Project Review

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Seismic Inverse Methods for Complex Structures
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Sponsor Mailing List
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Those reports in this book that were produced primarily under consortium support and have not been previously distributed or submitted for publication, will be available for general distribution after July 11, 1998. If you have independently generated results that duplicate or overlap these, and plan to submit them for publication under your own name before this date, please notify us immediately, so that misunderstandings do not arise.
Introduction

This, the fourteenth edition of the Project Review Report on the Consortium Project at the Center for Wave Phenomena, once again contains a large number of papers spanning a variety of topics.

Issues in processing and interpretation that take anisotropy into account continue to occupy a significant portion of the research effort. Building on work over the past few years, we address problems of moveout inversion and AVO analysis for models of increasing complexity (e.g., finely-layered, azimuthally anisotropic media, and transversely isotropic (TI) media with tilted symmetry axis). Included in the report are papers on dip moveout of, and parameter estimation for, converted waves in TI media. Also, a field-data example brings out the need to separate the influences of azimuthal anisotropy and lateral heterogeneity in 3-D inversion of moveout information, and shows an approach that accomplishes this goal.

Preservation of amplitude in imaging has been an area that has always received attention within CWP, and this year research in this general area has experienced a new spurt of activity. Also, new approaches to forward and inverse problems offer considerable promise for the development of modeling, migration, and multiple-suppression that take multi-pathing into account in a way that may be considerably more efficient than present alternatives.

Papers on these and many other topics are grouped in this report into the following categories: migration and inversion, traveltime inversion, data processing, modeling, scientific computation, theory, and optimization. Again this year, while most of the papers in the report will be presented orally during the Annual Project Review Meeting, some are offered in just their written version. Conversely, some of the oral presentations cover recent work that is not included in this volume, and will be distributed either at the Meeting or shortly afterward.

I especially would like to highlight that, once again, a CWP principal has received a major award of the SEG. Vladimir Grechka, who this past year moved from a post-doctoral position to assistant research professor in the Department of Geophysics, was recipient of the Society’s 1997 J. Clarence Karcher Award to an outstanding young geophysicist.

Following is an overview of other developments within CWP during the past year.

Center Support

The addition of TOTAL Petroleum as our newest sponsor brings the support base for CWP to its highest level ever — 35 industry sponsors. A full list of sponsors appears on the acknowledgment page at the beginning of this volume, and a list of technical contacts within those sponsor companies appears at the end of this volume.

Although our government support has also remained generally strong, we foresee struggle ahead in efforts to retain anything close to the current level of government support for research. A list of those sponsor agencies also appears in the acknowledgment page at the beginning of this volume.

Our industrial support and our government support for both research and education complement one another; each gains from, and strengthens, the other. As a net result, for the
present annual fee of $36,000, a company participates in a research project whose total funding level is close to $2M — a leverage factor of about 60.

Also, in addition to their support as CWP sponsors, Mobil, Shell, and Phillips have generously continued to provide much-needed and appreciated additional fellowship support for students and grants for faculty, both within CWP and in the Geophysics Department in general. This year has also continued a trend toward an increasing percentage of our CWP students working in industry during the summer. Such summer research within industry has been beneficial to both the sponsor companies and the students, many of whom have been able to pursue their Ph.D. research while working.

The SEG Foundation has continued to provide support for SU, under John Stockwell’s leadership, and for Samizdat Press, managed at CWP by John Scales and from a distance by Martin Smith, of New England Research. SU has also received support this past year from the Gas Research Institute.

CWP on the Web

With the addition of five titles and support from the SEG, Samizdat Press has grown to fifteen books and sets of lecture notes.

The most recent additions are

- “A Guided Tour of Mathematical Physics,” by Roel Snieder, Department of Geophysics, Utrecht University, The Netherlands.


- “Physics Formulary,” by J.C.A. Wevers.

- “Greek Seismology,” by Chris Liner, University of Tulsa.

- “Mathematical Programming Glossary,” by Harvey Greenberg.

Every few months, you may wish to check the latest offerings of Samizdat Press at samizdat@dix.mines.edu or www.cwp.mines.edu/bookshelf.html.

Computing Environment

We have phased out our old network of NeXT-stations for all technical work, replacing them with fast Intel-based workstations running Linux, and will soon replace the NeXTs used by our support staff with Windows NT-based Pentium workstations. Our latest batch of machines consists of 300 MHz dual-processor Pentium IIIs with .5 Gb of memory per machine, and now our fastest research machines (four of them) are dual-processor Pentium II PC’s, which are currently locally networked via a 100BaseT ethernet switch. Soon, all of our systems will be connected via fast ethernet. We are also planning on extending our data storage capacity by at least 30 gigabytes.
Software Releases

_CWP proprietary code_

The following codes were distributed this past year

- U41: REFANISO, Andreas Rueger's programs for computing reflection coefficients in VTI and HTI anisotropic media

- U42: SVUEL2DF, Tariq Alkalifa's program to compute stacking Velocity semblance for a single time over $V_{nmo}$ and $\eta$ in 2-D

- U43: TRIELAS, Andreas Rueger's programs for seismic model building and seismic modeling in Delaunay triangulated, 2D anisotropic elastic media

- U44: SUFRAYT3D, Meng Zhaobo's program for 3-D ray-tracing via wavefront construction in tetrahedrally gridded media characterized by linear variation in slowness-squared

_SU_

Release 31 of the Seismic Unix package was announced on 17 Oct 1997. Thanks to Stew Levin of Mobil, almost every code has been modified to agree more closely with ANSI C standards. Also, courtesy of Baoniu Han of CWP, several programs for prestack depth migration (by methods of phase-shift, split-step, finite-difference, and Fourier finite-difference) have been added to the package. All these programs are coded to operate in parallel on a heterogeneous distributed system of computers such as that within CWP. Also, some 3D graphics support has been added, thanks to Zhaobo Meng's efforts with the Mesa package. As always, numerous bug fixes and extensions have been made throughout the package. In addition to support from the Consortium, SU development continues to receive funding support from both the SEG and the Gas Research Institute.

Visitors

Continuing with the satisfying trend set a year ago, CWP was blessed with visits from a number of outstanding scientists and friends from other universities as well as from industry. Below is a list of those who spent some time here working with various CWP people.

- Art Gautesen, of the Applied Mathematics Department at the University of Iowa, spent three weeks interacting with Martijn de Hoop.

- Mathias Fink, from the University Denis Diderot, in Paris, spent a week giving highly stimulating lectures on physical experiments and theory of inversion of acoustic data, and interacting with John Scales and Roel Snieder.

- Roel Snieder, a visiting professor from Utrecht University, The Netherlands, completed his six months of a sabbatical with CWP, working on problems of wave propagation and inversion of global seismic data. Interacting primarily with John Scales, he also gave a number of thoroughly stimulating lectures.
• Jérôme Le Rousseau, visiting scholar sponsored by Elf Exploration, completed his 16-month visit in November. Jérôme developed PSPI migration algorithms for 2D and 3D migration in anisotropic media. Also, working with Martijn de Hoop, he has been extending applications of generalized phase screens and use of Bremmer series for modeling and migration, and will continue this research as a Ph.D. student.

• Andrew Norris, of Rutgers University met with Norm, Martijn, Ilya and many students for three days with the goal of gaining understanding of our approach to inversion and how it related to some inverse problems that he was working on.

• Sverre Brandsberg-Dahl, a graduate student of Bjorn Ursin’s at the Norwegian University of Science and Technology, Trondheim, spent six months working with Martijn de Hoop and will continue as a Ph.D. student in CWP.

• Kees Wapenaar, of Delft University, and Bjorn Ursin, from the Norwegian University of Science and Technology visited for a week, interacting with Norm Bleistein.

• Yves Le Stunff, of Total Exploration, spent a week getting an update on the range of CWP activities.

• Felix Hermann, of the Department of Mathematics at Stanford University and Rachel Kuske, from IMA, at the University of Minnesota, collaborated with Martin de Hoop for a week.

• Peter Pecholcs, of Saudi Aramco, spent the better part of a week working with Ken Larner and students and giving talks on statics estimation.

• Martin Tygel, of the University of Campinas, Brazil, worked with Herman Jaramillo and Norm Bleistein on problems related to seismic data mapping.

• Fusheng Yang, who is with Texaco in Houston, spent a bit more than a week interacting with various CWP faculty and students.

• Adrianus de Hoop, Professor Emeritus at Utrecht, gave some superb lectures and spent two weeks developing theory and writing papers with Martijn de Hoop.

• Mats Gustafsson, a Ph.D. student at the University of Lund, in Sweden, started a six-month stay with CWP in early April. He will be working with Martijn de Hoop on problems of imaging and inversion using Bremmer series.

Students

In a dramatic demonstration of a law of small numbers, a year ago we reported that nine students had graduated during the year, a result of which was a reduction in our numbers of graduate students. Herman Jaramillo, who just recently defended his Ph.D. thesis is this year’s only CWP graduate. Because only two new students, Albena Mateeva from Bulgaria, and Shogo Narahara supported by Japex Geosciences, joined us this past year, our numbers reduced to 10 students supported with funds from the Consortium project, and three students supported by their home national oil companies. Of the 13 students, 11 are studying for the Ph.D. and two for the M.S. degree.
We are most pleased with the prospects for the three new students who have accepted offers to join the Ph.D. program. We are especially pleased that Jérôme Le Rousseau and Sverre Brandsberg-Dahl, who have already spent considerable time with CWP as visiting scholars, have chosen to pursue their Ph.D. with CWP. Jerome was an Elf Exploration Visiting Scholar, and Sverre, an M.S. student of Bjorn Ursin at the Norwegian University of Science and Technology, in Trondheim, has been working with Martijn de Hoop and will continue with him for his Ph.D. studies. The third new student, Chris Robinson, comes with a B.S. degree in physics from Texas A&M.

Publications

Since last year's Project Review Meeting, 18 papers authored by CWP faculty and students have been published, and 32 publications are currently under review or in press for publication, in a variety of journals. These journals include *Geophysics, Geophysical Prospecting, Geophysical Journal International, Journal of Computational Physics, SIAM, Journal of the Acoustic Society of America, Science, Journal of Geophysical Research*, and *Proceedings on Mathematical, Physical, and Engineering Sciences of the Royal Society*. Many of these papers were presented to our sponsors during the 1997 Project Review meeting and appeared in the annual report (CWP-259). You can find a complete listing of distributed reports in the CWP list of "Available Papers."

Welcome

As always, it is with pleasure that we welcome representatives of our sponsor companies to the Annual Project Review Meeting. We look forward with eagerness to the opportunity to share results of this past year's research as well as our current thoughts about projects in progress. With equal enthusiasm, we look forward to the multi-path exchange of ideas that has been characteristic of past CWP Project Review Meetings (multi-pathing isn't always bad).

We hope that you will find the contents of this report to be stimulating and of interest. Likewise, we hope that you find that we attain our goal that the oral presentations convey information with clarity and provide motivation for sponsor representatives to subsequently dig in depth into the contents of particular interest within this written report.

Ken Larner, Interim Director
Center for Wave Phenomena
April 1998
Common scattering angle sections

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ABSTRACT
The Generalized Radon Transform (GRT) is employed to carry out a linearized asymptotic inversion of seismic data in a heterogeneous acoustic medium. From the inverse GRT operator, it is natural to define the common scattering-angle (CSA) domain. To obtain a complete pair of operators in the common scattering-angle domain, we introduce both a modified forward operator, which models data from the CSA sections, and an inverse operator. Further we show how the CSA sections can be used in a similar fashion as the common offset sections, i.e. giving a partial reconstruction of the scatterer for each fixed angle. From the forward, and inverse operator in the CSA-domain, we derive the sensitivity transform, which can give a measure of how good the current back-ground model is. This transform can give an update of medium parameters as long as the medium is described using a relatively sparse selection of parameters.

Key words: GRT, common scattering-angle domain, sensitivity transform

Introduction
We present a data domain in which we can do both modeling and imaging/inversion based on a micro-local coordinate representation of the seismic data. We use the approach to micro-local coordinates in inversion introduced by Beylkin et al. (1984). They show how one can use the generalized Radon transform (GRT) to invert seismic data, based on an integral over planes at the image point. However, in their approach the local geometry at the image point is transformed into the more conventional acquisition coordinates by introducing a Jacobian to change the integral variables in the inversion operator. The inverse operator in acquisition coordinates is matched to the volume scattering representation of the forward operator, the forward GRT operator. The forward operator is an integral over isochrons for a fixed source/receiver pair.

We consider a heterogeneous acoustic medium where a remote scattering domain is illuminated with acoustic waves generated by a point source. The scattering is linearized using the familiar perturbation representation, dividing the field into a known smooth background and an unknown singular part. The scattered field is recorded with point receivers on the surface. We invert for the unknown medium parameters using the generalized Radon transform for a common (fixed) scattering angle. This differs from the conventional approach (Miller et al., 1987) in the way we access the data, and the way the actual integration in the GRT is performed. Operating in this domain eliminates any Jacobian to relate the phase directions at the image point to sources and receivers. We do the ray-tracing from the image point and up to the surface, sorting the rays by scattering angle and azimuth and assigning them to be source- or receiver-ray, depending on where they hit the surface. The inversion then becomes an explicit integral over local variables. We also show how the inverse GRT in the common scattering-angle domain can be matched to a forward operator over isochrons for a fixed source-receiver pair.

Once we have established the two operators that can take us between the data and the scatterer domain, we introduce the sensitivity transform. This operator is treated in de Hoop & de Hoop (1997): it is an operator made from a combination of the forward and the inverse operators. The sensitivity transform can be used to analyze the coherency between different seismic experi-
ments, in our case the different common scattering-angle sections. It can also provide an update for the current medium parameters.

Finally, we show an example based on the 2D case of the theory, where we extract common image-point gatherers (CIG) from the common scattering-angle data volume. The CIG in scattering angle can be used to do velocity analysis employing the same techniques that are used on CIG's in offset.

Geometry and coordinates

The acquisition coordinates \((s, r)\) lie on the surfaces \(\partial S\) and \(\partial R\), respectively. The domain in which the scattering takes place is \(D \in \mathbb{R}^3\), and the acquisition surface, \(\partial S \times \partial R\) is homeomorphic* to a part of \(S^2 \times S^2\), where \(S^2\) is the unit sphere. Measurements are taken in a time interval \([0, T]\). We denote Cartesian coordinates in the configuration as

\[
\begin{align*}
x &= (x_1, x_2, x_3) &= \text{Cartesian position vector}, \\
s &= (s_1, s_2, s_3) &= \text{source point}, \\
r &= (r_1, r_2, r_3) &= \text{receiver point}.
\end{align*}
\]  

(1)

Ray geometry

We now define the ray geometry in our configuration. A general ray is a path connecting two points \(x\) and \(x'\) in space. The travel time between these two points along the ray is denoted by

\[
r = \tau(x, x').
\]  

(2)

We define the slowness vector at \(x\) associated with a ray emanating from a point source or a point receiver at \(x'\) in the surface \(\partial S \) or \(\partial R\), to a scattering point \(x \in D\) as

\[
\gamma_s = \nabla_x \tau(x, x').
\]  

(3)

The unit phase direction associated with the slowness vector at \(x\) along the ray is given by

\[
\alpha_s = c(x)\gamma_s(x),
\]  

(4)

where \(c(x)\) is the medium wave speed. According to equation (3), we can write the slowness vectors for the source ray and the receiver ray at \(x\) as

\[
\begin{align*}
\gamma_s^s &= \nabla_x \tau(x, s), \\
\gamma_r^r &= \nabla_x \tau(x, r),
\end{align*}
\]  

(5)

respectively, and the associated phase directions are

\[
\begin{align*}
\alpha_s^s &= c(x)\gamma_s^s(x), \\
\alpha_r^r &= c(x)\gamma_r^r(x).
\end{align*}
\]  

(6)

* homeomorphic: has the same shape as

![Figure 1. Ray geometry](image.png)

In acquisition coordinates, we define the total or two-way travel time \(T\) for the two rays connecting a source \(s\) with a receiver \(r\) via a scatterer \(x\), as

\[
T(r, x, s) = \tau(r, x) + \tau(x, s).
\]  

(7)

The mapping

\[
\begin{align*}
s &: (S^2)_s &\rightarrow & \partial S, &\quad \alpha_s^s &\rightarrow s(\alpha_s^s) \\
r &: (S^2)_r &\rightarrow & \partial R, &\quad \alpha_r^r &\rightarrow r(\alpha_r^r)
\end{align*}
\]  

(8)

transforms the acquisition variables \((s, r)\) into the image point, \(x\), related variables \((\alpha_s^s, \alpha_r^r)\). Here the subset \((S^2)_s\) of the unit sphere \(S^2\) is the area element which contains \(\alpha_s^s\), defining a ray that connects \(x\) with an actual source \(s\), that is \(\alpha_s^s \in (S^2)_s\). For the receiver we must have \(\alpha_r^r \in (S^2)_r\). Then we write the total travel time \(T\) as a function of the two phase directions \((\alpha_s^s, \alpha_r^r)\)

\[
T(\alpha_s^s, \alpha_r^r) = \tau(\alpha_s^s, x) + \tau(\alpha_r^r, x).
\]  

(9)

We also define the gradient of the total travel time

\[
\gamma_s^0 = \gamma_s^s + \gamma_r^r,
\]  

(10)

which, in acquisition coordinates, corresponds to

\[
\gamma_s^0 = \nabla_x T(r, x, s).
\]  

(11)

We introduce a short hand notation, setting \(\gamma_s^0\) to be the gradient of total travel time at \(x\) as defined in (11).

Dip, scattering angle and azimuth

In the framework of generalized Radon transforms (GRT) it is natural to work in local coordinates at the image point. We will use an indexing in migration dip \(\alpha_s^0\), scattering angle \(\theta\) and azimuth \(\psi\) (third Euler angle around \(\alpha_s^0\)). For an image point \(x \in D\) these variables
Isochron coordinates

We now reconsider the coordinates associated with $x \in \mathcal{D}$. We change those coordinates to curvi-linear ones associated with the acquisition geometry in accordance with the following construction. Let the level surface associated with the total travel time function be defined as

$$\Sigma(r) = \{x \in \mathcal{D} : T(r, x, s) = r\} \quad (15)$$

for $s \in \partial S$, $r \in \partial R$ fixed. This surface is the $r$ isochron for a fixed source and receiver pair $(s, r)$. The isochron is made up of all points $x \in \mathcal{D}$ where the travel time from source to scatterer to receiver is equal to the isochron time $r$. On the isochron we introduce the curvi-linear coordinates $\sigma = (\sigma_1, \sigma_2)$, $\mu = 1, 2$, such that $\sigma_1$ are coordinates in the surface of the isochron and $\sigma_2$ is the local coordinate normal to the isochron, in the $\alpha_n^s$ direction. Thus, $\sigma_2$ is the coordinate along the gradient of the total travel time, see Figure 2. If $\sigma_3$ represents the actual time of the isochron, we set $\sigma_3 = \tau$. To relate an infinitesimal volume in $x$, i.e. the volume form, to the one in isochron coordinates $\sigma$, we need the Jacobian

$$\frac{\partial(x)}{\partial(\sigma)} = \frac{\partial \varphi(x) \cdot \partial \varphi(x) \cdot \partial \varphi(x)}{\partial \varphi(x) \cdot \partial \varphi(x) \cdot \partial \varphi(x)} \quad (16)$$

For a homogeneous medium this change would be the one from Cartesian to elliptic coordinates.

To make the mapping from a source or a receiver at the surface to an image point along the rays unique, we introduce the projection of the slowness vectors at the source onto $\partial S$, and the projection of the slowness vectors at the receiver onto $\partial R$ (de Hoop, 1998) (Figure 1). These projections are given by

$$p_\varphi(x, \alpha_n^s) = \gamma_n^s((x, \alpha_n^s) - (\beta(\sigma) \cdot \gamma_n^s(x, \alpha_n^s))) \beta(\sigma),$$

$$p_\varphi(x, \alpha_n^r) = \gamma_n^r((x, \alpha_n^r) - (\beta(\tau) \cdot \gamma_n^r(x, \alpha_n^r))) \beta(r), \quad (17)$$

in which

$$\beta(\sigma) = \text{unit normal to } \partial S \text{ at source},$$

$$\beta(\tau) = \text{unit normal to } \partial R \text{ at receiver.} \quad (18)$$

$p_\varphi$ and $p_\varphi$ can be directly estimated from the data as a function of $s$ and $r$ (by slant stacking). Using these, we have the following injective mapping from a source $s$ and a receiver $r$ down to the image point $x$ on the $r$ isochron

$$(s, r, \tau, \omega p_\varphi, \omega p_\varphi, -\omega) \rightarrow (x, \omega \gamma_n^s), \quad (19)$$

where $x = x(\sigma)$ and $\sigma_3 = \tau$ : for a fixed source-receiver pair and with the two conditions

These angles and vectors are shown in Figure 1. We can uniquely go from acquisition-surface coordinates to local coordinates in the scattering domain.
the travel time $\tau$ is fixed

The mapping given by (19) induces a relation,

$$ (p_x, p_\nu) \rightarrow (\gamma^0_{\sigma_1, \sigma_2}) \rightarrow (\alpha^0_{\psi, \theta_\psi}). $$

(20)

This tells us that we know our position on the isochron from both the curvilinear coordinates $({\sigma_1, \sigma_2})$, and from the gradient of the total travel time $\gamma^0_{\sigma}$. We will later use the change

$$ (\sigma_1, \sigma_2) \rightarrow \frac{\partial(\sigma_1, \sigma_2)}{\partial(\alpha^0_{\sigma_1, \sigma_2, \tau})}. $$

(21)

When we know our position on the isochron as a function of $(\sigma_1, \sigma_2)$ we can use the mapping

$$ \gamma^0_{\sigma} = \frac{x \in \Sigma(\tau)}{(\alpha^0_{\sigma}, \psi_{\sigma})} $$

(22)

to give the scattering angle and azimuth.

For isotropic media, we obtain the migration dip and scattering angle from the gradient of total travel-time using equation (12), and introducing the Jacobian

$$ \frac{\partial(\gamma^0_{\sigma})}{\partial(\alpha^0_{\sigma}, \theta_{\sigma})}. $$

(23)

In the isotropic case the magnitude of the gradient vector is given by

$$ |\gamma^0_{\sigma}| = \frac{2\cos \theta/2}{c(x)} $$

(24)

Associated with $\gamma^0_{\sigma}$ is a ray with take-off direction $\alpha^0_{\sigma}$, that reaches the surface at $x^0(\alpha^0_{\sigma}) \in \partial S \cup \partial R$, with slowness vector $\gamma^0_{\sigma}(\alpha^0_{\sigma})$. Then, let

$$ p_0(x^0, \alpha^0_{\sigma}) = \gamma^0_{\sigma}(\alpha^0_{\sigma}) - (\beta(x^0) \cdot \gamma^0_{\sigma}(\alpha^0_{\sigma}) \beta(x^0)), $$

(25)

be the projection of this vector onto the acquisition surface.

Source and receiver Green’s functions

In the integral representation for the scattered field, we encounter the Green’s functions for the rays connecting the source and the receiver with a scattering point. The leading order asymptotic approximation for the Green’s function has the following structure

$$ G(x, x', \omega) = A(x, x') \exp[i\omega \tau(x, x')]. $$

(26)

Here $A$ denotes the pressure amplitude, satisfying the transport equation and originating from a point source at $x'$. We have

$$ A(x, x') = \frac{1}{4\pi \rho(x') \rho(x) M}, $$

$$ M = \frac{c(x)c(x') \left| \frac{\partial x}{\partial q_1} \times \frac{\partial x}{\partial q_2} \right|}{\left| \frac{\partial^2 x}{\partial q_1^2} \times \frac{\partial^2 x}{\partial q_2^2} \right|}, $$

(27)

where $(q_1, q_2)$ parameterize the rays originating at the source, and can be chosen to be $\alpha^0_{\tau}$. Further $c(x)$ is the medium wave speed, and $\rho(x)$ is density. The source and receiver Green’s functions are given as special cases of equation (26), setting $x' = s$ or $x' = r$. The Green’s function in time is found by taking the inverse Fourier transform of equation (26). The Green’s functions defined above satisfy the scalar wave equation for the medium given by $c(x)$.

Forward GRT operator in the CSA domain

In this section, we introduce the volume scattering representation in the Born approximation for the scattered field. Then, we show how this volume integral can be recast into an integral over isochron surfaces. First, we introduce the perturbation representation for the wave speed

$$ c^{-2}(x) = c_0^{-2}(x) + c^{(1)}(x) $$

(28)

where $c_0^{-2}(x)$ is the background model and $c^{(1)}(x)$ is the acoustic scattering coefficient. The scattered field is given by the forward generalized Radon transform (Miller et al., 1987)

$$ p(r, s, t) \simeq (L[c_0]c^{(1)}) (r, s, t) = \frac{\partial^2}{\partial t^2} \int_D dA(r, x, s)c^{(1)}(x)[t - T(r, x, s)] $$

(29)

This operator is a function of $c_0(x)$, and gives the scattered field at a time $t$ for a fixed source-receiver pair, as an integral over the support $D$ of $c^{(1)}$. The amplitude term is $A(r, x, s) = A(s, x)A(x, r)$, where the amplitudes from source to scatterer and scatterer to receiver are given as special cases of equation (27).

The hyper-surface over which the integral in the modeling operator (29) is taken is the isochron for a fixed travel time $\tau$ and a fixed source-receiver pair $(s, r)$ as defined in (15). We will use the same kind of isochron to re-model data from a scattering angle dependent scattering coefficient. On the isochron we introduce curvi-linear coordinates as above. Let $d\Sigma$ denote a surface element on the isochron. Then we can recast the volume integral taken over isochrons to a surface integral taken over...
isochron coordinates. Using eq.(16) we have
\[
dx = \frac{1}{|\nabla T(s, \tau)|} \left| \mathbf{v}(\sigma) \right| d\sigma_1 d\sigma_2.
\]  
with \(d\Sigma = |\partial_\sigma \mathbf{v} \times \partial_{\theta z} \mathbf{v}| d\sigma_1 d\sigma_2\).  

(30)

In carrying out this transformation, we are assuming that there are no singular points along the travel time curve. To rewrite the general expression (29), using eq. (30), we make use of the property,
\[
\int_D \cdots \delta(\tau - T(s, \tau, \sigma)) dx = \int_T(s, \tau, \sigma) = \cdots \left| \frac{1}{|\nabla T(s, \tau)|} \right| \mathbf{v}(\sigma) d\Sigma.
\]  

(31)

The forward operator in isochron coordinates then becomes
\[
p(s, r, \tau) \simeq -\frac{\partial^2}{\partial \tau^2} \int_T(s, \tau, \sigma) = \mathcal{A}(s, \tau, \sigma) \cdot c^{(1)}(\sigma) \left| \frac{\partial x}{\partial \sigma} \right| \mathbf{v}(\sigma) d\sigma_1 d\sigma_2.
\]  

(32)

Since \((s, \tau)\) is fixed, \(\alpha^0\) is the only degree of freedom along the isochron. By introducing the Jacobian in eq.(21), taking care of the change from isochron coordinates into dip, we can express the forward operator as
\[
p(s, r, \tau) \simeq -\frac{\partial^2}{\partial \tau^2} \int_{(S^2)_{\omega}} \mathcal{A}(s, \tau, \sigma) \cdot c^{(1)}(\sigma) \left| \frac{\partial x}{\partial \sigma} \right| \mathbf{v}(\sigma) d\sigma_1 d\sigma_2,
\]  

(33)

where \((S^2)_{\omega}\) is the subset of the unit sphere where the dip can vary. This last change of variables is only valid in the absence of singular points on the isochron. From this operator and the mapping in eq.(23), we can find the forward operator in the common scattering angle domain \(L^s\), where \(\theta = \theta(\alpha^0)\).

What then remains is to derive the expression for the two Jacobians. Later in the Appendix, we derive an explicit expression in the case of a homogeneous medium. For the inhomogeneous case, we need to find the expression numerically. Since the input scattering coefficient is a function of scattering angle, we need to relate the scattering angle to the coordinates along the isochron.

The dual GRT operator

The dual generalized Radon transform is
\[
\langle c^{(1)}(\psi) \rangle = -\int_{E_0} \int_{E_\omega} \int_{E_0} \cdots \left[ A(\alpha^0, \theta, \psi) \right] p(\alpha^0, \psi, \theta, \psi, t) \cdot \mathcal{H}^1 T(\alpha^0, \theta, \psi) - \mathcal{H}^1 = \int_{E_\omega} d\alpha^0 d\psi d\theta d\psi.
\]  

(34)

Equation (34) is a weighted Kirchhoff diffraction stack.

The inverse GRT operator

The inverse operator \(U^\theta\) for one single scattering angle \(\theta\), is given by
\[
\langle c^{(1)}(\psi, \theta) \rangle = \left( U^\theta [c_0] p(\psi, \theta) \right) = \frac{1}{\pi^2} \int_{E_\omega} \int_{E_0} \int_{E_0} \cdots \left| \mathcal{H}^1 T(\alpha^0, \theta, \psi) \right| \mathcal{A}(\alpha^0, \theta, \psi)
\]  

(35)

The inversion is performed in as a double integral over azimuth and migration dip. This operator is the inverse generalized Radon transform (Miller et al., 1987). The output from (35) is a partial reconstruction of \(c^{(1)}\) for a fixed scattering angle. To accomplish the full inversion we have to integrate over all \(\theta \in E_\theta\).

Common scattering-angle sections

A common scattering-angle section (CSA section) is a partial reconstruction of the scattering potential that uses one fixed angle \(\theta\) between the rays at all the image points. For the fixed scattering angle, we do the inversion by integrating over all the migration dips and azimuths using equation (35). Doing this, we will sweep all possible sources and receivers and we sum the contributions from the rays that hit in the vicinity of any of these data points. The output is a function of scattering angle: this is the angle dependent scattering coefficient or CSA section. If we compare this process with the more familiar common offset type inversion, we see that the scattering angle plays the same role as offset does there. We have sections representing a fixed angle at every image point, instead of a fixed offset at the surface. In this setting, both the forward and the inverse operator are functions of micro-local coordinates, not of acquisition parameters as in the offset case. The inversion is an explicit integration over the phase directions, or migration dip and scattering angle and azimuth, instead of an implicit integration through sources and receivers. By arranging the operators in this manner, we do not have to introduce any Jacobian for the change from phase directions at the image point to acquisition coordinates on the surface. We
access the data based on the ray end-point on the surface by tracing rays from the image point and up to the surface. The end point of each ray will be assigned to be either source or receiver, depending on where it hits the surface. Then, from two rays, one going to a source and one going to a receiver, and the total travel time, we can select the data.

Example

When we take a vertical section through a fixed surface position in the CSA data volume, we obtain a common image gather (CIG) in scattering angle and depth. We can perform the same kinds of analysis on these gathers as we usually do using CIG’s in offset. The CIG will show where the image front appears in depth for different scattering angles. To illustrate how the CSA-sections can be used in processing, we will perform a short analysis of the common image-point gather in scattering angle, and show it’s sensitive to errors in the velocity model. The relation between scattering angle and image depth can be used to do velocity analysis.

To test the concept we use a very simple 2D model. The model consists of two layers separated by a horizontal interface at 1500m. Each of the layers has a constant velocity gradient. The model parameters are for layer1: \( v_0 = 1500m/s, g_x = 0.0, g_z = 0.3 \), and layer2: \( v_0 = 2500m/s, g_x = 0.2, g_z = 0.5 \). We did an inversion with a background velocity given by \( v_0 = 900m/s, g_x = 0.3, g_z = 0.7 \). On the complete image, after summing all the scattering angles, the reflector is blurred and appears to be dipping to the right, see Figure 3. The fact that the reflector appears to be dipping is an effect caused by the lateral gradient introduced in the background model. To analyze the effect of the changing velocity we have extracted three common image gathers at different horizontal locations: \( x = 500m, x = 1500m, x = 2500m \).

At 500m the event appears to shallow and the image gather is curving upwards, indicating a too low velocity, Figure 4. As we move further to the right in the model the velocity increase due to the lateral gradient in the background model. We pick up the increasing velocity in the image gathers, where the event appears increasingly deeper as we move to the right. Also the curvature changes from upward to downward, Figure 4 and Figure 5. As we move further to the right the difference between the back-ground velocity model and the correct velocity increases, resulting in an increasing downward curvature in the CIG, see Figure 6.
Sensitivity transform

By combining several different experiments it is possible, to some extent, to determine spatial medium variations. In our case the different common scattering-angle sections are the different experiments, and we use the degree of coherency between them to derive the medium variations. To reveal the degree of coherency between the separate reconstructions made using eq.(35), we introduce the differential semblance

$$ \delta \left( \frac{c(1)}{(y, \theta)} \right) \frac{\partial}{\partial \theta}, \quad y \in \mathcal{D}, \quad (36) $$

For a medium update to be meaningful, the differential semblance should be minimized.

In order to get a global measure on how good the current computational state relates to the seismic data, we will do a comparison in the data domain. Then in order to compare the results from the inversion we re-generate data from the output of the inversion, after applying differential semblance. This is equivalent to applying the forward operator in the common scattering-angle domain to (36):

$$ L^\theta \left( \frac{\delta \left( c(1) \right)}{(y, \theta)} \right), \quad y \in \mathcal{D}. \quad (37) $$

Up to leading-order asymptotics, this operator follows

$$ \left( \partial_y T(\ldots, \ldots) \right) \partial \theta + \partial_y \quad (38) $$

Since the medium contrast $c(1)$ is the current output from the inversion process, we can describe the differential semblance as the derivative of the inverse operator (35) with respect to scattering angle $\theta$. Substituting the derivative of the inverse operator into (37) we obtain a new operator, the sensitivity transform, that yields a direct measure on how good the current inversion result is. This operator is

$$ S^\theta = [I - \Delta]^{-1/2} L^\theta \delta U^\theta, \quad (39) $$

where we introduce the resolvent of the Laplacian $\Delta$ in order to balance the derivative with respect to $\theta$. Let $\langle \ldots \rangle$ be the $L^2$ inner product over $(\theta, \psi, t)$. Then

$$ \epsilon [c_0] = \langle S^\theta, S^\theta \rangle, \quad (40) $$

can be employed as a measure to update the background wave speed $c_0$ in the zone illuminated by the rays emanating from $x$ to the acquisition surface. If we have the correct background model when doing the inversion, the sensitivity transform of the result will vanish or be close to zero.

Discussion

We have presented a set of coordinate systems that are convenient to use when we work with modeling and inversion using generalized Radon transforms. Since we choose to work in image-point related variables we can control the number of rays that actually contribute to each receiver/source at the surface, hence resolving the problem of multi-pathing. Using the phase directions and travel time as index of the rays will give a unique mapping from the image-point and up to either a source or a receiver.

The sensitivity transform is still in a very early development phase. However, used in the framework of the GRT and image-point variables, it has the potential to become a powerful tool for doing velocity analysis and model parameter updates.

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References


APPENDIX A: The forward operator in 2D homogeneous medium

In this appendix we present explicit formulas for some of the general expressions introduced in the text. We do this in 2D and a constant background medium. A typical isochron is illustrated in Figure A1. In this setting the isochron is an ellipse, with the source and receiver in each of the two foci, center in the origin, and the principal axis given by the travel time of the isochron $\tau$, source-receiver offset $2h = |r - s|$ and constant wave speed $c_0$.

First we describe the ellipse parametrically with a convenient variable $\sigma$, as follows

$$x(\sigma) = (a \cos(\sigma), b \sin(\sigma)) \quad \sigma \in [\pi, 2\pi],$$

(A1)

where

$$a = \frac{c_0 \tau}{2}, \quad \text{and} \quad b = \sqrt{\left(\frac{c_0 \tau}{2}\right)^2 - h^2}.$$  (A2)

The total travel time is a function of $\sigma_1$, and follows as

$$\tau = T(x(\sigma_1)) = T(s, x(\sigma_1), r)$$

$$= \tau(s, x(\sigma_1)) + \tau(r, x(\sigma_1)).$$

(A3)

Since we only have two spatial dimensions $x = (x_1, x_2)$, we can choose curvi-linear coordinates along the isochron, setting $\sigma_1$ as the arc-length, $\sigma_2 = 0$, and $\sigma_3$ as the normal direction. We then make use of (30), relating a line segment $dz$ to the isochron parameter $\sigma$

$$\frac{\partial z}{\partial (\sigma_1, \tau)} = \frac{1}{|\nabla T(x)|} \frac{\partial T}{\partial \sigma} d\sigma_1$$

$$= \frac{c_0}{2 \cos(\theta/2)} \frac{\partial (\sigma_1)}{\partial \sigma} d\tau d\sigma.$$  (A4)

Where the additional partial derivative gives the relation between the parametric parameter $\sigma$ and the arc-length $\sigma_1$

$$\frac{\partial (\sigma_1)}{\partial \sigma} = \frac{1}{2} \sqrt{-2h^2 + c_0^2 \tau^2 - 2h^2 \cos(2\sigma)}.$$  (A5)

Since we have a scattering angle dependent input, we need to relate the scattering angle to the position along the isochron. First we express the scattering angle as a function of $\sigma$

$$\theta = \cos^{-1} \left(\frac{-6h^2 + c_0^2 \tau^2 + 2h^2 \cos(2\sigma)}{c_0^2 \tau^2 - 4h^2 \cos^2(\sigma)}\right).$$  (A6)

Then the Jacobian is given by

$$\frac{\partial (\theta)}{\partial (\sigma)} = \frac{4h \sqrt{c_0^2 \tau^2 - 4h^2 \cos^2(\sigma)}}{2h^2 - c_0^2 \tau^2 + 2h^2 \cos(2\sigma)}.$$  (A7)

In the forward operator, we need the expression for $1/\cos(\theta/2)$, which can be found using (A6)

$$\frac{1}{\cos(\theta/2)} = \pm \sqrt{\frac{2}{1 + \cos(\theta)}}$$

$$= \pm \sqrt{\frac{c_0^2 \tau^2 - 4h^2 \cos^2(\sigma)}{4h^2 + c_0^2 \tau^2}}$$

$$= -\sqrt{\frac{c_0^2 \tau^2 - 4h^2 \cos^2(\sigma)}{4h^2 + c_0^2 \tau^2}},$$  (A8)

where we only need to keep the minus sign since $\sigma \in [\pi, 2\pi]$. If we then combine the Jacobians and the term with cosine given above, we obtain

$$\frac{\partial (\sigma_1)}{\partial (\sigma)} \frac{\partial (\theta)}{\partial (\sigma)} \frac{1}{\cos(\theta/2)} = -2h \cos(\sigma).$$  (A9)

The forward operator then becomes

$$p(s, r, \tau) = (L^2S)(s, r, \tau)$$

$$= \frac{\partial^2}{\partial \tau^2} \int_0^{2\pi} A(s, x(\sigma, \tau), r) \cdot \cdot \cdot \cdot \cdot \cdot \cdot (\theta(\sigma, \tau), x(\sigma, \tau)) c_0 h \cos(\sigma, \tau) d\sigma.$$  (A10)

Where $\theta(\sigma, \tau)$ is given by (A6).

Then, by applying differential semblance to (35), and substituting the whole operator into (A10) we have the sensitivity transform.
General formulation of screen methods for the scattering of acoustic waves

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**ABSTRACT**

The Bremmer series solution of the wave equation in generally inhomogeneous media, requires the introduction of pseudo-differential operators. In this paper, general screen representations of these pseudo-differential operators are derived. We focus on designing algorithms that essentially shuttle between the transverse space and transverse slowness domains, and hence are efficient though approximate. Our scheme adds to the standard phase-screen method in the following ways: we consider larger contrasts by introducing free parameters in the theory; we enhance the accuracy for wider scattering angles; we introduce (de)composition operators to incorporate any desired source- or receiver-type with the appropriate radiation characteristics; we take care of the backscattered field with the aid of the Bremmer coupling series.

**Introduction**

Directional wave field decomposition is a tool for analyzing and computing the propagation of waves in configurations with a certain directionality, such as waveguiding structures. The method consists of three main steps: (i) decomposing the field into two constituents, propagating upward or downward along a preferred direction, (ii) computing the interaction of the counterpropagating constituents and (iii) recomposing the constituents into observables at the positions of interest. The method is beneficial because it can be computationally efficient and it can be used to separate different propagation phenomena, which is of importance, for example, in the interpretation and inversion of seismic data.

