribution is constant 0 or 1 in those regions. In summary

$$w_{x,t} = u_x \cdot \overline{T_t^* v_x} = \begin{cases} \delta_x & \text{if } t < \bar{t} \\ \frac{1}{2}\delta_x & \text{if } t = \bar{t} \\ 0 & \text{if } t > \bar{t} \end{cases}$$

If we interpret $\langle w_{x,t}, 1 \rangle$ via the Fourier transform of $w_{x,t}$ as $\widehat{w_{x,t}}(0) = c(t)$ then we obtain $c(t)$ as the measurable function

$$c(t) = \begin{cases} 1 & \text{if } t < \bar{t} \\ \frac{1}{2} & \text{if } t = \bar{t} \\ 0 & \text{if } t > \bar{t} \end{cases}$$

As in the previous case, we observe that it is exactly the singular support of c – here, the point $\bar{t} = -x(1 - 1/\gamma)$ – that reveals the information of the correct shift.

Observe, however, that the travel time \bar{t} is in fact the only point where the (distributional) derivative $c'(t) = \delta_{\bar{t}}$ does *not* vanish. The previous evaluation based upon the implicit function theorem hence does not apply.

4.3 Wave equations with different medium constants

We now return to the wave equation (Section 2), assume constant coefficients, and invoke an exact solution representation rather than an asymptotic one. We consider propagation in one spatial dimension.

Let $\chi \in C^\infty(\mathbb{R})$ be real-valued, $\chi(-\xi) = \chi(\xi)$, $\chi \equiv 0$ in a neighborhood of 0 and $\chi \equiv 1$ for $|\xi| \geq 1$; γ a constant > 0 .

In the sense of OIs

$$\begin{aligned} u(x, t) &= \int e^{i(t|\xi| - x\xi)} \frac{\chi(\xi)}{|\xi|} d\xi \\ v(x, t) &= \int e^{i(t|\xi| - x\xi/\gamma)} \frac{\chi(\xi/\gamma)}{\gamma|\xi|} d\xi \end{aligned}$$

(u respectively v are the complex conjugates of $2\pi i$ times the subtrahends in OI representation of the fundamental solutions for the d'Alembert operator with wavespeed equal to 1 and γ , respectively.)

The general WF-bounds according to (11) give

$$\begin{aligned} \text{WF}(u) &\subseteq \{(t, \pm t, -\xi, |\xi|) \mid t \in \mathbb{R}, \pm\xi > 0\} \\ \text{WF}(v) &\subseteq \{(\gamma t, \pm t, -\xi/\gamma, |\xi|) \mid t \in \mathbb{R}, \pm\xi > 0\}. \end{aligned}$$

Observe that half rays in cotangent components are minimal closed cones in $\mathbb{R}^2 \setminus 0$. We will show that, in fact, the inclusion should be replaced by equality.

For symmetry reasons, we give detailed arguments in quadrant $x > 0, t > 0$ only. Since $(\partial_t^2 - \partial_x^2)u = 0$ and $(\partial_t^2 - \gamma^2 \partial_x^2)v = 0$, the theorem on the propagation of singularities ((Hörmander, 1990), Thm. 8.3.3), applies; in particular, if $(t, t) \in \text{singsupp}(u)$ (resp. $(\gamma t, t) \in \text{singsupp}(v)$) then the whole line through this point with directional vector $(1, 1)$ (resp. $(\gamma, 1)$) is in the singular support (with the same perpendicular cotangent component in the wave front set attached to it). Therefore, to prove equality in the above WF inclusion relations, it suffices to show that u (resp. v) is not smooth near $(0, 0)$.