In the Bremmer series approach to modelling, we encounter pseudo-differential operators in the directional (de)composition, in the downward and upward propagation or continuation, and in the reflections and transmissions due to variations in medium properties in the preferred direction (De Hoop (1)). To arrive at a fast algorithm to compute the Bremmer series, we will employ the ideas underlying the screen method, enforcing in the symbols of the pseudo-differential operators a separation of phase-space coordinates. We will refer to this approximate way of solving the wave equation as the 'general screen' method since it is a generalization of the phase-screen method. The separation of coordinates induces an adjustment of wave theory and the shapes of the wave fronts in particular. In the dual-domain, the phase-space path-integral representations of the propagators, and the associated vertical slowness symbol analysis, provide a logical framework for understanding the accuracy of our screen approximations.

The method of phase screens has been widely used for one-way wave propagation in **smoothly** varying, heterogeneous media. In particular, the method has been applied to light transmission through the atmosphere (Ratcliffe (2), Mercier (3), and Martin and Flatté (4)), propagation of light in optical fibers (Feit and Fleck (5)), propagation of radio signals through the ionosphere (Buckley (6), Bramley (7), and Knepp (8)), propagation of acoustic waves in the ocean (Flatté (9), Thomson and Chapman (10)), and propagation of seismic waves in the earth (Stoffa *et al.* (11)). More recently, the screen method has been introduced for elastic waves (Fisk and McCarter (12), Fisk *et al.* (13), and Wu (14)). The phase-screen approach has also been applied in media that are not smoothly varying at all: Berry (15) analyzed the intensity fluctuations of an incident plane wave scattered in a **fractal** medium.

Classically, the phase-screen method was designed for multiple downward scattering of waves, the downward direction being the preferred direction of propagation.
Thus it included phenomena like focussing and defocussing. The applicability of the phase-screen method generally requires that the screen interval satisfies the following criteria: small medium variations (weak scattering), transversely smooth medium variations (narrow angle scattering), and even smoother variations in the preferred direction (negligible backscattering). In this paper, we access the accuracy of the screen method, and generalize it to larger-contrast, wider-angle, and back-scattering. In the generalized screen approach we address the issue whether, possibly windowed, lateral Fourier bases can be employed to construct fast propagation algorithms, the complexity of such algorithms being proportional to $N \log_2 N$ where $N$ is the number of nodes at which the field is being downward continued. The Fourier bases are independent of the medium properties and hence fixed, unlike medium dependent eigen states of the characteristic operator that can be employed to diagonalize the propagator.

Directional wave field decomposition

For the details on the derivation of the Bremmer coupling series solution of the acoustic wave equation, we refer the reader to De Hoop (1). Here, we restrict ourselves to a summary of the method. Our configuration is three-dimensional.

Let $p = \text{acoustic pressure} [\text{Pa}], v_r = \text{particle velocity} [\text{m/s}], \rho = \text{volume density of mass} [\text{kg/m}^3], \kappa = \text{compressibility} [\text{Pa}^{-1}], q = \text{volume source density of injection rate} [\text{s}^{-1}], \text{and} f_k = \text{volume source density of force} [\text{N/m}^3]$. We assume that the coefficients $\kappa$ and $\rho$ are smooth, and constant outside a compact domain. This provision enables us to formulate the acoustic wave propagation, when necessary, as a scattering problem in a homogeneous embedding. The smoothness entails that the singularities of the wave field (in particular the ones in the neighborhood of the wave arrival) arise from the ones in the signatures of the source distributions. The formation of caustics, associated with multi pathing of characteristics, is captured in the approximation procedure developed in this paper.

To guarantee convergence of the generalized Bremmer series (1), we carry out our analysis in the time-Laplace domain. To show the notation, we give the expression for the acoustic pressure,

$$p(x_m, s) = \int_{t=0}^{\infty} \exp(-st)p(x_m, t) \, dt.$$  

(1)

Under this transformation, assuming zero initial conditions, we have $\partial_t \rightarrow s$. In the Laplace domain, the acoustic wave field satisfies the system of first-order equations

$$\partial_s \tilde{p} + sp \tilde{v}_k = \tilde{f}_k,$$  

(2)

$$s\kappa \tilde{p} + \partial_r \tilde{v}_r = \tilde{q}.$$  

(3)

The change of the wave field in space along a direction of preference can now be expressed in terms of the changes of the wave field in the direction transverse to it. The direction of preference is taken along the $x_3$-axis (or 'vertical' axis) and the remaining ('transverse' or 'horizontal') coordinates are denoted by $x_\mu, \mu = 1, 2$.

Since we allow the medium to vary with all coordinates and hence also with the preferred direction, we are forced to carry out the wave field decomposition from the system of first-order equations rather than the second-order scalar Helmholtz equation.

The reduced system of equations

The decomposition procedure requires a separate handling of the horizontal components of the particle velocity. From Eqs.(2) and (3) we obtain

$$\tilde{v}_\mu = -\rho^{-1}s^{-1}(\partial_s \tilde{p} - \tilde{f}_\mu),$$  

(4)

leaving, upon substitution, the matrix differential equation

$$(\partial_s \delta_{I,J} + s\delta_{I,J})\tilde{F}_J = \tilde{N}_I, \quad \tilde{A}_{I,J} = A_{I,J}(x_\mu, D_\nu; x_3), \quad D_\nu \equiv -\frac{1}{s}\partial_\nu,$$  

(5)

in which $\delta_{I,J}$ is the Kronecker delta, and the elements of the acoustic field matrix are given by

$$\tilde{F}_1 = \tilde{p}, \quad \tilde{F}_2 = \tilde{v}_3,$$  

(6)

the elements of the acoustic system’s operator matrix by

$$\tilde{A}_{1,1} = \tilde{A}_{2,2} = 0,$$  

(7)

$$\tilde{A}_{1,2} = \rho,$$  

(8)

$$\tilde{A}_{2,1} = -D_\nu(\rho^{-1}D_\nu) + \kappa,$$  

(9)
and the elements of the notional source matrix by
\[ \hat{N}_1 = f_3, \quad \hat{N}_2 = D_v (\rho^{-1} f_v) + \hat{q}, \]

(10)

It is observed that the right-hand side of Eq.(4) and \( \hat{A}_{I,J} \) contain the spatial derivatives \( D_v \) with respect to the horizontal coordinates only. \( D_v \) has the interpretation of horizontal slowness operator. Further, it is noted that \( \hat{A}_{1,2} \) is simply a multiplicative operator.

The coupled system of one-way wave equations

To distinguish up- and downgoing constituents in the wave field, we shall construct an appropriate linear operator \( \hat{L}_{I,J} \) with
\[ \hat{F}_I = \hat{L}_{I,J} \hat{W}_J, \]

(11)

that, with the aid of the commutation relation \((\partial_3 \hat{L}_{I,J}) = [\partial_3, \hat{L}_{I,J}]\), transforms Eq.(5) into
\[ \hat{L}_{I,J} (\partial_3 \delta_{I,M} + s \hat{A}_{J,M}) \hat{W}_M = - (\partial_3 \hat{L}_{I,J}) \hat{W}_J + \hat{N}_I, \]

(12)

as to make \( \hat{A}_{J,M} \), satisfying
\[ \hat{A}_{I,J} \hat{L}_{J,M} = \hat{L}_{I,J} \hat{A}_{J,M}, \]

(13)
a diagonal matrix of operators. We denote \( \hat{L}_{I,J} \) as the composition operator and \( \hat{W}_M \) as the wave matrix. The expression in parentheses on the left-hand side of Eq.(12) represents the two so-called one-way wave operators. The first term on the right-hand side of Eq.(12) is representative for the scattering due to variations of the medium properties in the vertical direction. The scattering due to variations of the medium properties in the horizontal directions is contained in \( \hat{A}_{J,M} \) and, implicitly, in \( \hat{L}_{I,J} \).

To investigate whether solutions of Eq.(13) exist, we introduce the column matrix operators \( \hat{F}_I^{(\pm)} \) according to
\[ \hat{F}_I^{(+)} = \hat{L}_{I,1}, \quad \hat{F}_I^{(-)} = \hat{L}_{I,2}. \]

(14)

Upon writing the diagonal entries of \( \hat{A}_{J,M} \) as
\[ \hat{A}_{1,1} = \hat{f}^{(+)} , \quad \hat{A}_{2,2} = \hat{f}^{(-)} , \]

(15)

Eq.(13) decomposes into the two systems of equations
\[ \hat{A}_{I,J} \hat{L}_J^{(\pm)} = \hat{L}_I^{(\pm)} \hat{f}^{(\pm)}. \]

(16)

By analogy with the case where the medium is translationally invariant in the horizontal directions, we shall denote \( \hat{f}^{(\pm)} \) as the vertical slowness operators. Notice that the operators \( \hat{L}_I^{(\pm)} \) compose the acoustic pressure and that the operators \( \hat{L}_2^{(\pm)} \) compose the vertical particle velocity, whereas the elements of \( \hat{W}_M \) may be physically 'non-observable'.

In De Hoop (1) an Ansatz procedure has been followed to solve the generalized eigenvalue-eigenvector problem (16) in operator sense: the acoustic-pressure normalization analog guarantees the commutation rule
\[ [\hat{L}_I^{(\pm)}, \hat{A}_{2,1} \hat{A}_{1,2}] = 0. \]

(17)

In this normalization we find the vertical slowness operator or generalized eigenvalues to be
\[ \hat{f}^{(+)} = - \hat{f}^{(-)} = \hat{f} = \hat{A}^{1/2}, \quad \hat{A} \equiv \hat{A}_{2,1} \hat{A}_{1,2}; \quad \hat{f}^2 = \hat{A} \]

(18)

is the characteristic operator equation, while the generalized eigenvectors constitute the composition operator
\[ \hat{L} = \begin{pmatrix} \hat{A}_{1,2} & \hat{A}_{1,2} \\ \hat{f} & -\hat{f} \end{pmatrix}. \]

(19)

Note that we have decomposed the pressure (up to a multiplication by density) viz. according to \( \hat{F}_1 = \hat{F}_I^+ + \hat{F}_I^- \), \( \hat{F}_I^+ = \hat{A}_{1,2} \hat{W}_1 \) and \( \hat{F}_I^- = \hat{A}_{1,2} \hat{W}_2 \). In terms of the inverse vertical slowness operator, \( \hat{f}^{-1} = \hat{A}^{-1/2} \), the decomposition operator then follows as
\[ \hat{L}^{-1} = \frac{1}{2} \begin{pmatrix} \hat{A}_{1,2}^{-1} & \hat{f}^{-1} \\ \hat{A}_{1,2}^{-1} & -\hat{f}^{-1} \end{pmatrix}. \]

(20)

Using the decomposition operator, Eq.(12) transforms into
\[ (\partial_3 \delta_{I,M} + s \hat{A}_{I,M}) \hat{W}_M = - (\hat{L}^{-1})_{I,M} (\partial_3 \hat{L}_{M,K}) \hat{W}_K + (\hat{L}^{-1})_{I,M} \hat{N}_M. \]

(21)
which can be interpreted as a coupled system of one-way wave equations. The coupling between the counter-propagating components, $\hat{W}_1$ and $\hat{W}_2$, is apparent in the first source-like term on the right-hand side, which can be written as

$$-\hat{L}^{-1}(\partial_3 \hat{L}) = \begin{pmatrix} \hat{T} & \hat{R} \\ \hat{R} & \hat{T} \end{pmatrix},$$  \hspace{1cm} (22)

in which $\hat{T}$ and $\hat{R}$ represent the transmission and reflection operators, respectively. In the acoustic-pressure normalization analog, we find

$$-\hat{T} = \frac{1}{2} \hat{A}_{1,2}^{-1}(\partial_3 \hat{A}_{1,2}) + \frac{i}{2} \hat{\Gamma}^{-1}(\partial_3 \hat{\Gamma}), \quad \hat{R} = -\frac{1}{2} \hat{A}_{1,2}^{-1}(\partial_3 \hat{A}_{1,2}) + \frac{i}{2} \hat{\Gamma}^{-1}(\partial_3 \hat{\Gamma}).$$  \hspace{1cm} (23)

### The one-way wave propagator

To arrive at a coupled system of integral equations that is equivalent to Eq.(21) and that can be solved in terms of a Neumann expansion, we have to invert the operator occurring on the left-hand side. We set $\hat{G}^{(\pm)} = (\partial_3 + i \hat{\Gamma}^{(\pm)})^{-1}$. The one-sided elementary kernels $\hat{G}^{(\pm)}(x_\mu, x_\nu; x_3')$ associated with these operators are the so-called one-way Green's functions. They satisfy the equations

$$(\partial_3 + i \hat{\Gamma}^{(\pm)}) \hat{G}^{(\pm)} = \delta(x_\nu - x_\nu') \delta(x_3 - x_3'),$$  \hspace{1cm} (24)

together with the condition of causality.

Now, consider the case $\hat{G} = \hat{G}^{(+)}$, $\hat{G}^{(+)}$ and $\hat{\Gamma}^{(+)}$. The operator $\hat{G}$ acts on a test field $\hat{u}$ as

$$(\hat{G} \hat{u})(x_\mu, x_3) = \int_{\mathbb{R}} \int_{x_\nu' \in \mathbb{R}} \hat{G}(x_\mu, x_3; x_\nu', \zeta) \hat{u}(x_\nu', \zeta) \, dx_\nu' \, dx_3 \, d\zeta. \hspace{1cm} (25)$$

Let us define the initial-value problem of determining the function $\hat{U}(x_\mu, x_3; \zeta)$ satisfying

$$(\partial_3 + i \hat{\Gamma}) \hat{U} = 0 \quad \text{for} \quad x_3 \geq \zeta, \quad \hat{U}(x_\mu, \zeta, \zeta) = \hat{u}(x_\mu, \zeta).$$  \hspace{1cm} (26)

Then it is observed that

$$\hat{G} \hat{u}(x_\mu, x_3) = \int_{\zeta = -\infty}^{x_3} \hat{U}(x_\mu, x_3; \zeta) \, d\zeta. \hspace{1cm} (27)$$

### The product integral

We note that the vertical slowness operators at different levels of $x_3$ do not necessarily commute with one another due to the heterogeneity of the medium. Thus we arrive at a 'time'-ordered product integral representation (16) of the one-sided propagators (cf. Eq.(26)) associated with the one-way wave equations (1, 17, 18) where 'time' refers to the vertical coordinate $x_3$,

$$\hat{U}^{(\pm)}(\cdot, x_3'; x_3) = \pm H(\mp[x_3' - x_3]) \left\{ \prod_{\zeta = x_3'}^{x_3} \exp[-s \hat{\Gamma}^{(\pm)}(\cdot, \zeta) \, d\zeta] \right\} \hat{u}(\cdot, x_3'), \hspace{1cm} (28)$$

where $H$ denotes the Heaviside function. In this expression, the operator ordering is initiated by $\exp[-s \hat{\Gamma}^{(\pm)}(\cdot, \zeta) \, d\zeta]$ acting on $\hat{u}(\cdot, x_3')$ followed by applying $\exp[-s \hat{\Gamma}^{(\pm)}(\cdot, \zeta) \, d\zeta]$ to the result, successively for increasing $\zeta$.

If the medium in the interval $[x_3', x_3]$ were weakly varying in the vertical direction, the Trotter product formula can be applied to the product integral in Eq.(28). This results in the Hamiltonian path integral representations (16) with measure $\hat{D}$ for the Green's functions,

$$\hat{G}^{(\pm)}(x_\mu, x_3; x_\nu', x_3') = \pm H(\mp[x_3' - x_3]) \int_{\mathcal{P}} \hat{D}(x_\alpha, x_\nu')$$  \hspace{1cm} (29)

$$\exp \left[ -s \int_{\zeta = x_3'}^{x_3} d\zeta \{ i \alpha_\nu' \, d\zeta \, x_\nu' + \gamma^{(\pm)}(x_\nu', \zeta, \alpha_\nu') \} \right] = \pm H(\mp[x_3' - x_3]) \hat{G}^{(\pm)}(x_\mu, x_3; x_\nu', x_3'),$$

$\mathcal{P}$ being a set of paths $(x_\nu''(\zeta), \alpha_\nu''(\zeta))$ in (horizontal) phase space satisfying $x_\nu''(\zeta = x_3') = x_\nu'$, $x_\nu''(\zeta = x_3) = x_\mu$. In Eq.(29), $\gamma^{(\pm)}$ is the left symbol of $\Gamma^{(\pm)}$, i.e.

$$\hat{\Gamma}^{(\pm)}(x_\mu, x_\nu; x_3) \exp(-i \sigma_\nu x_\sigma) = \gamma^{(\pm)}(x_\mu, x_3, \alpha_\nu) \exp(-i \sigma_\nu x_\sigma).$$  \hspace{1cm} (30)
The path integral in Eq.(29) is to be interpreted as the lattice multi-variate integral

\[
\hat{G}^{(k)}(x_\mu, x_3; x'_\mu, x'_3) = \pm H(\mp [x_3 - x_3]) \lim_{M \to \infty} \int \prod_{i=1}^{M} (s/2\pi)^2 \, d\alpha_i^{(i)} \, d\alpha_i^{(j)} \prod_{j=1}^{M-1} dx_j^{(j)} \, dx_2^{(j)} \\
\exp \left[ -s \sum_{k=1}^{M} \{ \alpha_k^{(k)}(x_k - x'_k) - \frac{1}{2} M^{-1} \Delta x_3, \alpha_k^{(k)} \} M^{-1} \Delta x_3 \right]
\]

(31)

with \( x_\mu^{(0)} = x'_\mu, x_\mu^{(M)} = x_\mu, \) and \( \Delta x_3 = x_3 - x'_3. \) All the integrations are taken over the interval \((-\infty, \infty), M^{-1} \Delta x_3 \) is the step size in \( \zeta, \) and \( (x_\mu^{(j)}, \alpha_j^{(j)}), \) are the coordinates of a path at the discrete values \( \zeta_j \) of \( \zeta \) as \( j = 1, \ldots, M. \)

The thin-slab propagator

If \( \Delta x_3 \) is sufficiently small (thin slab), the lattice multi-variate integral reduces to \( (M = 1) \)

\[
\tilde{g}^{(\pm)}(x_\mu, x_3; x'_\mu, x'_3) \simeq \int (s/2\pi)^2 \, d\alpha'_{\mu} \, d\alpha''_{\mu} \\
\exp[-is \alpha''_{\mu}(x_{\mu} - x'_{\mu})] \exp[-s \tilde{\gamma}^{(\pm)}(x_{\mu}, x_3 - \frac{1}{2} \Delta x_3, \alpha''_{\mu}) \Delta x_3].
\]

(32)

Thus, thin-slab propagation is composed of a forward Fourier transform, a multiplication by a phase factor (the phase is the vertical slowness left symbol) and an inverse Fourier transform. Switching from left to right symbols* yields

\[
\tilde{g}^{(\pm)}(x_\mu, x_3; x'_\mu, x'_3) \simeq \int (s/2\pi)^2 \, d\alpha'_{\mu} \, d\alpha''_{\mu} \\
\exp[-is \alpha'_{\mu}(x_{\mu} - x'_{\mu})] \exp[-s \tilde{\gamma}^{(\pm)}(x'_\mu, x_3 - \frac{1}{2} \Delta x_3, \alpha'_{\mu}) \Delta x_3].
\]

(33)

With the vertical slowness left symbol is associated a cokernel

\[
\tilde{\gamma}(\alpha, \alpha') = \int_{x_\mu \in \mathbb{R}} \exp(is \alpha x_\mu) \tilde{\gamma}(x_\mu, \alpha') \, dx_1 \, dx_2.
\]

(34)

Thus, in the Fourier domain, the thin slab propagator can written in the form (cf. Eq.(32)),

\[
\tilde{g}^{(\pm)}(\alpha, x_3; \alpha'', x'_3) \simeq \int dx_1 \, dx_2 \\
\exp(is(\alpha - \alpha'_{\mu}) x_\mu) \exp[-s \tilde{\gamma}^{(\pm)}(x_{\mu}, x_3 - \frac{1}{2} \Delta x_3, \alpha''_{\mu}, s) \Delta x_3],
\]

(35)

and may, for small \( \Delta x_3, \) be approximated by

\[
\tilde{g}^{(\pm)}(\alpha, x_3; \alpha'', x'_3) \simeq \delta(\alpha - \alpha''_{\mu}) - s \tilde{\gamma}^{(\pm)}(\alpha - \alpha''_{\mu}, x_3 - \frac{1}{2} \Delta x_3, \alpha''_{\mu}, s) \Delta x_3 \cdots .
\]

(36)

This representation shows the interaction of Fourier constituents \( \exp(-is \alpha x_\mu) \) explicitly.

The vertical slowness left symbol, contrast formulation

In the acoustic pressure normalization analog, the characteristic differential operator in Eq.(18) is given by

\[
\hat{A} = -D_\nu D_\nu + \kappa \rho - \rho^{-1}(D_\nu \rho) D_\nu - \rho^{-1}(D_\nu D_\nu \rho) - \rho^{-2}(D_\nu \rho)(D_\nu \rho),
\]

(37)

with left symbol \( \hat{a} = \hat{a}(x_\mu, \alpha_\nu). \) For the further analysis, we will employ a contrast formulation. Thus, in the interval \([x_3, x_3]\) we introduce a background medium with parameters \( \rho_0^{(0)}, \kappa^{(0)}; \) the medium perturbation is then given by

\[
\epsilon_{\nu}(x_\mu, \zeta) = \frac{\kappa^{(0)}(x_3)}{\kappa(x_\mu, \zeta)} - 1, \quad \epsilon_{\nu}(x_\mu, \zeta) = \frac{\rho(x_\mu, \zeta)}{\rho^{(0)}(x_3)} - 1; \\
c = (\kappa \rho)^{-1/2}, \quad c^0 = (\kappa^{(0)} \rho^{(0)})^{-1/2}.
\]

(38)

* The right or dual symbol \( \hat{\gamma}_R \) is related to the left symbol \( \hat{\gamma} \) according to \( \hat{\gamma}_R(x_\mu, \alpha_\nu) \sim \exp[i \delta_{\alpha_\nu} D_\nu x_\mu] \hat{\gamma}(x_\mu, \alpha_\nu). \) We will omit the subscript \( R \) whenever it is clear from the context which symbol is meant.
for $\zeta \in [\xi', x]$. This choice of perturbation functions leads to a decomposition of the left symbol of $\hat{A}$ according to $\hat{a} = \hat{a}^0 + \hat{a}^1 + \hat{a}^2$, with (the subscript refers to the order of the symbol)

$$\hat{a}^0 = \hat{a}_\mu^0 = \alpha_\nu^0 + (c^0)^{-2}$$

(40)

for the background, and

$$\hat{a}^1 = \hat{a}_\mu^1 + \hat{a}_\nu^1 + \hat{a}_0^1,$$

(41)

and

$$\hat{a}^2 = \hat{a}_\mu^2 = (\rho_0^1)^2 \rho^{-2} (D_s \epsilon_\rho)^2$$

(42)

for the contrast. Here,

$$\hat{a}_\mu^1 = (c^0)^{-2} \epsilon_\rho - c^{-2} \epsilon_\kappa,$$

$$\hat{a}_\nu^1 = \rho^0 \rho^{-1} (D_\nu \epsilon_\rho) i \alpha_\nu,$$

$$\hat{a}_0^1 = \rho^0 \rho^{-1} (D_0^2 \epsilon_\rho).$$

(43)

Using the composition rule for symbols of pseudo-differential operators, the operator equation (18) is transformed into a characteristic equation for the corresponding left symbols (1)

$$\exp \left[ -i \partial_{\alpha_\nu} D_{x^\nu} \right] \hat{\gamma}(x_\mu, x_\nu) \hat{\gamma}(x'_\mu, x'_\nu) \bigg|_{(x'_\mu, x'_\nu) = (x_\mu, x_\nu)} = \hat{a}(x_\mu, x_\nu).$$

(44)

This equation defines the generalized slowness surface and has solutions $\hat{\gamma}^{(\pm)}$. The two branches are $\hat{\gamma}^{(\pm)}(x_\mu, x_\nu)$ such that $\text{Re}\{\hat{\gamma}^{(+)}(x_\mu, x_\nu)\} \geq 0$ and $\text{Re}\{\hat{\gamma}^{(-)}(x_\mu, x_\nu)\} \leq 0$. Due to the local up/down symmetry of the medium we have $\hat{\gamma}^{(+)} = -\hat{\gamma}^{(-)}$. Note that as $s \to \infty$ the composition of symbols tends to an ordinary multiplication, and the solution of Eq.(44) reduces to the principal parts of the symbols. The principal part of the vertical slowness symbol corresponds to the vertical gradient of travel time, in accordance with the eikonal equation (which can be obtained from the high-frequency approximation of the path integral, see De Hoop (1)).

**Scaling**

We will control our medium contrast by two parameters, $\epsilon$ for the magnitude and $\Omega$ for the smoothness, i.e.

$$e_\mu(x_\mu, \zeta) = \epsilon e_\mu(\Omega x_\mu, \zeta),$$

$$e_\rho(x_\mu, \zeta) = \epsilon e_\rho(\Omega x_\mu, \zeta).$$

(45)

With

$$\rho^0 \rho^{-1} = 1 - \epsilon \epsilon_\rho + \epsilon^2 \epsilon_\rho^2,$$

$$\rho^0_\rho \rho^{-2} = 1 - 2 \epsilon \epsilon_\rho + 3 \epsilon^2 \epsilon_\rho^2,$$

$$\kappa^0 \rho^{-1}_\kappa = 1 - \epsilon \epsilon_\kappa + \epsilon^2 \epsilon_\kappa^2,$$

$$\epsilon^0 \rho^{-2} = 1 + \epsilon (\epsilon_\rho - \epsilon_\kappa) - \epsilon^2 (\epsilon_\rho - \epsilon_\kappa),$$

we find expansions for the symbols composing the characteristic operator:

$$\hat{a}_\mu^1 = \epsilon (c^0)^{-2} \{e_\rho - e_\kappa \{1 + \epsilon (e_\rho - e_\kappa) - \epsilon^2 (e_\rho - e_\kappa) + \cdots\}\},$$

$$\hat{a}_\nu^1 = \epsilon \Omega \{1 - \epsilon e_\rho + \epsilon^2 e_\rho^2 + \cdots\} (D_\nu \epsilon_\rho) i \alpha_\nu,$$

$$\hat{a}_0^1 = \epsilon \Omega^2 \{1 - \epsilon e_\rho + \epsilon^2 e_\rho^2 + \cdots\} (D_0^2 \epsilon_\rho),$$

(47)

while

$$\hat{a}_\mu^2 = \epsilon^2 \Omega^2 \{1 - 2 \epsilon e_\rho + \cdots\} (D_\nu \epsilon_\rho)^2.$$

(48)
Generalized screens

We have introduced the scaled horizontal slowness operators $D_\nu$ which differentiates with respect to $\Omega z_\nu$.

The symbol $a_2^{\Delta}$ is directly related to the screen function $S_{c-1}$ (19) which we will encounter in the later analysis. Let $\epsilon_{c-1}(x_\nu, \zeta)$ denote the relative medium slowness perturbation for $\zeta \in [x_3, x_3]$, then

$$\frac{1}{2} a_2^{\Delta} \simeq (c^2)^{-2} \epsilon_{c-1}, \quad \epsilon_{c-1} \simeq \frac{1}{2} (e_\rho - e_\kappa); \quad S_{c-1} = \frac{1}{\Delta x_3} \int_{x_3}^{x_3} \epsilon_{c-1} d\zeta. \quad (49)$$

The approximations are $O(\epsilon^2)$.

The generalized Bremmer coupling series

The coupled system of integral equations

Applying the operators with kernels Eq.(29) to Eq.(21) we obtain a coupled system of integral equations. In operator form, they are given by

$$(\delta_{I,J} - \tilde{K}_{I,J}) \tilde{W}_J = \tilde{W}_1^{(0)}, \quad (50)$$

in which $\tilde{W}^{(0)}$ denotes the incident field. In our configuration the domain of heterogeneity will be restricted to the slab $(0, x_3^{\text{exit}})$, and the excitation of the waves will be specified through an initial condition at the level $x_3 = 0$, viz.

$$\tilde{W}_1^{(0)}(x_\mu, x_3) = \int_{x_3' \in \mathbb{R}} \tilde{G}^{(+)}(x_\mu, x_3; x_3', 0) \tilde{W}_1(x_3', 0) \, dx_3' \, dx_3, \quad (51)$$

$$\tilde{W}_2^{(0)}(x_\mu, x_3) = 0, \quad (52)$$

in the range of interest, $x_3 \in [0, x_3^{\text{exit}}]$; the second equation reflects the assumption that there is no excitation below the heterogeneous slab. The integral operators in Eq.(50) are given by

$$(\tilde{K}_{1,1} \tilde{W}_1)(x_\mu, x_3) = \int_{x_3' \in \mathbb{R}} \int_{\zeta' = 0}^{x_3'} \tilde{G}^{(+)}(x_\mu, x_3; x_3', \zeta')(\tilde{T} \tilde{W}_1)(x_3', \zeta) \, dx_3' \, d\zeta', \quad (53)$$

$$(\tilde{K}_{1,2} \tilde{W}_2)(x_\mu, x_3) = \int_{x_3' \in \mathbb{R}} \int_{\zeta' = 0}^{x_3'} \tilde{G}^{(+)}(x_\mu, x_3; x_3', \zeta')(\tilde{R} \tilde{W}_1)(x_3', \zeta) \, dx_3' \, d\zeta', \quad (54)$$

$$(\tilde{K}_{2,1} \tilde{W}_1)(x_\mu, x_3) = \int_{x_3' \in \mathbb{R}} \int_{\zeta' = x_3}^{x_3^{\text{exit}}} \tilde{G}^{(-)}(x_\mu, x_3; x_3', \zeta')(\tilde{R} \tilde{W}_1)(x_3', \zeta) \, dx_3' \, d\zeta', \quad (55)$$

$$(\tilde{K}_{2,2} \tilde{W}_2)(x_\mu, x_3) = \int_{x_3' \in \mathbb{R}} \int_{\zeta' = x_3}^{x_3^{\text{exit}}} \tilde{G}^{(-)}(x_\mu, x_3; x_3', \zeta')(\tilde{T} \tilde{W}_2)(x_3', \zeta) \, dx_3' \, d\zeta'. \quad (56)$$

They describe the interaction between the counter-propagating constituent waves.

We can represent the action of the one-sided Green's kernels by product integrals, viz.,

$$\tilde{W}_1^{(0)}(., x_3) = \left\{ \prod_{\zeta' = 0}^{x_3} \exp[-s \tilde{G}^{(+)}(., \zeta') \, d\zeta'] \right\} \tilde{W}_1(., 0), \quad (57)$$

while

$$\tilde{K}_{1,1} \tilde{W}_1(., x_3) = \int_{\zeta = 0}^{x_3} \left\{ \prod_{\zeta' = \zeta}^{x_3} \exp[-s \tilde{G}^{(+)}(., \zeta') \, d\zeta'] \right\} (\tilde{T} \tilde{W}_1)(., \zeta) \, d\zeta, \quad (58)$$

and so on.

Bremmer series

If $s$ is real and sufficiently large, the Neumann expansion can be employed to invert $(\delta_{I,J} - \tilde{K}_{I,J})$ in Eq.(50). Such a procedure leads to the Bremmer coupling series,

$$\tilde{W}_J = \sum_{j=0}^{\infty} \tilde{W}_j^{(j)}, \quad \text{in which } \tilde{W}_J^{(j)} = \tilde{K}_{I,J} \tilde{W}_J^{(j-1)} \text{ for } j \geq 1, \quad (59)$$
can be interpreted as the \( j \)-times reflected or scattered wave. This equation indicates that the solution of Eq.(50) can be found with the aid of a iterative scheme. On the basis of Lerch’s theorem (20) we can then transform the individual terms back to the time domain.

**An iterative scheme**

To arrive at an iterative scheme (see Van Stralen et al. (21)), consider the \( j \)-times reflected constituent wave. We split the interval \([0, z_{n+1}^{\text{tot}}]\) into \( M \) thin slabs with thickness \( \Delta x_3 \). Set

\[
\hat{W}_j^{(j)}(., k \Delta x_3) = \hat{f}_j^{(j)}(., k) + \hat{j}_j^{(j)}(., k),
\]

\( j = 1, 2, \ldots \) and \( k = 0, 1, \ldots, M \), where (cf. Eq.(59))

\[
\hat{f}_j^{(j)}(., k) = \{ \hat{K}_{f_1} \hat{W}_j^{(j-1)}(., k \Delta x_3) \},
\]

\[
\hat{j}_j^{(j)}(., k) = \{ \hat{K}_{f_2} \hat{W}_j^{(j-1)}(., k \Delta x_3) \}.
\]

Upon comparison with Eq.(58) we find that

\[
\hat{f}_j^{(j)}(., k) = \int_{\zeta=0}^{\Delta x_3} \left\{ \prod_{\zeta'=\zeta}^{k \Delta x_3} \exp[-s \hat{f}^{(\cdot)}(., \zeta') \, d\zeta'] \right\} \hat{x}_j^{(j)}(., \zeta) \, d\zeta,
\]

with

\[
\hat{x}_j^{(j)}(., \zeta) = (\hat{T} \hat{W}_j^{(j-1)})(., \zeta).
\]

Similarly,

\[
\hat{j}_j^{(j)}(., k) = \int_{\zeta=0}^{\Delta x_3} \left\{ \prod_{\zeta'=\zeta}^{k \Delta x_3} \exp[-s \hat{f}^{(\cdot)}(., \zeta') \, d\zeta'] \right\} \hat{x}_j^{(j)}(., \zeta) \, d\zeta,
\]

\[
\hat{j}_j^{(j)}(., k) = -\int_{\zeta=0}^{\Delta x_3} \left\{ \prod_{\zeta'=\zeta}^{k \Delta x_3} \exp[-s \hat{f}^{(\cdot)}(., \zeta') \, d\zeta'] \right\} \hat{x}_j^{(j)}(., \zeta) \, d\zeta,
\]

with

\[
\hat{x}_j^{(j)}(., \zeta) = (\hat{T} \hat{W}_j^{(j-1)})(., \zeta),
\]

\[
\hat{x}_j^{(j)}(., \zeta) = (\hat{T} \hat{W}_j^{(j-1)})(., \zeta),
\]

\[
\hat{x}_j^{(j)}(., \zeta) = (\hat{T} \hat{W}_j^{(j-1)})(., \zeta).
\]

To construct the iteration scheme, we carry out the following steps. Let \( \hat{P} \) denote the thin-slab propagator (cf. Eq.(32))

\[
\hat{P}(., k) = \left\{ \prod_{\zeta'=(k-1) \Delta x_3}^{k \Delta x_3} \exp[-s \hat{f}^{(\cdot)}(., \zeta') \, d\zeta'] \right\}.
\]

Then, using the semi-group property,

\[
\hat{f}_j^{(j)}(., k) = \left\{ \prod_{\zeta'=(k-1) \Delta x_3}^{k \Delta x_3} \exp[-s \hat{f}^{(\cdot)}(., \zeta') \, d\zeta'] \right\}

\int_{\zeta=0}^{(k-1) \Delta x_3} \left\{ \prod_{\zeta'=\zeta}^{(k-1) \Delta x_3} \exp[-s \hat{f}^{(\cdot)}(., \zeta') \, d\zeta'] \right\} \hat{x}_j^{(j)}(., \zeta) \, d\zeta

+ \int_{\zeta=(k-1) \Delta x_3}^{k \Delta x_3} \left\{ \prod_{\zeta'=\zeta}^{k \Delta x_3} \exp[-s \hat{f}^{(\cdot)}(., \zeta') \, d\zeta'] \right\} \hat{x}_j^{(j)}(., \zeta) \, d\zeta,
\]

which can be written as

\[
\hat{f}_j^{(j)}(., k) = \hat{P}(., k) \hat{f}_j^{(j)}(., k-1) + \hat{Q}_j^{(j)}(., k),
\]
where
\[
\hat{Q}^{(j)}_{1,1}(\cdot, k) = \int_{\zeta=(k-1)\Delta x_3}^{k\Delta x_3} \left\{ \prod_{\zeta' = \zeta}^{k\Delta x_3} \exp[-s \hat{\Gamma}^{(+)}(\cdot, \zeta', \zeta')] \right\} \hat{X}^{(j)}_{1,1}(\cdot, \zeta) \, d\zeta .
\]  
(73)

Recursion relations similar to the one in Eq.(72) can be found for the other elements of \( \hat{I} \), viz.,
\[
\begin{align*}
\hat{I}^{(j)}_{1,j}(\cdot, k) &= \hat{P}(\cdot, k) \hat{I}^{(j)}_{1,j}(\cdot, k-1) + \hat{Q}^{(j)}_{1,j}(\cdot, k) \quad \text{for} \quad k = 1, 2, \ldots, M, \\
\hat{I}^{(j)}_{2,j}(\cdot, k) &= \hat{P}(\cdot, k+1) \hat{I}^{(j)}_{2,j}(\cdot, k+1) + \hat{Q}^{(j)}_{2,j}(\cdot, k) \quad \text{for} \quad k = M - 1, M - 2, \ldots, 0.
\end{align*}
\]  
(74)

Here
\[
\begin{align*}
\hat{Q}^{(j)}_{1,j}(\cdot, k) &= \int_{\zeta=(k-1)\Delta x_3}^{k\Delta x_3} \left\{ \prod_{\zeta' = \zeta}^{k\Delta x_3} \exp[-s \hat{\Gamma}^{(+)}(\cdot, \zeta', \zeta')] \right\} \hat{X}^{(j)}_{1,j}(\cdot, \zeta) \, d\zeta ,
\end{align*}
\]  
(75)

\[
\begin{align*}
\hat{Q}^{(j)}_{2,j}(\cdot, k) &= \int_{\zeta=(k+1)\Delta x_3}^{k\Delta x_3} \left\{ \prod_{\zeta' = k\Delta x_3}^{\zeta} \exp[s \hat{\Gamma}^{(-)}(\cdot, \zeta', \zeta')] \right\} \hat{X}^{(j)}_{2,j}(\cdot, \zeta) \, d\zeta .
\end{align*}
\]  
(76)

The initial values for the recursion scheme (74) are given by
\[
\begin{align*}
\hat{I}^{(j)}_{1,j}(x_\mu, 0) &= 0 , \\
\hat{I}^{(j)}_{2,j}(x_\mu, M) &= 0 ,
\end{align*}
\]  
(77)

(78)

again, for \( j = 1, 2, \ldots \).

**Numerical issues**

The implementation of the iterative scheme is as follows. It is initiated by the calculation of the incident field, \( \hat{W}^{(0)}_1 \), according to
\[
\hat{W}^{(0)}_1(\cdot, k\Delta x_3) = \hat{P}(\cdot, k) \hat{W}^{(0)}_1(\cdot, (k-1)\Delta x_3) \quad \text{for} \quad k = 1, 2, \ldots, M,
\]  
(79)

with initial condition
\[
\hat{W}^{(0)}_1(\cdot, 0) = \hat{W}_1(\cdot, 0),
\]  
(80)

according to Eq.(51). During the forward propagation, at each of the discrete levels, \( \hat{X}^{(1)}_{1,1} \) are computed and stored; \( \hat{X}^{(1)}_{2,2} \) are set to zero. The procedure is continued by the backward propagation defined by the second iteration in Eq.(74). At each of the discrete levels, \( \hat{W}^{(1)}_2 \) is computed (Eq.(60)) and used to calculate \( \hat{X}^{(2)}_{2,2} \); the latter quantity is stored as before. The scheme continues to switch from backward to forward propagation based on the first iteration in Eq.(74), and so on.

To evaluate the elements of \( \hat{Q}^{(j)} \), we apply the trapezoidal rule. Then
\[
\begin{align*}
\hat{Q}^{(j)}_{1,1}(\cdot, k) &\approx \frac{1}{2} \Delta x_3 \left[ \hat{X}^{(j)}_{1,1}(\cdot, k\Delta x_3) + \hat{P}(\cdot, k) \hat{X}^{(j)}_{1,1}(\cdot, (k-1)\Delta x_3) \right], \\
\hat{Q}^{(j)}_{2,2}(\cdot, k) &\approx -\frac{1}{2} \Delta x_3 \left[ \hat{X}^{(j)}_{2,2}(\cdot, k\Delta x_3) + \hat{P}(\cdot, k+1) \hat{X}^{(j)}_{2,2}(\cdot, (k+1)\Delta x_3) \right],
\end{align*}
\]  
(81)

(82)

which formulas are accurate \( O((\Delta x_3)^2) \).

In the remainder of this paper, we will derive approximate, screen representations for \( \hat{\Gamma}, \hat{P}, \hat{\Gamma}^{-1} \) and \( \hat{R} \). This will lead to an overall multiple screen representation of the full solution of the acoustic wave equation.

**Screen representation of the propagator**

There exists a wide variety of techniques to approximate the vertical slowness symbol to speed up the computation of the one-sided propagator. Amongst them is the perturbative, split-step Fourier approach of Stoffa et al. (11). Here, we pursue this approach to arrive at a general screen-type replacement of the propagator.

Consider a thin slab, \([x'_3, x_3]\), with thickness \( \Delta x_3 = x_3 - x'_3 \). Introduce a background medium in the slab with parameters
\[ \rho^0, \kappa^0. \] The background medium is constant in the slab, but may vary from one slab to another. We express this by letting \( \rho^0 = \rho^0(x_3) \) and \( \kappa^0 = \kappa^0(x_3) \). We set (cf. Eq.(40))
\[ \gamma^0(\zeta, \alpha, \nu) = \sqrt{\delta_2^0(x_3, \alpha)} = \sqrt{\alpha^2 + [\kappa^0(x_3)]^{-2}} = \gamma^0(x_3, \alpha) \text{ if } \zeta \in [x_3', x_3]. \]
(83)

**Perturbation expansion of the vertical slowness left symbol**

We suppose that the vertical slowness symbol can be approximated by a small perturbation, \( \gamma^1 \) say, superimposed on the vertical slowness in the laterally homogeneous embedding,
\[ \gamma(x_\mu, \zeta, \alpha, \nu, s) = \gamma^0(x_3, \alpha) + \gamma^1(x_\mu, \zeta, \alpha, \nu, s) \text{ if } \zeta \in [x_3', x_3]. \]
(84)

To avoid that an artificial branch point would enter the propagating-wave domain, we assume that
\[ c(x_\mu, \zeta) \geq c^0(x_3). \]
(85)

First, let us expand the vertical slowness perturbation into the magnitude of the medium perturbation, i.e.
\[ \gamma^1(x_\mu, \zeta, \alpha, \nu, s) \sim \sum_{n=1}^{\infty} \varepsilon^n \eta^{[n]}(x_\mu, \zeta, \alpha, \nu, s); \]
(86)

second, we expand the series further in terms of the smoothness of the medium perturbation,
\[ \eta^{[n]}(x_\mu, \zeta, \alpha, \nu, s) \sim \sum_{m=0}^{\infty} \Omega^m \eta^{[n]}_{[1-m]}(x_\mu, \zeta, \alpha, \nu, s). \]
(87)

The leading term of this expansion represents the high-frequency approximation or the principal part. We will suppress the dependencies on \( \zeta \) and \( s \) in our notation.

While substituting these expansions into Eq.(44), we observe that for \( k \geq 1 \),
\[ (-i \partial_{x_\mu} D_{x_\nu}^0)^k \gamma(x_\mu, \alpha, \nu) \gamma'(x_\nu, \alpha') \bigg|_{(x_\mu', \alpha') = (x_\mu, \alpha)} = \]
\[ \sum_{\ell=0}^{k} \left( \begin{array}{c} k \\ \ell \end{array} \right) (-i)^k \left( \partial_{x_\mu}^{k-\ell} \partial_{x_\nu}^\ell \gamma^0(x_\mu, \alpha, \nu) + \left( \partial_{x_\mu}^{k-\ell} \partial_{x_\nu}^\ell \gamma^1(x_\mu, \alpha, \nu) \right) (D_{x_\mu}^{k-\ell} D_{x_\nu}^\ell \gamma^1)(x_\mu, \alpha, \nu) \right), \]
(88)

hence
\[ (-i \partial_{x_\mu} D_{x_\nu}^0)^k \gamma(x_\mu, \alpha, \nu) \gamma'(x_\nu, \alpha') \bigg|_{(x_\mu', \alpha') = (x_\mu, \alpha)} = \sum_{\ell=0}^{k} \left( \begin{array}{c} k \\ \ell \end{array} \right) (-i)^k \Omega^k \sum_{n'=1}^{\infty} \varepsilon^{n'} \sum_{m'=0}^{\infty} \Omega^{m'} \]
\[ \left( \partial_{x_\mu}^{k-\ell} \partial_{x_\nu}^\ell \gamma^0(x_\mu, \alpha, \nu) + \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=0}^{\infty} \Omega^m \left( \partial_{x_\mu}^{k-\ell} \partial_{x_\nu}^\ell \eta^{[n]}_{[1-m]}(x_\mu, \alpha, \nu) \right) \right) \left( D_{x_\mu}^{k-\ell} D_{x_\nu}^\ell \eta^{[n']}_{[1-m']}(x_\mu, \alpha, \nu) \right). \]
(89)

Upon substituting the expansions (86)-(87) into Eq.(44), we will collect terms of equal order in \( \varepsilon \). These terms are then separated in orders of \( \Omega \). We will carry out our analysis up to \( O(\Omega^2) \).

The terms \( O(\varepsilon^0) \) yield Eq.(83), viz.
\[ (\gamma^0)^2 = \hat{\alpha}_2^0 \]
(90)

\[ [k = 0] \text{ which is } O(\Omega^0). \] The terms \( O(\varepsilon) \) decompose into
\[ 2\gamma^0 \eta^{[1]}_{[1]} = \hat{\alpha}_1^1 \]
(91)

\[ [k = 0] \text{ which is } O(\Omega^0), \] and
\[ 2\gamma^0 \eta^{[1]}_{[1-m]} - i \left( \partial_{x_\mu} \gamma^0 \left( D_{x_\mu} \eta^{[1]}_{[1-m]} \right) \right) = \hat{\alpha}_{2-m}^1, \quad m = 1, 2 \]
(92)

\[ [k = 1] \text{ which is } O(\Omega^m). \] The terms \( O(\varepsilon^2) \) decompose into
\[ (\eta^{[1]}_{[1]})^2 + 2\gamma^0 \eta^{[2]}_{[1]} = 0 \]
(93)
[k = 0] which is $O(\Omega^0)$,
\begin{equation}
2\eta_{[1]}^{[1]} \eta_{[0]}^{[1]} + 2\gamma^0 \eta_{[0]}^{[2]} = 0,
\end{equation}

[k = 0, 1] which is $O(\Omega)$, and
\begin{equation}
2\gamma^0 \eta_{[-1]}^{[2]} + (\eta_{[0]}^{[1]})^2 + 2\eta_{[-2]}' \eta_{[-1]}'
- i [(\partial_{\alpha'}\gamma^0)(D_{\alpha'}^{[1]}) + (\partial_{\alpha'}\eta_{[1]}^{[1]}) (D_{\alpha'} \eta_{[0]}^{[1]}) + (\partial_{\alpha'} \eta_{[0]}^{[1]}) (D_{\alpha'} \eta_{[1]}^{[1]})]
- \frac{1}{2} [(\partial_{\alpha'} \gamma^0)^2 (D_{\alpha'}^{[2]}) + 2(\partial_{\alpha'} \partial_{\alpha''} \gamma^0)(D_{\alpha'} D_{\alpha''} \eta_{[1]}^{[1]}) + (\partial_{\alpha'} \gamma^0)(D_{\alpha'} D_{\alpha''} \eta_{[1]}^{[1]})]
+ (\partial_{\alpha} \eta_{[1]}^{[1]}) (D_{\alpha} \eta_{[1]}^{[1]}) + 2(\partial_{\alpha} \partial_{\alpha''} \eta_{[1]}^{[1]})(D_{\alpha} D_{\alpha''} \eta_{[1]}^{[1]}) + (\partial_{\alpha} \eta_{[0]}^{[1]}) (D_{\alpha} \eta_{[1]}^{[1]})]^2
= a^3
\end{equation}

[k = 0, 1, 2] which is $O(\Omega^2)$. Etc.

Solving the equations $O(\Omega^0)$, i.e. for $\eta_{[1]}^{[n]}$, observe that the constituent symbols generate an expansion of the type
\begin{equation}
\gamma_1 = \sqrt{a^2 + \hat{a}^2}_0 = \gamma^0 \sqrt{1 + (\hat{a}^2_0/a^2)} \sim \gamma^0 + \sum_{n=1}^{\infty} e^n \eta_{[1]}^{[n]},
\end{equation}

which could also be rewritten as
\begin{equation}
\sqrt{a^2 + \hat{a}^2} = \sqrt{a^2 + a^2} \sqrt{1 + (a^2_0/a^2)}
\end{equation}

and expanded in a Taylor series accordingly. The first three terms of expansion (96) are illustrated in Figure 1. In this figure, we have carried out the rotations in the frequency and slowness domains
\begin{equation}
s = i\omega, \quad \alpha_{\nu} = -i\nu_{\nu}.
\end{equation}

Solving the equations $O(\Omega)$, we find that the constituent symbols represent an expansion of the type
\begin{equation}
\frac{1}{2(\gamma^0 + (\hat{a}^2_0/a^2))^3/2} \left[ a^2_0 \left[ 1 + (\hat{a}^2_0/a^2) \right] + \frac{3}{4} i\alpha_{\sigma} (D_{\alpha'}^2 \hat{a}^2) \right]
= \frac{1}{2(\gamma^0)^3(1 + (\hat{a}^2_0/a^2))^3/2} \left[ (\gamma^0)^2 a^2_0 \left[ 1 + (\hat{a}^2_0/a^2) \right] + \frac{5}{4} i\alpha_{\sigma} (D_{\alpha'}^2 \hat{a}^2) \right] \sim \sum_{n=1}^{\infty} e^n \eta_{[0]}^{[n]},
\end{equation}

which takes into account the rate of change of the medium properties.

**Screen reduction**

As a consequence of the expansion in $\epsilon$, the constituent symbols of Eq.(87) are all separable in the phase space coordinates, $x_{\mu}$ and $\alpha_{\nu}$:
\begin{equation}
\eta_{[1-m]}^{[n]}(x_{\mu}, \alpha_{\nu}) = \sum_{\lambda} \xi_{[1-m]}^{[n]}(x_{\mu}) \alpha_{[1-m]}^{[n]}(\alpha_{\nu}), \quad m = 0, 1, \ldots
\end{equation}

(If $n = 1$ and $m = 0$ we have one term only in which case we omit the $\lambda$.) This separation of variables is why our lattice integral Eq.(31) will reduce to a screen propagator (19). Equation (98) implies the following structure for the cokernel (cf. Eq.(36)) of the vertical slowness operator,
\begin{equation}
\eta_{[1-m]}^{[n]}(\alpha_{\mu} - \alpha_{\mu''}) = \xi_{[1-m]}^{[n]}(\alpha_{\mu} - \alpha_{\mu''}) a_{[1-m]}^{[n]}(\alpha_{\mu''}), \quad m = 0, 1, \ldots
\end{equation}

which reveals how Fourier constituents ('plane' waves) interact. Asymptotically, we have
\begin{equation}
\xi_{[1-m]}^{[n]}(\alpha_{\mu} - \alpha_{\mu''}) \sim s^{-2} \left\{ \xi_{[1-m]}^{[n]}(0) \delta(\alpha_{\mu} - \alpha_{\mu''}) + i\Omega (D_{\alpha'0} \xi_{[1-m]}^{[n]}(0) \delta(\partial_{\alpha'}^0)(\alpha_{\mu} - \alpha_{\mu''}) + \cdots \right\}.
\end{equation}
Let us consider the expansion for the vertical slowness left symbol up to $O(\varepsilon)$ and $O(\Omega^0)$. Using Eq.(96), we arrive at

$$\eta_{[1]}^{[1]}(x, \alpha) = \xi_{[1]}^{[1]}(x, \alpha) \, a_{[1]}^{[1]}(\alpha), \quad \xi_{[1]}^{[1]} = \frac{1}{2} \hat{a}^2, \quad a_{[1]}^{[1]} = \frac{1}{\gamma^0}. \quad (100)$$

We denote this first-order expansion as the \textit{wide-angle screen} approximation. Observe the occurrence of the branch point associated with the background medium, which is artificial. Ignoring the artificial branch point, we now Taylor expand the factor $a_{[1]}^{[1]}$ about $\alpha = 0$. Up to zero-order,

$$a_{[1]}^{[1]}(\alpha) \approx a_{[1]}^{[1]}(0) = \epsilon^0, \quad (101)$$

which leads to the \textit{phase-screen} or split-step approximation, which is also shown in Figure 1. For most practical purposes, the second-order expansion in $\alpha \, \alpha_\nu$ is sufficiently accurate,

$$a_{[1]}^{[1]}(\alpha) \approx \epsilon^0 - \frac{1}{2 (\epsilon^0)^3} \alpha_\nu \alpha_\nu + \frac{3}{8 (\epsilon^0)^5} (\alpha_\nu \alpha_\nu)^2. \quad (102)$$

These expansions are illustrated in Figure 2. Note that, upon carrying out the Taylor expansion of $a_{[1]}^{[1]}$ in $\alpha_\nu$, we can replace the background from \textit{minimum} value in the thin slab to \textit{average} value, thus enhancing the accuracy. This replacement also represents the transition from the phase-screen to the split-step Fourier approach. Some wave fronts associated with Figures 1 and 2 are shown in Figure 3. Using Huygens principle, we can now anticipate the evolution of a wave front in a heterogeneous medium. In all the approximations, independent of order, the propagation speed in the horizontal direction is $\epsilon^0$, which causes any of our approximate instantaneous wave fronts to fold inwards away from the true instantaneous wave front. As such, the generalized screen approximation differs, for example, from the paraxial approximation where the accuracy with propagation angle is independent of the medium.

The contribution to the left symbol $O(\varepsilon)$ and $O(\Omega)$ follows from Eq.(97) as

$$\eta_{[0]}^{[1]} = \frac{1}{2} \hat{a} \, \hat{a}^1 + \frac{2}{\gamma^0} (D_\varepsilon \hat{a}^0) \frac{i \alpha_\nu}{(\gamma^0)^3}, \quad (103)$$

which extends the wide-angle approximation Eq.(100).
**Optimization**

Rather than accepting a given order of expansion, we will discuss a way—introducing intermediate parameters—to smear out the error of our approximation in phase space. In particular, we will be concerned about the errors in travel times of the wave constituents.

As an example, consider the wide-angle approximation (100). By shifting the background medium branch points in this expression, we can bend the principal part of the slowness curve. We will exploit this bending in an optimization procedure. In Eqs. (84), (100) set

\[
\gamma^0(x_3, \alpha_\nu) \to \sqrt{\beta_0 \alpha_\nu \alpha_\nu + [c^0(x_3)]^{-2}}, \quad a_{[1]}^{[1]}(\alpha_\nu) \to \frac{1}{\sqrt{\beta_1 \alpha_\nu \alpha_\nu + [c^0(x_3)]^{-2}}}.
\]

(104)

If the range of scattering angles is limited, by adjusting \(\beta_0, \beta_1\), indeed, the branch points can be moved inwards. However, upon applying the Taylor series expansion (102) the troublesome branch point in \(a_{[1]}^{[1]}\) is removed; then we set

\[
\gamma^0(x_3, \alpha_\nu) \to \gamma^0(x_3, \alpha_\nu), \quad a_{[1]}^{[1]}(\alpha_\nu) \to c^0 - \frac{1}{2(c^0)^3} \beta_0 \alpha_\nu \alpha_\nu + \frac{3}{8(c^0)^5} \beta_1 \alpha_\nu \alpha_\nu^2.
\]

(105)

instead.

For the purpose of optimization, we will turn the composition equation (44) into a weak form, viz., we introduce the error function

\[
\text{err}(x_\mu, \zeta, s) = (s/2\pi)^2 \int \left| \exp \left[ -i \partial_{\alpha_\nu} D_{x_\mu} \right] \bar{\gamma}(x_\mu, \zeta, \alpha_\sigma, s) \bar{\gamma}(x_\mu', \zeta, \alpha_\nu, s) \right|_{(x_\mu', \alpha_\nu) = (x_\mu, \alpha_\nu)} - \tilde{a}(x_\mu, \zeta, \alpha_\nu, s) \right|^2 \, d\alpha_1 d\alpha_2,
\]

measuring an \(L^2\) average over the relevant propagating horizontal slownesses. Restricting the error function to \(O(\Omega^0)\), i.e. the vertical gradients of travel time, yields

\[
\text{err}_{[1]}^N(x_\mu, \zeta; \beta) = (s/2\pi)^2 \int
\]

(107)
Figure 3. Wave fronts associated with the various generalized screen approximations.

\[
\left[ \gamma^0(x_3, \alpha_\nu) + \sum_{n=1}^{N} e^{n \eta_{i[n]}(x_\mu, \zeta, \alpha_\nu)} \right]^2 - \hat{\alpha}_{z}^2(x_\mu, \zeta, \alpha_\nu) \right]^{2} \, \mathrm{d} \alpha_1 \, \mathrm{d} \alpha_2.
\]

The misfit in the optimization for multi-parameter \( \beta \) is then simply the integral of the error function over horizontal space. This integration reflects the variation of the medium in the horizontal directions. Thus, for the \( \mathcal{O}(\Omega) \) optimization problem, we might integrate over the associated range of medium slownesses instead.

Enforcing the parameters \( \beta_0, \beta_1 \) to be equal ensures that the approximate expression (104) contains a single branch point only. To ensure that the approximate expression reduces to a screen representation, the parameters should be independent of \( x_\mu \).

The screen propagator

We will now return to the basic symbol structure Eqs.(84)-(100) and analyse what this structure implies for the thin-slab propagator given in Eq.(32). In general, with Eq.(84) we get

\[
\tilde{g}^{(s)}(x_\mu, x_3, x'_\mu, x_3 - \Delta x_3) \simeq \int (s/2\pi)^2 \mathrm{d} \alpha'_1 \, \mathrm{d} \alpha'_2 \exp[-is \alpha'_2(x_\mu - x'_\mu)] \exp[-s \{ \gamma^0(x_3 - \Delta x_3, \alpha''_3) + \hat{\gamma}^1(x_\mu, x_3 - \frac{1}{2} \Delta x_3, \alpha''_3) \} \Delta x_3].
\]

(108)

Invoking the first-order generalized screen approximation (100) yields

\[
\exp[-s \{ \gamma^0(x_3 - \Delta x_3, \alpha''_3) + \hat{\gamma}^1(x_\mu, x_3 - \frac{1}{2} \Delta x_3, \alpha''_3) \} \Delta x_3]
\]

\[
\simeq \exp[-s \varepsilon \xi_{i[1]}(x_\mu, x_3 - \frac{1}{2} \Delta x_3) a_{i[1]}(x_3 - \frac{1}{2} \Delta x_3, 0) \Delta x_3] \exp[-s \gamma^0(x_3 - \Delta x_3, \alpha''_3) \Delta x_3]
\]

\[
\exp[-s \varepsilon \xi_{i[1]}(x_\mu, x_3 - \frac{1}{2} \Delta x_3) a_{i[1]}(x_3 - \frac{1}{2} \Delta x_3, 0) - a_{i[1]}(x_3 - \frac{1}{2} \Delta x_3, 0)] \Delta x_3]
\]

\[
\simeq \exp[-s \varepsilon \xi_{i[1]}(x_\mu, x_3 - \frac{1}{2} \Delta x_3) a_{i[1]}(x_3 - \frac{1}{2} \Delta x_3, 0) \Delta x_3] \exp[-s \gamma^0(x_3 - \Delta x_3, \alpha''_3) \Delta x_3]
\]

\[
\left\{ 1 - s \varepsilon \xi_{i[1]}(x_\mu, x_3 - \frac{1}{2} \Delta x_3) a_{i[1]}(x_3 - \frac{1}{2} \Delta x_3, 0) - a_{i[1]}(x_3 - \frac{1}{2} \Delta x_3, 0) \right\} \Delta x_3
\]

if

\[
\frac{s \Delta x_3}{c_0} \left\| \varepsilon \xi_{i[1]}(x_\mu, \cdot) [a_{i[1]}(\cdot, \alpha''_3) - a_{i[1]}(\cdot, 0)] \right\| \ll 1.
\]
Note that $c^0\xi_{[1]}^{(1)} = (c^0)^{-1}S_{-1}$ represents the absolute medium slowness perturbation, while $[a_{[1]}^{(1)}(\omega) - a_{[1]}^{(1)}(0)]$ vanishes at normal incidence ($\omega_0 = 0$) and is bounded by $c^0$ as $\omega_0 \rightarrow \infty$.

Substituting Eq.(109) into Eq.(108) results in

$$
\hat{\gamma}^{(+)}(x_\mu, x_\nu, x_\nu', x_3 - \Delta x_3) \approx \exp[-s \xi_{[1]}^{(1)}(x_\mu, x_3 - \frac{1}{2}\Delta x_3) a_{[1]}^{(1)}(x_3 - \frac{1}{2}\Delta x_3, 0) \Delta x_3] \\
\{ \int (s/2\pi)^2 da_1'' da_2'' \exp[-is \omega''_\sigma x_\sigma] \exp[-s \gamma^0(x_3 - \Delta x_3, \omega'_\nu) \Delta x_3] \exp[is \omega''_\sigma x_\sigma] \\
- se_{[1]}^{(1)}[x_\nu, x_3 - \frac{1}{2}\Delta x_3] \Delta x_3 \int (s/2\pi)^2 da_1'' da_2'' \exp[-is \omega''_\sigma x_\sigma] \\
\exp[-s \gamma^0(x_3 - \Delta x_3, \omega'_\nu) \Delta x_3] \{ \int d\nu_1 d\nu_2 \exp[is \omega''_\nu x_\nu'] \exp[-se_{[1]}^{(1)}(x_\nu', x_3 - \frac{1}{2}\Delta x_3)a_{[1]}^{(1)}(x_3 - \frac{1}{2}\Delta x_3, 0)] \\
\exp[is \omega''_\nu x_\nu'] \{ \int d\nu_1 d\nu_2 \exp[is \omega''_\nu x_\nu'] \exp[-se_{[1]}^{(1)}(x_\nu', x_3 - \frac{1}{2}\Delta x_3)a_{[1]}^{(1)}(x_3 - \frac{1}{2}\Delta x_3, 0)] \\
\int d\nu_1 d\nu_2 \exp[is \omega''_\nu x_\nu'] \exp[-se_{[1]}^{(1)}(x_\nu', x_3 - \frac{1}{2}\Delta x_3)a_{[1]}^{(1)}(x_3 - \frac{1}{2}\Delta x_3, 0)] \}
\}
$$

which, switching from left to right symbols, can also be written as

$$
\int d\nu_1 d\nu_2 \hat{\gamma}^{(+)}(x_\mu, x_3; x_\nu', x_3 - \Delta x_3)(\omega) \approx \int (s/2\pi)^2 da_1'' da_2'' \\
\exp[-is \omega''_\nu x_\nu] \exp[-s \gamma^0(x_3 - \Delta x_3, \omega'_\nu) \Delta x_3] N_{\sigma''_\nu, \omega''_\nu} \\
\{ \int d\nu_1 d\nu_2 \exp[is \omega''_\nu x_\nu'] \exp[-se_{[1]}^{(1)}(x_\nu', x_3 - \frac{1}{2}\Delta x_3)a_{[1]}^{(1)}(x_3 - \frac{1}{2}\Delta x_3, 0)] \\
- s \Delta x_3 [a_{[1]}^{(1)}(x_3 - \frac{1}{2}\Delta x_3, \omega''_\nu) - a_{[1]}^{(1)}(x_3 - \frac{1}{2}\Delta x_3, 0)] \int d\nu_1 d\nu_2 \\
\exp[is \omega''_\nu x_\nu'] \exp[is \omega''_\nu x_\nu'] \exp[-se_{[1]}^{(1)}(x_\nu', x_3 - \frac{1}{2}\Delta x_3)a_{[1]}^{(1)}(x_3 - \frac{1}{2}\Delta x_3, 0)] \}
\}
$$

In this expression we have introduced a (windowed) normalizing operator $N$ to restore the appropriate amplitude behavior which was destroyed by the Taylor expansion of the exponential in Eq.(109),

$$
N(y(1 + p + iq)) = y \exp(ip) \left[ 1 + \frac{p}{1 + iq} \right]^{-1} \left[ 1 + \frac{p}{1 + iq} \right], \ y = 1.
$$

Every additional order in the generalized screen expansion (96), in Eq.(110) requires an additional inverse Fourier transform.

The key simplification of the screen propagator compared with the ‘exact’ thin-slab propagator is, that the inverse scaled Fourier transforms with respect to $\omega''_\nu$ no longer have to be evaluated for each $x_\mu$ separately. The application of the thin-slab propagator simply yields a shuttling between the horizontal space and horizontal slowness domains.

The phase-screen approximation is obtained by omitting the second terms – proportional to $\Delta x_3$ – in expressions (110), (111).

In Eq.(110) we recognize the Laplace-domain Green’s functions in the background medium,

$$
\int (s/2\pi)^2 da_1'' da_2'' \exp[-s \{ ia''_\sigma(x_\sigma - x''_\sigma) + \gamma^0(x_3 - \Delta x_3, \omega''_\nu) \Delta x_3\}]
$$

[the pressure response due to a vertical point force] and

$$
\int (s/2\pi)^2 da_1'' da_2'' a_{[1]}^{(1)}(x_3 - \frac{1}{2}\Delta x_3, \omega''_\nu) \exp[-s \{ ia''_\sigma(x_\sigma - x''_\sigma) + \gamma^0(x_3 - \Delta x_3, \omega''_\nu) \Delta x_3\}]
$$

[the pressure response due to a point injection source]. This is no surprise, as is elucidated in Appendix A. As well known, the transformation of those functions back to the time domain can be carried out analytically. With this interpretation, the path of integration in the complex radial horizontal slowness $\omega''_\nu$-domain is chosen appropriately.

The zero-order term of the Bremmer series now follows from cascading the thin-slab propagators (110) into the multi-screen representation using the multi-variate integral Eq.(31)

$$
\hat{\Phi}_{GPS}^{(+)}(x_\mu, x_3; x_\nu', x_3) = H[(-x''_3 - x_3)] \\
\int \prod_{k=2}^{M} (s/2\pi)^2 da_1^{(k)} da_2^{(k)} dx_1^{(k-1)} dx_2^{(k-1)} \exp[-is a_\sigma^{(k)}(x_\sigma^{(k)} - x''_\sigma^{(k)})] \\
\exp[-s \{ \zeta_h - M^{-1} \Delta x_3, \omega''^{(k)}(x_\mu^{(k)}) \} + \gamma^1(x_\mu^{(k)}, \zeta_1 - \frac{1}{2}M^{-1} \Delta x_3, \omega''^{(k)}) \} M^{-1} \Delta x_3]
\[
\int (s/2\pi)^2 d^2 \alpha^{(1)}_\nu \exp[-is \alpha^{(1)}_\nu (x'_\nu - x'_\nu)] \\
\exp[-is \{\gamma^0 (\zeta_1 - M^{-1} \Delta x_3, \alpha^{(1)}_\nu) + \gamma^1 (x'_\mu, \zeta_1 - \frac{1}{2} M^{-1} \Delta x_3, \alpha^{(1)}_\nu) \} M^{-1} \Delta x_3].
\]

(112)

The numerical procedure, in two spatial dimensions, is illustrated in Figure 4. This figure is the counterpart of Figure 3.

The windowed screen propagator

To improve the accuracy of our propagator, we briefly consider the extension of the generalized screen approach based on a single reference or background medium to one with multiple reference media. Thus, per window, the medium contrast is reduced. To this end, we will have to localize our analysis in the horizontal directions. For this we will employ the windowed Fourier transform:

\[
\int dx'_1 dx'_2 \exp[is \alpha''_\nu x'_\nu] \chi(x'_\nu - \bar{x}_\nu)
\]

with the associated inverse transform

\[
\int d\bar{x}_1 d\bar{x}_2 \chi^*(x_\nu - \bar{x}_\nu) \int (s/2\pi)^2 d\alpha''_1 d\alpha''_2 \exp[-is \alpha''_\nu x_\nu]
\]

such that

\[
\int d\bar{x}_1 d\bar{x}_2 |\chi(x_\nu - \bar{x}_\nu)|^2 = 1.
\]

Equation (111) in windowed form, for example, becomes

\[
\int dx'_1 dx'_2 g^{(b)}(x_\mu, x_3; x'_1, x_3 - \Delta x_3)(\cdot) =
\int d\bar{x}_1 d\bar{x}_2 \chi^*(x_\nu - \bar{x}_\nu) \int (s/2\pi)^2 d\alpha''_1 d\alpha''_2 \exp[-is \alpha''_\nu x_\nu]
\]

\[
\exp[-s \gamma^0 (x_3 - \Delta x_3, \alpha''_\nu) \Delta x_3] N_{\alpha''_\nu, \Delta x_3} \left\{ \int dx'_1 dx'_2 \\
\chi(x'_\nu - \bar{x}_\nu) \exp[is \alpha''_\nu x'_\nu] \exp[-s e \xi^{(1)}_{(1)}(x'_\mu, x_3 - \frac{1}{2} \Delta x_3) \alpha^{(1)}_{(1)}(x_3 - \frac{1}{2} \Delta x_3, 0) \Delta x_3] \cdot \\
-s \Delta x_3 [a^{(1)}_{(1)}(x_3 - \frac{1}{2} \Delta x_3, \alpha''_\nu) - a^{(1)}_{(1)}(x_3 - \frac{1}{2} \Delta x_3, 0)] \int dx'_1 dx'_2 \chi(x'_\nu - \bar{x}_\nu) \exp[is \alpha''_\nu x'_\nu]
\right\}
\]
Generalized screens

$$\varepsilon \xi^{[1]}(x', x - \frac{1}{2} \Delta x_3) \exp[-s \xi^{[1]}(x', x - \frac{1}{2} \Delta x_3) a^{[1]}(x - \frac{1}{2} \Delta x_3, 0) \Delta x_3](\cdot)$$

Each window is associated with a particular reference medium.

The P(ase) S(hift) P(lus) I(nterpolation) propagator

If, in each window, the medium is supposed to be constant, and if the windowing is replaced by interpolation, we arrive at the P(ase) S(hift) P(lus) I(nterpolation) procedure (22). Let us return to the thin-slab propagator in Eq.(32). In P(SIP), we restrict our vertical slowness left symbol to its principal part (cf. Eq.(96)).

$$\tilde{\gamma}^{(+)}(x, x - \frac{1}{2} \Delta x_3, x') \sim \tilde{\gamma}^{(i)}(x, x - \frac{1}{2} \Delta x_3, 0) \Delta x_3 = \sqrt{a^{[0]}(x - \Delta x_3, x') + a^{[1]}(x, x - \frac{1}{2} \Delta x_3)}$$

whereas we sample the thin-slab propagator at the coordinate values \(\{x_m^{(i)}\}_{i=1}^f\) such that the set \(\{\tilde{\gamma}^{(i)}(x, x - \frac{1}{2} \Delta x_3, 0)\}_{i=1}^f\) regularly samples the range of medium velocities encountered across the current slab. Let

$$\tilde{g}^{(i)}(x, x' - \frac{1}{2} \Delta x_3) = \int (s/2\pi)^2 \alpha \alpha' \exp[is \alpha''(x, 0)] \exp[is \alpha''(x, \Delta x_3) \Delta x_3],$$

which are simply homogeneous medium propagators. The thin-slab propagator \(\tilde{g}^{(+)}(, x, x', x')\) at any \(x, x'\) is then obtained by carrying out an interpolation based on \(\{\tilde{g}^{(i)}\}_{i=1}^f\).

Screen representations of the (de)composition operators

The composition operator

The screen representation of the composition operator (cf. Eq.(19)) requires a screen approximation of the vertical slowness operator itself. In accordance with Eq.(111) the Schwartz kernel for the vertical slowness is thus approximated by

$$\tilde{c}^{(+)}(x, x'; x_3) \approx \left\{ \int (s/2\pi)^2 \alpha \alpha' \exp[is \alpha''(x, 0)] \exp[is \alpha''(x', 0)] \delta(x - x') \right\} \times$$

$$+ \varepsilon \xi^{[1]}(x, x_3) \xi^{[1]}(x, x_3, 0) \delta(x - x')$$

(114)

Switching from left to right symbols yields

$$\tilde{c}^{(+)}(x, x'; x_3) \approx \left\{ \int (s/2\pi)^2 \alpha \alpha' \exp[is \alpha''(x, 0)] \exp[is \alpha''(x', 0)] \right\} \times$$

$$+ \varepsilon \xi^{[1]}(x, x_3) \xi^{[1]}(x, x_3, 0) \delta(x - x')$$

(115)

In the wide-angle approximation, we substitute Eq.(100) in the equations above. With optimization it would be Eq.(104) or Eq.(105). Every additional order requires an additional inverse Fourier transform.

The phase-screen approximation is obtained by omitting the third terms on the right-hand sides of Eqs.(114)-(115).

The decomposition operator

The screen representation of the decomposition operator (cf. Eq.(20)) requires a screen approximation of the inverse vertical slowness operator. To this end, the reciprocal vertical slowness symbol must be expanded as in Eqs.(86)-(87). The reciprocal vertical slowness operator is order -1 and hence we denote its constituent symbols as \(\hat{\eta}^{-1}_{-1-m}\).
As a consequence of the expansion in $\epsilon$, the constituent symbols are all separable in the phase space coordinates, $x_\mu$ and $\alpha_\nu$,

$$
\eta^{-1[n]}_{-1, m}(x_\mu, \alpha_\nu) \equiv \sum_{\lambda} \lambda^{-1[n]} \xi_{-1, m}^{-1}[x_\mu, \alpha_\nu], \quad m = 0, 1, \ldots
$$

(If $n = 1$ and $m = 0$ we have one term only in which case we omit the $\lambda$.) The expansion for the reciprocal vertical slowness left symbol up to $O(\Omega^0)$ (the principal parts) follows the series

$$
\frac{1}{\gamma_1} = \frac{1}{\sqrt{\Delta_2^{\Delta_2} + \Delta_2^{\Delta_2}}} = \frac{1}{\gamma_1^0 \sqrt{1 + (\Delta_2^{\Delta_2}/\Delta_2^{\Delta_2})}} \sim \frac{1}{\gamma_1^0} + \sum_{n=1}^{\infty} \epsilon^n \eta^{-1[n]}_{-1, 1}.
$$

Up to $O(\epsilon)$ we arrive at the wide-angle approximation,

$$
\eta^{-1[1]}_{-1, 1}(x_\mu, \alpha_\nu) = \xi^{-1[1]}_{-1, 1}(x_\mu) a^{-1[1]}_{-1, 1}(\alpha_\nu), \quad \xi^{-1[1]}_{-1, 1} = -\frac{1}{2} \Delta_2^{\Delta_2}, \quad a^{-1[1]}_{-1, 1} = \frac{1}{(\gamma_1^0)^3}.
$$

Carrying out a Taylor series expansion about $\alpha_\nu = 0$ to zero order yields

$$
a^{-1[1]}_{-1, 1}(\alpha_\nu) \approx a^{-1[1]}_{-1, 1}(0) = (\epsilon_1^0)^3,
$$

which leads to the phase-screen or split-step approximation.

In analogy with Eq.(117), the $O(\Omega)$ correction term is found to be

$$
-\frac{1}{2} \left[ \Delta_2^{\Delta_2} + \Delta_2^{\Delta_2} \right] \left[ \Delta_2^{\Delta_2} \left[ 1 + (\Delta_2^{\Delta_2}/\Delta_2^{\Delta_2}) \right] + \frac{1}{2} \Delta_2^{\Delta_2} \left( D_{\alpha_\nu} \Delta_2^{\Delta_2} \right) \right] \sim \sum_{n=1}^{\infty} \epsilon^n \eta^{-1[n]}_{-1, 1}.
$$

Up to $O(\epsilon)$ we get

$$
\eta^{-1[1]}_{-1, 1} = -\frac{1}{2} \Delta_2^{\Delta_2} \left( \frac{1}{(\gamma_1^0)^3} - \frac{\Delta_2^{\Delta_2}}{3} \right) \left( D_{\alpha_\nu} \Delta_2^{\Delta_2} \right) \frac{1}{(\gamma_1^0)^3},
$$

which extends the wide-angle approximation Eq.(118).

The Schwartz kernel of the inverse vertical slowness operator has a representation like in Eq.(114) or (115).

**Screen representations of the reflection/transmission operators**

The interaction operator in Eqs.(22)-(23) allows a screen representation as well. In the leading order interaction, the vertical derivatives of the background medium will be neglected.

The symbol of the multiplication operator $\frac{1}{2} \hat{A}_1 \hat{A}_2 (\partial_3 \hat{A}_1 \hat{A}_2)$ (density variation) on the right-hand side of Eq.(23) follows as

$$
\frac{1}{2\rho} (\partial_3 \rho) = \frac{1}{2} \frac{1}{1 + \epsilon \epsilon_\rho} (\partial_3 \epsilon_\rho) \sim \frac{1}{2} \epsilon (\partial_3 \epsilon_\rho) + \cdots
$$

which reveals an inherent separation of phase space coordinates.

The symbol of the operator $\frac{1}{2} \hat{A}_1^{-1} (\partial_3 \hat{A}_1)$ (wave slowness variation) on the right-hand side of Eq.(23) requires the composition of symbols

$$
\exp \left[ -i \partial_{\alpha_\nu} D_{\alpha_\nu} \right] \tilde{\gamma}^{-1} (x_\mu, \alpha'_\nu) (\partial_3 \tilde{\gamma})(x'_\nu, \alpha'_\nu) \bigg|_{(x'_\nu, \alpha'_\nu) = (x_\mu, \alpha_\nu)}
$$

in which we employ the expansions in accordance with Eqs.(98) and (116). The principal part is thus found to be (cf. Eqs.(96), (117))

$$
\frac{1}{2\gamma_1} (\partial_3 \tilde{\gamma}) = \frac{\partial_3 \Delta_2^{\Delta_2} + \Delta_2^{\Delta_2}}{4 \Delta_2^{\Delta_2} \left[ 1 + (\Delta_2^{\Delta_2}/\Delta_2^{\Delta_2}) \right]} \sim \frac{\partial_3 \Delta_2^{\Delta_2}}{4 (\gamma_1^0)^2} + \cdots.
$$

Again, the expansion in $\epsilon$ reveals a separation of phase space coordinates in the constituent terms. Carrying out a Taylor series expansion about $\alpha_\nu = 0$ to zero order yields

$$
\frac{1}{4 (\gamma_1^0)^2} \sim \frac{(\epsilon_1^0)^2}{4}.
$$
which leads to the phase-screen or split-step approximation of Eq.(123).

In analogy with Eq.(117), the $O(\Omega^2)$ correction term is found to be

$$\frac{1}{(a_2^0 + a_2^1)^3} \left[ -\frac{1}{8} (a_2^0 + a_2^1) (\partial_3 (a_2^0 + a_2^1)) a_1^1 + \frac{1}{8} (a_2^0 + a_2^1)^2 (\partial_3 a_1^1) \\
- \frac{1}{8} i \alpha_0 (\partial_3 (a_2^0 + a_2^1)) (D_{x_2} a_1^1) + \frac{1}{8} i \alpha_0 (a_2^0 + a_2^1) (D_{x_2} \partial_3 a_1^1) \right]$$

(125)

Up to $O(\epsilon)$ we get

$$\frac{1}{(\gamma^0)^6} \left[ -\frac{1}{8} (\gamma^0)^2 (\partial_3 (a_2^0 + a_2^1)) a_1^1 + \frac{1}{8} (\gamma^0)^4 (\partial_3 a_1^1) \\
- \frac{1}{8} i \alpha_0 (\partial_3 (a_2^0 + a_2^1)) (D_{x_2} a_1^1) + \frac{1}{8} i \alpha_0 (\gamma^0)^2 (D_{x_2} \partial_3 a_1^1) \right]$$

(126)

which extends the wide-angle approximation in Eq.(123).

The Schwartz kernel of the interaction operator has a representation like in Eq.(114) or (115).

**Comparison with the De Wolf approximation**

According to Eq.(59) the leading order backscattered field is given by

$$\mathcal{W}^{(1)}_{21} = \hat{K}_{21} \mathcal{W}^{(0)}$$

(127)

which can be cast into a recursion for $\mathcal{W}^{(1)}_{21}$ using Eq.(74). This equation bears resemblance with the De Wolf approximation (23) though the interaction operator follows from scattering across level surfaces rather than from thin volumes. The algorithm is described by Eq.(76) ($J = 2$) for the slab $[k \Delta x_3, (k + 1) \Delta x_3]$. The symbol of our interaction operator relates to the linearized plane-wave reflection/transmission coefficients, see Appendix B.

**Concluding remarks**

We have developed a screen-type algorithm for the Bremmer coupling series. The key ingredient was a factorization of the vertical slowness in phase space, based on a small and smooth contrast approximation. Our scheme has added to the standard phase-screen method in the following ways: we considered larger contrasts by introducing free parameters in the theory; we enhanced the accuracy for larger scattering angles; we introduced (de)composition operators to incorporate any desired source- or receiver-type with the appropriate radiation characteristics; we have taken care of the backscattered field with the aid of the Bremmer coupling series. The backscattered field includes phenomena like turning ray waves; the screen propagator accounts, for example, for focussing and defocussing effects. Also, our generalized screen solution captures some of the evanescent wave contributions.