Assuming the contrary, would imply that the function $x \mapsto \partial_t u(x, 0)$ is smooth; but it is also equal to the Fourier transform of $i\chi$, which cannot be smooth since $\mathcal{F}\chi = \mathcal{F}(\chi - 1 + 1) = \mathcal{F}(\chi - 1) + 2\pi\delta$ where the first term is a smooth function (of rapid decay) since $\chi - 1$ is smooth and of compact support. The argument for $\partial_t v(x, 0)$ is the same. We conclude that

$$\text{WF}(u) = \{(t, \pm t, -\xi, |\xi|) \mid t \in \mathbb{R}, \pm\xi > 0\} \quad (23)$$

$$\text{WF}(v) = \{(\gamma t, \pm t, -\xi/\gamma, |\xi|) \mid t \in \mathbb{R}, \pm\xi > 0\}. \quad (24)$$

It follows immediately that both u and v are restrictable to S_x and u_x and v_x are represented as the one-dimensional OIs where x appears as parameter in the phase only. Assuming $x > 0$, we clearly have

$$\text{WF}(u_x) = \{(\pm x, \xi) \mid \xi > 0\} \quad (25)$$

$$\text{WF}(v_x) = \{(\pm x/\gamma, \xi) \mid \xi > 0\}. \quad (26)$$

But then the time shift is given by

$$t_1 - t_0 = \mp x(1 - 1/\gamma), \quad (27)$$

as expected from physical intuition.

In the remainder of this section we analyze the correlation in detail, and investigate how the time shift appears. In the correlation we have to multiply the distributions

$$u_x(s) = \int e^{i(s|\xi| - x\xi)} \frac{\chi(\xi)}{|\xi|} d\xi$$

and

$$\overline{T_t v_x(s)} = \int e^{-i((s+t)|\xi| - x\xi/\gamma)} \frac{\chi(\xi/\gamma)}{\gamma|\xi|} d\xi,$$

which have wave front sets

$$\begin{aligned} \text{WF}(u_x) &= \{-x, x\} \times \mathbb{R}_+ \\ \text{WF}(\overline{T_t v_x}) &= \{-\frac{x}{\gamma} - t, \frac{x}{\gamma} - t\} \times \mathbb{R}_+. \end{aligned}$$

Hence, whenever $t \neq \mp x(1 \mp 1/\gamma)$, the distributions have disjoint singular supports and in case $t = \mp x(1 \mp 1/\gamma)$ the cotangent vectors in their wave front set cannot add up to 0. We conclude that for all t the wave front sets are in favorable position and the product $w_{x,t} = u_x \cdot \overline{T_t v_x} \in \mathcal{D}'(\mathbb{R})$ can be defined in the sense of (1990), Thm. 8.2.10. The following lemma states that we are even allowed to use the naive product of the OI expressions.

Lemma 3. $w_{x,t}$ is (essentially) an OI given by

$$w_{x,t}(s) = \iint e^{i(s(|\xi| - |\eta|) - t|\eta| - x(\xi - \eta/\gamma))} \frac{\chi(\xi)\chi(\eta/\gamma)}{\gamma|\xi||\eta|} d\xi d\eta \quad (28)$$

and $t \mapsto w_{x,t}$ is weakly continuous $\mathbb{R} \rightarrow \mathcal{D}'(\mathbb{R})$. We introduce the following notation:

$$\begin{aligned}\phi_t(s; \xi, \eta) &= s(|\xi| - |\eta|) - t|\eta| - x(\xi - \eta/\gamma) \\ a(\xi, \eta) &= \frac{\chi(\xi)\chi(\eta/\gamma)}{\gamma|\xi||\eta|}\end{aligned}$$

for the phase function and the amplitude.