The methodology followed in this paper provides a platform from which one can evaluate a screen-like representation for wave propagation and scattering up to any desired level of accuracy. However, the validity of the methodology is restricted to the time-Laplace domain. Transforming the screen-like representation to the time-Fourier domain implies a breakdown of the vertical slowness symbol at the branch points, associated with the background medium, in the horizontal slowness domain. Still, the propagator can be given meaning in the time-Fourier domain by considering the screen expansion of the symbol term wise.

Also, we have investigated the influence of the rate of change of the medium properties. Our formulae can be used to quantify the leading-order contribution due to those rates to the scattering process.

The computational complexity of our algorithm is given by $(2 + m) (N \log N + N)$, where $m$ is the order and $N$ is the number of nodes in the horizontal directions, multiplied by the number of vertical steps and by the number of terms in the Bremmer series. For all practical purposes, this complexity is comparable to the one associated with the P(arabolic) E(quation) algorithms. However, the horizontal sampling of the field in our generalized screen approach can be as coarse as the Nyquist rate, unlike the PE approach.

There are various fields of application of the method presented in this paper. One concerns the numerical implementation of wave equation solvers: the algorithm is efficient. Otherwise, in the study of propagation of waves in smoothly varying random media, the method has been proven to be useful. Finally, the lowest order approximations discussed in this paper found already application in seismic imaging (24, 25, 26). A first attempt to extend the procedure to elasticity can be found in (27).
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Appendix A. Comparison of the wide-angle-screen with the local-Born approximations

Dividing up the propagation range into thin slabs, and applying a Born approximation to each of them, Wu and Huang (19) found the following spectral-domain propagator,

\[
\hat{g}^{(\infty)}(\alpha_{\mu}, x_3; \alpha'_{\mu}, x'_3) \simeq \exp[-s\gamma^0(x_3, \alpha_{\mu})x_3] \left\{ \delta(\alpha_{\mu} - \alpha'_{\mu}) + \frac{(s - \frac{e^{0}(x_3)}{2})}{2[e^{0}(x_3)]^2\gamma^0(x_3, \alpha_{\mu})} \Delta x_3 \tilde{f}(\alpha_{\mu}, x_3, \alpha'_{\mu}, x'_3) \right\} \exp[s\gamma^0(x_3, \alpha'_{\mu})x'_3],
\]

where

\[
\tilde{f}(\alpha_{\mu}, x_3, \alpha'_{\mu}, x'_3) = -\tilde{e}_a(\alpha_{\mu} - \alpha'_{\mu}, \gamma^0(x_3, \alpha_{\mu}) - \gamma^0(x_3, \alpha'_{\mu}))
+ [e^{0}(x_3)]^2 [\alpha_{\mu} \alpha'_{\mu} + \gamma^0(x_3, \alpha_{\mu})\gamma^0(x_3, \alpha'_{\mu})] \tilde{e}_p(\alpha_{\mu} - \alpha'_{\mu}, \gamma^0(x_3, \alpha_{\mu}) - \gamma^0(x_3, \alpha'_{\mu}))
\]

arising from the contrast-source radiation patterns, and in which

\[
\tilde{e}_a = \frac{1}{\Delta x_3} \int_{\zeta = \zeta_5}^{\zeta_3} d\zeta \exp[i\alpha_{\mu} \zeta] \tilde{e}_a(\alpha_{\mu}, \zeta), \quad \tilde{e}_p = \frac{1}{\Delta x_3} \int_{\zeta = \zeta_5}^{\zeta_3} d\zeta \exp[i\alpha_{\mu} \zeta] \tilde{e}_p(\alpha_{\mu}, \zeta)
\]

are windowed Fourier transforms with respect to the vertical coordinate.

Applying a lowest order difference paraxial approximation to the perturbation term in the thin slab propagator, viz.,

\[
\gamma^0(x_3, \alpha_{\mu}) - \gamma^0(x_3, \alpha'_{\mu}) = O(\alpha_{\mu} - \alpha'_{\mu})
\]

\[
\alpha_{\mu} \alpha'_{\mu} + \gamma^0(x_3, \alpha_{\mu})\gamma^0(x_3, \alpha'_{\mu}) = [e^{0}(x_3)]^2 + O(\alpha_{\mu} - \alpha'_{\mu})
\]

we arrive at

\[
\hat{f} \simeq \tilde{e}_p(\alpha_{\mu} - \alpha'_{\mu}, 0) - \tilde{e}_a(\alpha_{\mu} - \alpha'_{\mu}, 0).
\]

Upon comparing now the spectral-domain propagator with Eqs.(36)-(36), we find an expression,

\[
\gamma^0(x_3, \alpha_{\mu}) + \frac{1}{\gamma^0(x_3, \alpha_{\mu})} \frac{\tilde{f}(\alpha_{\mu}, x_3, \alpha'_{\mu}, x'_3)}{2[e^{0}(x_3)]^2}
\]

which, in view of Eq. (A.4), is directly related to the cokernel (Eq.(99)) associated with the wide-angle approximation Eq.(100). Out of the expression for the cokernel, we can extract the screen \(S_{c-1}\) (cf. Eq.(49))

\[
\gamma(\alpha_{\mu} - \alpha'_{\mu}, x_3 - \frac{1}{2} \Delta x_3, \alpha'_{\mu}, s) \simeq \gamma^0(x_3, \alpha'_{\mu}) + \frac{1}{\gamma^0(x_3, \alpha'_{\mu})} \frac{\tilde{e}_p(\alpha_{\mu} - \alpha'_{\mu}, x_3)}{2[e^{0}(x_3)]^2}.
\]

Appendix B. Comparison of the wide-angle-screen approximation with linearized transmission coefficients

To account for horizontal medium variations, we introduced the leading-order scattering correction Eq.(100):

\[
\gamma_1 \sim \frac{\bar{e}_{\mu}}{2 \gamma^0} \sim \frac{(e^{0})^{-1}}{2 \gamma^0} \tilde{e}_p.
\]

In comparison, the scattering due to vertical medium variations is given by Eq.(123), i.e.,

\[
\frac{1}{2 \gamma_1} (\partial_3 \gamma_1) \sim \frac{\partial_3 (\bar{e}_2^2 + \bar{e}_2^2)}{4 (\gamma^0)^2} \sim \frac{(\partial_3 e^{-1})}{2 \gamma^0 (\gamma^0)^2},
\]

which constitutes the linearized transmission/reflection coefficient.

Upon discretizing the vertical derivative, the two expressions above become directly related.
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A comparison of four depth-migration methods

Baoniu Han

ABSTRACT

Much research has focused on ray-based migration methods, such as Kirchhoff and Gaussian beam, with special ray-tracing effort to overcome the multi-pathing problems that arise in complicated media. Despite advances that enhance the ray-based methods, most problems related to complex structures have not been fully solved as yet. Though relatively expensive compared with the ray-based methods, migration methods based on wavefield extrapolation exhibit advantages in handling complex structures. Although ray-based methods are still the dominant imaging tool for 3-D prestack depth migration, with the rapid advancement of computer hardware technology, we can expect migration algorithms based on wavefield extrapolation to play a more important role in the near future.

This paper describes implementation of four such related one-way algorithms for depth migration of prestack and poststack data: phase-shift-plus-interpolation, split-step Fourier, implicit $\omega - \tau$ finite-difference, and Fourier finite-difference. All the algorithms work in the frequency domain, resulting in natural and similar parallelization of the codes. These straightforward implementations of one-way methods yield accurate imaging in complicated geological structures such as the Marmousi model, where ray-based methods require complicated ray-tracing efforts to image the target zone.

Key words: Phase shift, Finite difference, Depth migration, Wave equation.

Introduction

Many migration methods have been developed and put into practice over the years. We usually put them into either of two categories — ray-based methods and methods based on wavefield extrapolation, although all of them are derived from the basic acoustic or elastic wave equation and depend more or less on a high-frequency approximation.

Ray-based migration methods are popular currently, mainly because of their target-oriented feature, flexibility in handling irregularly sampled 3-D data, generally good imaging results and relative computational efficiency. Considering the high computation cost in processing 3-D data sets, Kirchhoff and Gaussian beam methods are presently the only viable ways to do 3-D prestack depth migration. The defining characteristic of ray-based methods is their dependence on ray tracing to get the traveltime needed for the migration mapping. In complicated geological structures, however, multiple arrivals are a complication that often causes ray tracing to fail to obtain proper traveltimes, hence leading to poor subsurface imaging. Recently, several methods have been developed to help overcome such problems. Examples include semi-recursive Kirchhoff migration (Bevc, 1997) and maximum-energy traveltime methods (Nichols, 1996), but they only partially solve the multi-pathing problem.

In contrast, methods based on wavefield extrapolation can naturally handle the multi-pathing problem, leading to more accurate subsurface imaging. So, though relatively expensive compared with ray-based methods, wavefield-extrapolation migration methods have often been preferred for 2-D and 3-D poststack migrations where cost is a less serious issue.

Among these wavefield-extrapolation migration methods, some, such as the reverse-time migration, are
derived from the two-way wave equation, while others depend on approximate solutions to the one-way acoustic wave equation. Two-way solutions are most accurate but also most expensive, so in practice most migration algorithms are based on the less costly one-way solution. With certain optimization efforts, the one-way solution can reach an accuracy approaching that of the two-way solution (Jenner et al., 1997).

Here, I compared four one-way depth-migration methods: phase-shift-plus-interpolation (PSPI), implicit $\omega$-x finite-difference (FD), Fourier finite-difference (FFD), and split-step Fourier (SSF). I selected these four algorithms among other one-way algorithms because of their close relationship to one another (Rühl & Ristow, 1995). It is of interest to review their inter-relationship as well as their migration performance.

One further motivation for this work is that we need accurate migration and inverse-migration tools to continue the work of Tjan (1995) on static correction using prestack depth migration.

I have implemented each of the four methods in 2-D poststack migration and 2-D prestack common-shot migration and applied them to zero-offset synthetic data as well as the multi-offset Marmousi data, comparing their imaging accuracy and computation efficiency. Following a review of the four different migration algorithms and their inter-relationship, I show migration results on the synthetic data sets.

A common basis

Let us start with the 2-D acoustic wave equation. Given a homogeneous velocity structure, in the frequency-wavenumber domain we have

$$k_z^2 P - \frac{\partial^2 P}{\partial z^2} = \frac{\omega^2}{v^2} P,$$

(1)

where $P$ is the pressure wavefield, $z$ is the depth, $\omega$ is frequency and $v$ is velocity. Also $k_z$ is the wavenumber in the lateral (i.e., $x$-) direction. Taking the spatial Fourier transform in the $z$-direction, gives the dispersion relationship

$$k_z^2 + k_x^2 = \frac{\omega^2}{v^2},$$

(2)

or

$$k_z = \pm \sqrt{\frac{\omega^2}{v^2} - k_x^2},$$

(3)

and its corresponding one-way wave equation,

$$\frac{\partial P}{\partial z} = \pm ik_z P.$$

(4)

Here the $\pm$ signs are for the downward and upward wavefields, respectively. Analytic solutions of this one-way wave equation are

$$P(z + dz, k_x, \omega) = P(z, k_x, \omega) \exp^{\pm ik_z dz}.$$  

(5)

Thus, wavefield extrapolation involves just a simple phase shift in the frequency-wavenumber domain. To obtain the final image, we evaluate the wavefield when $t=0$. This is just a summation of all the frequency components for position $(k_x, z)$. Then the image is inverse Fourier transformed from the wavenumber domain to the space domain.

$$\text{Image}(x, z) = (\text{FFT})^{-1} \left[ \sum_{\omega_i} P(x, k_x, \omega_i) \right].$$

(6)

This formula is the essence of the phase-shift migration method often called Gazdag migration (1978). The phase-shift migration method, however, requires laterally homogeneous [i.e., $v(x)$] media, making it essentially a time-migration method. Advantages of this method are that it is stable with no special requirement for the grid spacing, and it is accurate, capable of migrating reflectors with dip up to and beyond 90°. Given these advantages, people have attempted to extend it to depth migration, that is, to deal with lateral velocity variation in a $v(x, z)$ medium.

Phase-shift-plus-interpolation migration (PSPI)

When the velocity field has lateral variation, the pure phase-shift migration method fails because it requires computation in the spatial wavenumber domain. Gazdag and Sguazzero (1984) offered a method to treat lateral variation in velocity that uses the central features of the phase-shift method. Instead of only one velocity at each depth step, he introduced several reference velocities to account for the lateral velocity variation. At each depth step, the wavefield is propagated in the $(\omega, k_x)$ domain with each of these reference velocities, yielding multiple reference wavefields. Then an inverse Fourier transform brings these reference wavefields back to the $(\omega, z)$ domain. The true wavefield is obtained by linearly interpolating the reference wavefields based on the relationship of the local velocity to the reference velocities.

To maintain high accuracy for small dip, Gazdag (1984) introduced a time-shift in the $(\omega, z)$ domain as a preprocessor for the input data. Specifically, he defined a modified field

$$P^*(z) = P(z) \exp \left[ \pm i \frac{\omega}{v(z)} dz \right],$$

(7)

and

$$P(z + dz) = \hat{P}^*(z) \exp \left[ \pm i (k_z \mp \frac{\omega}{v(z)} dz \right].$$

(8)
Here \( v_r \) is the reference velocity, and the \( \hat{P}^* \) is the Fourier transform of \( P^* \) from \( (\omega, z) \) to \( (\omega, k_z) \). Such a time-shift term is important in the PSPI method. Later, when we look at the split-step Fourier method (SSF), we will recognize that SSF has the same term and is only a special case of PSPI, wherein we use only one reference velocity in PSPI.

As to the linear interpolation, since it works on the complex wavefield, we have two choices for ways to do it. One is to interpolate the amplitude and phase individually, and the other is to interpolate the real and imaginary parts individually. In my implementation, I choose the latter because it introduces less noise in the interpolation and thus gives a more clean image.

The accuracy of PSPI is directly related to the number of reference velocities used at each depth step in PSPI, that number depending on the amount of lateral velocity variation at that depth. Bagani (1995) proposed an adaptive choice for the reference velocities determined by the statistical distribution of the velocity over the depth step. With his method, we obtain the most important reference velocities whose statistical distribution is comparable to that of the original velocity field; consequently, the total computation cost in PSPI is controlled. Figure 1 shows, the number of reference velocities used for migrating the Marmousi data as a function of depth. Interested readers may refer to Bagani et al. (1995) for a full description of the procedure for the velocity selection.

Another issue for the PSPI method is the wrap-around introduced by the spatial Fourier transformation from the \( (\omega, z) \) domain to the \( (\omega, k_z) \) domain. The wrap-around is not serious if the number of reference velocities is not large. In the first poststack migration example, when the number of reference velocities is less than five, the wrap-around is negligible, but as a result, the steep reflectors are totally attenuated. When we introduce more reference velocities, however, image quality is degraded by bringing 'noise' to the deeper part of the section. To attack this problem, following Cerjan (1985), I implemented a damping region on each side of the depth section. For example, on the left side for \( 1 \leq i_z \leq 20 \) (\( i_z \) is the output grid-point index in \( z \)-direction), I multiply the wavefield by a Gaussian-shaped spatial function \( g = \exp[-(20 - i)^2] \), with \( \sigma \) set to 0.015 and the damping region empirically set to 20 traces. The damping region also acts to reduce unwanted reflections from the side-boundary. Since split-step Fourier (SSF) and Fourier finite-difference (FFD) methods also work in the dual-domain, (\( \omega, z \)) and (\( \omega, k_z \)), I apply this damping technique to them as well. Also, this technique could be used as a damped boundary region for the finite-difference (FD) method.

Split-step Fourier migration (SSF)

The split-step Fourier method, used since the 1970s, has been applied to fields such as underwater acoustics and light propagation in optical fibers (Lee et al., 1991). Stoffa (1990) reintroduced this method into seismic imaging. In more recent years, this method has been implemented in both 2-D prestack migration (Roberts et al., 1997) and 3-D poststack migration (Tauis & Stoffa, 1997).

Like PSPI, SSF also involves a wavefield extrapolation in the frequency-wavenumber domain and a phase shift in the frequency-space domain, but its sequence of operations is just opposite to that of PSPI. For PSPI, a phase shift is done first, prior to the wavefield extrapolation. Instead of using multi-reference velocities to propagate the wavefield, SSF uses only one reference velocity, which can be either an average velocity, the minimum velocity, the root-mean-square velocity or an inverse of the average slowness at each depth step. The choice of the reference velocity among these alternatives is not critical for the migration results.

SSF is based on perturbation theory, according to which we can split the laterally varying velocity field into a constant term and a small perturbation term, \( \nu(z, x) = \nu_0(z) + \delta \nu(z, x) \),

\[

\nu(x, z) = \nu_0(z) + \delta \nu(x, z), \tag{9}

\]

First we use \( \nu_0(z) \) to propagate wavefields in the \( (\omega, k_z) \) domain,
\[ P^* = P(z, k_z, \omega) \exp \left[ \pm i \sqrt{\left( \frac{\omega}{v_0(z)} \right)^2 - k_z^2} dz \right]. \]  

(10)

Then \( P^* \) is transformed back to the \((w, x)\) domain, and a phase correction is applied to it to account for the lateral velocity variation,

\[ P(z + dz, x, w) = \hat{P}^* \exp \left[ \pm i \left( \frac{\omega}{v(x, z)} - \frac{\omega}{v_0(x)} \right) dz \right]. \]  

(11)

Lee et al. (1991) further showed that the accuracy of SSF can be improved by making the propagator symmetric, that is, by splitting the phase-shift term into two identical parts and applying them before and after the wave propagation in the \((\omega, k_z)\) domain,

\[ \pm i \left( \frac{\omega}{v(x, z)} - \frac{\omega}{v_0(x)} \right) dx = \pm i (2 \pi \frac{1}{2}) \left( \frac{\omega}{v(x, z)} - \frac{\omega}{v_0(x)} \right) dx \]  

(12)

For strong lateral variation of the velocity field, however, the perturbation theory fails, and more than one reference velocity has to be used for SSF. Logic for use of multi-reference velocities (MRVL) was introduced into SSF by Kessinger (1992), necessitating an interpolation for the overlapping region between two reference velocities (Wu & Jin, 1997). With multi-reference velocities, however, the cost of SSF increases to that of PSPI. In my implementation, I chose not to use MRVL in SSF because there is no need to repeat the job of PSPI. Also in a later example of prestack migration, we shall see that the strong lateral velocity variation is not a serious problem for SSF because we can choose a difference single reference velocity for each shot gather at no greater computational effort than that for a single velocity across the entire line.

**Implicit \((\omega-x)\) finite-difference migration (FD)**

Since Clearbou (1985) developed the FD method in the early 1970s, it has been extensively used in seismic imaging. Unlike the phase-shift migration method, the FD method works in the frequency-space domain, thus making it suitable to handle the lateral velocity variation. In deriving the FD method, one starts again from the one-way acoustic wave equation [equation (3)] in the frequency-wavenumber domain, then substitutes \(-i \frac{\partial}{\partial x}\) for \( k_x \), and approximates the square-root appearing in \( k_x \) with continued fractions. The resulting second-order approximation of the one-way wave equation is solved by a method of finite differences. Here I use an implicit (as opposed to the less costly explicit) finite-difference scheme because of its unconditional stability.

The original FD method is accurate up to only about 45° dip. By cascading the diffraction terms, the accuracy of FD can be extended up to 90°. The accuracy can also be improved by optimizing the coefficients \(a\) and \(b\) in the diffraction term of the FD algorithm (Lee & Suh, 1985).

**Fourier finite-difference migration (FFD)**

Ristow (1994) introduced the Fourier finite-difference method as an extension of the split-step Fourier method. In addition to the SSF operator, an adaptive FD operator is introduced into the wave-propagator. Below, without derivation, is the expression for wave propagator in FFD.

\[ \sqrt{\frac{\omega^2}{v(x, z)^2} + \frac{\partial^2}{\partial x^2}} = I + II + III, \]  

(13)

\[ I = \sqrt{\frac{\omega^2}{v_0^2} + \frac{\partial^2}{\partial x^2}}, \]

\[ II = \frac{\omega}{v(x, z)} - \frac{\omega}{v_r}, \]

\[ III = \frac{\omega}{v(x, z)} \left( 1 - \frac{v_r}{v(x, z)} \right) \frac{S_2^2}{a + bS_2^2}. \]

Here, as before, \( S_2 = \frac{v(x, z)^2}{\omega^2} \cdot \frac{\partial^2}{\partial x^2} \), and \( v_r \) is a reference velocity. For stability, \( v_r \) should be chosen to be the minimum velocity along the layer \((x, z + dz)\). Following Ristow and Rühl, \( a \) and \( b \) are adjustable coefficients, with \( a \) set to 2 and \( b = \frac{1}{2} [(v_r/v)^2 + v_r/v + 1] \). The combination \( I + II \) is just the SSF operator discussed above. The extrapolation of the wavefield can be determined analytically in the frequency-wavenumber and frequency-space domains. The \( III \) term is a FD operator, the main difference between this operator and the traditional one being the \( b \) coefficient, which is adapted to the velocity variation.

For a \( v(x) \) medium, FFD is the same as a pure phase-shift method. When the lateral variation in velocity is strong, FFD has action somewhere between that of SSF and FD. In fact, if we choose \( v_r \ll v(x, z) \), FFD reverts back to a pure FD method. In general, FFD is a hybrid of SSF and FD, combining features of each.

**Relationship among the various methods**

When Stoffa (1990) reintroduced the SSF method, he separated the velocity field into a constant background term and a small perturbation term. Using the perturbation term as a secondary source to the wave equation, he derived the SSF method from the phase-shift method.

Rühl and Ristow (1995) further showed that SSF and FFD form a link between the phase-shift method and the finite-difference method. Similar to Stoffa’s derivation, they also use the perturbation term as a secondary source in the wave equation. They then derived the FD method from the phase-shift method in the presence of strong lateral velocity variation. The SSF and FFD methods are just intermediate steps of that derivation.
Performance comparison in 2-D poststack migration

Example 1

In the first comparison, I apply the four algorithms to zero-offset synthetic data (Figure 3) generated by the Kirchhoff modeling code "susynlv" in SU (Seismic Unix). The model (Figure 2) for this data set includes reflectors with dips ranging from 0° to 63°. The velocity model is extreme in that it has a large lateral change in velocity, varying linearly along the horizontal direction from a minimum velocity 2000 m/s on the left to a maximum velocity 5150 m/s on the right. This model has no vertical velocity gradient since we are concerned about the ability of these algorithms to handle the lateral velocity variation. The grid spacings for migration are \( dx = 25 \) m and \( dz = 10 \) m, and the total size of the 2-D depth section is 420 by 600 samples.

The comparison of computation time shown in Figure 4 pertains to the migrated depth sections shown in Figures 5 through 8. I used the 65° FD method to migrate these synthetic data because the maximum dip is 63°. We see that PSPI, FD and FFD all produce correct images of this model; all the dipping reflectors are positioned at the correct locations. The FD and FFD results, however, suffer numerical dispersion, which, at increased cost, can be eliminated by reducing the horizontal grid size. Note that the FFD result has less numerical dispersion than does that of the 65° FD.

All the dipping reflectors above 30° have been mispositioned by the SSF method. It cannot handle such an extreme model with only one reference velocity (the velocity error between the reference velocity and the local velocity is more than 50%). With this huge error, the small-perturbation assumption no longer holds. As described above, the accuracy of SSF can be improved by introducing multi-reference velocities with computation cost that may become as large as that of PSPI. Though SSF is less accurate than other methods for this extreme model, it would be the most efficient one for less demanding models, for which, as we will see, SSF can have good performance.

PSPI with ten reference velocities for this model gave the best image among the various algorithms. Actually even as few as seven velocities gave a comparable result (at 30% reduction in computation time). When I used only two velocities in PSPI, the process could image dips up to only 40°, with larger dips attenuated.

Considering both accuracy and cost, none of the above algorithms is satisfactory for this extreme model. The PSPI is the most accurate, but too expensive to be affordable. SSF is the fastest, but it gives a poor subsurface image. FD and FFD are between them, with the correctly positioned images degraded by numerical problems. Given a choice in the tradeoff between accuracy and cost, I would select FFD and FD, which give an acceptable image at affordable computation cost.

Example 2

Figure 9 is a 2-D slice of the SEG/EAEG 3-D salt model provided by Huang (1997), and Figure 10 shows the zero-offset data generated by a finite-difference scheme based on the exploding-reflector model. These data have 1024 traces with 1024 samples at a sampling interval of 4 ms. For the migration, 12.2-m spacing interval is used both vertically and horizontally, and the velocity profile contains 320 by 1024 points with \( v_{min} = 1524 \) m/s and \( v_{max} = 4480 \) m/s.

The comparison computing times shown in Figure 4 still hold, with the exception of the PSPI algorithm. Its computing cost is reduced to slightly more than three times of that of SSF because an average of 3.7 velocities for each depth step is used to migrate this less-extreme data set.

The results obtained with the different algorithms are shown in Figures 11 through 14. All four algorithms give good results in the shallow region where the velocity structure is simple. Differences lie along the boundary of the high-velocity salt body. The SSF algorithm fails to image the triangle reflector (\( x=7.9 \) km, \( z=1.8 \) km) sinking into the salt body. The other three methods give an improved imaging of that region, but the image of PSPI is somewhat broadened compared to that of FD and FFD. FD shows the most narrow shape for that triangle region, which is close to that in the true depth model. FFD is between PSPI and FD. As to the area below the salt body, all four algorithms obtain similar images. Some of the steep faults shown in the velocity model are missing in the migrated sections. The reflections from these missing faults likely have not been recorded in the zero-offset data.

Huang (1997) showed migration results on the same data set using other algorithms, such as extended pseudo-screen, Kirchhoff, and FX-migration. The results obtained by PSPI, FD and FFD here are comparable to that of Huang’s extended pseudo-screen algorithm, considerably better than that of the Kirchhoff algorithm, and exhibit less numerical noise than that of FX-migration.

Performance comparison in 2-D prestack migration

I also tested the four algorithms on the Marmousi synthetic data (Versteeg & Grau, 1990). The velocity model
Figure 2. Model with reflectors having dip ranging from $0^0$ to $63^0$. It has only lateral velocity variation, with $v_{\text{min}} = 2000$ m/s and $v_{\text{max}} = 5150$ m/s.

Figure 3. Zero-offset synthetic data for the model generated by a Kirchhoff modeling code.
of the Marmousi data contains 369 by 750 grid points with lateral spacing intervals of 25 m and vertical step size of 4 m. The synthetic data contain 240 shot gathers, each of which includes 96 traces sampled at 4-ms interval. Each trace has 726 samples, and both the shot spacing and group interval are 25 m. A main difficulty in imaging the Marmousi data (Figure 15) is the target
Figure 6. SSF migration with one velocity.

Figure 7. Second-order 65° FD migration.
zone, a low-velocity wedge located at \((x=6.5 \text{ km}, z=2.5 \text{ km})\).

To reduce the computation time, I re-sampled the original Marmousi velocity model vertically, changing the number of depth samples from 750 to 350, and used a frequency range from 15 to 35 Hz. Also, for the cross-correlation imaging condition, I used a 25-Hz Ricker wavelet as the source function. The computations were done with parallel versions of the prestack migration codes implemented in the PVM environment (Han, 1998). Using 19 Pentium PCs, the total processing time for the Marmousi data was about 25 minutes for SSF (the fastest of the methods).

As seen in the migrated depth sections shown in Figure 16 through 20, all four algorithms provided good imaging of the Marmousi model. Specifically, all of them did well in imaging the anticline in the deeper section.
Some of them have problems in the shallow part. The problem related to SSF is the ambiguity in imaging the shallow region of the rightmost steep fault ($x=5.5$ km, $z=0.8$ km) and the small triangular area ($x=5.8$ km, $z=1.7$ km). For the FD method, the $65^\circ$ algorithm gives an acceptable image, but has some low-frequency noise in the shallow section ($x$ around 4-5.5 km, $z$ about 0.7 km); the $80^\circ$ algorithm shows a significant improvement in the imaging quality. The amplitude of the shallow section for PSPI is slightly weaker than that of other algorithms, perhaps due to the interpolation, which attenuates some energy. The result of FFD is close to that of the $80^\circ$ FD method.

Comparative computation times for the various algorithms are shown in Figure 21. Because an average of 4.3 reference velocities was used for the PSPI migration, the PSPI method is about four times more costly than SSF. The cost for FFD is slight less than twice that of the SSF, and the cost of FD65 is between that of SSF and FFD. As to FD80, its cost is 20% higher than that of FFD.
Conclusions

Of the four one-way depth-migration methods tested on the Marmousi data here, SSF was the fastest. Although in the first example of the poststack migration on a model with more extreme lateral variation, the SSF method was distinctly less accurate than other methods, its performance on the Marmousi data is comparable to that of other three methods. This good performance for prestack migration is understandable, because in shot-gather migration, different reference velocities can be chosen for each gather at no additional cost since each shot gather has limited illumination area; thus prestack SSF has less velocity error than it would for poststack SSF.

The other three methods gave similar results for the migration of the Marmousi data. The PSPI method is the most expensive of the methods, but it is also accurate and has less numerical noise. The FD and FFD methods also achieve satisfactory performance. Between
these two, the FFD is superior to the conventional 45° and 65° FD, and nearly equivalent to the 80° FD. Of course, detailed differences in speed and accuracy depend also on specific choices for parameters in the various algorithms.

Except for FD, all approaches are dual-domain algorithms that shuttle between the frequency-wavenumber and frequency-space domain. One advantage of this shuttle between domains is that it is convenient to design filters in the wavenumber domain to attenuate postcritical inhomogeneous waves. As pointed out by Li (1991) and Graves & Clayton (1989), these dual-domain methods consequently have more flexibility in eliminating numerical noise compared with the conventional FD algorithm, thus yielding clearer images.

Generally, we can expect the relative accuracy seen here to hold for 3-D data. The extension from 2-D to 3-D is straightforward for PSPI and SSF, involving only an additional Fourier transform in the y-direction. It is more difficult to take FD and FFD to the 3-D domain because of the numerical anisotropy caused by splitting the FD operator in the inline and crossline directions. To compensate for such numerical noise, multi-direction splitting (Ristow & Rühl, 1995) may be required, but at increased computation cost for FD and FFD. It seems, therefore, that 3-D imaging will put more preference on the PSPI and SSF methods.

An outcome of this study is that we now have a number of depth-migration codes and their parallel versions. In the future I will extend them to the 2-D prestack common-offset domain and to the 3-D domain.

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References


2.5D downward continuation using data mapping theory

Steven D. Sheaffer & Norman Bleistein

ABSTRACT

Data mapping is a procedure for transforming seismic or acoustic data collected with one set of source/receiver locations and wavespeed model, to data equivalent to that collected in a different source/receiver configuration and wavespeed model. A major application of this general process is downward continuation of the wavefield, or wave-equation datuming. Unlike many previous implementations of the downward continuation process, this study proposes a procedure that is not only kinematically correct, but also dynamically correct in heterogeneous media, at least in a model-consistent sense.

Here the general, "true-amplitude" data mapping platform of Bleistein & Jaramillo (1997) is specialized to the process of downward continuation of receivers given a common source, and to downward continuation of sources given a common receiver. This is done by stationary phase analysis of the general data mapping integral, as specialized to the downward continuation problem for common-source or common-receiver geometries. The asymptotic analysis produces a stationary point for each set of input and output locations, and amplitudes that depend on parameters along the ray paths connecting this stationary point with the source and receiver locations. In general heterogeneous media, a large investment in external ray tracing is required to locate this point and calculate the associated ray quantities. The procedure may be greatly simplified, however, when assumptions about the wavespeed structure of the medium can be made, and analytic results can be used in conjunction with, or in place of, ray tracing results.

Key words: data mapping, downward continuation, wave-equation datuming

Introduction

Reverse propagation of the surface wavefield is an important tool in seismic imaging, processing, and interpretation. In areas of irregular topography or complex near-surface structure, datum correction of seismic surveys via wave-theoretical downward continuation may be preferable to an approximate, vertical static correction (Zhu, et al., 1995, Salinas, 1997). This is especially true when the near surface contains high velocities or other structures that make the assumption of vertical propagation suspect. The ability to perform datuming with accurate amplitudes is also highly desirable for any application involving amplitude-variation-with-offset (AVO) and amplitude-variation-with-angle (AVA) analysis of data collected in geologically challenging areas. More specialized applications have also recently arisen in seismic migration. Most notable is the method of Bevc (1995), which alleviates multipathing problems when first-arrival times are used in Kirchhoff migration by alternating steps of migration and downward continuation.

Expressions for Kirchhoff-type wave-equation datuming in homogeneous media have previously been derived and implemented by Berryhill (1979). More recently, these results have been modified by Bevc (1995) to account for 3D geometrical spreading in 2D media, using stationary phase analysis. Further, kinematic issues in wave-equation datuming have been addressed using these results by Salinas (1997). These previous works,
however, are all based on the assumption of constant wavespeed, and are therefore not dynamically accurate in variable media. In this study, a method is proposed to obtain expressions for “true-amplitude” downward continuation in heterogeneous media, at least in the sense that the amplitude mapping is consistent with the assumed wavespeed model.

This method involves the derivation and analysis of integral expressions for 2.5D downward continuation of receivers and sources for offset data based on the 2.5D seismic data mapping theory of Bleistein & Jaramillo (1997). This approach is desirable for several reasons. First, this is a general, dynamically correct theory for the transformation of data from one survey configuration to another. Second, 2.5D theory provides correct amplitudes for data that are truly 3D but collected in a linear survey, provided that the survey line is oriented normal to the dominant structural axis. Finally, the general data mapping platform allows the development of specific methods for repositioning of data to suit subsequent processing or particular applications. More explanation of these issues will be given later.

The result of the basic derivations are formulas for the downward continuation of both receivers and sources, using the data mapping platform as a starting point. The kinematics will be shown to be a function of the traveltime between the input and output locations of the points being datumed. This can be found for general media via ray tracing between the recording surface and the datum, so a kinematically accurate result can be obtained easily in any implementation.

Amplitude, however, depends on the evaluation of various Jacobians on the paths to and from a particular stationary point in the medium. This stationary point must be evaluated for each combination of input and output locations and times. In this paper, stationary phase analysis locates this point at the intersection of an isochron related to a particular input time and the ray connecting the input and output locations. This input isochron has the property that it is tangent to the isochron associated with the desired output time at this intersection point. However, only the times related to travel paths through this point are initially known, not the location of the point itself. In a general heterogeneous medium, the determination of this point and the evaluation of the integrals can be expensive, demanding a great deal of information from external ray tracing. While a specific procedure to accomplish this is not described here, it is evident that the large cost makes extrapolation in a completely general medium somewhat impractical. This arises from the fact that little or nothing can be assumed about the rays and isochrons. In more simple wavespeed models, though, enough generalizations about the isochron geometry may be possible to greatly simplify this process.

So, after the derivation of general expressions for 2.5D downward continuation, the example of a constant wavespeed medium is offered. For constant wavespeed, isochrons are elliptical and rays are straight lines. Therefore, analytic expressions can be developed for the location of the stationary point, given the traveltime. These expressions are derived for both a horizontal recording surface and a general, topographic recording surface. With respect to field data, where the subsurface is usually far from homogenous, the constant wavespeed case is primarily for illustrative purposes. However, given that correct kinematics can be easily installed, one can envision using the fast, constant wavespeed dynamical calculation as an approximation in applications where true-amplitude is not required.

A simple wavespeed model that is usually more representative of the actual subsurface is that of a depth-dependent wavespeed. In many field data problems, assuming a depth-dependent, $v(z)$ medium for the amplitude calculation provides a reasonable approximation of the subsurface wavespeed structure. This important case is left to a future paper.

**General derivations for 2.5D downward continuation**

In this section, the 2.5D assumption and the general data mapping platform are presented. Then, a common-source geometry is applied to produce a general method for downward continuation of receivers, via stationary phase analysis of the data mapping integral. Following that, the analogous expression for the continuation of the sources given a common-receiver geometry is developed. Results in this section are completely general with respect to velocity heterogeneity, as well as to the shapes of the recording and datum surfaces.

**2.5D data mapping formulation**

To adequately characterize the 3D nature of the subsurface, 3D data acquisition and processing is generally required. Given the expense of conducting fully 3D surveys, however, it is often still attractive to collect data in linear surveys. The problem with this is that 2D processing is generally applied to such data. Truly 2D methods, however, assume that the source is a line source, and therefore undergoes 2D geometrical spreading. Since the seismic source is generally a point source of waves that
spread in 3D, 2D processing will not produce correct amplitudes.

The 2.5D approach is one alternative that is applicable when the linear survey is conducted in the direction of the dominant structural dip of the subsurface. Mathematically, the \( x_1 \)-direction in a Cartesian coordinate system can be oriented parallel to this survey line, allowing the velocity structure to be assumed constant in the normal, \( x_2 \)-direction. Under these assumptions, no energy is reflected back into the vertical plane extending below the survey (the survey plane) from any out-of-plane reflectors, making the data independent of \( x_2 \). Then, the \( x_2 \)-dependence of the appropriate 3D integral expression for propagation can be integrated by the method of stationary phase in the \( x_2 \)-direction to eliminate that dependence. The resulting equation depends only on the in-plane variables, but will preserve amplitude variation due to 3D geometrical spreading. This is known as the two-and-one-half dimensional (2.5D) approximation. A full discussion can be found in Bleistein, et al. (1997).

The seismic data mapping formula of Bleistein & Jaramillo (1997) is a general integral expression that allows sources and receivers of some arbitrary input configuration to be mapped to some other configuration. In their paper, a general 3D mapping integral is derived by cascading an inversion formula and a modeling formula. This is achieved by replacing the spatially dependent reflectivity in the modeling formula with an appropriate expression for data inversion. This is equivalent to inverting the input data for reflectivity and then remodelling through this reflectivity distribution to the output locations, all in one process. Subsequent analysis yields a simplified true-amplitude data mapping platform in 3D. They then derive the 2.5D version by the above procedure, yielding a true-amplitude data mapping platform for parallel, linear input and output survey configurations, under the assumption of constant structure in the direction normal to the data lines.

Following Bleistein & Jaramillo (1997), the general 2.5D data mapping integral is

\[
\omega_0 (\xi_0, \omega_0) \approx \frac{\sqrt{|\omega_0|}}{4\pi^2} e^{-i\pi/4} e^{i\pi/4} \int \sqrt{|\omega_1|} \cdot d\omega_1 d\xi_1 d^2 x, \tag{1}
\]

where \( x \) is the integration variable for the \( d^2 x \) integral, and \( \omega_0 \) represents scattering points in the survey plane. The subscripts \( I \) and \( O \) denote quantities associated with the input and output configurations, respectively. The subscripts \( S \) and \( G \) indicate source and geophone, with all vectors evaluated in the survey plane. Source locations in the input data are given by \( \xi_0 \), receiver locations by \( \xi_1 \), and both are described by a parameter \( \xi \). Analogous quantities in the output data are given by \( \xi_0 \), \( \xi_1 \), and \( \xi \). The ratio,

\[
a_I (x, \xi_0) + \frac{A(x, \xi_0 (\xi_o), \xi)}{A(x, \xi_0 (\xi_o), x)} A(x, \xi_0 (\xi_o), x)
\]

is that of the output to input amplitudes, where each \( a \) is the product of the Green’s function amplitudes, \( A \), evaluated in the survey plane, as required by the 2.5D theory. The denominator is the product of the amplitudes associated with the paths from \( x \) to the scattering point \( x \), and from \( x \) to \( x \), in the input data. The numerator is the product of amplitudes associated with the paths from \( x \) to the scattering point \( x \), and from \( x \) to \( x \), in the output data. It is important to note that these are ray theoretical Green’s functions, and are not valid in the presence of a caustic. The factor,

\[
H(x, \xi) = \det \left| \frac{\partial \varphi}{\partial \xi} \frac{\partial \varphi}{\partial \xi} \right|,
\]

is the 2D Beylkin determinant for the problem, evaluated in plane as prescribed by the 2.5D result. A full discussion of this can be found in Bleistein, et al. (1997). The symbol \( \sigma \) represents a parameter measured along the raypath from a scatterer to a source or receiver location, as

\[
\left( \frac{\partial x}{\partial \sigma} \right)^2 = p^2 = |\nabla r|^2 = \frac{1}{c^2(x)},
\]

where \( p \) is the ray vector, and \( r \) is travel time along the ray. As before, the subscripts \( S \) and \( G \) indicate a parameter evaluated on the source-to-scatterer and scatterer-to-geophone paths, respectively.

The data mapping formulation is referred to here as a platform. This choice of terminology reflects the fact that this general integral expression can be modified, and hopefully simplified, for specific data mapping applications. One of the most important is downward extrapolation of the wavefield. This is achieved by asymptotic
analysis of the 2.5D data mapping expression under the application of the source and receiver configurations specific to this procedure. Extrapolation of prestack data requires the downward continuation of both receivers and sources, and is approached here as a two-step, cascaded process.

Downward continuation of receivers

Consider the problem of the downward continuation of receivers with a fixed source (common-shot gather). This implies

\[ x_{OS} = x_{IS} = x_S = \text{constant}. \]  

(6)

Assume a Cartesian coordinate system, where \( x_3 \) represents depth and \( x_1 \) is the direction of the linear survey on the surface. The vertical \( (x_1, x_3) \) plane will be referred to as the survey plane. Recall that in the 2.5D assumption, only propagation in this plane is relevant to the input data. In this case, integral (1) is over the survey plane, parameterized by \( d^2x = dx_1dx_3 \).

To obtain an expression useful for application of the theory, the spatial integration in equation (1) should be reduced to one over the input data only. Therefore, the integration over the survey plane is performed by the method of stationary phase. This is simplified by a change of variables from \( (x_1, x_3) \) to coordinates based on reflection isochrons. In this context, an isochron denotes any surface of scattering points, that, for a given source and receiver, produces reflected arrivals with equal traveltimes \( \tau \). (For example, in constant wavspeed media these curves are ellipses.)

So, for a given source and receiver pair in the input, and a given receiver location in the output, every depth point \( x \) lies on the intersection of two of these isochronal surfaces of constant reflected traveltimes. One is associated with the traveltimes \( \tau_I \) from the source to the input receiver location, and the other is associated with the traveltimes \( \tau_O \) from the source to the output receiver location. Therefore, \( d^2x = dx_1dx_3 \) can be transformed to \( d\gamma d\tau \), where \( \tau \) is an isochron and \( \gamma \) is a parameter along \( \tau \). This geometry is outlined in Figure 1. Since each subsurface point lies on both an input and an output isochron, the integration can be performed along either one. For reasons that become obvious in the following analysis, the input isochron, \( \tau_I \) will be used. This means

\[ x = f(\tau_I, \gamma_I), \]  

(7)

\[ d^2x = dx_1dx_3 = \left| \frac{\partial(x)}{\partial(\tau_I, \gamma_I)} \right| d\gamma_I d\tau_I. \]  

(8)

Figure 1. General geometry for downward continuation. Any depth point \( x \) can be expressed in coordinates \((\tau_I, \gamma_I)\), referencing the traveltime isochron for the scattered raypath from the source to the input receiver.

In terms of these new variables, equation (1) becomes

\[ u_O(x_0, \omega_0) \approx \frac{\sqrt{\omega_0}}{4\pi^2} e^{-i\frac{\pi}{4} \kappa \gamma \tau} \int \frac{\sqrt{\omega_I}}{4\pi^2} \frac{e^{i\pi/4 \kappa \gamma \tau} u_I(\xi_I, \omega_I) a_O(x, \xi_O)}{a_I(x, \xi_I)} \left| \frac{\nabla e^{\tau_0(x, \xi_O)} \cdot \nabla e^{\tau_1(x, \xi_I)}}{\nabla e^{\tau_0(x, \xi_O)} \cdot \nabla e^{\tau_1(x, \xi_I)}} \right| d\gamma_I d\tau_I d\omega_I d\xi_I. \]  

(9)

The next step is to approximate the \( \gamma_I \) integral by the method of stationary phase. The phase in equation (9) is

\[ \Phi = \omega_0 \tau_0(x, \xi_O) - \omega_1 \tau_1(x, \xi_I). \]  

(10)

The \( \gamma_I \) integration is an integration of \( x \) along a particular input isochron, so that \( \gamma_I \) is constant with respect to differentiation over \( \gamma_I \). Thus, stationarity is defined by the condition

\[ \frac{\partial \Phi}{\partial \gamma_I} = \omega_0 \frac{\partial \tau_0(x, \xi_O)}{\partial \gamma_I} = \omega_0 \frac{\partial \tau_0}{\partial x_i} \frac{\partial x_i}{\partial \gamma_I} = \omega_0 \left( \nabla e^{\tau_0} \cdot \nabla e^{\tau_1} \right) = 0. \]  

(11)

Note that \( \nabla e^{\tau_0} \) is a vector normal to the \( \tau_0 \) isochron. Since \( x \) is constrained to be on the \( \tau_I \) isochron, \( (\tau_I = \)
constant), in this integration, \( \partial z / \partial \tau \) is, therefore, a vector that is tangent to the \( \tau \) isochron.

At stationarity, the dot product is zero, requiring that these vectors be orthogonal. This occurs at all \( x \) where the tangents of the isochrons \( \tau_O \) and \( \tau_I \) are colinear, or equivalently, where the gradients of these isochrons are colinear. For a given source, input, and output locations, assuming no multipathing, there is a unique \( x \) on each \( \tau_I \) isochron where this is true. For a fixed source, both the input and output configurations share the same ray from the source to \( x \). When the isochrons are tangent at \( x \), the scattered rays to the input and output receivers leave \( x \) at the same angle, and therefore the ray from \( x \) to \( x_{OG} \) overlays the ray from \( x \) to \( x_{IG} \). This means that stationarity occurs at all \( x \) along the ray passing through both \( x_{IG} \) and \( x_{OG} \), as shown in Figure 2.

An important feature of this result is that both the input and the output paths scatter at the same angle. Therefore, the angularly-dependent reflection coefficients are the same for both the input and output configurations. So, in this case, preservation of the input reflection coefficients in the output correctly maps the amplitude.*

Since the ray path from the source to the integration point \( x \) at stationarity is the same path, and the source location is fixed for both the input and output geophone locations, it follows that

\[
\sigma_{OS} = \sigma_{IS} = \sigma_S, \tag{12}
\]

\[
\frac{A(x, x_{OS}(\xi_O))}{A(x, x_{IS}(\xi_I))} = \frac{\sqrt{\sigma_{OS}}}{\sqrt{\sigma_{IS}}} = 1. \tag{13}
\]

Also, along the stationary ray, the gradients of \( \tau_I \) and \( \tau_O \) are equal. So by the stationarity condition,

\[
\frac{\sqrt{\sigma_{\tau_O}(x, \xi_O)}}{\sqrt{\sigma_{\tau_I}(x, \xi_I)}} = 1. \tag{14}
\]

Evaluation of equation (9) under the stationary phase condition therefore yields

\[
u_O(\xi_O, \omega_O) \approx \frac{\sqrt{2\pi}}{4\pi^2} \int \left| \frac{\omega_O}{\omega''} \right| e^{-i\pi/4 \operatorname{sgn}(\omega_O) + i\pi/4 \operatorname{sgn}(\omega'')} \cdot \sqrt{|\omega_I|} e^{i\pi/4 \operatorname{sgn}(\omega_I) u_I(\xi_I, \omega_I)} \cdot \frac{A_0(x_{OG}(\xi_O), x)}{A_I(x_{IG}(\xi_I), x)}
\]

* Note that this result is characteristic of the downward continuation problem, and does not generally occur in data mapping.

Figure 2. Geometry at stationarity for the \( \gamma_I \) integration. All points along the ray through both \( x_{IG} \) and \( x_{OG} \) are stationary. The delta function that results from the \( \gamma_I \) integration picks out a single stationary point on this ray, namely where it crosses the \( \tau_O = t_O \) isochron. The input and output raypaths are coincident at stationarity.

\[
\frac{\sqrt{\sigma_S + \sigma_{IG}}}{\sqrt{\sigma_S + \sigma_{OG}}} \cdot \frac{\sqrt{\sigma_{IG}}}{\sqrt{\sigma_{OG}}} \cdot H(x, \xi_I) \cdot e^{i\omega_O \tau_O(x, \xi_O) - i\omega_I \tau_I(x, \xi_I)}
\]

\[
\left. \frac{\partial}{\partial (\tau_I, \xi_I)} \right|_{\tau_I = t_I, \xi_I = \xi_I} d\tau_I d\omega_I d\xi_I.
\]

The second derivative of the phase and the Jacobian remain to be evaluated. The second derivative calculation is identical in many data mapping applications, and has already been derived in Bleistein [3]. This result is

\[
\frac{\partial^2 \Phi}{\partial \gamma^2} = \omega_O \cos^2 \theta \left( \frac{\partial s}{\partial \theta} \right)^2 (\kappa_{OG} - \kappa_{IG}), \tag{16}
\]

where \( \kappa \) is the curvature of the wavefront along the stationary ray at the given receiver location due to a source at depth point \( x \), as depicted in Figure 3. The angle \( \theta \) is half of the angle between the incoming ray from the source to \( x \), and the stationary ray between \( x \) to the receiver locations, shown in Figure 2. Since \( \gamma_I \) is measured along the isochron, but is not necessarily equal to the arclength, \( s \), the derivative \( \partial s / \partial \gamma_I \) appears. From equation (16), the sign of the second derivative depends on that of frequency, as well as that of the difference in the curvatures.

In the absence of velocity anomalies that result in lensing, given two wavefronts propagating along the same ray, the maximum wavefront divergence will always be displayed by the wavefront that travels the longest distance along the ray. The downward continuation prob-
Figure 3. Example of the definitions of the curvatures $k_1$ and $k_0$. These can be viewed as wavefront curvatures at the stationary point $x_{ST}$ due to sources at either $x_{IOO}$ or $x_{OO}$. 

The problem is defined by the condition that $x_{OO}$ be at a greater depth than $x_{IO}$. So, for any point on the stationary ray beyond $x_{OO}$, the path length to the datuming surface is always shorter than that to the recording surface, meaning $k_{OO} > k_{IO}$ and the difference of curvatures in (16) is always positive. Since stationary points between the datum and the recording surface generally represent energy scattered from features above the datum that does not belong in the downward continued data, the integral is evaluated only for stationary points below $x_{OO}$, strictly, and the above condition holds. Of course, velocity anomalies can occur along the stationary ray, between the recording and datuming surfaces, that change the sign of the curvature difference. This, however, requires that the ray pass through a caustic. As previously noted, the data mapping expression (1) is based on the use of ray-theoretical Green's functions that assume no multipathing. Therefore, our downward continuation results are subject to the same restriction, and are not valid in the presence of a caustic. Under these limitations, then, $k_{OO} > k_{IO}$, and therefore

$$\text{sgn} \left( \frac{\phi''}{\omega_0} \right) = \text{sgn} \left( \omega_0 \right).$$

Similarly, the expression for the Jacobian at stationarity is identical to that in other data mapping applications, and is derived in Bleistein [3]. This result is

$$\left| \frac{\partial (x)}{\partial (\tau_1, \tau_1)} \right| = \frac{c (x)}{2 \cos \theta} \left( \frac{\partial s}{\partial \tau_1} \right).$$

So, using equations (16), (17), and (18), and assuming no caustics, the final form of integral (15) at stationarity becomes

$$u_0 (\xi_0, \omega_0) \approx \frac{\sqrt{2 \pi}}{8 \pi^2} \int \frac{\sqrt{c^2 (x) | \omega_1 |}}{\cos^3 \theta} e^{i \pi / 4 \text{sgn} (\omega_1)}$$

$$\cdot \frac{\xi_1 (\xi_1, \omega_1)}{\sqrt{k_{OO} - k_{IO}}} \frac{A_0 (x_{OO} (\xi_0), x) \sqrt{\sigma_0 + \sigma_0} \sqrt{\sigma_{OO}}}{A_1 (x_{IO} (\xi_1), x) \sqrt{\sigma_0 + \sigma_0} \sqrt{\sigma_{IO}}}$$

$$\cdot \frac{|H (x, \xi_1)|}{\sqrt{k_{OO} - k_{IO}}} e^{i \omega_0 \tau_0 (x, \xi_0) - i \omega_1 \tau_1 (x, \xi_1)}$$

$$d \tau_1 d \xi_1.$$  \hspace{1cm} (19)

Now, take the inverse Fourier transform of equation (19) from $\omega_0$ to the output time $\tau_0$. Since the only $\omega_0$ dependence is in the phase, a delta function results, giving

$$u_0 (\xi_0, \tau_0) \approx \frac{\sqrt{2 \pi}}{8 \pi^2} \int \frac{\sqrt{c^2 (x) | \omega_1 |}}{\cos^3 \theta} e^{i \pi / 4 \text{sgn} (\omega_1)}$$

$$\cdot \frac{\xi_1 (\xi_1, \omega_1)}{\sqrt{k_{OO} - k_{IO}}} \frac{A_0 (x_{OO} (\xi_0), x) \sqrt{\sigma_0 + \sigma_0} \sqrt{\sigma_{OO}}}{A_1 (x_{IO} (\xi_1), x) \sqrt{\sigma_0 + \sigma_0} \sqrt{\sigma_{IO}}}$$

$$\cdot \frac{|H (x, \xi_1)|}{\sqrt{k_{OO} - k_{IO}}} e^{i \omega_1 \tau_1 (x, \xi_1) \delta (\tau_0 - \tau_0 (x, \xi_0))}$$

$$d \tau_1 d \xi_1.$$  \hspace{1cm} (20)

At stationarity, the gradients of the input and output isochrons are equal, so the integration in equation (20) over $\tau_1$ can be transformed to an integral over $\tau_0$ by simply making the substitution $d \tau_1 = d \tau_0$. Using the sifting property of the delta function, the integration is straightforward, yielding the result,

$$u_0 (\xi_0, \tau_0) \approx \frac{\sqrt{2 \pi}}{8 \pi^2} \int \frac{\sqrt{c^2 (x) | \omega_1 |}}{\cos^3 \theta} e^{i \pi / 4 \text{sgn} (\omega_1)}$$

$$\cdot \frac{\xi_1 (\xi_1, \omega_1)}{\sqrt{k_{OO} - k_{IO}}} \frac{A_0 (x_{OO} (\xi_0), x) \sqrt{\sigma_0 + \sigma_0} \sqrt{\sigma_{OO}}}{A_1 (x_{IO} (\xi_1), x) \sqrt{\sigma_0 + \sigma_0} \sqrt{\sigma_{IO}}}$$

$$\cdot \frac{|H (x, \xi_1)|}{\sqrt{k_{OO} - k_{IO}}} e^{i \omega_1 \tau_1 (x, \xi_1) \delta (\tau_0 - \tau_0 (x, \xi_0))}$$

$$d \tau_0 d \xi_1,$$  \hspace{1cm} (21)

evaluated at the point $x$ on the stationary ray where,

$$\tau_0 (x, \xi_0) = \tau_0.$$  \hspace{1cm} (22)

Note that integration over the delta function has chosen a single stationary point on what could previously only be constrained to a stationary ray. However, equation (21) still contains the isochron $\tau_1$ in the phase. With the condition (22), though, $\tau_1$ can be determined given the traveltime along the stationary ray from $x_{OO}$ to $x_{IO}$, or $\tau_0$. Given this, $\tau_0$ can be defined as

$$\tau_1 (x, \xi_1) = \tau_0 (x, \xi_0) + \tau_0 (x_{IO}, x_{OO})$$
Two additional simplifications to equation (21) that are a result of the 2.5D dimensionality are derived in Bleistein et al. (1997) and are

\[
|H(x, \xi_t)| = \frac{2 \cos^2 \theta}{c(x)} \left| \frac{\partial}{\partial \xi_t} \mathbf{v}_{\eta I G} \right|,
\]

where the gradient is that of the traveltime to the input receiver location, and

\[
|A(x_o, x)| = \frac{\text{constant}}{\sqrt{\sigma_o} \left| J(x_o, x) \right|}.
\]

Finally, then, the 2.5D downward continuation of receivers can be performed via the integral expression,

\[
u_o(\xi_o, t_o) \approx \frac{1}{\sqrt{2\pi}} \int \frac{\sqrt{c(x)}}{\sqrt{|\kappa_{OG} - \kappa_{IG}|}} \left| J(x_{IG}, x) \right| \frac{\sqrt{\sigma_s + \sigma_{IG}}}{\sqrt{\sigma_s + \sigma_{OG}}} \left| \frac{\partial \mathbf{v}_{\eta I G}}{\partial \xi_t} \right| D_f(\xi_t(x, \xi_t)) d\xi_t,
\]

\[
D_f(\xi_t, t) = \frac{1}{2\pi} \int \sqrt{|\omega|} \left| u_I(\xi_t, \omega_I) \right| e^{-i\omega t + i\pi/4 \text{sgn}(\omega_I)} d\omega_I.
\]

This is a general result, but it is vital to recognize that in practice the receiver and source extrapolations will be a cascaded process, so care must be taken in implementation of the second process to account for the fact that part of the data has already been downward continued. This alters the values of the \(\sigma\) terms in the integral.

**Implementation in General Media**

In evaluation of the above integrals, only the desired output time, the input and output geometries, and the wavespeed model, are known initially. To obtain the correct kinematics, ray tracing between the recording and datuming surfaces can be used to provide \(\tau_{IO}\) and therefore \(\tau_{I}\), giving accurate travel times and phase. However, evaluation of the amplitude factors requires determination of the isochrons and rays so that the stationary points can be located, and the relevant raypaths, Jacobians, curvatures, and ray parameters can be calculated. Given what is known initially, this can only be accomplished in general, heterogeneous media by the extensive application of ray tracing. Since these factors must be calculated for every input-output configuration at every output time, this process is extremely expensive, possibly prohibitively. So, further work is required to assess the practicality of implementation for the general case.

Another important concern is operator aliasing. Since this type of downward continuation procedure is often applied to datuming from topography, or for static correction, the datuming surface is often close to the recording surface. The shape of the summation operator for the problem is determined by the difference in depth between the recording and datuming surfaces. When this difference is small with respect to the problem dimensions, the operator is steep at small offsets, and operator aliasing is a significant problem. Therefore, a summa-
tion aperture must be implemented and kept small for shallow datuming applications.

**Constant wavespeed media**

Implementation in general media is problematic because nothing can be assumed about the isochron or ray geometries. In constant wavespeed media, these geometries are known, and analytic expressions for the coordinates of the stationary point, as well as for the relevant ray quantities, can be derived. So, in this section, the general expressions for downward continuation are simplified for the case of a constant wavespeed medium, with arbitrary recording and datuming surfaces.

**Receiver continuation in a constant wavespeed medium**

For a constant wavespeed medium, several simplifications to integral (26) can be made. Assuming an arbitrary recording surface, let the source and receiver locations be given by

\[ x_{1G} = (\xi_1, z_1), \quad x_{OG} = (\xi_0, z_0), \quad x_S = (x_S, z_S), \]

and define the stationary point as

\[ x = (\bar{x}_1, \bar{x}_3). \]  

As shown in Figure 4, rays are straight and isochrons are ellipses, tilted with respect to the coordinate system such that the source and the appropriate receiver location are at the foci. As a result, analytic expressions can be derived for the location of the stationary point that are functions of only the source and receiver locations and the traveltimes.

In a constant wavespeed medium, the \( \sigma \)-factors are simply \( rc \), where \( r \) is the linear path length, and at stationarity, the Jacobians are simply the path lengths between \( x \) and the receiver locations, or

\[ |J(x_{1G}, x)| = r_{1G} = \sqrt{(\bar{x}_1 - \xi_1)^2 + (\bar{x}_3 - z_1)^2} \] \hspace{1cm} (32)

\[ |J(x_{OG}, x)| = r_{OG} = \sqrt{(\bar{x}_1 - \xi_0)^2 + (\bar{x}_3 - z_0)^2}. \] \hspace{1cm} (33)

For homogeneous media, the radii of curvature of the wavefronts emitted from \( x \) and arriving at the input and output locations along the stationary ray are simply the distances between the points, and the wavefronts are spherical, so

\[ \frac{1}{\kappa_{1G}} = \frac{r_{1G}}{c} = \frac{\sigma_{1G}}{c}, \quad \frac{1}{\kappa_{OG}} = \frac{r_{OG}}{c} = \frac{\sigma_{OG}}{c}. \] \hspace{1cm} (34)

Finally, the gradient of the scattered traveltimes is a vector pointing in the direction of \( r_{1G} \) with magnitude \( 1/c \). In the Appendix, its derivative is shown to be

\[ \left| \frac{\partial V_x}{\partial \xi} \right| = G \frac{(\bar{x}_1, \bar{x}_3)}{c r_{1G}^3}, \] \hspace{1cm} (35)

where \( G \) is a factor that contains the influence of topographic variations in the recording surface, and is given by the expression

\[ G(\bar{x}_1, \bar{x}_3) = \left[ (\bar{x}_3 - z_1)^4 + \left( r_{1G} \dfrac{\partial z_1}{\partial \xi} - (\bar{x}_1 - \xi_1)(\bar{x}_3 - z_1) \right)^2 \right]^{1/2}. \] \hspace{1cm} (36)

These results allow equation (26) to be written

\[ u_o(\xi_0, t_0) \approx \frac{1}{\sqrt{2\pi} c} \int d\xi G(\bar{x}_1, \bar{x}_3) \frac{\sqrt{r_s + r_{1G}}}{\sqrt{r_s + r_{OG}} \sqrt|r_{1G} - r_{OG}|} D_f(\xi_1, t_1(\bar{x}_1, \bar{x}_3)). \]

\[ D_f(\xi_1, t) = \int \frac{1}{2\pi} \sqrt{\left| \omega I \right|} u_I(\xi_1, \omega I)e^{-i\omega t + i\omega I/4 \text{sgn}(\omega I)} d\omega I. \] \hspace{1cm} (37)

The integral still contains functions of the stationary point, which requires determination of the location of \( x \). As previously described, this point lies at the intersection...
of the ray through \( x_{IG} \) and \( x_{OG} \) and the output isochron \( \tau_O = \tau_O \), or equivalently, with the input isochron \( \tau_I = \tau_I + \tau_{IO} \), where \( \tau_{IO} \) is independently determined.

In the general derivation, it was noted that under the assumption of no caustics, downward continuation requires that the stationary point must always be below the datum. In the constant wavespeed case, this is equivalent to the condition

\[
\tau_{IG} > \tau_{OG}. \tag{38}
\]

Since values of \( \tau_O \) exist that violate this condition when evaluating the integral (37) over all output times, requirement (38) must be applied. Not only does this exclude scattered energy associated with features above the datum, but it also avoids the singularity in the denominator. Indeed, for the stationary ray method to be valid here, it is required that \( 2\omega \tau_{IG} \tau_{OG}/c >> 1 \), usually \( \geq \pi \).

**Location of the stationary point for receiver continuation**

Using the geometry shown in Figure 4, analytic expressions can be derived for the location of the stationary point by considering the tilted ellipse of the \( \tau_O \) isochron and straight ray between the input and output locations. To facilitate the calculation for any general set of source and receiver locations placed on arbitrary recording and datuming surfaces, proceed as follows: For every combination of source and input-output receiver locations,

(i) shift the origin in both coordinate directions to the source location \((x_s, z_s)\):

(ii) rotate the coordinate axes about the new origin so that the new horizontal axis is coincident with the major axis of the \( \tau_O \) ellipse;

(iii) calculate the location of the stationary point and the required paths in the rotated coordinates.

The shift-rotation can be expressed in terms of the rotation angle and the primed quantities,

\[
x_s' = 0, \quad z_s' = 0, \quad z_o' = 0, \tag{39}
\]

\[
\varphi = \tan^{-1} \left( \frac{z_o - z_s}{\xi_o - \xi_s} \right), \tag{40}
\]

\[
\xi_o' = (\xi_o - x_s) \cos \varphi + (z_o - z_s) \sin \varphi, \tag{41}
\]

\[
\xi_i' = (\xi_i - x_s) \cos \varphi + (z_i - z_s) \sin \varphi, \tag{42}
\]

\[
z_s' = (\xi_i - x_s) \sin \varphi + (z_i - z_s) \cos \varphi, \tag{43}
\]

\[
h' = \dfrac{\xi_o'}{2}. \tag{44}
\]

Given this change in reference, the major axis of the ellipse representing the \( \tau_O \) isochron is aligned with the \( z_1 \) axis. It is centered at \( z_1' = h' \), with \( h' \) being the signed half-offset between the source and the output receiver location in the primed coordinate frame. From the general derivation, \( \tau_O = \tau_O \), and using these facts, the ellipse is defined by the equation

\[
x_3'^2 = Q \left[ (cto)^2 - 4 \left( z_1' - h' \right)^2 \right] . \tag{45}
\]

\[
Q = \left( \frac{(cto)^2 - 4h'^2}{4(cto)^2} \right) . \tag{46}
\]

The stationary ray is the line through \( x_{IG} \) and \( x_{OG} \), and is described by

\[
x_1' = \left( \frac{\xi_i' - \xi_o'}{z_1'} \right) x_3' + \xi_o'. \tag{47}
\]

The stationary point is one of the two intersections of these two curves. In our case of a straight ray that crosses the major axis of the ellipse, these two points are distinguished by the fact that one must lie above the \( z_3 = 0 \) surface, the other below. Because this construction exists only in the \( z_3 > 0 \) halfspace, the simplest approach to the problem is to solve equations (45) and (47) for \( x_3' \), then choose only the positive solution. After some algebra, this produces

\[
\bar{x}_3' = \sqrt{S + P^2} - P, \tag{48}
\]

where

\[
S = \left( \frac{Q \left( (cto)^2 - \xi_o'^2 \right) z_3'^2}{z_i'^2 + 4Q \left( \xi_i' - \xi_o' \right)^2} \right), \tag{49}
\]

and

\[
P = \left( \frac{2Q \left( \xi_i' - \xi_o' \right) \xi_o' z_i'}{z_i'^2 + 4Q \left( \xi_i' - \xi_o' \right)^2} \right), \tag{50}
\]

with \( Q \) defined in equation (47). Equation (48) produces a value that is both positive and real, since, by the problem geometry, \( S \) is always positive.

Now, \( z_i' \) is easily found using this result and either equation (45) or (47). Since rotation of the coordinate system does not change the path lengths between the source and receiver locations and the stationary point, these lengths
can be calculated directly in the primed coordinates using
\[
\begin{align*}
\tau_S &= \tau_S' = \sqrt{\bar{x}_1'^2 + \bar{x}_3'^2}, \quad (51) \\
\tau_{OG} &= \tau_{OG}' = \sqrt{(\bar{x}_1' - \bar{\xi}_O')^2 + \bar{x}_3'^2}, \quad (52) \\
\tau_{IG} &= \tau_{IG}' = \sqrt{(\bar{x}_1' - \bar{\xi}_I')^2 + (\bar{x}_3' - \bar{\xi}_I')^2}. \quad (53)
\end{align*}
\]

To evaluate the depth coordinate of the stationary point appearing in the integral, rotate the depth in the primed frame back to the unprimed frame, and undo the shift, via
\[
\bar{x}_3 = \bar{x}_1 \sin \varphi + \bar{x}_3 \cos \varphi + z_S. \quad (54)
\]

**Receiver continuation from a horizontal surface**

The previous discussion is applicable to data collected on any arbitrary recording surface. If data are collected on a horizontal recording surface, however, several simplifications can be made. Define the flat surface by letting
\[
z_I = \frac{\partial z_I}{\partial \xi_I} = 0. \quad (55)
\]
Given this, equation (36) becomes
\[
G(\bar{x}_1, \bar{x}_3) = \bar{x}_3 \tau_{IG}, \quad (56)
\]
making the integral for the flat surface case,
\[
\begin{align*}
\omega_0 (\xi_O, t_O) &\approx \frac{1}{\sqrt{2\pi}} \int d\xi_I \frac{\bar{x}_3}{\tau_{IG}} \\
&\times \frac{\sqrt{\tau_G + \tau_{IS}}}{\sqrt{\tau_I + \tau_{IS}} \sqrt{\tau_{IG} - \tau_{OG}}} D_I (\xi_I, \tau_I (\bar{x}, \xi_I)). \\
\end{align*}
\]

Location of the stationary point follows the same procedure as for the general case.

**Source continuation in a constant wavespeed medium**

Now complete the process by implementing a source continuation for constant wavespeed. This derivation parallels that for receiver continuation, where the roles of the source and receiver locations are interchanged. So define
\[
\begin{align*}
x_{IS} &= (\eta_I, z_I), \quad x_{OS} = (\eta_O, z_O), \quad x_G = (z_G, z_G). \\
\end{align*}
\]

The resulting geometry is shown in Figure 5.

Applying simplifications analogous to those for receiver continuation to the general integral (28), gives the constant wavespeed expression
\[
\begin{align*}
\omega_0 (\eta_O, t_O) &\approx \frac{1}{\sqrt{2\pi}} \int d\eta_I \frac{G(\bar{x}_1, \bar{x}_3)}{t_{IS}^2} \\
&\times \frac{\sqrt{\tau_G + \tau_{IS}}}{\sqrt{\tau_I + \tau_{IS}} \sqrt{\tau_{IG} - \tau_{OG}}} D_S (\eta_I, \tau_I (x, \eta_I)), \\
D_S (\eta_I, t) &= \\
&\frac{1}{2\pi} \int \left[ \sqrt{\omega_I} u_I (\eta_I, \omega_I) e^{-i \omega_I t + i \omega_I / 4 \sigma_0 (\omega_I)} d\omega_I, \right]^{1/2} \quad (59)
\end{align*}
\]
where the factor containing the topographic variation in the survey surface is
\[
G(\bar{x}_1, \bar{x}_3) = \\
\left[ \frac{1}{(\bar{x}_3 - z_I)^4 + \left( \tau_{IS}^2 \frac{\partial z_I}{\partial \eta_I} - (\bar{x}_1 - \eta_I) (\bar{x}_3 - z_I) \right)^2} \right]^{1/2}. \quad (60)
\]

As for the receiver case, the integral contains factors that depend on knowing the location of the stationary point, \(x\). Equation (59) is also subject to the same validity conditions as for expression (37), and under the assumption of no caustics, the condition analogous to requirement (38) in this case is
\( r_{IS} > r_{OS}. \) 

(61)

As before, this condition eliminates scattered energy in the input data that should not appear in the downward continued data. An analogous condition for validity of the stationary phase approximation is also relevant.

**Location of the stationary point for source continuation**

Analytic expressions for the location of the stationary point in source continuation are analogous to those for the receiver continuation, and follow the geometry outlined in Figure 5. The same shift-rotation is performed for each set of source and receiver locations, into a coordinate system aligned with the axes of the \( \tau_0 \) isochron. This time, however, the origin will be shifted to the common-receiver location, \((x_G, z_G)\). As expected, the results are those of the receiver continuation with the roles of the source and receiver locations interchanged. Using definitions (58), the shift-rotation is expressed in terms of the rotation angle and the primed quantities,

\[
x'_G = 0, \quad z'_G = 0, \quad z'_O = 0, \quad \varphi = \tan^{-1} \left( \frac{z_O - z_G}{\eta_O - x_G} \right),
\]

(62)

\[
\eta'_O = (\eta_O - x_G) \cos \varphi + (z_O - z_G) \sin \varphi,
\]

(63)

\[
\eta'_G = (\eta_G - x_G) \cos \varphi + (z_G - z_G) \sin \varphi,
\]

(64)

\[
z'_G = - (\eta_G - x_G) \sin \varphi + (z_G - z_G) \cos \varphi,
\]

(65)

\[
h' = \frac{\eta'_G}{2}.
\]

(66)

As before, this change in reference means the major axis of the ellipse representing the \( \tau_0 \) isochron is aligned with the \( x'_1 \) axis. It is centered at \( x'_1 = h' \), as \( h' \) is the signed half-offset between the common-receiver location and the output source location in the primed frame. Again, use the fact that \( \tau_0 = t_0 \). The equation of the ellipse is the same as in the previous case,

\[
x'^2 = Q \left[ (c t_0)^2 - 4 \left( z'_1 - h' \right)^2 \right],
\]

(68)

but interpreted under the source continuation geometry. The stationary ray is the line through \( x_{IS} \) and \( x_{OS} \), and is described by,

\[
x'_1 = \left( \frac{\eta'_1 - \eta'_O}{z'_1} \right) x'_3 + \eta'_O.
\]

(70)

As before, the stationary point is the intersection of these two curves in the \( x'_3 > 0 \) half-space. This produces the solution

\[
x'_3 = \sqrt{S + P^2} - P,
\]

(71)

where

\[
S = \frac{Q \left( (c t_0)^2 - \eta'_O^2 \right) z'_1^2}{z'_1^2 + 4Q \left( \eta'_1 - \eta'_O \right)^2},
\]

(72)

and

\[
P = \frac{2Q \left( \eta'_1 - \eta'_O \right) \eta'_O z'_1}{z'_1^2 + 4Q \left( \eta'_1 - \eta'_O \right)^2}.
\]

(73)

where, again, the result in equation (71) is real and positive since \( S \) is always positive. \( x'_3 \) is easily found using this result and either equations (68) or equation (70), and the path lengths are computed in the primed coordinates using

\[
r_G = r'_G = \sqrt{\left( x'_1 \right)^2 + \left( x'_3 \right)^2},
\]

(74)

\[
r_{OS} = r'_{OS} = \sqrt{\left( x'_1 - \eta'_O \right)^2 + \left( x'_3 \right)^2},
\]

(75)

\[
r_{IS} = r'_{IS} = \sqrt{\left( x'_1 - \eta'_I \right)^2 + \left( x'_3 - x'_1 \right)^2}.
\]

(76)

Rotating back to the unprimed frame and reversing the shift, the depth coordinate of the stationary point appearing in the integral can be determined via

\[
x_3 = x'_3 \sin \varphi + x'_3 \cos \varphi + x_G.
\]

(77)

**Source continuation from a horizontal surface**

Assuming a horizontal recording surface,

\[
z_1 = \frac{\partial z_1}{\partial \eta_I} = 0,
\]

(78)

and the topographic factor \( G \) becomes

\[
G(x_1, x_3) = x_3 r_{IS},
\]

(79)

making the integral for the flat surface case
\[ u_0(\eta_0, t_0) \approx \frac{1}{\sqrt{2\pi}} \int d\eta \frac{\bar{x}_3}{r_{13}} \frac{\sqrt{r_{13} + r_{1S}}}{\sqrt{r_{13} + r_{1S}} \sqrt{r_{1S} - r_{0S}}} D_s(\eta, \tau_{1}(x, \eta_1)). \] (80)

The location of the stationary point can be found by the analytic method used in the topographic case.

Implementation options for constant wavelens speed

While the assumption of constant wavelens speed is not generally representative of real data, the case is not purely illustrative. Obtaining the correct kinematics in the continuation is simply a matter of having the correct values for \( \tau_{10} \), as previously noted, obtained by ray-tracing between the recording surface and the datum. Given this, an implementation that is kinematically correct, but uses the constant wavelens speed assumption for the dynamics, can be developed. This involves using the constant wavelens speed form of the integral, approximating all path lengths by straight rays, but using ray-traced values for \( \tau_{10} \) to determine \( \tau_{1} \) in the phase. In applications where true-amplitudes are not required, this is a fast, kinematically accurate option. The efficiency arises from the ability to calculate the straight raypaths on-the-fly using the analytic expressions developed in this section.

References


APPENDIX A: Derivation of the factor \( G \) for constant wavelens media

This appendix contains the derivation of the topographic factor \( G(\bar{x}_1, \bar{x}_3) \), for receiver continuation, as given in equations (35) and (36). First, assume the same definitions for the source and receiver locations as in (30),

\[ x_{1G} = (\xi_1, z_1), \quad x_{0G} = (\xi_0, z_0), \quad x_S = (x_S, z_S), \quad (A1) \]

as well as for the stationary point,

\[ x = (\bar{x}_1, \bar{x}_3). \quad (A2) \]

Referring to Figure 4, the path \( r_{1G} \) is represented by the vector

\[ r_{1G} = ((\bar{x}_1 - \xi_1), (\bar{x}_3 - z_1)). \quad (A3) \]

The gradient of the isochron associated with waves propagating along this path is a vector pointing in the direction of \( r_{1G} \) with magnitude \( 1/c \), or

\[ \nabla \cdot r_{1G} = \frac{r_{1G}}{c r_{1G}} = \frac{1}{c r_{1G}} ((\bar{x}_1 - \xi_1), (\bar{x}_3 - z_1)). \quad (A4) \]

The derivative of this gradient with respect to the parameter \( \xi_1 \) is then

\[ \frac{\partial \nabla \cdot r_{1G}}{\partial \xi_1} = \frac{\partial}{\partial \xi_1} \left( \frac{\bar{x}_1 - \xi_1}{c r_{1G}} \right) \hat{z}_1 + \frac{\partial}{\partial \xi_1} \left( \frac{\bar{x}_3 - z_1}{c r_{1G}} \right) \hat{z}_3, \quad (A5) \]

where \( \hat{z}_1 \) and \( \hat{z}_3 \) are unit vectors along the corresponding coordinate axes. Performing the differentiations yields

\[ \frac{\partial \nabla \cdot r_{1G}}{\partial \xi_1} = \frac{(\bar{x}_3 - z_1)^2}{c r_{1G}^3} \hat{z}_1 + \frac{1}{c} \left( \frac{(\bar{x}_1 - \xi_1)(\bar{x}_3 - z_1)}{r_{1G}^3} - \frac{r_{1G}^2}{r_{1G}^3} \frac{\partial z_1}{\partial \xi_1} \right) \hat{z}_3. \quad (A6) \]

which has magnitude

\[ \left| \frac{\partial \nabla \cdot r_{1G}}{\partial \xi_1} \right| = \frac{1}{c r_{1G}^3} \left[ (\bar{x}_3 - z_1)^4 + \left( \frac{r_{1G}^2}{r_{1G}^3} \frac{\partial z_1}{\partial \xi_1} - (\bar{x}_1 - \xi_1)(\bar{x}_3 - z_1) \right)^2 \right]^{1/2}. \quad (A7) \]

In the text, the second factor is expressed as \( G \), so that
\[ \frac{\partial V_x \tau_{1G}}{\partial \xi} = \frac{G(\xi_1, \xi_3)}{c \tau_{1G}^3}, \]  

(A8)

where \( G \) is defined as in equation (36).

The expression for this factor for source continuation follows an analogous derivation, where only the source and receiver parameters are interchanged, yielding equation (57).
Modeling and imaging with the generalized screen algorithm

Jérôme H. Le Rousseau and Maarten V. de Hoop

ABSTRACT

Here, we analyze the extension of the phase-screen or split-step Fourier method (Stoffa et al., 1990) to wave propagation in media with large and rapid lateral variations, and including wider scattering angles and back-scattering. The structure of the associated algorithm is similar to the one of the phase-screen algorithm. We demonstrate the accuracy of our algorithm by modeling snapshots and imaging synthetic data in the Marmousi model.

Introduction

In realistic geological models, the heterogeneity in the medium properties is such that the phenomenon of multiple scattering is significant. We distinguish two classes of multiple scattering: one where the multiples are identified with respect to the projection of their propagation paths onto the vertical direction (depth), and one where the multiples are identified with respect to the projection of their propagation paths onto the horizontal or lateral plane. In the asymptotic framework of wavefront analysis, paths are rays. The first class of multiple scattering is associated with ‘turning rays’, the second possibly combined with the first class of multiple scattering, is associated with ‘multi-pathing’. A ray-theoretic treatment of these phenomena is not straightforward and is algorithmically rather involved.

A scattering theory that follows the ray picture but accounts for full-wave behavior has been developed by De Hoop (1996). It is based on an extension of the Bremmer coupling series to multi-dimensionally varying media. Bremmer’s method decomposes the wave field into a recursion of one-way propagation operators each using the previous wavefield as a source. Thus, the method first generates a wavefield dominated by downward propagation, then generates a ‘first’ upward propagating wavefield, then a ‘second’ downward propagating field, etc. In this manner, multiple reflections are accounted for in a controlled manner.

The propagator in the generation of this series is based on a Hamiltonian path-integral representation that accounts for not only the energy travelling along the ray but also for the transport along non-stationary paths. These path integrals reveal any possible multi pathing. The path integral generalizes Gazdag’s phase-shift migration operator: the second is valid in laterally homogeneous background media, while the first allows lateral (and vertical) medium variations.

In the path integral, ‘time’ is identified with depth, and ‘momenta’ are identified with the horizontal wave slownesses which, in the ray-theoretic limit, coincide with the horizontal components of the gradient of travel time. The (square-root) Hamiltonian is identified with vertical wave slowness which, in the ray-theoretic limit, coincides with the vertical component of the gradient of travel time (De Hoop (1996)).