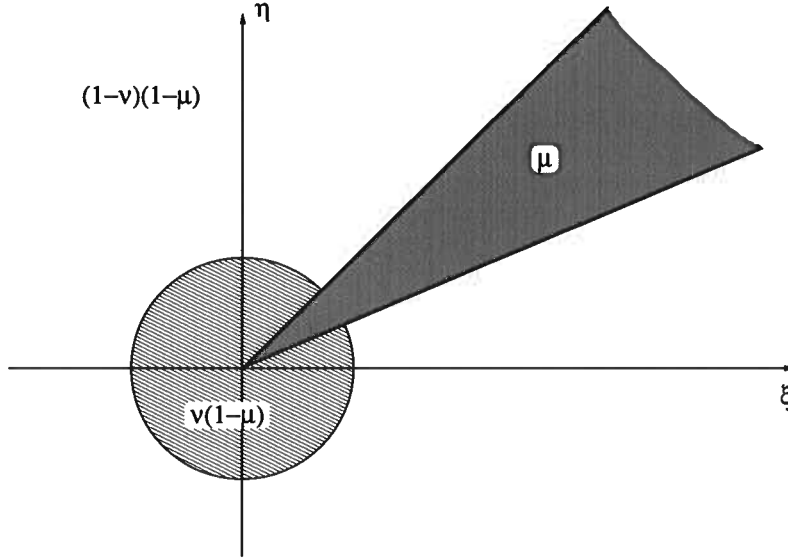
Proof. For the justification of (28) we use the construction of the distributional product in Hörmander (1990), Thm. 8.2.10 via the pullback of the tensor product on \mathbb{R}^2 under the map $\iota(s) = (s, s)$ which embeds \mathbb{R} as the diagonal into \mathbb{R}^2 . In doing so the original OIs may be approximated by smooth regularizations (e.g., amplitude cut-offs in the integrands) the tensor products thereof being pulled back simply as smooth functions (meaning restriction to (s, s) in this case).

It is easily seen then that the smooth functions obtained thereby converge weakly (as OI regularizations) to the OI given in (28). By continuity of the pullback (under the given wave front set conditions) this limit equals the pullback of the tensor product of the corresponding limits and therefore, in turn, is the distributional product $w_{x,t} = u_x \cdot \overline{T_t v_x}$.

Note that a is smooth in (ξ, η) (due to the cut-off χ) and homogeneous of degree -2 outside the set $\{|\xi| \geq 1, |\eta| \geq \gamma\}$ and is therefore a symbol of order -2 . The function ϕ_t is smooth on $\text{supp}(a)$ and homogeneous of degree 1 in (ξ, η) . If $|t| \neq |x(1 \pm 1/\gamma)|$ then the gradient $\partial_{(s,\xi,\eta)}\phi_t \neq (0, 0, 0)$ for all (s, ξ, η) and hence ϕ_t is a phase function.

In case $|t| = |x(1 \pm 1/\gamma)|$ the gradient vanishes exactly along one half-ray component of the set $\{(\xi, \eta) \mid |\xi| = |\eta|\}$ (e.g., along $\xi = \eta > 0$ if $t = -x(1 - 1/\gamma)$). Although it is no longer a phase function in the strict sense, the distribution $w_{x,t}$ is then defined as the sum of a classical integral, an OI, and a Fourier transform of an L^2 -function. We discuss this for the case $t = -x(1 - 1/\gamma)$ in detail, the other cases are completely analogous.

Let $\mu(\xi, \eta)$ be a smooth function that is equal to 1 near $\xi = \eta > 1$, has support in $\{\xi > 0, \eta > 0\}$, and satisfies $0 \leq \mu \leq 1$. Let $\nu(\xi, \eta)$ be smooth with compact support and $\nu(\xi, \eta) = 1$ when $\xi^2 + \eta^2 \leq 1$.



We can split the integral defining $w_{x,t}$ into three terms according to $1 = \nu(1 - \mu) + (1 - \nu)(1 - \mu) + \mu$. The first integral, then, is a classical one defining a smooth function, the second is an OI since the gradient of ϕ_t does not vanish on the support of the integrand. In the third integral, we have

$$\phi_t(s; \xi, \eta) = (s - x)(\xi - \eta) = -\langle (x - s, s - x) | (\xi, \eta) \rangle$$

(insert $t = -x(1 - 1/\gamma)$ and use the fact that $|\xi| = \xi > 0$ and $|\eta| = \eta > 0$ on the support of the integrand) and hence the last term is equal to

$$\int e^{-i\langle (x-s, s-x) | (\xi, \eta) \rangle} \mu(\xi, \eta) a(\xi, \eta) d\xi d\eta,$$

which we interpret via the Fourier transform of the L^2 -function μa on \mathbb{R}^2 as $s \mapsto \mathcal{F}(\mu a)(x - s, s - x)$ in the sense of locally integrable functions — hence it is distribution on \mathbb{R} .

The weak continuity w.r.t. t follows from the smooth dependence of the phase function in the OI representation (cf. (Duistermaat, 1996), before Thm. 2.2.2) and the continuity of the Fourier transform on L^2 . \square

Remark 4. From the last part of the proof it follows that $t \mapsto w_{x,t}$ is weakly smooth on $\mathbb{R} \setminus \{\pm x(1 \pm 1/\gamma)\}$.

In order to define the correlation function, we need to check whether the action of $w_{x,t}$ on 1 is well defined. We will do so by showing that $w_{x,t}$ is tempered with Fourier transform $\widehat{w_{x,t}}$ being in fact a continuous function. This function can be evaluated at 0 yielding the interpretation $\langle w_{x,t}, 1 \rangle = \widehat{w_{x,t}}(0)$.

We use an OI regularization of $w_{x,t}$ via the symmetric cut-off function $\rho(\xi, \eta) = \rho_0(\xi)\rho_0(\eta)$ where $\rho_0 \in \mathcal{D}(\mathbb{R})$ with $\rho_0(r) = 0$ when $|r| \geq 1$, $\rho_0(r) = 1$ when $|r| \leq 1/2$ and $0 \leq \rho_0 \leq 1$. Writing $\rho_j(\xi, \eta) = \rho(\xi/j, \eta/j)$ ($j = 1, 2, \dots$) we obtain $\text{supp}(\rho_j) \subseteq [-j, j]^2$ and $\rho_j \rightarrow 1$ uniformly over compact subsets of \mathbb{R}^2 as $j \rightarrow \infty$. Hence

$$w_{x,t} = \mathcal{D}' - \lim_{j \rightarrow \infty} \int_{-j}^j \int_{-j}^j e^{i\phi_t(\cdot; \xi, \eta)} a_j(\xi, \eta) d\xi d\eta = \mathcal{D}' - \lim_{j \rightarrow \infty} w_{x,t}^j$$

where

$$a_j(\xi, \eta) = \rho_j(\xi, \eta) a(\xi, \eta) = \rho_j(\xi, \eta) \frac{\chi(\xi)\chi(\eta/\gamma)}{\gamma|\xi||\eta|}.$$

Since $\text{supp}(a_j) \subseteq [-j, j]^2$ is compact $s \mapsto w_{x,t}^j(s)$ is smooth and by differentiating inside the integral we see that for all $l \in \mathbb{N}_0$ $(\frac{d}{ds})^l w_{x,t}^j(s)$ is bounded by some constant (depending on l and a_j). Hence $(w_{x,t}^j)_{j \in \mathbb{N}}$ is a sequence in the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions. Therefore, to prove that $w_{x,t}$ is in $\mathcal{S}'(\mathbb{R})$, it suffices to show that $(w_{x,t}^j)_{j \in \mathbb{N}}$ converges weakly in $\mathcal{S}'(\mathbb{R})$, i.e., for all rapidly decaying smooth functions $\varphi \in \mathcal{S}(\mathbb{R})$ the sequence $\langle w_{x,t}^j, \varphi \rangle$ is convergent.