The problem with the path integrals is the computational complexity of their numerical evaluation. De Hoop et al. (1998) have developed a method that reduces the computational complexity of such evaluation dramatically at the cost of adjusting the acoustics (the shape of wave fronts). The result is an algorithm that, per propagation step, is built from a multiplication - forward Fourier transform - multiplication - inverse Fourier transform, where the transform is in the horizontal directions and may be windowed. Since this algorithmic structure coincides with the one of the classical phase-screen propagator, we denote our approximations as generalized screens. We have designed a hierarchy of increasingly accurate approximations. Underlying these approximations is an expansion of the background medium simultaneously into magnitude and smoothness of variation.

The original phase-screen method was designed for multiple downward scattering of waves, the downward direction being the preferred direction of propagation. Thus it included phenomena such as focussing and defocussing. The applicability of the phase-screen method generally requires that the screen interval satisfies the following criteria: small medium variations (weak scat-
tering), laterally smooth medium variations (narrow angle scattering), and even smoother variations in the preferred direction (negligible backscattering). With the generalized screen approach, we access the accuracy of the phase-screen method, and generalize it to larger-contrast, wider-angle, and back-scattering.

Our approach accounts for the first class of multiple scattering through the generalized Bremmer series (De Hoop, 1996), and accounts for the second class of multi pathing through the generalized screen propagation (De Hoop et al., 1998).

The one-way wave propagator

Let \( x_3 \) denote depth. Consider a thin vertical slab, \([x_3', x_3]\), with thickness \( \Delta x_3 = x_3 - x_3' \). Denote horizontal or lateral coordinates by \( x_1 \) (two-dimensional configuration) or \( (x_1, x_2) \) (three-dimensional configuration). We will show the analysis for the two-dimensional configuration.

In the complex-frequency (s-\( \cdot \)) domain, for a vertical step \( \Delta x_3 \) sufficiently small, the Hamiltonian path-integral representation for the one-way wave propagator reduces to (De Hoop, 1996)

\[
\hat{g}(x_1, x_3; x_1', x_3') \simeq \int \left(s/(2\pi)^2 \exp[-is \omega \gamma_0(x_1 - x_1')] \right) \\
\exp[-s \gamma_1(x_1, x_3, \omega_1) \Delta x_3] d\omega_1,
\]

where
\[
x_3 = x_3 - \frac{1}{2}\Delta x_3.
\]

The quantities \( s \) and \( \omega_1 \) relate to circular frequency \( \omega \) and horizontal wave slowness \( p \) as \( s = i\omega \) and \( \omega_1 = -ip \). The phase factor in expression (1) contains a symbol \( \gamma_1 \) related to vertical wave slowness, which, in the high-frequency approximation, reduces to (De Hoop et al., 1998)

\[
\gamma_1(x_1, x_3, \omega_1) = \sqrt{c^{-2}(x_1, \bar{x}_3) + \alpha_1^2}.
\]

This approximation, mathematically, represents the principal part of the vertical wave slowness symbol. This principal part corresponds to the vertical component of the gradient of travel time, in accordance with the solution of the eikonal equation. Note that upon multiplying vertical wave slowness by circular frequency we obtain vertical wave number \( k_3 = \omega \gamma_1 \); horizontal wave number is given by \( k_1 = \omega p \). With these transformations of variables, in the limit of a laterally homogeneous thin slab, our propagator reduces to Gazdag's phase-shift operator (Gazdag, 1978).

The phase: a symbol expansion

To arrive at a screen-style representation of the one-way wave propagator, we will expand the symbol in Eq. (3). In the thin slab, we introduce a background medium with wave speed \( c^0 \). The background medium is assumed to be constant in the slab, but may vary from one slab to another. We express this by letting \( c^0 = c^0(x_3) \). In the background, the vertical slowness is given by

\[
\gamma_1^0(\zeta, \alpha_1) = \sqrt{(c^0)^{-2} + \alpha_1^2} = \gamma_0(x_3, \alpha_1) \quad \text{if} \quad \zeta \in [x_3', x_3].
\]

To avoid that an artificial branch point would enter the propagating-wave domain, we assume that

\[
c(x_1, \zeta) \geq c^0(x_3).
\]

Now, introduce the contrast function as

\[
a_1^2(x_1, \zeta, x_3) = c(x_1, \zeta)^{-2} - (c^0(x_3))^{-2}.
\]

Then Eq. (3) can be rewritten as

\[
\gamma_1 = \sqrt{(\gamma_0^0)^2 + a_1^2} = \gamma_0^0 \sqrt{1 + a_1^2/(\gamma_0^0)^2}.
\]

Expanding this expression into a Taylor series yields

\[
\gamma_1 = \gamma_0^0 + \gamma_1^1,
\]

with

\[
\gamma_1^1 = \sum_{l=1}^{n} a_1^l \frac{(a_1^l)^l}{(\gamma_0^0)^{2l}} + o((a_1^l)^n),
\]

representing the perturbation in vertical wave slowness. The relevant property of this series expansion is that each term factorizes in \( x_1 \) and \( \alpha_1 \).

The phase-screen approximation follows from expansion (7) by setting \( n = 1 \) and approximating \( 1/\gamma_0^0 \) by its zero-order Taylor expansion in \( \alpha_1 \) about 0 (vertical propagation). The split-step Fourier approximation (Stoffa et al., 1990) equals the phase-screen approximation subject to replacing the background wave speed from minimum value to average value in the thin slab.

The generalized screen propagator

Substituting Eq. (6) into the propagator in Eq. (1) yields

\[
\hat{g}(x_1, x_3; x_1', x_3') \simeq \int \left(s/(2\pi)^2 \exp[-is \omega \gamma_0(x_1 - x_1')] \right) \\
\exp[-s \gamma_1^0(x_3, \alpha_1) \Delta x_3] d\omega_1.
\]

In this representation we extract propagation along vertical according to

\[
\gamma_1^0(x_1, x_3, \alpha_1) = \gamma_1^0(x_1, x_3, 0)
\]

\[
+ [\gamma_1^0(x_1, x_3, \alpha_1) - \gamma_1^0(x_1, x_3, 0)] \Delta x_3
\]

and expand the exponential \( \exp[-s \gamma_1^0(x_3, \alpha_1) \Delta x_3] \) into a Taylor series. Thus

\[
\hat{g} \simeq \hat{g}^0 + \hat{g}^1,
\]
where $\tilde{g}^0(x_1, x_3; z')$ has the structure of the split-step Fourier approach,

$$\tilde{g}^0(x_1, x_3; z') = \exp[-s \gamma^0(x_1, z_3, 0) \Delta z_3]$$

$$\int (s/2\pi)^2 d\alpha_1 \exp[-is \alpha_1 (x_1 - z')]$$

$$\exp[-s \gamma^0(x_3, \alpha_1) \Delta z_3],$$

and $\tilde{g}^1(x_1, x_3; z')$ represents the generalized screen contribution,

$$\tilde{g}^1(x_1, x_3; z') = -s \Delta z_3 \sum_{l=1}^{n} a_l (a^*_l(x_1, z_3, x_1)) f$$

$$\exp[-s \gamma^0(x_1, z_3, 0) \Delta z_3]$$

$$\int (s/2\pi)^2 d\alpha_1 \exp[-is \alpha_1 (x_1 - z')]$$

$$\exp[-s \gamma^0(x_3, \alpha_1) \Delta z_3]$$

$$\left[ \frac{1}{(\gamma^0(x_3, \alpha_1))^{2l-1}} - \frac{1}{(\gamma^0(x_3, z_3))^{2l-1}} \right].$$

The order of the generalized screen approximation is $n$. The higher the order, the greater the accuracy for wide-angle propagation. The order $n = 1$ propagator simply yields a shuttling between the horizontal space and horizontal slowness domains together with a multiplication in each domain. Each additional term of the generalized screen expansion (7) requires an additional Fourier transform in space. As the computational complexity of the downward continuation in the split-step Fourier method is proportional to $2N \log_2 N$, the complexity of our nth order generalized screen approach is proportional to $(2 + n)N \log_2 N$.

The Taylor expansion of the exponential in Eq.(12) destroys the unitarity of the propagator and hence the amplitude characteristics. To restore, approximately, the amplitude behavior we apply a normalization and obtain the generalized screen propagator (GPS)

$$\tilde{g}_{GPS} = \tilde{g}^0 \mathcal{N} \left[ 1 + \frac{\tilde{g}^1}{\tilde{g}^0} \right].$$

(13)

The normalizing operator $\mathcal{N}$ is given by

$$\mathcal{N}[1 + p + iq] = \exp(iq) \left[ 1 + \frac{p}{1 + iq} \right]^{-1} \left[ 1 + \frac{p}{1 + iq} \right],$$

where $p$ and $q$ are the real and imaginary parts of a complex number. This normalization restores the amplitude behavior exactly for of a constant medium perturbation.

![Figure 1. Contour deformation.](image)

The generalized screen algorithm

Here, we discuss the generalized screen algorithm associated with the generalized screen propagation described in Eqs.(8)-(12). We denote the (one-way) wavefield by $\tilde{W}$ and we will apply the transformation of variables $(s, \alpha_1) \rightarrow (\omega, k_1)$. Note that each frequency $\omega$ can be evaluated independently, so computations can be performed in parallel. The algorithm consists of four steps.

Let the current depth be set to $z' = z$. Following Eq.(12), we introduce the intermediate field quantities $w_0, \cdots, w_n$ according to (step 1)

$$w_0(x_1) = \exp[-i \omega \Delta x_3 (c^{-1}(x_1, z_3) - (c^0(z)^{-1}))] \tilde{W}(x_1, x, \omega),$$

$$w_I(x_1) = -i \omega \Delta x_3 a_I (a^*_I(x_1, z_3, x_1)) f w_0(x_1),$$

$$I = 1, \cdots, n.$$

These intermediate fields quantities are then Fourier transformed to the horizontal-wave-number domain, $w_I(x_1) \rightarrow \hat{w}_I(k_1), I = 0, \cdots, n$ (step 2). The wavefield at depth $x_3 = z + \Delta x_3$ then follows as

$$\tilde{W}(k_1, x + \Delta x_3, \omega) = \hat{w}_0(k_1) \exp[-i \omega \Delta x_3 \gamma^0(z, k_1, \omega)]$$

$$\mathcal{N} \left[ 1 + \frac{\hat{w}_I(k_1)}{\hat{w}_0(k_1)} \left( \frac{1}{\gamma^0(z, k_1, \omega)} - c^0(z) \right) + \cdots \right]$$

$$+ \frac{\hat{w}_n(k_1)}{\hat{w}_0(k_1)} \left( \frac{1}{(\gamma^0(z, k_1, \omega))^{2n-1}} - (c^0(z))^{2n-1} \right)$$

(step 3). Finally, we carry out the inverse Fourier transform, $\tilde{W}(k_1, x + \Delta x_3, \omega) \rightarrow \tilde{W}(x_1, x + \Delta x_3, \omega)$ (step 4).

Some numerical issues

In Eq.(5), by separating $\gamma^0$, we have introduced artificial branch points in the complex $\alpha_1$-plane, viz. at $\pm(c^0)^{-1}$. These branch points appear also in the expansion (7) for the principal part of the vertical wave slowness symbol, and in expansion (12) for the generalized screen propagator. As a consequence, in Eqs.(11)-(12), the path of integration in the $\alpha_1$-plane should be chosen appropriately around the branch cuts. In a numerical scheme, in addition, we have to stay far enough away from the artificial branch points to prevent loss of precision: the
Figure 2. Wavefield snapshots associated with the various generalized screen (1st to 4th order) and the split-step Fourier approximations for a constant medium perturbation.

higher the order $n$ of our generalized screen algorithm, the higher the power in reciprocal background vertical wave slowness, the further we have to stay away from the artificial branch points. (The artificial branch points are associated with propagation in the horizontal directions.) The procedure is illustrated in Fig. 1.

In practice, we apply the contour deformation to all factors appearing in the generalized screen algorithm, except for the intermediate field quantities in the wave number domain. These are assumed to vary smoothly away from the real axis and hence are approximated by their values on the real axis.

Accuracy analysis: modeling

We illustrate the accuracy of our generalized screen algorithm for two cases: a constant medium perturbation, and the Marmousi model. The case of a constant medium perturbation provides us insight in how wave fronts evolve based on Huygens' principle. Let the background medium be characterized by a wavespeed $c_0$ that is $2/3$ of the true wavespeed. We generated impulse responses of the one-way propagator for different orders of generalized screens, as well as for the split-step Fourier method. The results are shown in Fig. 2, in which the singular support of the exact response is plotted dashed. Note that the accuracy varies with propagation angle or dip, and that this accuracy also varies with (local) medium contrast. In all the approximations, independent of order $n$, the propagation speed in the horizontal

Figure 3. a. Part of the Marmousi model. The asterisk indicates the position of the source. b. Snapshot obtained with the full acoustic wave equation (finite differences). c. Snapshot obtained with the split-step Fourier method. d. Snapshot obtained with the fourth-order generalized screen method. All snapshots at time $t = 0.95$ s.
directions is φ, which causes any approximate instantaneous wave front to fold inwards away from the true instantaneous wave front. As such, the generalized screen approximation differs, for example, from the paraxial approximation where the accuracy with dip is independent of the medium. From Fig. 2 we conjecture that, as a rule of thumb, the split-step Fourier method is accurate up to 17°; the first-order generalized screen is accurate up to 34°, the second-order up to 48°, the third-order up to 55°, and the fourth-order up to 62°. The rate of convergence with order is also apparent in Fig. 2.

The Marmousi model offers a realistic structure with complexities that can be encountered in the real world (Bourgeois et al., 1991). We computed snapshots generated by a point source located at the reservoir horizon, below a complex part of the model (anticline, unconformity, faults). In Fig. 3 the snapshots of the upgoing wavefield are shown for the fourth-order generalized screen algorithm and the split-step Fourier method. Both snapshots are compared with a finite-difference computation of the two-way wavefield. The differences between the snapshots are pronounced only in the secondary arrivals, which are associated with multi-pathing and scattering at relatively wide angles. The secondary arrivals constitute key contributions to the imaging of data generated in the Marmousi model.

Application: pre-stack depth migration

We incorporated our generalized screen algorithm in pre-stack depth, shot-gather migration. In Fig. 4 close-ups of images of the Marmousi data are shown. The differences between the images obtained with the split-step Fourier method and the second-order generalized screen algorithm, though subtle, are apparent at the unconformity and the reservoir. In particular, the image of the top of the reservoir obtained with the generalized screen algorithm follows the model grid wise. If the waves in the model would have been scattered at larger angles, the results would have been more pronounced.

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References


Inversion of azimuthally dependent NMO velocity in transversely isotropic media with a tilted axis of symmetry

Vladimir Grechka & Ilya Tsvankin

ABSTRACT

Just as the transversely isotropic model with a vertical symmetry axis (VTI media) is typical for horizontally layered sediments, transverse isotropy with a tilted symmetry axis (TTI) describes dipping TI layers, such as upfilled shale beds near salt domes. Here, we show that all parameters of TTI media responsible for P-wave velocity, including the tilt and azimuth of the symmetry axis, can be obtained from azimuthally-varying P-wave normal-moveout (NMO) velocities measured for two reflectors with different dips and/or azimuths (one of the reflectors can be horizontal). The shear-wave velocity $V_{50}$ in the symmetry direction, which has negligible influence on P-wave kinematic signatures, can be found only from the moveout of SV-waves.

Using the exact NMO equation, we examine the propagation of errors in observed moveout velocities into estimated values of the anisotropic parameters and establish the necessary conditions for a stable inversion procedure. Since the azimuthal variation of the NMO velocity has been shown to be elliptical, each reflection event provides us with up to three constraints on the model parameters. P-wave NMO velocity in TTI media is controlled by the velocity $V_{p0}$ in the symmetry direction, Thomsen’s anisotropic coefficients $\epsilon$ and $\delta$, and the orientation (tilt $\nu$ and azimuth $\beta$) of the symmetry axis. Generally, all five parameters can be obtained from two P-wave NMO ellipses, but the feasibility of the moveout inversion strongly depends on the tilt $\nu$. If the symmetry axis is close to vertical (small $\nu$), the P-wave NMO ellipse is largely governed by the NMO velocity from a horizontal reflector $V_{pnm}(0)$ and the “anellipticity” coefficient $\eta$, which is close to the difference between $\epsilon$ and $\delta$; in this case, the parameters $\epsilon$ and $\delta$ cannot be determined separately. Only if the tilt exceeds 30-40° (e.g., the symmetry axis can be horizontal), it is possible to resolve all P-wave moveout parameters individually. Another condition required for a stable parameter estimation is that the medium be sufficiently different from elliptical (i.e., $\epsilon$ cannot be close to $\delta$). This limitation, however, can be overcome by including the SV-wave NMO ellipse from a horizontal reflector in the inversion procedure.

Although most of the analysis is carried out for a single TTI layer, we extend the inversion algorithm to vertically heterogeneous TTI media above a dipping reflector using the generalized Dix equation. A synthetic example for a strongly anisotropic, stratified TTI medium demonstrates a high accuracy of the inversion for models with sufficient tilt of the symmetry axis. The results of moveout inversion in TTI media are sufficient for performing anisotropic depth imaging, as opposed to time imaging for vertical transverse isotropy.

Introduction

Transverse isotropy (TI) is the most common anisotropic model of the subsurface typical for massive shale formations or thin-bed sedimentary sequences. If sediments are horizontally layered, the symmetry axis of the corresponding TI medium is vertical (the so-called VTI model, or vertical transverse isotropy). For dipping TI layers, often found in overthrust areas or near flanks of
salt domes and volcanic intrusions, the symmetry axis usually remains orthogonal to the layering and, therefore, becomes tilted (TTI media). TI media with a horizontal (HTI) or near-horizontal symmetry axis are often associated with vertical or steeply dipping fracture systems (e.g., Thomsen, 1988).

Reflection traveltime, in general, and normal-moveout (NMO) velocity, in particular, provide the most reliable information about the anisotropic parameters of TI media. For vertical transverse isotropy, as shown by Alkhaliifah and Tsvankin (1995), P-wave NMO velocity in the dip plane of the reflector (and time-domain processing as a whole) are controlled by only two parameter combinations: the zero-dip NMO velocity (i.e., the NMO velocity for a horizontal reflector)

$$V_{P_{nmo}}(0) = V_{P_0} \sqrt{1 + 2\delta},$$

and the anisotropic parameter

$$\eta = \frac{\epsilon - \delta}{1 + 2\delta}.$$

Here $V_{P_0}$ is the P-wave velocity in the symmetry (vertical) direction, and $\epsilon$ and $\delta$ are Thomsen’s (1986) anisotropic coefficients; $V_{P_0}$, $\epsilon$ and $\delta$ are responsible for all P-wave kinematic signatures in VTI media (Tsvankin, 1996). The parameter $\eta$ goes to zero in elliptically anisotropic media ($\epsilon = \delta$) and, therefore, describes the “anellipticity” of the P-wave slowness surface. For elliptical anisotropy, normal-moveout velocity is the same function of the ray parameter $p$ and zero-dip NMO velocity as in isotropic media. Provided $V_{P_{nmo}}(0)$ has been obtained using semblance analysis of horizontal events, $\eta$ can be estimated from the NMO velocity for a dipping reflector.

The result of Alkhaliifah and Tsvankin (1995) was based on the 2-D NMO equation of Tsvankin (1995) restricted to a common-midpoint (CMP) line in the dip plane of the reflector. Grechka and Tsvankin (1996) developed a more general 3-D NMO equation valid for CMP reflections recorded over arbitrary anisotropic heterogeneous media. They showed that the azimuthal variation of NMO velocity is described by an ellipse in the horizontal plane and is controlled by the spatial derivatives of the ray parameter at the CMP location. Applying their equation to VTI media, they proved that P-wave NMO velocity is fully governed by $V_{P_{nmo}}(0)$ and $\eta$ for arbitrary orientations of the CMP line and reflector strike. Hence, for vertical transverse isotropy the parameters $V_{P_0}$, $\epsilon$ and $\delta$ cannot be resolved individually from P-wave NMO data.

For TI media with a horizontal symmetry axis (HTI), the P-wave kinematic parameters include the azimuth $\beta$ of the symmetry axis, in addition to $V_{P_0}$, $\epsilon$, and $\delta$. ($V_{P_0}$ remains the velocity in the symmetry direction, which is horizontal in HTI media, while $\epsilon$ and $\delta$ are defined with respect to the symmetry axis.) Despite the increase in the number of unknowns, all four moveout parameters can be estimated from azimuthally dependent P-wave NMO velocities measured for a horizontal and dipping reflector (Tsvankin, 1997a; Contreras et al., 1997). Note that moveout inversion in HTI media provides enough information to perform depth processing, in contrast to time processing in VTI media.

The TI model with a tilted symmetry axis can be considered as intermediate between VTI and HTI. This does not mean, however, that the moveout inversion in HTI media can be understood just by examining the results for the two extreme orientations of the symmetry axis. Indeed, the tilt of the symmetry axis represents an extra parameter to be recovered from moveout data. Tsvankin (1997b) performed an analysis of P-wave NMO velocity in the vertical symmetry plane of TTI media that contains the symmetry axis. To make the problem two-dimensional, he also assumed that the symmetry axis is confined to the dip plane of the reflector. He concluded that the NMO velocity is rather sensitive to the tilt $\nu$ and is not fully controlled by $\eta$ (for fixed $\nu$). Also, Tsvankin’s (1997b) results show that the dip dependence of NMO velocity in a single vertical symmetry plane is not sufficient to resolve the medium parameters.

Here, we carry out a 3-D (azimuthal) analysis of normal moveout in TI media with arbitrary tilt and azimuth of the symmetry axis. By employing the 3-D NMO equation of Grechka and Tsvankin (1996), we develop an inversion procedure to obtain all five relevant parameters ($V_{P_0}$, $\epsilon$, $\delta$, $\beta$ and $\nu$) from P-wave NMO ellipses for two different reflection events. We show that for noelliptical media and the practically important case of a horizontal and a dipping reflector, the inversion procedure becomes stable if reflector dip reaches at least 30° and the tilt of the symmetry axis exceeds 30–40°. The generalized Dix differentiation (Tsvankin et al., 1997) allows us to extend the parameter-estimation methodology to vertically heterogeneous TTI media. To estimate the shear-wave velocity $V_{S_0}$ in the symmetry direction and increase the overall stability of the inversion procedure, the P-wave NMO velocities for horizontal and dipping events can be supplemented with the SV-wave NMO ellipse from a horizontal reflector.

**Basic theory of azimuthally varying NMO velocity**

**Equation of the NMO ellipse**

Here, we give an overview of the analytic representation of NMO velocity in anisotropic media. In conventional acquisition design, the maximum source-receiver offset
Inversion of NMO velocity in TTI media

is close to the distance between the common midpoint (CMP) and the reflector. For these moderate offsets \( z \), reflection moveout \( t \) is usually close to a hyperbola (e.g., Taner and Koehler, 1969; Tsvankin and Thomsen, 1994),

\[
t^2(z, \alpha) \approx t_0^2 + \frac{z^2}{V_{\text{nmo}}(\alpha)^2},
\]

where \( t_0 \) is the two-way zero-offset traveltime and \( V_{\text{nmo}}(\alpha) \) is the normal-moveout velocity, which generally depends on the azimuth \( \alpha \) of the CMP line. As shown by Grechka and Tsvankin (1996), NMO velocity of any pure mode can be expressed as

\[
V_{\text{nmo}}^2(\alpha) = W_{11} \cos^2 \alpha + 2 W_{12} \sin \alpha \cos \alpha + W_{22} \sin^2 \alpha,
\]

where \( W_{ij} = \partial p_i / \partial x_j \) (\( i, j = 1, 2 \)), \( \theta_0 = t_0 / 2 \) is the one-way zero-offset traveltime, \( p_i \) are the horizontal components of the slowness vector for one-way rays emanating from the zero-offset reflection point, and \( x_i \) are the horizontal spatial coordinates; the derivatives are evaluated at the common midpoint. (For brevity, below we will not show the azimuth \( \alpha \) explicitly as an argument of the NMO-velocity function; the value of \( \alpha \) in each case will be clear from the context.)

Equation (4) is valid for pure modes in arbitrary anisotropic heterogeneous media as long as the reflection traveltime can be expanded in a Taylor series in \( x_i \) near the CMP location. Unless reflection traveltime decreases with offset in a certain direction leading to so-called "reverse moveout" (i.e., there is at least one azimuth \( \alpha \) for which \( V_{\text{nmo}}^2(\alpha) < 0 \)), the symmetric matrix \( W \) is positive definite, and equation (4) describes an ellipse. Clearly, the NMO ellipse is fully determined by the three elements of the matrix \( W \).

Homogeneous arbitrary anisotropic layer

For a homogeneous anisotropic layer above a dipping reflector, the matrix \( W \) can be represented as the following function of the slowness components \( p_i \) (Tsvankin et al., 1997):

\[
W = \begin{pmatrix}
p_{01} q_{11} + p_{02} q_{22} - q_{12} & q_{12} \\
q_{12} & q_{11} - q_{12} & -q_{12} & q_{11}
\end{pmatrix}
\]

(5)

where \( q \equiv q(p_1, p_2) \equiv p_3 \) denotes the vertical component of the slowness vector, \( q_i \equiv \partial q / \partial p_i \), and \( q_{ij} \equiv \partial^2 q / \partial p_i \partial p_j \); the horizontal slowness components \( p_1 \) and \( p_2 \) and all derivatives are evaluated for the zero-offset ray.

Note that \( p_1 \) and \( p_2 \) control the reflection slope on the zero-offset (stacked) section and can be obtained directly (unlike reflector dip) from reflection data. The vertical slowness \( q = q(p_1, p_2) \) in a known anisotropic model can be found by solving the Christoffel equation for given values of \( p_1 \) and \( p_2 \). Then, implicit differentiation of Christoffel equation yields the derivatives \( q_i 

and \( q_{ij} \) (Tsvankin et al., 1997). Therefore, equation (5) provides a simple and numerically efficient recipe for obtaining NMO velocity of pure modes in a layer with any anisotropic symmetry.

NMO velocity in TTI media: special cases

Normal moveout for steep reflectors

Generation of a specular zero-offset reflection requires that some portion of the incident wavefront be parallel to the reflecting interface or, in other words, that the slowness (or phase-velocity) vector for this segment be orthogonal to the reflector. If the wavefront of the downgoing wave is symmetric with respect to the horizontal plane, specular reflections in a homogeneous layer exist for the whole range of dips from 0° to 90° (Tsvankin, 1997b). Obviously, this condition is always satisfied in models with a horizontal symmetry plane, such as vertical and horizontal transverse isotropy. A tilt of the symmetry axis, however, makes the wavefront asymmetric with respect to horizontal, and its cross-sections in some azimuthal directions may contain only a limited range of phase angles. As illustrated in Figure 1, the phase (wavefront) vector corresponding to a horizontal ray may point down, and the reflection traveltime from the non-vertical interface normal to the phase vector will not change with offset in the ray direction (i.e., the NMO velocity in this direction becomes infinite). Even steeper interfaces reflect all incident rays downward and, therefore, become invisible on surface seismic data. If the medium is heterogeneous and velocity increases with depth, the "missing" dips may produce reflected arrivals at the surface, but these reflections would represent turning rays.

Tsvankin (1997b) studied the existence of specular reflections for the "2-D" TTI model with the symmetry axis confined to the dip plane of the reflector. He showed that if the symmetry axis is tilted towards the reflector, the maximum phase (wavefront) angle in the dip plane typically is smaller than 90°, and steeper interfaces do not generate reflected arrivals at the surface. In contrast, for symmetry axis tilted away from the reflector, it is possible to record reflections even from interfaces with dips exceeding 90° (overhang structures) with rays that are not turning.

Here, we treat a more general situation of the symmetry axis making an arbitrary azimuth with the dip plane of the reflector. Figure 2 shows the dependence of the maximum dip that generates a zero-offset reflected ray on the azimuth of the symmetry axis for a particular TTI model. For the maximum dip, the zero-offset ray is horizontal, and the NMO velocity in the ray direction is infinite, which means that the NMO ellipse degenerates into two parallel straight lines (Grechka and Tsvankin, 1996). If the symmetry axis is confined to the dip plane...
and points towards the reflector ($\beta = 0$), the maximum dip is just 76° for this model, which agrees with the result of Tsvankin (1997b, Figure 3). As the symmetry axis deviates from the dip plane, the range of “missing” dips becomes more narrow and vanishes altogether for $\beta = 90^\circ$, when the horizontal projection of the symmetry axis coincides with the reflector strike. In this case, the horizontal ray in the dip direction belongs to the isotropy plane and, therefore, is orthogonal to the wavefront. As a result, the segment of the wavefront propagating horizontally in the dip plane should generate a zero-offset reflection from a vertical interface, and the maximum visible dip is indeed equal to $90^\circ$. For azimuth $\beta$ larger than $90^\circ$, the maximum dip exceeds $90^\circ$, so it is possible to record surface reflections from overhang structures. For the symmetry axis back in the dip plane but, now tilted away from the reflector ($\beta = 180^\circ$), the maximum dip reaches $103^\circ$.

It should be emphasized that it does not take a large tilt of the symmetry axis for the maximum dip to deviate considerably from $90^\circ$ (in Figure 2, $\nu = 25^\circ$). As discussed by Tsvankin (1997b), the dependence of the maximum dip on $\nu$ is not symmetric with respect to $\nu = 45^\circ$; the most pronounced variations in the maximum dip correspond to tilts well below $45^\circ$.

**Elliptical anisotropy**

A transversely isotropic model with any orientation of the symmetry axis becomes elliptically anisotropic if $\epsilon = \delta$. For elliptical anisotropy, the $P$-wave slowness surface and wavefront (group-velocity surface) have an ellipsoidal shape, while the $SV$-wave velocity is equal to $V_{50}$ in all directions (i.e., is independent of angle). Although the condition $\epsilon = \delta$ is seldom satisfied for subsurface formations (Thomsen, 1986), elliptical models require a special treatment in the inversion procedure and, therefore, have to be considered separately.

Tsvankin (1997b) showed that for elliptical anisotropy the dip-dependence of NMO velocity, expressed through the ray parameter $p$ of the zero-offset ray, is exactly the same as in isotropic media:

$$V_{nmo}(p) = \frac{V_{nmo}(0)}{\sqrt{1 - p^2V_{0}(0)}};$$

(6)

$V_{nmo}(0)$ corresponds to a horizontal reflector. This result, however, is limited to the dip plane of the reflector, under the assumption that this plane contains the symmetry axis of the medium.

Analysis of equation (5) shows that in elliptical media the dependence of the $P$-wave NMO velocity on $p_1$ and $p_2$ remains isotropic for arbitrary orientations of the symmetry axis and reflector normal, and any azimuth of the CMP line. The $P$-wave phase-velocity function for elliptical anisotropy is given by

$$V_p(\theta) = V_{p0} \sqrt{1 + 2\delta \sin^2 \theta},$$

(7)

where $\theta$ is the angle between the phase-velocity (slowness) vector and the symmetry axis. Representing $\theta$ through the slowness components $p_1$, $p_2$ and $q$ and the symmetry-axis orientation (defined by the angles $\nu$ and $\beta$) yields

$$\cos \theta = \frac{V_p}{(p_1 \sin \nu \cos \beta + p_2 \sin \nu \sin \beta + q \cos \nu)}.$$
Combining equations (7) and (8) and taking into account that \( V_F^2 = (p_1^2 + p_2^2 + q^2)^{-1} \) leads to a quadratic equation for \( q \) as a function of \( p_1 \) and \( p_2 \). Substituting \( q(p_1, p_2) \) into equation (5), we find the following equation for the matrix \( W \) in elliptically anisotropic media:

\[
W_{ij}(p_1, p_2) = W_{ij}(0, 0) - p_ip_j.
\]  

(9)

It is easy to verify that equation (9) is identical to the dependence of \( W_{ij} \) on \( p_1 \) and \( p_2 \) for isotropic media. Hence, the influence of elliptical anisotropy on NMO velocity in equation (9) is hidden in the NMO ellipse for horizontal events \( W_{ij}(0, 0) \). This result has serious implications for the inversion procedure discussed below. Equation (9), as with any isotropic kinematic relationship, can be also used for \( SV \)-waves in elliptical media.

The isotropic form of the function \( W_{ij}(p_1, p_2) \) also means that conventional dip-moveout algorithms developed for isotropic media are valid for elliptical anisotropy with any orientation of the axes. Since reflection moveout in elliptical media is purely hyperbolic (Uren et al., 1990), all isotropic time-related processing methods (NMO and DMO corrections, pre-stack and post-stack time migration) can be used for elliptical anisotropy without any modification. These conclusions, as well as equation (9), always apply to \( SH \)-waves because their propagation is any TI medium is governed by the elliptical dependencies.

**TI media with the symmetry axis perpendicular to the reflector**

Before considering transversely isotropic media with arbitrary tilt of the symmetry axis, it is useful to discuss the important special case of the symmetry axis orthogonal to the dipping reflector. Such a model is typical for TI formations with the symmetry axis perpendicular to the layering (e.g., shales) that were tilted due to tectonic processes after sedimentation. The simplicity of this model makes it possible to obtain concise exact expressions for the NMO ellipse in terms of the medium parameters.

Since the dip plane of the reflector contains the symmetry axis, it becomes a vertical symmetry plane for the whole model and determines the orientation of the NMO ellipse. The semi-axis of the ellipse that lies in the dip plane (the “dip component” of the NMO velocity) is given by (Tsvankin, 1995)

\[
V_{nmo}^{(1)}(\phi) = \frac{V_{nmo}(0)}{\cos \phi}.
\]  

(10)

Here \( \phi \) is the reflector dip and \( V_{nmo}(0) \) is the NMO velocity from a horizontal reflector obtained under the assumption that the symmetry axis remains perpendicular to the reflector (i.e., it is vertical for a horizontal reflector). Equation (10) is identical to the cosine-of-dip dependence of normal-moveout velocity in isotropic media (Levin, 1971), with \( V_{nmo}(0) \) replacing the medium velocity.

Introducing the ray parameter \( p = \sqrt{p_1^2 + p_2^2} \) into equation (10), we obtain

\[
V_{nmo}^{(1)}(p) = \frac{V_{nmo}(0)}{\sqrt{1 - p^2 V_0^2}},
\]  

(11)

where \( V_0 = \sin \phi/p \) is the symmetry-direction velocity of the mode under consideration (it may be either a \( P \)- or an \( S \)-wave). Since in anisotropic media \( V_{nmo}(0) \) and \( V_0 \) generally are different, equation (11) does not coincide with the corresponding isotropic expression. It is interesting to examine whether the strike component of the NMO velocity \( V_{nmo}^{(2)} \) provides any additional information about the medium. Straightforward, but tedious, algebraic transformations of equation (5) give

\[
V_{nmo}^{(2)} = V_{nmo}(0).
\]  

(12)

For \( P \)-waves this result can be more easily obtained in the weak-anisotropy approximation (discussed in detail below) by substituting the relation \( \tan \nu = \tan \phi = p_1/p_2 \) into equation (A7) for \( W_{22} \). Therefore, \( V_{nmo}^{(2)} \) is simply equal to the zero-dip NMO velocity and is completely independent of dip, comparable to the result of Levin (1971) for isotropic media.

Combining the two semi-axes of the NMO ellipse [equations (11) and (12)] and obtaining \( p \) from the zero-offset section, we can find \( V_{nmo}(0) \) and the symmetry-direction velocity \( V_0 \) (if the dip is not too mild). Since for the \( P \)-wave \( V_{Pnmo}(0) = V_P \sqrt{1 + 2\delta} \) [equation (1)], the \( P \)-wave NMO ellipse for a dipping event yields the anisotropic parameter \( \delta \) in addition to the symmetry-direction velocity \( V_P \). Likewise, the NMO ellipse of the \( SV \)-wave provides the symmetry-direction velocity \( V_S \) and the anisotropic parameter \( \sigma = [V_{S0}^2/V_{S0}^2(\epsilon - \delta)] \).

Both parameters can be determined even from 2-D data acquired in the dip plane if the zero-dip velocity \( V_{Pnmo}(0) \) was recovered from a horizontal event. Note that for the NMO velocity from a horizontal reflector to be equal to \( V_{nmo}(0) \) in equation (11), the symmetry axis for the horizontal event should be vertical. We can imagine, for instance, that the reflecting interface may have a gradually changing slope with the symmetry axis remaining orthogonal to the reflector. In the inversion procedure described below, however, we fix the orientation of the symmetry axis and use the NMO ellipses for horizontal and dipping events to obtain the medium parameters.
Figure 3. Quasi-circular P-wave NMO ellipses in a horizontal TTI layer. Solid – the ellipse computed using equation (5); dotted – the ellipse reconstructed using the hyperbolic equation (3) from ray-traced traveltime computed along four CMP lines at azimuths 0°, 45°, 90°, and 135° (the azimuth is shown around the plot); the spreadlength is equal to the thickness of the layer. The relevant medium parameters are \( v_P = 2.0 \text{ km/s} \), \( \epsilon = 0.2 \), \( \delta = 0.1 \), \( \beta = 30° \), and \( \nu = 45° \).

P-wave NMO ellipse in weakly anisotropic TTI media

Horizontal layer

Although equations (4) and (5) give a concise representation of azimuthally-varying NMO velocity, the dependence of \( V_{nmo} \) on the model parameters is hidden in the components of the slowness vector. To understand the influence of the axis orientation and anisotropic parameters on the P-wave normal moveout, we linearize equation (5) with respect to \( \epsilon \) and \( \delta \) assuming that \( |\epsilon| \ll 1 \) and \( |\delta| \ll 1 \) (Appendix A). The final expressions (A7) and (A8) correspond to the coordinate system in which the azimuth of the symmetry axis \( \beta = 0 \) (i.e., the symmetry axis is in the \( [x_1, x_3] \)-plane, see Figure A1). Note that the shear-wave velocity \( V_S \) in the symmetry direction does not appear in the linearized NMO equations because all P-wave kinematic signatures in the weak-anisotropy approximation are independent of \( V_S \) (Tsvankin, 1996).

For a horizontal reflector \( (p_1 = p_2 = 0) \) equations (A7) and (A8) can be simplified to

\[
W_{11} = \frac{1}{V^2_P} \left[ 1 + 2\delta + 2\epsilon \sin^2 \nu - 14 (\epsilon - \delta) \sin^2 \nu \cos^2 \nu \right],
\]

\[
W_{12} = 0,
\]

\[
W_{22} = \frac{1}{V^2_P} \left[ 1 - 2\delta - 2 (\epsilon - \delta) \sin^2 \nu (1 + \cos^2 \nu) \right].
\]

Since \( W_{12} = 0 \), the semi-axes of the P-wave NMO ellipse for a horizontal reflector are parallel to the coordinate axes \( x_1 \) and \( x_2 \). This result could be expected because the \( [x_1, x_3] \)-plane contains the symmetry axis and, therefore, represents a symmetry plane of the horizontal TTI layer. In the case of vanishing \( W_{12} \), the quantities \( W_{11} \) and \( W_{22} \) are reciprocal to the squared NMO velocities along the semi-axes of the ellipse (Grechka and Tsvankin, 1996):

\[
W_{ii} = \frac{1}{[V^2_{nmo}(0)]^2}, \quad (i = 1, 2).
\]

The velocity \( V^2_{nmo}(0) \) defined by equations (13) and (16) corresponds to the CMP line in the vertical plane that contains the symmetry axis. Equations (13) and (16) are equivalent to the result of Tsvankin (1997b) [his equation (23)] who studied normal moveout only in this vertical symmetry plane. The second semi-axis of the NMO ellipse, \( V^2_{nmo}(0) \) [equations (15) and (16)], is generally different from \( V^2_{nmo}(0) \). Therefore, equations (13) and (15) provide us with three constraints for the layer parameters: the orientation of the NMO ellipse depends on the azimuth \( \beta \) of the symmetry axis, while the values of the semi-axes give two more equations for all five parameters. Note that the azimuth of the symmetry axis cannot be unambiguously found from the NMO ellipse from a horizontal reflector, because \( \beta \) can be equal to the azimuth of either semi-major or semi-minor axis depending on the medium parameters. For horizontal transverse isotropy \( (\nu = 0) \) due to parallel penny-shaped cracks, the symmetry axis typically coincides with the semi-major axis of the P-wave NMO ellipse (Tsvankin, 1997a).