We have

$$\begin{aligned} \langle w_{x,t}^j, \varphi \rangle &= \int \varphi(s) \int e^{i\phi_t(\cdot; \xi, \eta)} a_j(\xi, \eta) d(\xi, \eta) ds \\ &= \int e^{-i(t|\eta| + x(\xi - \eta/\gamma))} a_j(\xi, \eta) \int e^{is(|\xi| - |\eta|)} \varphi(s) ds d(\xi, \eta) \\ &= \int e^{-i(t|\eta| + x(\xi - \eta/\gamma))} a_j(\xi, \eta) \widehat{\varphi}(|\eta| - |\xi|) d(\xi, \eta). \end{aligned}$$

Here, the integrand tends pointwise to $e^{-i(t|\eta| + x(\xi - \eta/\gamma))} a(\xi, \eta) \widehat{\varphi}(|\eta| - |\xi|)$ as $j \rightarrow \infty$ and is dominated by $|a(\xi, \eta) \widehat{\varphi}(|\eta| - |\xi|)|$. It remains to show that $(\xi, \eta) \mapsto a(\xi, \eta) \widehat{\varphi}(|\eta| - |\xi|)$ is in $L^1(\mathbb{R}^2)$; then an application of Lebesgue's dominated convergence theorem will provide us with existence of an explicit integral expression for the limit $\langle w_{x,t}, \varphi \rangle$.

Since $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^2)$, using the explicit structure of a , we have for any $k \in \mathbb{N}$ a bound of the form

$$|a(\xi, \eta) \widehat{\varphi}(|\eta| - |\xi|)| \leq C_k (1 + |\xi|)^{-1} (1 + |\eta|)^{-1} (1 + ||\eta| - |\xi||)^{-k} \quad \forall (\xi, \eta) \in \mathbb{R}^2.$$

While integrating the right-hand side of this inequality over \mathbb{R}^2 , we split the integration into four parts according to the sign combinations of ξ and η . By symmetry, this boils down to estimating only the two kinds of integrals

$$I_- = \int_0^\infty \int_0^\infty \frac{d\xi d\eta}{(1 + \xi)(1 + \eta)(1 + |\eta - \xi|)^k}, \quad I_+ = \int_0^\infty \int_0^\infty \frac{d\xi d\eta}{(1 + \xi)(1 + \eta)(1 + \eta + \xi)^k}.$$

In I_+ we only have to note that $(1 + \xi + \eta)^{-k} \leq (1 + \xi)^{-k/2} (1 + \eta)^{-k/2}$ which together with the remaining factors gives a finite integral as soon as $k > 0$. In I_- we change variables to $\nu = \eta - \xi$, $\mu = \eta$ to obtain

$$I_- = \int_{-\infty}^\infty \frac{1}{(1 + |\nu|)^k} \int_{\max(0, \nu)}^\infty \frac{d\mu}{(1 + \mu)(1 + \mu - \nu)} d\nu.$$

In the inner integral we use $1 + \mu - \nu = (1 + \mu)(1 - \nu/(1 + \mu)) \geq (1 + \mu)/(1 + |\nu|)$ yielding an upper bound $(1 + |\nu|) \int_0^\infty (1 + \mu)^{-2} d\mu$ and hence

$$I_- \leq \int_0^\infty \frac{d\mu}{(1 + \mu)^2} \int_{-\infty}^\infty \frac{d\nu}{(1 + |\nu|)^{k-1}}$$

which is finite if $k > 2$. This proves the assertion that $(\xi, \eta) \mapsto a(\xi, \eta) \widehat{\varphi}(|\eta| - |\xi|)$ is indeed in $L^1(\mathbb{R}^2)$ and establishes the following result.

Proposition 5. $w_{x,t} \in \mathcal{S}'(\mathbb{R})$ and for any $\varphi \in \mathcal{S}(\mathbb{R})$

$$\langle w_{x,t}, \varphi \rangle = \lim_{j \rightarrow \infty} \langle w_{x,t}^j, \varphi \rangle = \int e^{-i(t|\eta| + x(\xi - \eta/\gamma))} a(\xi, \eta) \widehat{\varphi}(|\eta| - |\xi|) d(\xi, \eta). \quad (29)$$

We are now in a position to determine the Fourier transform of $w_{x,t}$ explicitly.