The number of equations, however, reduces to two if the NMO ellipse degenerates into a circle, i.e.,

\[
V^2_{nmo}(0) = V^2_{nmo}(0).
\]

Obviously, for VTI media \( (\nu = 0) \) NMO velocity is always independent of azimuth, and \( V^2_{nmo}(0) = V^2_{nmo}(0) = V^2_{nmo}(0) = V^2_P \sqrt{1 + 2\delta} \) [equation (1)]. It is interesting that condition (17) can also be satisfied for azimuthally anisotropic media with a tilted symmetry axis. Substituting equations (13), (15), and (16) into equation (17) and assuming \( \nu \neq 0 \), we find

\[
2\epsilon - \delta - 6 (\epsilon - \delta) \cos^2 \nu = 0.
\]

For instance, if the symmetry axis is horizontal \( (\cos \nu = 0) \), relation (18) gives

\[
2\epsilon - \delta = 0.
\]
which represents a known condition for the P-wave NMO ellipse in weakly anisotropic HTI media to degenerate into a circle. The ellipticity of P-wave NMO velocity in HTI media is governed by the coefficient $\delta^{(V)} \approx 2\epsilon - \delta$ (Tsvankin, 1997a), which goes to zero if equation (19) is satisfied.

An example of azimuthally-independent NMO-velocity for TTI media with an intermediate tilt angle of 45° is shown in Figure 3. Equation (18) with $\nu = 45^\circ$ yields $\epsilon - 2\delta = 0$, so for the model from Figure 3 the semi-axes of the NMO ellipse should be equal to each other. Although equation (18) is an approximation valid only for weakly anisotropic media, the exact NMO ellipse calculated using equation (5) is, indeed, almost circular.

Another important point illustrated by Figure 3 is that the theoretical NMO ellipse (solid) is close to the ellipse reconstructed from ray-traced traveltimes for spreadlength equal to the reflector depth (dotted). The difference between the two ellipses, caused by the deviation of the moveout curve from the analytic hyperbola, is limited by 1.1%. Clearly, the influence of nonhyperbolic moveout on P-wave moveout velocity can be ignored, as long as the maximum offset does not exceed reflector depth. The same conclusion was drawn by Tsvankin (1997b) in his study of the 2-D moveout problem in TTI media.

Dipping TTI layer

The P-wave NMO ellipse from a horizontal reflector provides either two or three constraints on the parameters of the TTI layer, depending on whether or not the ellipse degenerates into a circle. To resolve all five relevant layer parameters ($V_{p0}$, $\epsilon$, $\delta$, $\beta$, and $\nu$), we suggest also including the NMO ellipse from a dipping reflector (i.e., three more equations) in the inversion procedure.

Although there seems to be enough equations to recover all unknowns, in certain situations the parameters cannot be resolved individually. The weak-anisotropy approximation of the P-wave NMO ellipse [equations (A7) and (A8)] helps to identify the trade-offs between model parameters and study the stability of the inversion procedure. For instance, equation (A7) shows that the anisotropic coefficients in all dip-dependent terms appear only as the difference $\epsilon - \delta$. Still, $\epsilon$ and $\delta$ can be resolved individually using the NMO ellipse for horizontal events [equations (13) and (15)], unless the tilt is relatively small. For mild tilts $\nu$ the medium approaches VTI, and the NMO velocity from a horizontal reflector yields a single quantity ($V_{p0} \sqrt{1 + 2\delta}$), while the dip dependence of NMO velocity is controlled by $\eta \approx \epsilon - \delta$ (Alkalifah and Tsvankin, 1998). Thus, $\epsilon$ and $\delta$ can be estimated separately, but only for a sufficiently large tilt of the symmetry axis.

A similar constraint applies to the reflector dip, which should not be too mild for the dip-dependence of NMO velocity to be sensitive enough to the anisotropic parameters. [The anisotropic dip terms $\tilde{W}_{ij}$ in the weak-anisotropy approximation (A8) can be used to make appropriate estimates.] On the whole, determination of the five relevant parameters of TTI media from the P-wave NMO ellipses for a horizontal and a dipping reflector should be feasible if both the reflector dip and the tilt of the symmetry axis are not "too small." In the next section, we quantify this statement by performing the actual inversion based on the exact NMO equation.

Parameter estimation in a TTI layer using P waves

Here, we present results of the numerical inversion of P-wave NMO velocity for a single homogeneous TTI layer. The NMO ellipses for a horizontal ($\tilde{W}_{ij}^{phor}$) and a dipping ($\tilde{W}_{ij}^{dip}$) event are inverted for five medium parameters – the velocity $V_{p0}$ in the symmetry direction, Thomsen's anisotropic coefficients $\epsilon$ and $\delta$, the tilt $\nu$ and the azimuth $\beta$ of the symmetry axis. The parameters are found by minimizing the least-squares objective function

$$F_P = \sum_{i,j=1}^{2} \left( W_{ij}^{phor} - \tilde{W}_{ij}^{phor} \right)^2$$

$$+ \sum_{i,j=1}^{2} \left( W_{ij}^{dip} - \tilde{W}_{ij}^{dip} \right)^2 ,$$

(20)

where the matrices $W_{ij}^{phor}$ and $W_{ij}^{dip}$ are calculated using the exact equation (5) for a particular set of parameters. The weights

$$w_h = 4 \left( \tilde{W}_{11}^{phor} + \tilde{W}_{22}^{phor} \right)^{-2}$$

and

$$w_d = 4 \left( \tilde{W}_{11}^{dip} + \tilde{W}_{22}^{dip} \right)^{-2}$$

(21)

(22)

are used to equalize the contributions of the two ellipses in equation (20). The minimization of the objective function was carried out using the simplex method. Although in principle the nonlinear system (20) might have multiple solutions, extensive numerical testing shows that for error-free input data and a realistic starting model (i.e., $|\epsilon| < 1$, $|\delta| < 1$) the inversion algorithm always converges towards the correct set of parameters.

For numerical testing, we computed the matrices $\tilde{W}_{ij}^{phor}$ and $\tilde{W}_{ij}^{dip}$ from equation (5) and added Gaussian noise with a variance of 2% to simulate errors in velocity picking. Figure 4 shows the results of parameter estimation for TTI layers with different tilts $\nu$ of the symmetry axis. Each dot on the plots corresponds to the inversion result for a particular realization of random errors in the NMO ellipses; the inversion procedure
for each model was repeated 200 times. The mean values of all parameters were recovered accurately despite the fact that we used an incorrect shear-wave velocity $V_{S0} = V_{P0}/2$ to perform the inversion. Hence, in agreement with the results of Tsvankin (1996, 1997b) and the weak-anisotropy approximations (A7) and (A8), $V_{S0}$ has a negligible influence on $P$-wave moveout in TI media.

Figure 4 also confirms the conclusion of the previ-
Inversion of NMO velocity in TTI media

Figure 5. Same as Figure 4, but for variable reflector dip. The parameters $V_{P0}$, $\epsilon$, $\delta$, $\beta$ and the reflector azimuth are the same as those in Figure 4; the tilt of the symmetry axis $\nu = 60^\circ$. The reflector dip $\phi$ is equal to 20$^\circ$ in (a) and (b), 30$^\circ$ in (c) and (d), and 40$^\circ$ in (e) and (f). The solid line on plot (a) corresponds to the correct value of $\eta = 0.167$.

ous section that the inversion becomes more stable with increasing tilt of the symmetry axis. While the inverted values of $\epsilon$ and $\delta$ are close to the correct ones for the tilts $\nu = 60^\circ$ and $\nu = 80^\circ$ (the variance $\epsilon$ and $\delta$ in Figures 4e and 4g is less than 0.02), the deviations visibly increase for $\nu = 40^\circ$ (Figure 4c) and, especially, for $\nu = 20^\circ$ (Figure 4a). The inversion results for $\nu < 30 - 40^\circ$ are so sensitive to errors in the input parameters that $P$-wave NMO velocity cannot be used to resolve $\epsilon$ and $\delta$ individually. Nonetheless, the inverted values of $\epsilon$ and $\delta$ are tightly clustered near the line corresponding to the correct value of $\eta$ [see equation (2)]. Therefore, $\eta$ is the only combination of the anisotropic parameters constrained by the $P$-wave NMO velocity for mild tilts $\nu$, which agrees with the known result of Alkhalifah and Tsvankin (1995) for vertical transverse isotropy.

The accuracy in the $P$-wave velocity $V_{P0}$ in the symmetry direction (not shown in Figure 4) exhibits a similar dependence on the tilt, with the scatter in the inversion results monotonically increasing with decreasing $\nu$. The variance in $V_{P0}$ changes from a small value of 0.01 km/s (0.5% of the correct $V_{P0}$) for a near-horizontal symmetry axis ($\nu = 80^\circ$) to 0.23 km/s (11.5%) for a tilt of 20$^\circ$. For a near-vertical symmetry axis, $P$-wave normal moveout can be used to obtain the zero-dip NMO velocity ($V_{P0}\sqrt{1+2\delta}$), but not $V_{P0}$ or $\delta$ separately (see the analysis for VTI media in Alkhalifah and Tsvankin, 1995).

Another interesting observation, this time from the right-column plots in Figure 4, is that the azimuth $\beta$ of the symmetry axis is well-constrained for all four tilts $\nu$, while the variance in the tilt itself is more significant.
Overall, the accuracy in both $\beta$ and $\nu$ is quite satisfactory for a wide range of tilts; the only exception is "quasi-VTI" models with small $\nu < 10^\circ$ (not shown here) for which the orientation of the symmetry axis does not have much influence on normal moveout.

The inversion results in Figure 4 were obtained for a rather favorable (large) reflector dip $\phi = 60^\circ$. As expected, the inversion becomes less stable as the dipping reflector tilts toward horizontal (Figure 5). Each row of plots in Figure 5 should be compared with Figures 4e,f, which were generated for the same $\nu = 60^\circ$. While the scatter in the inverted parameters for the dips $\phi = 40^\circ$ (Figures 5e,f) and $\phi = 60^\circ$ (Figures 4e,f) is comparable, the results deteriorate for a smaller $\phi = 30^\circ$ (Figures 5c,d) and become quite unstable for $\phi = 20^\circ$ (Figures 5a,b). The inverted values of $\epsilon$ and $\delta$ in Figure 5a still cluster around a straight line, but this line is no longer described by the actual parameter $\eta$. It is interesting that the scatter in the parameters responsible for the symmetry-axis orientation is much more sensitive to the reflector dip (the right column of plots in Figure 5) than to the tilt of the axis (Figure 4). The variance in the symmetry-direction velocity $V_{p0}$ (not shown) also increases from 1.4% for $\phi = 40^\circ$ to 2.8% for $\phi = 30^\circ$ to more than 7% for $\phi = 20^\circ$.

Therefore, the dip should reach at least $30^\circ$ for the inversion to be reasonably stable. Similar values of the minimum dip were given by Alkhalifah and Tsvankin (1995), who used the dip dependence of NMO velocity in the inversion for the anisotropic parameter $\eta$ in VTI media. On the other hand, the reflector should not be too steep, because, for dips approaching $90^\circ$, specular reflections and the NMO ellipse may not exist at all (see the discussion above).

We conclude that the moveout inversion based on the P-wave NMO ellipses is not possible for a horizontal and a dipping reflector yields sufficiently stable results if both the reflector dip and tilt of the axis exceed $30 - 40^\circ$.

In addition to the constraints on $\nu$ and $\phi$, stable moveout inversion requires that the parameters $\epsilon$ and $\delta$ be sufficiently different from one another. If $\epsilon = \delta$ (the medium is elliptical), the dip dependence of the P-wave NMO velocity is purely isotropic [equation (9)], and the only information contained in the NMO ellipse for the dipping event is that $\epsilon = \delta$. This implies that a single P-wave NMO ellipse for a horizontal event (three equations) has to be used to recover the four remaining parameters ($V_{p0}$, $\epsilon$, $\nu$, and $\beta$). Obviously, this inversion is ambiguous, and it is possible to find a family of elliptical models with identical NMO ellipses from horizontal and dipping reflectors (Figure 6). The inversion becomes feasible only if one of the parameters is known a priori.

**Figure 6.** P-wave NMO ellipses for horizontal and dipping reflectors calculated from the exact equation (5) in an elliptical TTI layer for two different sets of medium parameters. The solid ellipses are computed for $V_{p0} = 3.0$ km/s, $\epsilon = \delta = 0.15$, $\nu = 50^\circ$; for the dipping reflector, $\phi = 50^\circ$. For the dotted ellipses, $V_{p0} = 2.443$ km/s, $\epsilon = \delta = 0.4806$, $\nu = 25.34^\circ$, $\phi = 39.98^\circ$. The azimuth of the symmetry axis and that of the dipping reflector are equal to zero for both models; the horizontal sloveness of the zero-offset ray for the dipping event is $p_1 = 0.255$.

For instance, if the symmetry axis is assumed to be horizontal and $\nu = 90^\circ$ (HTI media), the remaining three parameters can be obtained from the NMO ellipse from a horizontal reflector. Below, we show that the parameters of elliptical media can be resolved for arbitrary orientation of the symmetry axis of a P-wave moveout data are supplemented with the SV-wave NMO ellipse from a horizontal reflector.

**Joint inversion of P- and SV-wave moveout in a TTI layer**

P- and SV-wave propagation in transversely isotropic media is controlled by the same set of the stiffness coefficients $C_{ij}$. If the stiffnesses are replaced with Thomson parameters, P-wave kinematics becomes independent of the shear-wave symmetry-direction velocity $V_{s0}$, but both P- and SV-wave velocities are still influenced by $V_{p0}$, $\epsilon$, $\delta$, and the orientation of the symmetry axis. Hence, we can expect to increase the stability of the inversion procedure discussed in the previous section by combining P-wave data with SV-wave NMO ellipses.

In this section, we modify our parameter-estimation algorithm by adding the NMO velocity of SV-waves from a horizontal reflector to the input data. The SV-wave NMO ellipse for horizontal events in the weak-anisotropy
approximation can be obtained from the corresponding P-wave equations (13)-(15) by making the following parameter substitutions (Tsvankin, 1995, 1997b): the velocity \( V_{P0} \) should be replaced with \( V_{S0} \), \( \epsilon \) set to zero, and \( \delta \) replaced with the parameter \( \sigma \) defined as

\[
\sigma = \left( \frac{V_{P0}}{V_{S0}} \right)^2 (\epsilon - \delta).
\]  

(23)

This yields the following expressions for the semi-axes of the SV-wave ellipse from a horizontal reflector (the axis \( x_1 \) is parallel to the horizontal projection of the symmetry axis):

\[
V_{SV_{\text{nmo}}(0)}^{(1)} = V_{S0} \sqrt{1 + 2\sigma (1 - 7 \sin^2 \nu \cos^2 \nu)}
\]  

(24)

and

\[
V_{SV_{\text{nmo}}(0)}^{(2)} = V_{S0} \sqrt{1 + 2\sigma \cos^4 \nu},
\]  

(25)

For vertical transverse isotropy (\( \nu = 0 \)), equations (24) and (25) reduce to the azimuthally-independent SV-wave NMO velocity given by Thomsen (1986),

\[
V_{SV_{\text{nmo}}(0)} = V_{S0} \sqrt{1 + 2\sigma}.
\]  

(26)

The SV-wave NMO ellipse degenerates into a circle not only for VTI media, but also if the above trigonometric functions multiplied with \( 2\sigma \) are identical, i.e., \( \cos^4 \nu = \frac{1}{6} \) (\( \nu = 65.9^\circ \)). It is interesting that for a tilt of 65.9\(^\circ\) the P-wave NMO ellipse can also become a circle, but only for \( \epsilon = 0 \) [see equation (18)].

The NMO velocities of P- and SV-waves in a horizontal VTI layer are insufficient for the determination of the vertical velocities and anisotropic coefficients (Tsvankin and Thomsen, 1986). Although the vertical-velocity ratio \( V_{P0}/V_{S0} \) can be found from the zero-offset traveltimes, the two NMO velocities still contain three unknown parameters (\( V_{P0}, \delta, \) and \( \sigma \) or \( \epsilon \)). The only exception is elliptical anisotropy, for which the SV-wave NMO velocity is independent of angle, and the NMO velocity is simply equal to \( V_{S0} \). Then \( V_{P0} \) can be found from the zero-offset traveltimes, and the P-wave NMO velocity yields the anisotropic parameter \( \epsilon = \delta \). This inversion procedure, however, is based on the assumption that the medium is elliptical, which cannot be verified from the data, unless dipping events or SH data are available.

It seems, however, that for a tilted symmetry axis it might be possible to resolve all medium parameters in a horizontal layer even for nonelliptical anisotropy, because we can take advantage of the azimuthal dependence of normal moveout. Indeed, the NMO ellipses of P- and SV-waves in a horizontal TTI layer provide six equations for the medium parameters. Only five of these equations, however, are independent because the two ellipses have the same orientation determined by the azimuth of the symmetry axis. In principle, the number of equations can go down to four if both ellipses degenerate into circles, but the above analysis shows that this situation is highly unlikely*. Including SV-waves also allows us to obtain the ratio of the zero-offset traveltimes \( r = t_{P0}/t_{SV0} \) and add a sixth equation into the inversion procedure.

Thus, combining P- and SV-data in a horizontal TTI layer results in a system of six nonlinear equations for six unknowns \( (V_{P0}, V_{S0}, \epsilon, \delta, \nu, \beta) \). Unfortunately, numerical analysis of this system shows that it does not have a unique solution. As illustrated by Figure 7, it is possible to find at least two different realistic TTI models which yield practically identical NMO ellipses of P- and SV-waves. Also, the ratio of the zero-offset traveltimes for both models from Figure 7 is the same \( (r = 0.546) \).

Since the P- and SV-wave NMO ellipses from a horizontal reflector do not provide enough information for unambiguous parameter estimation, we add the P-wave NMO velocity for a dipping event to the input data and construct the following objective function [compare with

\[
\mathcal{L} = \left( \frac{V_{P0}}{V_{S0}} \right)^2 (\epsilon - \delta).
\]  

* For weak anisotropy, both ellipses become circles only if the conditions \( \cos^2 \nu = 1/6 \) and \( \epsilon = 0 \) are satisfied simultaneously.
Figure 8. Inverted values of the parameters $\epsilon$, $\delta$, $\beta$, $\nu$, $V_{P0}$, and $V_{S0}$ in a homogeneous TTI layer. For input data, we use the $P$-wave NMO ellipses from a horizontal and a dipping

\[
\mathcal{F}_{P,SV} = \left( \frac{r}{r_0} - 1 \right)^2 + \sum_{i,j} w_h \left( W_{ij}^{\text{hor}} - \tilde{W}_{ij}^{\text{hor}} \right)^2 + w_d \left( W_{ij}^{\text{dip}} - \tilde{W}_{ij}^{\text{dip}} \right)^2
\]

\[+ w_e \left( W_{ij}^{SV\text{hor}} - \tilde{W}_{ij}^{SV\text{hor}} \right)^2.\]

As before, $\tilde{W}$ denote the ellipses measured from the data, while $W$ are calculated from the exact NMO equation (5). The weighting coefficients $w_h$ and $w_d$ are defined in equations (21) and (22), and $w_e$ is the weight for $SV$-wave NMO ellipse given by
\[ w_s = 4 (W_{11}^{SV_{\text{hor}}} + W_{22}^{SV_{\text{hor}}})^{-2}. \]

Compared to the pure \(P\)-wave inversion in the previous section, the objective function contains four more equations and only one additional unknown parameter \((V_S)^0\), so the inversion procedure should become more stable. Overall, the joint inversion of \(P\) and \(SV\) data involves solving nine nonlinear equations (provided that none of the ellipses degenerates into a circle) for six unknowns \(- V_{P0}, V_S, \epsilon, \delta, \nu, \) and \(\beta\).

Typical results of parameter estimation for a TTI layer obtained by minimizing the objective function (27) are presented in Figure 8. As before, we added Gaussian noise with a variance of 3\% to all NMO ellipses and performed the inversion 200 times for different realizations of the input data. Figure 8 exhibits the same general trend as that for the \(P\)-wave results in Figure 4: the inversion procedure becomes more stable with increasing tilt \(\nu\) of the symmetry axis. Comparison of the corresponding plots in Figures 4 and 8 shows, however, that the scatter in the inverted values is higher for the pure \(P\)-wave inversion (Figure 4), especially for a tilt of 40°. Therefore, as expected, the addition of the \(SV\)-wave NMO ellipse from a horizontal reflector to \(P\)-wave data increases the stability of parameter estimation.

 Nonetheless, even the combination of \(P\) and \(SV\) data is not sufficient for resolving all medium parameters in a stable fashion if the tilt of the symmetry axis is small (Figure 8a–8c). It seems that in the limit of \(\nu = 0\) (VTI media) we should be able to find \(\epsilon, \delta\) and the symmetry-direction (vertical) velocities separately because, after obtaining \(\eta \approx \epsilon - \delta\) from the dip-dependence of \(P\)-wave NMO velocity and the \(V_{P0}/V_S^0\) ratio from the vertical traveltimes, we can determine \(V_S^0\) (and then all other parameters) using the \(SV\)-wave velocity for a horizontal event [equation (26)]. However, while this inversion works well on noise-free data, small errors in \(\eta\) propagate with significant amplification into the value of \(\sigma\) [equation (23)] and the vertical velocities. Hence, despite the parameter \(\eta\) being well-resolved (Figure 8a), we observe an increased scatter in \(V_{P0}, V_S, \epsilon\) and \(\delta\) for mild tilt angles.

The addition of the \(SV\)-wave NMO ellipse for a horizontal reflector does help, however, to overcome the ambiguity of \(P\)-wave inversion for elliptical anisotropy. In this case, the dip dependence of the \(P\)-wave NMO ellipse is sufficient only to establish the fact that \(\epsilon = \delta\), and the parameters \(V_{P0}, \epsilon, \nu, \) and \(\beta\) cannot be found from a single \(P\)-wave NMO ellipse for a horizontal event (see the discussion above). The \(SV\)-wave phase velocity in elliptical media is independent of angle, so the NMO velocity from a horizontal reflector is simply equal to \(V_S^0\) in any direction. After obtaining \(V_S^0\) from the \(SV\)-wave moveout, we can use the ratio of the zero-offset \(P\) and \(SV\) traveltimes as one more equation for \(V_{P0}, \epsilon, \nu, \) and \(\beta\), which allows us to resolve all medium parameters. Our numerical analysis shows that this inversion procedure is reasonably stable.

**Parameter estimation for vertically heterogeneous TTI media**

The above discussion was limited to moveout inversion for a single homogeneous TI layer with a tilted symmetry axis. Here, we extend the parameter-estimation methodology to vertically-heterogeneous TTI media composed of a stack of horizontal TTI layers (the orientation of the symmetry axis may be arbitrary) above a dipping reflector. Azimuthally varying NMO velocity for this model can be expressed through the interval NMO ellipses using the generalized Dix equation of Tsvankin et al. (1997). The interval NMO ellipse \(W_\ell\) in layer \(\ell\) can be found by means of the Dix-type differentiation:

\[
W_\ell^{-1} = \frac{\tau(\ell)W^{-1}(\ell) - \tau(\ell - 1)W^{-1}(\ell - 1)}{\tau(\ell) - \tau(\ell - 1)},
\]

where \(W(\ell - 1)\) and \(W(\ell)\) describe the NMO ellipses for the reflections from the top and bottom of the layer, and \(\tau(\ell - 1)\) and \(\tau(\ell)\) are the corresponding zero-offset travel-times. Although this equation looks similar to the well-known Dix (1955) formula [one may think about formally replacing the matrices \(W^{-1}\) by the squared NMO velocities], it is much more general because it fully accounts for the simultaneous influence of arbitrary anisotropy and reflector dip on the azimuthally dependent NMO velocity. Equation (28) is not limited to any particular anisotropic symmetry and can be used for all pure reflected modes (\(P\)-waves or \(S\)-waves). For example, Grochla and Tsvankin (1997) applied this equation to parameter estimation in vertically-heterogeneous orthorhombic media.

Here, we use the generalized Dix equation to invert
the reflection traveltimes from horizontal and dipping reflectors for the interval parameters of layered TTI media. The effective and interval NMO ellipses in equation (28) are evaluated for the slowness components of the zero-offset ray. In the case of horizontal events, the zero-offset slowness vector is vertical for all reflections (i.e., its horizontal components vanish), and the interval ellipse can be found directly from the effective ellipses for the reflections from the top and bottom of the layer. For dipping reflectors, however, the ray parameters of the zero-offset ray change from layer to layer depending on the reflector dip in a particular layer. Therefore, after carrying out the inversion in the subsurface layer using the technique described in the previous sections, we need to calculate the matrix $W$ in this layer for the slowness components of the zero-offset reflection from the dipping interface in the second layer. Then we obtain the interval NMO ellipse for the dipping event in the second layer from equation (28), combine it with the corresponding ellipse from the horizontal reflector, carry out the inversion in the second layer and continue the layer-stripping procedure downward.

We applied our inversion algorithm to synthetic data generated for a three-layer TTI model with a dipping reflector (e.g., a fault plane) shown in Figure 9. Note that the azimuth $\beta$ of the symmetry axis varies from layer to layer (Table 1), so the model does not have a throughgoing vertical symmetry plane. We performed 3-D anisotropic ray tracing and computed P-wave reflection traveltimes from the horizontal and dipping reflectors along four CMP lines oriented at azimuths $0^\circ$, $45^\circ$, $90^\circ$, and $135^\circ$ with respect to the orientation of the dipping reflector. Then, we fit hyperbolas (3) to the computed traveltimes on conventional-length spreads and obtained azimuthally-varying effective moveout velocities. After approximating these velocities by the best-fit NMO ellipses (4), we used the generalized Dix formula (28) to calculate the interval ellipses (as described above) and carried out parameter estimation in each layer.

The inversion results are shown in Table 2. Since SV-wave reflections were not used in this test, we could not obtain the shear-wave symmetry-direction velocity $V_{S0}$. The errors in the medium parameters for all layers are rather small and mostly due to the influence of nonhyperbolic moveout on the finite-spread moveout velocity. Although we used an incorrect (“the best-guess”) value of $V_{S0}$ by assuming $V_{S0} = V_{P0}/2$, $V_{S0}$ has a practically negligible influence on P-wave moveout. To verify this conclusion again, we carried out the inversion for the third layer using the exact NMO ellipses [given by equation (5)], rather than the ones obtained from the exact traveltimes, and an incorrect $V_{S0} = 1.5$ km/s. The inversion result in this case was almost perfect: $V_{S0} = 2.98$ km/s, $\epsilon = 0.31$, $\delta = 0.16$, $\nu = 39.8^\circ$, and $\beta = 60.3^\circ$ (compare to the last lines in Tables 1 and 2).

Nonhyperbolic moveout, caused by both vertical heterogeneity and anisotropy, introduces distortions into the moveout velocity, which propagate into the interval NMO ellipses after being amplified by the Dix differentiation (28). The maximum deviations of the interval NMO velocities from the exact ellipses, however, are quite moderate [0.5%, 3.0%, and 1.9% for the first (subsurface), second, and third layer, respectively] and correspond to the horizontal events. Clearly, it is justified to use the hyperbolic moveout approximation for P-waves on conventional spreads close to the distance between CMP and reflector.

It is interesting that while the largest error in the NMO velocity (3.6%) was observed in the second layer, the inversion results are least accurate for the bottom (third) layer (see Table 2). This is explained by the larger tilt $\nu$ of the symmetry axis in the second layer ($60^\circ$ compared to $40^\circ$ in the third layer), which makes the inverted parameters less sensitive to errors in the input data. Indeed, the scatter of the inversion results in Figures 4c-

<table>
<thead>
<tr>
<th>Layer</th>
<th>$V_{P0}$ (km/s)</th>
<th>$V_{S0}$ (km/s)</th>
<th>$\epsilon$</th>
<th>$\delta$</th>
<th>$\nu$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
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<td>2.00</td>
<td>1.20</td>
<td>0.15</td>
<td>0.10</td>
<td>50.0</td>
<td>20.0</td>
</tr>
<tr>
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<td>0.20</td>
<td>0.10</td>
<td>60.0</td>
<td>40.0</td>
</tr>
<tr>
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<td>1.80</td>
<td>0.30</td>
<td>0.15</td>
<td>40.0</td>
<td>60.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Layer</th>
<th>$V_{P0}$ (km/s)</th>
<th>$\epsilon$</th>
<th>$\delta$</th>
<th>$\nu$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
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<td>0.09</td>
<td>53.6</td>
<td>15.6</td>
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<td>0.10</td>
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<td>38.9</td>
</tr>
<tr>
<td>3</td>
<td>2.91</td>
<td>0.38</td>
<td>0.14</td>
<td>43.7</td>
<td>64.5</td>
</tr>
</tbody>
</table>

† The azimuthal variation of the fourth-order (quartic) moveout term is not an ellipse; in a horizontal layer, it is described by a quartic oval curve (Sayers and Ebrom, 1997).
4f is substantially smaller for the tilt $\nu = 60^\circ$ than that for $\nu = 40^\circ$.

Overall, the accuracy of the inversion in vertically heterogeneous TTI media can be predicted on the basis of the single-layer error analysis with a correction that takes into account the amplification of errors inherent in the Dix-type layer stripping.

**Discussion and conclusions**

We showed that the NMO ellipses of reflection events obtained from 3-D (azimuthal) moveout analysis can be used to determine the parameters of layered transversely isotropic media with a tilted axis of symmetry (TTI). For $P$-waves, reflection moveout and all other kinematic signatures in TTI media depend on five parameters – the symmetry-direction $P$-wave velocity $V_p0$, anisotropic coefficients $\epsilon$ and $\delta$, and the orientation (tilt $\nu$ and azimuth $\beta$) of the symmetry axis. Since each NMO ellipse provides us with up to three equations for the medium parameters, the inversion procedure requires at least two ellipses for different reflection events. By developing the weak-anisotropy approximation and performing numerical inversion based on the exact NMO equation, we showed the feasibility of parameter estimation for the most common case of a horizontal and a dipping reflector. It should be emphasized that since the moveout inversion yields all parameters responsible for $P$-wave kinematics, conventional-spread surface reflection data contain enough information to perform depth processing in TTI media.

Stable inversion for all parameters, however, is impossible unless the symmetry-axis orientation, $\epsilon$ and $\delta$ satisfy several constraints. For instance, for vertical transverse isotropy ($\nu = 0$) $P$-wave moveout inversion depends on just two combinations of medium parameters – the zero-dip NMO velocity and the anisotropic coefficient $\eta$ close to the difference between $\epsilon$ and $\delta$ (Alkhalifah and Tsvankin, 1995; Grechka and Tsvankin, 1996). Therefore, it is not surprising that the $P$-wave NMO ellipses become insensitive to the individual values of $V_p0$, $\epsilon$, and $\delta$ for relatively mild tilts of the symmetry axis. Our numerical results show that if $\nu < 35 - 40^\circ$, the dip dependence of NMO velocity allows us to estimate only a combination of $\epsilon$ and $\delta$ close to $\eta$.

Also, the medium parameters cannot be found in a stable way if the reflector dip $\phi$ is less than about $30^\circ$ (Figure 5) and the dip-dependent terms are not large enough for the anisotropic inversion. A similar minimum dip requirement in the $P$-wave moveout inversion was obtained for VTI media by Alkhalifah and Tsvankin (1995) and for HTI media by Contreras et al. (1997). Although the inversion becomes more stable with increasing dip, for dips approaching 70-80$^\circ$ specular reflections in a homogeneous medium may not exist at all. Yet another constraint should be satisfied by $\epsilon$ and $\eta$: if the difference between $\epsilon$ and $\delta$ is small, the medium approaches elliptically anisotropic ($\epsilon = \delta$), and the moveout data can provide only the NMO ellipse for horizontal events and an estimate of ($\epsilon - \delta$).

Thus, determination of the individual values of all parameters of TTI media responsible for $P$-wave velocity is possible only if both the tilt of the symmetry axis and reflector dip exceed 30-40$^\circ$, and the medium is not close to elliptical. Even if some of these conditions are not satisfied, we may still be able to obtain certain parameter combinations (e.g., the parameter $\eta$ for small tilts $\nu$), but the inversion results will not be sufficient to carry out depth imaging.

A natural way to enhance the stability of the parameter estimation is to include NMO velocities of shear waves into the inversion procedure. The addition of the $SV$-wave NMO ellipse from a horizontal reflector helps to increase the accuracy of the inversion since the $SV$-wave moveout depends on only one extra parameter (the symmetry-direction shear velocity $V_{s0}$). Also, combining $SV$ reflections from a horizontal interface with the $P$-wave NMO ellipses for a horizontal and a dipping reflector makes it possible to determine the parameters of elliptically anisotropic media. Still, including horizontal $SV$ events is not sufficient to overcome the above constraints on the tilt of the symmetry axis and reflector dip. For instance, to determine the medium parameters for vertical transverse isotropy or mild tilts $\nu$, it is necessary to use reflection moveout of $SV$ or converted waves from dipping reflectors; this will be discussed in detail in a sequel paper.

Extension of our inversion scheme to vertically heterogeneous TTI media is based on the generalized Dix equation (Tsvankin et al., 1997), which expresses the NMO velocity through the matrices responsible for the interval NMO ellipses. As a result of the Dix-type layer stripping, we obtain the interval NMO ellipse, which can be inverted for the medium parameters using the single-layer algorithm. This inversion methodology was successfully tested on a synthetic data set generated by 3-D anisotropic ray tracing in a multilayered TTI model with depth-varying azimuth and tilt of the symmetry axis.

The TI model with a tilted symmetry axis discussed here should be rather common in overthrust areas (such as the Canadian Foothills) and near salt domes. Our inversion methodology provides an efficient way of estimating the anisotropic parameters needed for seismic imaging in these important exploration areas.

**Acknowledgments**

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Figure A1. In the derivation of the weak-anisotropy approximation, the symmetry axis (described by the unit vector $\mathbf{a}$) is assumed to be confined to the $[x_1,x_3]$-plane.


APPENDIX A: $P$-wave NMO ellipse in a weakly anisotropic TTI layer

The matrix $W$ [equation (5)] that determines the NMO ellipse can be linearized with respect to the anisotropic coefficients $\epsilon$ and $\delta$ under the assumption of weak anisotropy ($|\epsilon| \ll 1$ and $|\delta| \ll 1$). Without losing generality, we assume that the symmetry axis (unit vector $\mathbf{a}$) lies within the coordinate plane $[x_1,x_2]$ (Figure A1). Then

$$\mathbf{a} \equiv [a_1,0,a_3] = [\sin \nu,0,\cos \nu],$$  \hspace{2cm} (A1)

where $\nu$ is the tilt of the axis.

To obtain the elements of the matrix $W$, we need to solve the Christoffel equation for the vertical component of the slowness vector $q \equiv p_3$ as a function of the horizontal slowness components $p_1$ and $p_2$. The Christoffel
equation for TI media can be written as

\[ F \equiv (c_{11} s^2 + c_{44} t^2 - 1) (c_{44} s^2 + c_{33} c^2 - 1) - (c_{13} + c_{44}) s^2 t^2 = 0, \quad (A2) \]

where \( s = |p| \sin \theta \) and \( c = |p| \cos \theta \); \( \theta \) is the angle between the slowness vector \( p \) and the symmetry axis \( a \). Expressing \( s \) and \( c \) through the components of the vectors \( p \) and \( a \), we find

\[ s^2 \equiv (a \times p) \cdot (a \times p) = p_z^2 + (a_3 p_1 - a_1 p_3)^2 \]

and

\[ c^2 \equiv (a \cdot p)^2 = (a_1 p_1 + a_3 p_3)^2. \]

Next, we replace the stiffness coefficients \( c_{ij} \) with Thomsen (1986) parameters defined with respect to the symmetry axis:

\[ c_{33} = \rho V_{p0}^2, \quad c_{44} = \rho V_{s0}^2, \quad c_{11} = \rho V_{p0}^2 (1 + 2 \epsilon), \]
\[ c_{13} = \rho \sqrt{(V_{p0}^2 - V_{s0}^2)(V_{p0}^2 + V_{s0}^2)} - \rho V_{s0}^2, \quad (A3) \]

and \( \rho \) is the density.

The linearized solution of the Christoffel equation (A2) for the vertical slowness \( q \) can be represented as the sum of the "isotropic" value \( \bar{q} \) and the correction term \( \Delta q \) due to the influence of anisotropy:

\[ q \equiv p_3 = \bar{q} + \Delta q. \quad (A4) \]

For P-waves, the vertical slowness in isotropic media is given by

\[ \bar{q} = \sqrt{\frac{1}{V_{p0}^2} - p_1^2 - p_2^2}. \quad (A5) \]

\( \Delta q \) can be considered as the linear term in a Taylor series expansion of \( q \) in \( \epsilon \) and \( \delta \) for fixed horizontal slownesses \( p_1 \) and \( p_2 \):

\[ \Delta q = -\frac{1}{\partial F/\partial p_3} \left( \frac{\partial F}{\partial \epsilon} \epsilon + \frac{\partial F}{\partial \delta} \delta \right), \quad (A6) \]

with the partial derivatives obtained by differentiating the Christoffel equation (A2), \( F = 0 \). Combining equations (A4), (A5) and (A6) yields \( q \) as a function of \( p_1 \) and \( p_2 \) in weakly anisotropic TTI media.

Now we can obtain the derivatives \( q_{ij} = \partial q/\partial p_i \) and \( q_{ij} = \partial q/\partial p_i \partial p_j \), \( (i, j = 1, 2) \) from equation (A4) and substitute them into the exact equation (5) for the matrix \( W \). Further linearization of \( W \) in the isotropic parameters using symbolic software Mathematica leads to the following final result:

\[ W_{11} = \frac{1}{V_{p0}^2} \left( 1 - 2 \delta - 2 \epsilon \delta - 14 \epsilon \epsilon - 14 \delta \delta \right) \]
\[ - p_1^2 + \bar{W}_{11}(\epsilon - \delta) \]
\[ W_{12} = -p_1 p_2 + \bar{W}_{12}(\epsilon - \delta), \quad (A7) \]

and

\[ W_{22} = \frac{1}{V_{p0}^2} \left( 1 - 2 \delta - 2 \epsilon \delta - 14 \epsilon \epsilon - 14 \delta \delta \right) \]
\[ - p_2^2 + \bar{W}_{22}(\epsilon - \delta). \]
3-D moveout inversion in azimuthally anisotropic media with lateral velocity variation: Theory and a case study

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ABSTRACT

Reflection moveout recorded over an azimuthally anisotropic medium (e.g., such as in the presence of vertical or dipping fractures) varies with the azimuth of the source-receiver line. Normal-moveout (NMO) velocity, responsible for the reflection traveltimes on conventional-length spreads, forms an elliptical curve in the horizontal plane. While this result remains valid in the presence of arbitrary anisotropy and heterogeneity, inversion of the NMO ellipse for the medium parameters has been discussed so far for only laterally homogeneous models above a horizontal or dipping reflector.

Here, we develop an analytic moveout correction for weak lateral velocity variation in horizontally layered azimuthally anisotropic media. The correction term is proportional to the curvature of the zero-offset traveltime surface at the common midpoint and, therefore, can be estimated from surface seismic data. After the influence of lateral velocity variation on the NMO ellipses has been stripped, the generalized Dix equation can be used to compute the interval velocity ellipses and evaluate the magnitude of azimuthal anisotropy within the layer of interest.

We applied this methodology to a 3-D "wide-azimuth" data set acquired over a fractured reservoir in the Powder River Basin, Wyoming. The processing sequence included 3-D semblance analysis (based on the elliptical NMO equation) for a grid of common-midpoint "supergathers," spatial smoothing of the NMO ellipses and zero-offset traveltimes, correction for lateral velocity variation and generalized Dix differentiation. Our estimates of depth-varying fracture trends in the survey area, based on the interval P-wave NMO ellipses, are in good agreement with the results of outcrop measurements, FMI/FMS borehole scans, and the rotational analysis of four-component S-wave data.

Key words: azimuthally anisotropic media, lateral velocity variation, velocity analysis, fracture characterization

Introduction

The azimuthal variation of normal-moveout velocity for pure (non-converted) modes is described by an ellipse in the horizontal plane, even if the medium is generally anisotropic and heterogeneous (Grechka and Tsvankin, 1996). This conclusion breaks down only for subsurface models in which common-midpoint reflection traveltimes cannot be described by a series expansion or does not increase with offset. The orientation of the NMO ellipse and the values of its semi-axes are determined by the orientation of the reflecting interface at the zero-offset reflection point and the medium properties above the reflector.

Conventional stacking-velocity analysis in 3-D surveys has often ignored the azimuthal dependence of normal moveout from horizontal reflectors, which leads to distortions in seismic processing (Lynn et al., 1996). A single value of stacking (normal-moveout) velocity at a
given spatial location will generally cause underestimation of NMO velocity for source-receiver azimuths near the "fast" direction and overestimation of \( \nu_{\text{ave}} \) near the "slow" direction. Hence, mixing of data from different azimuths may impair the performance of moveout correction and the quality of the stacked section. Recognizing the elliptical variation in NMO velocity makes it possible to avoid these distortions by reconstructing the best-fit NMC ellipse from the data and obtain the correct stacking velocity for all azimuthal directions.