Proposition 6. $\widehat{w_{x,t}}$ is the continuous function on \mathbb{R} given by (the classical integral)

$$\widehat{w_{x,t}}(r) = 4\pi e^{itr} \int_{\{|\xi| \geq r\}} e^{-i(x\xi+t|\xi|)} \cos\left(\frac{x}{\gamma}(|\xi| - r)\right) a(\xi, |\xi| - r) d\xi. \quad (30)$$

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R})$ then $\widehat{\varphi}(s) = 2\pi\varphi(-s)$ and from (29) we obtain

$$\begin{aligned} \langle \widehat{w_{x,t}}, \varphi \rangle &= \langle w_{x,t}, \widehat{\varphi} \rangle \\ &= 2\pi \int e^{-i(t|\eta|+x(\xi-\eta/\gamma))} a(\xi, \eta) \varphi(|\xi| - |\eta|) d(\xi, \eta) \\ &= 2\pi \int e^{-ix\xi} \left(\int_{-\infty}^0 e^{-i(-t\eta-x\eta/\gamma)} a(\xi, \eta) \varphi(|\xi| + \eta) d\eta \right. \\ &\quad \left. + \int_0^{\infty} e^{-i(t\eta-x\eta/\gamma)} a(\xi, \eta) \varphi(|\xi| - \eta) d\eta \right) d\xi, \end{aligned}$$

where in the last line we have made use of the symmetry properties of $a(\xi, \eta)$. Changing coordinates in the inner integrals to $r = |\xi| \pm \eta$ and again by the symmetry of $a(\xi, \eta)$ this reads

$$2\pi \int e^{-i(x\xi+t|\xi|)} \int_{-\infty}^{|\xi|} e^{itr} a(\xi, |\xi| - r) \varphi(r) \underbrace{\left(e^{i(r-|\xi|)x/\gamma} - e^{-i(r-|\xi|)x/\gamma} \right)}_{2 \cos\left(\frac{x}{\gamma}(|\xi| - r)\right)} dr d\xi.$$

Finally, since $\varphi \in \mathcal{S}$ and $|a(\xi, |\xi| - r)| \leq p(r)(1 + |\xi|)^{-2}$ for some polynomial in r , we may interchange the order of integration and arrive at

$$\langle \widehat{w_{x,t}}, \varphi \rangle = \int \varphi(r) \cdot 4\pi e^{itr} \int_{\{|\xi| \geq r\}} e^{-i(x\xi+t|\xi|)} \cos\left(\frac{x}{\gamma}(|\xi| - r)\right) a(\xi, |\xi| - r) d\xi dr.$$

Since φ was arbitrary and the above upper bound for $a(\xi, |\xi| - r)$ shows that the inner integrand is in L^1 w.r.t. ξ , the proposition is proved. \square

From (30) we immediately obtain the correlation by setting $c(t) = \widehat{w_{x,t}}(0)$, in the form

$$\begin{aligned} c(t) &= 4\pi \int_{-\infty}^{\infty} e^{-i(x\xi+t|\xi|)} \cos\left(\frac{x\xi}{\gamma}\right) a(\xi, \xi) d\xi \\ &= \frac{2\pi}{\gamma} \left(\int_{-\infty}^{\infty} e^{-i(x\xi(1-1/\gamma)+t|\xi|)} \frac{\chi(\xi)\chi(\frac{\xi}{\gamma})}{|\xi|^2} d\xi \right. \\ &\quad \left. - \int_{-\infty}^{\infty} e^{-i(x\xi(1+1/\gamma)+t|\xi|)} \frac{\chi(\xi)\chi(\frac{\xi}{\gamma})}{|\xi|^2} d\xi \right). \end{aligned} \quad (31)$$

This shows that $t \mapsto c(t)$ is continuous and can be represented as the difference of two (classically convergent) OIs with symbols of order -2 , and hence $c \in L^1(\mathbb{R})$. Note that the (distributional) derivative $c'(t)$ can be obtained by differentiating w.r.t. t inside the OI raising the order of the symbol by one. Therefore c' will not be continuous on the whole line.

Finally, we observe that again the information about the singularity shift is revealed by the singular support of $c(t)$. By the stationarity condition on the phase functions, we find

$$\text{WF}(c) \subseteq \{\pm x(1 + 1/\gamma), \pm x(1 - 1/\gamma)\} \times \mathbb{R}_+,$$

where $\pm x(1 - 1/\gamma)$ represent the true shifts from $\pm x/\gamma$ to $\pm x$ whereas $\pm x(1 + 1/\gamma)$ are the distances from $\mp x/\gamma$ to $\pm x$. It is easily seen that $c(t)$ cannot be smooth at the points $t = \pm x(1 \pm 1/\gamma)$, e.g., by noting that each time derivative brings down a new factor of $|\xi|$ in each integrand, and at the t values in question one of the phase functions vanishes identically along a half-line in ξ . Hence, we have in fact the exact information

$$\text{singsupp}(c) = \{\pm x(1 + 1/\gamma), \pm x(1 - 1/\gamma)\}, \quad (32)$$

which also fits nicely with remark 4 on the weak smoothness of $w_{x,t}$.

5 MICROLOCALIZATION OF THE CORRELATION

From the case studies, we conjecture that the singular support of the correlation of two wave fields reveals the relative shift in wave front sets between them. As we pointed out, in general, the critical point set of the correlation need not be compatible with this shift. Here, we propose an alternative approach to extract the shift from the correlation, viz., by detecting its singular support. We design a pseudodifferential operator that enables this detection. Our approach can be applied invariably to any derivative of the correlation also.

In the generic case, the correlation $c \in \mathcal{S}'$ with Fourier transform \widehat{c} . Let ϕ be the Gaussian in one dimension, define

$$\psi_{r,t}(s) = \frac{1}{r} \phi\left(\frac{s-t}{r}\right). \quad (33)$$

Introduce

$$W_{\psi_{r,t}c}(\tau) = \widehat{\psi_{r,t}} * \widehat{c}(\tau) \quad \text{for } \tau = \pm 1, \quad (34)$$

a continuous wavelet transform that can be written as the action of a pseudodifferential operator $\psi_{r,t}(D_\tau)$ (in $\text{Op } \mathcal{S}^{-\infty}$) on \widehat{c} . The growth properties reveal the wave front set at t in the direction ± 1 . In fact, $(t, \pm 1) \notin \text{WF}c$ if for any $N \in \mathbb{N}$,

$$|W_{\psi_{r,t}c}(\pm 1)| \leq C_N r^N \quad \text{for } r \in]0, 1] \quad (35)$$

(see Folland (1989)). Effectively, this leads to a scanning procedure over t : whenever the condition is not satisfied, $t \in \text{singsupp}(c)$. In particular, this applies if $|W_{\psi_{r,t}c}(\pm 1)| \approx r^M$ for some fixed M .

If c would allow an OI representation, as is the case in the examples of Section 4, we could apply a stationary phase argument instead, as in (11).

6 DISCUSSION

Starting from the microlocal representation, we analyzed the measurement process of wave fields. Such process can be described by a restriction operator. We then addressed the issue of how the detection of wave front sets propagates through the measurement process. Then we focused on the detection of (base) shifts in wave front sets due to perturbation of the wave field within the measurement. We introduced the distributional cross-correlation as a tool for this purpose, and analyzed its properties. In a series of case studies, we investigated in what way the cross-correlation reveals the shifts. In the first case the correlation was a measure, in the second case it was a bounded measurable function, and in the third case it was a continuous function. It was conjectured that the time shift coincides with the singular support of the correlation. We proposed a procedure (a pseudodifferential operator) to detect the shift based on microlocalization. Such procedure would comprise the foundation for wave-equation tomography.

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