Inversion of the NMO ellipse is a much more complicated issue because the ellipticity can be caused not just by azimuthal anisotropy, but also by reflector dip and lateral heterogeneity. If the medium above the reflector is horizontally homogeneous, reflector dip manifests itself through the reflection slope on the zero-offset section and can be accounted for in the parameter-estimation procedure (Grechka and Tsvankin, 1997). Depending on the symmetry of the medium, the recovery of the anisotropic parameters in horizontally homogeneous models may require NMO ellipses measured for several different modes or reflector orientations. For instance, in orthorhombic media with a horizontal symmetry plane, the \( P \)-wave NMO velocity is controlled by the directions of the vertical symmetry planes, two symmetry-plane NMO velocities from a horizontal reflector and three "anellipticity" coefficients similar to the Alkhalifah-Tsvankin (1996) parameter \( \eta \) for vertical transverse isotropy (Grechka and Tsvankin, 1997). All these parameters can be determined from two \( P \)-wave NMO ellipses corresponding to different reflector dips and/or azimuths. Grechka and Tsvankin (1997) extended azimuthal moveout inversion to vertically heterogeneous (i.e., horizontally stratified) orthorhombic media above a dipping reflector using the generalized Dix equation of Tsvankin et al. (1997). This methodology is also valid for the simplest azimuthally anisotropic model – transverse isotropy with a horizontal symmetry axis (HTI media), where the ellipticity becomes even more stable due to the smaller number of the anisotropic parameters (Contreras et al., 1997).

Reflection moveout and NMO velocity, however, are also influenced by lateral velocity variation, which can make the azimuthal dependence of NMO velocity elliptical even for a horizontal reflector beneath an azimuthally isotropic medium. Since the anisotropy-induced ellipticity of the NMO velocity usually is relatively small (up to 10%), it may well be comparable to the distortions caused by mild lateral velocity variations. Here, we introduce a moveout correction for lateral velocity variation in horizontally layered media by extending the approach of Grechka (1998) originally developed for vertical transverse isotropy. The distortion of the NMO ellipse due to lateral variation in the elastic constants turns out to be controlled by the curvature of the zero-offset traveltimes surface at the common-midpoint (CMP) location and, therefore, can be evaluated using surface data. After removing the influence of lateral velocity variation on the NMO (stacking) velocity, we obtain the interval NMO ellipses from the generalized Dix equation. For such common azimuthally anisotropic models as HTI and orthorhombic media with a single vertical fracture system, the azimuth of one of the axes of the interval NMO ellipse corresponds to the fracture orientation, and the fractional difference between the semi-axes is related to the fracture density (Tsvankin, 1997a; Grechka and Tsvankin, 1997).

After discussing the theory and methodology, we present an application of our algorithm to a 3-D data set acquired by ARCO (with funding from the Gas Research Institute) over a fractured reservoir in the Powder River Basin, Wyoming. Our processing sequence made it possible to obtain the depth-varying fracture orientation and estimate the magnitude of azimuthal anisotropy (measured by \( P \)-wave moveout velocity) over the survey area. The direction of the semi-major axis of the interval NMO ellipse in all layers is in agreement with the fracture trends detected by other methods.

Theory

Equation of the NMO ellipse

Grechka and Tsvankin (1996) considered azimuthally dependent reflection moveout of pure (non-converted) modes around a fixed CMP location over an arbitrary anisotropic heterogeneous medium. For source-receiver offsets that do not exceed the distance between the CMP and the reflector, the reflection traveltimes \( t \) in most cases can be accurately approximated by the hyperbolic equation,

\[
t^2(h, \alpha) = t_0^2 + \frac{4h^2}{V_{\text{ave}}^2(\alpha)},
\]

where \( h \) is half the source-receiver offset, \( \alpha \) is the azimuth of the CMP line, \( t_0 \) is the zero-offset traveltimes, and \( V_{\text{ave}}(\alpha) \) is the normal-moveout velocity analytically defined in the zero-spread limit. Since NMO velocity is not influenced by reflection-point dispersal (Hubral and Krey, 1980; Grechka and Tsvankin, 1996), the hyperbolic portion of the moveout curve can be described in terms of the one-way traveltimes \( r \) from the zero-offset reflection point to the surface. Under the assumption that \( r \) is sufficiently smooth near the common midpoint to be expanded in a Taylor series in the horizontal coordinates \( x_1 \) and \( x_2 \), Grechka and Tsvankin (1996) obtained the
following expression for the azimuthally varying NMO velocity:

\[ V_{\text{NMO}}^2(\alpha) = W_{11} \cos^2 \alpha + 2W_{12} \sin \alpha \cos \alpha + W_{22} \sin^2 \alpha, \]  

(2)

where the symmetric matrix \( W \) depends on the derivatives of \( \tau \) with respect to the horizontal coordinates \((x_1, x_2)\):

\[ W_{ij} = \tau_0 \left. \frac{\partial^2 \tau}{\partial x_i \partial x_j} \right|_{x=x_{\text{CMP}}} = \tau_0 \left. \frac{\partial p_i}{\partial x_j} \right|_{x=x_{\text{CMP}}} \]  

\( (i, j = 1, 2) \).

Here \( \tau_0 = t_0/2 \) and \((p_1, p_2)\) are the horizontal components of the slowness vector for rays excited at the zero-offset reflection point and recorded at the surface.

To show that the azimuthal variation in NMO velocity is elliptical, we introduce the angle

\[ \beta = \tan^{-1} \left( \frac{W_{22} - W_{11} + \sqrt{(W_{22} - W_{11})^2 + 4W_{12}^2}}{2W_{12}} \right) \]  

(4)

\((W_{12} \neq 0)\),

which corresponds to one of the principal directions of the matrix \( W \). Expressing \( W \) in terms of its eigenvalues

\[ \lambda_{1,2} = \frac{1}{2} \left[ W_{11} + W_{22} \pm \sqrt{(W_{11} - W_{22})^2 + 4W_{12}^2} \right] \]  

(5)

and the angle \( \beta \), we can rewrite equation (2) as

\[ V_{\text{NMO}}(\alpha) = \lambda_1 \cos^2(\alpha - \beta) + \lambda_2 \sin^2(\alpha - \beta). \]  

(6)

Unless the model has reverse moveout wherein reflection traveltimes decreases with offset in some directions (i.e., \( V_{\text{NMO}}^2 < 0 \)), which implies that \( \lambda_1 < 0 \) and/or \( \lambda_2 < 0 \), equation (6) describes an ellipse in the horizontal plane (Grechka and Tsvankin, 1996). The NMO velocities in the directions of the elliptical axes (we denote them \( v_{\ell 1} \) and \( v_{\ell 2} \)) can be expressed through the eigenvalues as

\[ v_{\ell 1} = 1/\sqrt{\lambda_1} \quad \text{and} \quad v_{\ell 2} = 1/\sqrt{\lambda_2}. \]  

(7)

**Generalized Dix equation**

To obtain normal-moveout velocity from equations (2) and (3), it is necessary to evaluate the spatial derivatives of the slowness vector at the CMP location. For general anisotropic heterogeneous media, the parameters of the NMO ellipse can be determined numerically using the dynamic ray-tracing equations for the zero-offset ray (Tsvankin et al., 1997). However, if the model consists of a stack of horizontal homogeneous layers above a dipping reflector, NMO velocity can be found analytically from the generalized Dix equation of Tsvankin et al. (1997).

The "effective" matrix \( W(L) \) [equation (3)] that defines the NMO ellipse (2) for reflections from the bottom of layer \( L \) can be expressed through a Dix-type average of inverses of the interval matrices \( W_\ell \) weighted by the interval zero-offset traveltimes \( \tau_\ell \):

\[ W^{-1}(L) = \frac{1}{\sum_{\ell=1}^{L} \tau_\ell} \sum_{\ell=1}^{L} \tau_\ell W^{-1}_\ell. \]  

(8)

Equation (8) can be solved explicitly for the interval quantities in the Dix-type differentiation form:

\[ W^{-1}_\ell = \frac{\tau(\ell)W^{-1}(\ell) - \tau(\ell-1)W^{-1}(\ell-1)}{\tau(\ell) - \tau(\ell-1)}. \]  

(9)

The matrices \( W(\ell-1) \) and \( W(\ell) \) are responsible for the NMO ellipses for reflections from the top and the bottom of the \( \ell \)-th layer; \( \tau(\ell-1) \) and \( \tau(\ell) \) are the corresponding zero-offset traveltimes. Equation (9) allows one to calculate the matrix \( W_\ell \), which determines the interval NMO ellipse in the \( \ell \)-th layer.

Equations (8) and (9) represent a 3-D generalization, for arbitrary anisotropic media, of the well-known Dix (1955) formula. The interval matrices \( W_\ell \) in equation (3) are computed for the horizontal components of the slowness vector of the zero-offset ray. Therefore, if the reflector is dipping and the zero-offset slowness direction is not vertical, layer-stripping by means of equation (9) involves recalculating each interval matrix \( W_\ell \) from one value of the slowness vector (corresponding to a certain real reflector in a given layer) to another — that of the zero-offset ray.

It should be emphasized that equation (9) cannot be used if the horizontal components of the slowness vector vary along the one-way ray under the influence of lateral heterogeneity and/or dipping (or irregular) interfaces above the \( \ell \)-th reflector. For these laterally heterogeneous models, Hubral and Krey (1980) suggested implementing the Dix-type differentiation using downward continuation of the wavefront curvature associated with the zero-offset ray. In principle, this methodology can be generalized to construct the NMO ellipses in anisotropic heterogeneous media (Tsvankin et al., 1997). However, such an approach involves numerical ray tracing through heterogeneous anisotropic models, and equation (9) loses its simplicity. Note that the numerical solution is required even for relatively simple isotropic models containing homogeneous layers separated by curved interfaces (Hubral and Krey, 1980). Also, the downward continuation requires knowledge of the medium parameters in the overburden.

**Influence of weak lateral heterogeneity on NMO ellipses**

The discussion above suggests that reconstruction of the interval NMO ellipses in heterogeneous anisotropic me-
dia is impossible without determination of the properties
of the overburden and employing numerical algorithms.
The only way to preserve the relative simplicity of the
Dix differentiation [equation (9)] in the presence of lat-
eral heterogeneity is to impose certain restrictions on
model complexity. Below we implement this approach by
assuming that the medium consists of horizontal layers
with weak lateral velocity variation.

**Horizontal layer with lateral velocity variation**

Let us consider the simplest model of a single anisotrop-
ic layer above a horizontal reflector. We assume that
the horizontal plane is a plane of symmetry, which implies
that the medium may be monoclinic, orthorhombic, or
transversely isotropic with a vertical or horizontal sym-
metry axis. In general, the type of symmetry may change
laterally provided the slowness surface remains symmetric
with respect to the horizontal plane.

Suppose the NMO ellipse (2) for the reflection from
the bottom of the reference homogeneous layer (with the
same parameters as at the CMP location) is described
by a matrix denoted as \( W^{\text{hom}} \) [equation (3)]. In accordance
with the general result of Grechka and Tsvankin
(1996), the azimuthal variation of NMO velocity in the
presence of lateral velocity variation remains elliptical,
but the corresponding matrix \( W \) is different (\( W^{\text{het}} \)).
Assuming that lateral velocity variation (or lateral hetero-
geneity, "LH") is weak and retaining only linear terms in
the spatial derivatives of the velocity function yields
the following relationship between the two NMO ellipses
(Appendix A):

\[
W_{ij}^{\text{het}} = W_{ij}^{\text{hom}} - \frac{\tau_0^2}{3V_0} \left. \frac{\partial^2 V_0}{\partial y_i \partial y_j} \right|_{y=y_{\text{CMP}}} ,
\]

\( i, j = 1, 2 \),

where \( \tau_0 = \tau(y) \) is the one-way zero-offset (vertical)
traveltime and \( V_0 = V_0(y) \) is the vertical velocity at
CMP location \( y = (y_1, y_2) \). Equation (10) indicates that,
in the linear approximation with respect to lateral hetero-
geneity, influence of the heterogeneity on the NMO
ellipse is proportional to the curvature of the vertical-
velocity surface \( V_0(y) \) at the common midpoint.

Figure 1 displays a numerical test of equation (10)
for a transversely isotropic medium with a horizontal
symmetry axis parallel to the axis \( y_1 \). Ray-traced P-
wave reflection traveltimes were computed at a common-
midpoint \( y_1 = y_2 = 0 \) for two anisotropic models: an
HTI layer with laterally-varying vertical velocity and a
reference homogeneous HTI layer with parameters corre-
sponding to the CMP location (Figure 1a). After calculat-
ing the traveltime along four differently oriented CMP
lines, we used the hyperbolic moveout equation (1) to
obtain azimuthally-dependent NMO velocity and recon-
structed the NMO ellipses for both the homogeneous and
LH layers (Figure 1b). The difference between the two el-
ipses can be understood by analyzing the saddle-shaped
velocity surface \( V_0(y) \) in Figure 1a. The positive curva-
ture of \( V_0(y) \) in approximately the \( y_2 \)-direction (Fig-
ure 1a) increases the velocity for rays propagating in this
direction and, therefore, leads to a higher NMO velocity
for azimuths close to 90° (Figure 1b). Likewise, there is
a small decrease in the NMO velocity at azimuths near 0°
due to the negative curvature of \( V_0(y) \) in the
\( y_1 \)-direction.

To test the analytic correction for lateral velocity
variation, we calculated the derivatives \( \partial^2 V_0/\partial y_i \partial y_j \)
at the CMP location and used equation (10) with the ex-
act expression for \( W_{ij}^{\text{hom}} \) to obtain the NMO velocity in
the LH layer. This analytic NMO ellipse (solid line in
Figure 1b) is virtually identical to the one generated num-
erically (dotted line), with the maximum difference less
than 0.5%. Note that the quasi-linear increase in vertical
velocity in the \( y_1 \)-direction (Figure 1a) has almost
no influence on NMO velocity, which is controlled by the
curvature of \( V_0(y) \) at the common midpoint. This
observation agrees with the result of Grechka (1998) who
proved that NMO velocity for vertical transverse iso-
tropy is independent of constant lateral velocity gradient
(in the linear approximation).

Even though equation (10) provides an adequate ap-
proximation for the NMO ellipse in an LH layer, it is
difficult to use in reflection data processing because the
vertical velocity \( V_0 \) in anisotropic media cannot be deter-
mined from P-wave reflection traveltimes. [The only ex-
ception is the HTI model, for which one of the semi-axes
of the NMO ellipse is equal to the true vertical velo-
city (Tsvankin, 1997a; Contreras et al., 1997.)] However,
since the layer is horizontal and \( V_0 \tau_0 = \text{const} \), the spa-
tial derivatives of the vertical velocity \( V_0 \) can be replaced
with those of the vertical traveltime \( \tau_0 \). Specifically, in
the linear approximation with respect to lateral velocity
variation,

\[
\frac{\partial^2 V_0}{\partial y_i \partial y_j} \tau_0 + \frac{\partial^2 \tau_0}{\partial y_i \partial y_j} V_0 = 0 .
\]

(11)

Substituting equation (11) into equation (10) gives the
following representation of \( W^{\text{hom}} \):

\[
W_{ij}^{\text{hom}} = W_{ij}^{\text{het}} - \frac{\tau_0}{3} \left. \frac{\partial^2 \tau_0}{\partial y_i \partial y_j} \right|_{y=y_{\text{CMP}}} ,
\]

\( i, j = 1, 2 \).

Note that all terms in the right-hand side of equation
(12) can be found from the traveltimes of a given
pure-mode reflection recorded over an area around the CMP location.

Two layers with lateral velocity variation

The methodology discussed in the previous section can be extended to the case of reflector beneath any number of horizontal anisotropic layers with a horizontal symmetry plane. For multilayered media, however, we assume that not only lateral velocity variation, but also azimuthal anisotropy is weak. Without the latter assumption, the correction term involves interval vertical velocities, which cannot be found from reflection data.

If the model consists of two horizontal layers, the NMO ellipse for the reflection from the bottom of the second layer can be represented as (Appendix B)

\[ W^\text{hom}_i = W^{\text{het}}_i - \frac{\tau_0}{3} \left[ k^2 \frac{\partial^2 \tau_0}{\partial y_i \partial y_j} + (1 + k) \frac{\partial^2 \tau_{01}}{\partial y_i \partial y_j} \right], \quad (i,j = 1,2). \]

(13)

with

\[ k = 1 - \frac{\tau_{01} V^2_{\text{cir}1}}{\tau_0 V^2_{\text{cir}}}. \]

(14)

Here \( \tau_{01} \) and \( \tau_0 \) are the one-way zero-offset traveltimes for the reflections from the bottom of the first and the second layer, respectively; all derivatives are evaluated at the CMP location \( y = y_{\text{CMP}} \). \( V_{\text{cir}1} \) and \( V_{\text{cir}} \) are the "circular" approximations of the NMO ellipses for the reflections from the bottom of the first and second layer. They are obtained by averaging the azimuthally dependent NMO velocity [equation (2)] over all azimuths:

\[ V_{\text{cir}}^{-2} = \frac{1}{2\pi} \int_0^{2\pi} V_{\text{norm}}^{-2}(\alpha) \, d\alpha = \frac{W^{\text{het}}_{11} + W^{\text{het}}_{22}}{2}. \]

(15)

To examine how the lateral velocity variation in a certain interval contributes to the overall correction term in equation (13), we consider a model with a laterally homogeneous upper layer, i.e.,

\[ \frac{\partial^2 \tau_{01}}{\partial y_i \partial y_j} = 0. \]

(16)

In this case, equation (13) simplifies to

\[ W^\text{hom}_i = W^{\text{het}}_i - \frac{k^2 \tau_0}{3} \frac{\partial^2 \tau_0}{\partial y_i \partial y_j}, \quad (i,j = 1,2). \]

(17)

As expected, the correction due to lateral velocity variation becomes smaller if we add a laterally homogeneous layer on top of an LH layer. Indeed, the difference between the correction terms in equation (17) and in the single-layer equation (12) is in the factor \( k^2 = (1 - \tau_{01} V^2_{\text{cir}1}/\tau_0 V^2_{\text{cir}})^2 \), which is always smaller than unity. More importantly, equation (17) shows that, if \( V^2_{\text{cir}1} = V^2_{\text{cir}} \), the correction for LH is proportional to the squared relative thickness of the LH layer. If, for example, the relative thickness of the LH layer \([\tau_0 - \tau_{01}]/\tau_0]\) is equal to 0.5, the term \((1 - \tau_{01}/\tau_0)^2\) goes down to 0.25.
\( k^2 = 0.25 \) compared to unity for a single layer. Such a nonlinear dependence means that the influence of an LH layer with fixed thickness on the NMO ellipse rapidly decreases with layer’s depth. This fact can be explained by considering the lateral position of reflected rays in a CMP gather: as the LH layer moves deeper, the rays crossing it sample a smaller vicinity of the common midpoint and, therefore, are less dependent on the lateral velocity variation.

**Field-data example**

The correction for lateral velocity variation combined with the generalized Dix formula provides an analytic basis for interval moveout velocity analysis in arbitrary anisotropic media. We tested the formalism described above on a 3-D data set acquired by ARCO (with funding from the Gas Research Institute) in the Powder River Basin, Wyoming. This “wide-azimuth” survey was designed to map fracture networks using the azimuthal dependence of P-wave signatures. To enhance the signal-to-noise ratio, the data were collected into 169 “superbins,” each with an irregular distribution of azimuths and offsets (Figure 2). A description of the geology of the area, data acquisition, and preliminary processing results can be found in Corrigan et al. (1996) and Withers and Corrigan (1997).

Our main goal was to recover the interval P-wave NMO ellipses associated with azimuthal anisotropy and use them to characterize any laterally varying fracturing in different subsurface layers. The processing flow included the following main steps:

- 3-D semblance analysis to obtain the NMO ellipses for each superbin;
- Spatial smoothing of the NMO ellipses;
- Correction of the NMO ellipses for lateral velocity variation;
- Generalized Dix-type layer stripping to obtain the interval NMO ellipses.

Below, we describe each processing step in detail and discuss our final results.

**3-D semblance analysis**

First, it was necessary to find the NMO ellipses for reflections from the most prominent boundaries over the survey area. There are at least two possible ways of performing moveout velocity analysis of 3-D multi-azimuth P-wave reflection data and obtaining the NMO ellipses.

**Figure 2.** Plan view of the source and receiver positions (dots) for a single superbin. The superbin contains approximately 400 source-receiver pairs with common-midpoint scatter of up to about 80 m (2% of the maximum offset). The maximum offset is approximately equal to the depth of the deepest reflector.

First, one can divide the data for a given superbin into several azimuthal sectors and perform conventional hyperbolic velocity analysis for source-receiver pairs within each sector. Then, the best-fit moveout (stacking) velocities \( V_{nmo}(\alpha) \) for a given reflector are determined by approximating the moveout-velocity measurements with an ellipse using equation (2). While azimuthal sectoring makes it possible to use conventional software for semblance analysis, we found that the number and size of the sectors may influence the results of the velocity estimation because the distribution of offsets and azimuths is fairly erratic.

Therefore, we implemented another approach by treating all azimuths simultaneously and performing semblance analysis for the whole superbin at each zero-offset time \( \tau_0 \). This “global” semblance analysis is based on the hyperbolic moveout equation (1) and requires scanning over the three components of the matrix \( W \) describing the NMO ellipse [equation (2)]. Note that since the maximum offset was a little smaller than the depth of the basement (the deepest reflector), the moveout for all superbins was close enough to hyperbolic.

To make this search more efficient and avoid the full-scale 3-D semblance scan, we use an equivalent representation of the NMO ellipse (2) in terms of the average velocity \( V_{cir} \) [equation (15)] and two quantities \( e_1 \) and \( e_2 \):

\[
V_{nmo}^{-2}(\alpha) = V_{cir}^{-2} (1 + e_1 \cos 2\alpha + e_2 \sin 2\alpha),
\]

where the parameters \( e_1 \) and \( e_2 \) control the deviation of

* A composite CMP gather that includes all source-receiver pairs with close midpoints.
the NMO ellipse from an average "NMO circle." Combining equations (2), (15) and (18) yields

\[ e_1 = \frac{W_{11} - W_{22}}{W_{11} + W_{22}} \quad \text{and} \quad e_2 = \frac{2W_{12}}{W_{11} + W_{22}}. \] (19)

Although both equation (18) and the original NMO ellipse (2) contain three unknown parameters, the introduction of \( e_1 \) and \( e_2 \) allows us to speed up the semblance analysis by dividing it into two stages. Indeed, \( e_1 \) and \( e_2 \) are dimensionless and small compared to unity if azimuthal anisotropy is relatively weak and the NMO ellipses are not far different from a circle (the most common case). Therefore, at first we assume \( e_1 = e_2 = 0 \) and carry out a conventional 1-D scan over \( V_{\text{circ}} \). This procedure, routinely applied in 3-D processing, ignores the azimuthal dependence of NMO velocity and yields an average NMO circle that can be considered as an initial guess for the NMO ellipse. Then we iterate \( V_{\text{circ}} \) and the coefficients \( e_1 \) and \( e_2 \) to obtain the best-fit NMO ellipse (18) that provides the highest value of semblance. This search is performed by an efficient minimization technique (Powell’s method, see Press et al., 1987) that usually converges in 5–10 iterations, making an extensive 3-D semblance scan unnecessary.

Figure 3 shows typical semblance curves obtained from the conventional and azimuthal velocity analysis for one of the superbins. (By conventional velocity analysis we mean the first stage of our semblance search, which provides the best-fit azimuthally independent velocity \( V_{\text{circ}} \)). While the two curves are close to each other over most of the time interval (i.e., the NMO ellipses are close to a circle), for the reflections at the vertical times of 1.54 s and 1.84 s, the azimuthal velocity analysis improves the fit to moveout and provides higher semblance values. The improvement is especially pronounced for the reflection event at 2.57 s, where moveout ellipticity is the greatest. For this event, the azimuthal velocity analysis increases the semblance value by 10% compared to that produced by conventional velocity analysis.

For these reflections, the conventional algorithm smears the azimuthal velocity variations and results in a distorted value of moveout velocity for any given azimuth, thus leading to a lower quality of stack. Also, of course, conventional moveout analysis cannot be used to extract information about azimuthal anisotropy from reflection traveltimes.

According to Withers and Corrigan (1997), the reflection at a two-way vertical time of 2.14 s corresponds to the bottom of the Frontier/Niobrara formations, and the event at 2.57 s is the basement reflection. The producing fractured reservoir (the main target of the survey) covers the time interval approximately between 1.84 s and 2.14 s.

![Figure 3. Semblance curves obtained by the conventional velocity analysis, which ignores the azimuthal dependence of moveout velocity (dashed) and by our azimuthal velocity analysis (solid). Arrows indicate the reflections used in the generalized Dix differentiation.](image)

The results of the azimuthal velocity analysis for reflections marked in Figure 3 over the whole survey area are shown in Figure 4. Each tick corresponds to the NMO ellipse at a certain superbin. The direction of a tick indicates the azimuth of the semi-major axis of the NMO ellipse, while the tick's length is proportional to the ellipticity \( e \) (i.e., the elongation of the NMO ellipse). We define \( e \) as the fractional difference between the semi axes \( v_{e1} \) and \( v_{e2} \) [equation (7)],

\[ e = 2 \frac{v_{e1} - v_{e2}}{v_{e1} + v_{e2}}. \] (20)

The maximum value of \( e \) for the whole survey area does not exceed 0.05. We will see, however, that the interval NMO ellipses in certain layers may have much higher values of \( e \). The ellipticity patterns in Figure 4 are sufficiently close to the results of Withers and Corrigan (1997), who used a somewhat different (interactive) algorithm for azimuthal velocity analysis of the same data.

Smoothing of the NMO ellipses

An important issue is to what extent we can trust the rapid lateral variations in the parameters of the NMO ellipses in Figure 4. Note that the size of the whole survey area is not far different from the maximum offset for a single superbin (compare the scales in Figures 2 and 4). This means that the reflected rays corresponding to adjacent superbins propagate through almost the same subsurface volume. Hence, a 90° rotation of the semi-major axis of the NMO ellipse between adjacent superbins (e.g., between the point with the crossline coordinate \( y_1 = 0.6 \) km and inline coordinate \( y_2 = 2.0 \) km and an adjacent point \( y = (0.6, 2.2) \) km in Figure 4d)
most likely is due to noise in input data that leads to errors in azimuthal velocity analysis. Clearly, reliable estimation of the interval ellipticities is impossible without spatial smoothing of the NMO ellipses.

Our design of the smoothing procedure is based on the size of the first Fresnel zone at the reflector. Figure 5 shows that the whole survey area is only about 6–7 times larger than the area of the Fresnel zone computed for the deepest reflector. Since it is reasonable to assume that each Fresnel zone can yield a single NMO ellipse; no more than 6–7 independent ellipses can be obtained for the whole area. Therefore, the spatial variation in the smoothed matrix $W^{\text{het}}(y)$ should be represented by a function with 6–7 independent parameters. In essence, by applying spatial smoothing we suppress short-wavelength spatial variations in the NMO velocity that cannot be resolved from the data.

We seek $W^{\text{hst}}(y)$ as a quadratic polynomial

$$W^{\text{het}}(y_1, y_2) = \sum_{k, l=0}^{k+l \leq 2} W^{(k)}_{ij} y_1^k y_2^l, \quad (i, j = 1, 2),$$  \hspace{1cm} (21)$$

where the six coefficients $W^{(k)}_{ij}$ (for each element $W_{ij}$ of the matrix $W^{\text{het}}$) are found by least-squares fitting of equation (21) to the raw NMO ellipses from Figure 4. The choice of the quadratic polynomial (21) means that the NMO ellipses are approximated with surfaces $W^{\text{hst}}(y)$ of constant curvature over the whole survey area.

The ellipticities after the spatial smoothing are displayed in Figures 5b and 6. The smoothing led to an overall decrease in the magnitude of azimuthal NMO-velocity
variation, especially for the events at 1.54, 1.84, and 2.14 s. For the deepest reflection at 2.57 s, the smoothed ellipticity is more substantial and has a predominant N-S orientation. The results of smoothing indicate that the level of errors in picking azimuthally-dependent NMO velocities is about 1–2%, and small ellipticities on the order of 2-2.5% can hardly carry useful information about the medium.

Correction for lateral velocity variation

The formalism of Grechka and Tsvankin (1996) shows that the ellipticities in Figure 6 may be caused by some combination of the following three factors:
- reflector dip;
- lateral heterogeneity above the reflector;
- azimuthal anisotropy.

Correcting for the influence of reflector dip and lateral heterogeneity is essential where the goal is fracture characterization using azimuthal moveout analysis. Below, we demonstrate that the reflector dips in the area are negligible small, and the subsurface model can be adequately represented by a stack of horizontal layers. Then, we use the analytic expressions developed above [equations (12)–(14)] to correct the NMO ellipses for lateral velocity variation. Applicability of our approximations, based on the assumption of weak lateral velocity variation and weak anisotropy, is justified by the fact that the ellipticity is relatively small (Figure 6), and, therefore, all factors responsible for it have to be small as well.

Figure 7 shows smoothed surfaces of the two-way zero-offset reflection traveltime \( t(y) = 2\tau_0(y) \) for the four reflection events. The smoothing function had the same form as the one used for NMO ellipses [equation (21)]. The maximum apparent time-dip in Figure 7 reaches 30–40 ms over a distance of about 3 km, giving an apparent horizontal slowness \( p = \frac{1}{2} \frac{dt}{dy} \) of about 0.005 s/km. Attributing this spatial variation in the zero-offset time to reflector dip \( \phi \) yields the relative correction in the NMO velocity that can be roughly estimated (in the isotropic limit) as \( \frac{1}{\cos \phi - 1} \approx \frac{1}{\sqrt{1 - p^2 V_{circ}^2} - 1} \).

Substituting \( p = 0.005 \) s/km and \( V_{circ} = 4.0 \) km/s (see Figure 8), we find that the distortion due to reflector dip is close to just 0.02% of \( V_{circ} \). This distortion is an order of magnitude smaller than the estimated errors in the NMO ellipses. Even though the apparent dip of the zero-offset traveltime surface is also influenced by lateral velocity variation (see Figure 8), it is clear that the contribution of dip to NMO velocity can be ignored for all reflection events. The absence of dip enables us to correct for lateral velocity velocity variation using the theory for horizontally layered media developed above.

The zero-offset traveltime surfaces \( t(y) \) (Figure 7), together with the surfaces of the average NMO velocity \( V_{circ}(y) \) shown in Figure 8, can be used to strip the influence of lateral velocity variation from the NMO ellipses. Note that the curvature of the traveltime surface, responsible for the influence of lateral velocity variation on the NMO ellipse, is directly determined by the coefficients of our smoothing function [equation (21)] and is thus constrained to be constant over the survey area. The correction for lateral velocity variation for the shallow reflection (at 1.54 s) was carried out using the single-layer
equation (12), while for the event at 1.84 s we applied the two-layer equation (13). In principle, the two deepest events (at 2.14 s and 2.57 s) should be treated by means of a more complicated multiple-layer correction formula (not given here). However, in order to simplify the processing algorithm, we used the same equation (13) for both deeper reflections under the assumption that the whole stratified overburden may be described as a single effective layer. For instance, in calculating the correction term for the basement reflection (2.57 s), the first-layer parameters in equation (13) ($\tau_0$ and $V_{cir}$) were assumed to correspond to the event at 2.14 s.

Comparing the corrected NMO ellipses in Figure 9 with the input ones in Figure 6 illustrates the possible distortions caused by lateral velocity variation in the azimuthal dependence of NMO velocity. As an example, for the event at 1.54 s the surface $t(y)$ has a negative curvature in approximately the E–W direction (Figure 7a). Hence, the lateral velocity variation leads to an increase in the NMO velocity in the E–W azimuth [see equation (12)], making the uncorrected NMO ellipses in Figure 6a almost circular. By removing the influence of lateral velocity variation, we produce the ellipses extended in the orthogonal (S–N) direction, thus restoring the signature caused by the azimuthal anisotropy (Figure 9a). For the other three reflections, the interpretation of the results becomes more complicated because the correction term involves two traveltime surfaces ($\tau_0$ and $\tau_0$).

It should be emphasized that the correction for lateral velocity variation is highly sensitive to the shape of the surfaces $t(y)$. Since these surfaces are supposed to be differentiated twice [see equations (12) and (13)], the
form of the smoothing function can substantially change the corrected NMO ellipse. We attempted to reduce the degree of smoothing of NMO ellipses by using a higher-order bi-cubic polynomial \(|k + l \leq 3\) in equation (21)] or the running average over the area of the first Fresnel zone (see Figure 5). In both cases, we obtained unrealistic, large values of the corrected ellipticity (exceeding 0.3), which is indicative of overfitting the data and amplifying errors in the raw NMO ellipses.

Generalized Dix differentiation of the NMO ellipses

After removing the influence of lateral velocity variation, we can perform the last step of our processing sequence — apply the generalized Dix equation (9) to the ellipticities (Figure 9) and obtain the interval NMO ellipses (Figure 10). Azimuthal variation in the interval NMO velocity in Figure 10 is associated exclusively with azimuthal anisotropy and, therefore, carries information about fracturing. Assuming that each layer contains a single set of vertical fractures in an isotropic or VTI matrix, which is consistent with the available geologic information (Corrigan et al., 1996; Withers and Corrigan, 1997), the azimuth of the semi-major axis of the NMO ellipse gives the fracture orientation, while the ellipticity \(e\) is related to the fracture density (Tsvankin, 1997a).

In the layer between 1.54 s and 1.84 s, the semi-major axis of the NMO ellipse is predominantly oriented E—W, but the ellipticity is rather small — up to 3% (Figure 10a). The pattern changes significantly in the producing interval between 1.84 and 2.14 s. The azimuth of the NMO ellipse varies substantially over the area, with the maximum ellipticity reaching 8% (Figure 10b);
higher ellipticities are observed in the north-west and south-east corners of the survey area. In the deepest layer above the basement, the NMO ellipses are oriented close to the S-N direction, with the ellipticity somewhat increasing towards the south (Figure 10c).

The generalized Dix differentiation, as its conventional counterpart, suffers from amplification of errors for layers that are thin compared to their depth. The average relative thickness of the intervals used in our analysis reaches about 15%, which well might be considered close to the limit of applicability of the Dix differentiation. The NMO velocities, however, were obtained by two sequential averaging procedures, which should have increased the stability of the interval moveout estimation. Indeed, the NMO ellipse was built by a 3-D semblance search for the whole superbin (which amounts to azimuthal averaging of NMO velocity) followed by the spatial smoothing of the NMO ellipses over the survey area. As discussed below, the results of our interval moveout analysis (Figure 10) are supported by independent borehole measurements and shear-wave data.

Discussion and conclusions
We have developed and tested on field data a method of interval moveout analysis capable of separating the influence of azimuthal anisotropy and weak lateral velocity variation on normal-moveout velocity. The correction for lateral velocity variation in horizontally layered media depends on the shape (curvature) of the vertical-traveltime surface at the CMP location. Hence, removing the influence of weak lateral velocity variation does not require any information in addition to surface data.
Figure 9. Ellipticities corrected for lateral velocity variation for the reflections at 1.54 s (a), 1.84 s (b), 2.14 s (c), and 2.57 s (d).

Our processing methodology consists of the following main steps:

- **Azimuthal velocity analysis**
  We compute 3-D semblance as a function of vertical time at a fixed CMP location for all available data (i.e., including all source-receiver offsets and azimuths) using the hyperbolic moveout equation (1) parameterized by the NMO-velocity ellipse. The semblance maximum at a certain vertical time corresponds to the NMO ellipse for the corresponding reflection event. Azimuthal velocity analysis is not a model-dependent procedure and can be applied to any pure-mode reflection data.

- **Smoothing of the vertical traveltimes and NMO ellipses**
  Smoothing is needed to reduce picking errors and compute vertical-traveltime surfaces and NMO ellipses suitable for the subsequent processing steps. Without careful smoothing we would not have been able to account for lateral velocity variation in our case study and would have no hope of obtaining meaningful interval NMO ellipses.

- **Correction for lateral heterogeneity**
  The presence of lateral velocity variation significantly complicates the recovery of the interval moveout parameters from surface reflection data. Generally, moveout inversion in laterally heterogeneous anisotropic media would require tomographic algorithms based on anisotropic ray tracing. Here, we found a simple approximate way of correcting for weak lateral velocity variation in horizontally layered anisotropic models without ray tracing. To do this, however, we had to assume that not only the lateral variation in the elastic constants, but also the
anisotropy is weak in order to be able to apply our equations to vertically inhomogeneous media. We also had to substantiate that the influence of reflector dip was negligible. Our approximate solutions [equations (12) – (14)] indicate that the influence of lateral velocity variation on the NMO ellipses is controlled by the curvature of the vertical-traveltime surface $\tau(y)$, where $y$ are lateral (CMP) coordinates. Stable estimation of the curvature is impossible without smoothing the traveltime surface.

- **Generalized Dix differentiation**
  After correcting the NMO ellipses for lateral velocity variation, one can obtain the *interval* NMO velocity by the generalized Dix differentiation (9). For most common azimuthally anisotropic models caused by a single system of vertical fractures, the semi-major axis of the $P$-wave interval NMO ellipse coincides with the fracture direction, while the fracture density is related to the difference between the semi-axes.

Applying this processing sequence to a “wide-azimuth” 3-D data set from the Powder River Basin, Wyoming, we obtained the interval $P$-wave NMO ellipses shown in Figure 10. Undoubtedly, the final result is influenced to a certain extent by a number of assumptions and approximations discussed above. Also, small ellipticities for all reflection events made the recovery of the anisotropic signature in moveout velocity rather difficult. Therefore, it is important to compare our conclusions with information about the fracture orientation obtained by other methods (D. Corrigan, personal communication):

- The shallow section (above $t = 1.54$ s) has fractures oriented approximately N110E. This is supported by out-
crop measurements and FMI/FMS borehole scans, and is recognizable on the rotational analysis of 4-component S-wave data. The N110E orientation is in general agreement with our results.

- Outcrop studies and analysis of a nine-component VSP survey indicate that the fractures in the interval between \( t = 1.84 \) and 2.14 s are oriented approximately N70E. The same orientation was obtained by layer stripping analysis of the surface S-wave data. These are producing fractures whose orientation and density are of most interest from an exploration standpoint. The average azimuth of the semi-major axis of the NMO ellipse (Figure 10b) is indeed between N50E and N70E; note, however, that moveout data are indicative of a substantial variability in both the direction and density of fractures.

- Fracture orientation in the interval between \( t = 2.14 \) and 2.57 s is less well established. However, there is some outcrop evidence for a S–N orientation, which agrees with the moveout-inversion results.

One of the most important issues related to this analysis is which anisotropic coefficients can be estimated from the interval P-wave NMO ellipses. The answer is model-dependent and becomes increasingly complex for lower anisotropic symmetries. For horizontal transverse isotropy (the simplest azimuthally anisotropic model), the difference between the semi-axes is roughly proportional to the anisotropic coefficient \( \delta^{(y)} \), which is close to the fracture density for thin fluid-filled cracks (Tsvankin, 1997a). The semi-major axis of the NMO ellipse for HTI media is equal to the vertical P-wave velocity and points in the direction of fractures (see the ticks in Figure 10).

If the model is orthorhombic with a horizontal symmetry plane, the ellipticity is determined by the difference between the two anisotropic coefficients, \( \delta^{(1)} \) and \( \delta^{(2)} \), defined in the vertical symmetry planes of the medium (Tsvankin, 1997b). For orthorhombic media with a single fracture system, this difference is related to the crack density (it goes to zero if there are no fractures), and one of the semi-axes is parallel to the crack orientation. However, orthorhombic symmetry can also be caused by two orthogonal fracture systems, or two nonorthogonal systems with equal crack density. For these more complicated physical models the interpretation of the NMO ellipse becomes more ambiguous. Therefore, it is important to combine P-wave moveout with additional data, such as the traveltimes of split shear and converted (PS) waves, Prestack amplitudes, etc.

It should be mentioned that for both HTI and orthorhombic media the axes of the P-wave NMO ellipse are parallel to the polarization vectors of two split shear waves at vertical incidence. More complex fracture models may have monoclinic symmetry (e.g., due to two unequal nonorthogonal sets of cracks), for which the orientation of P-wave NMO ellipse deviates from the polarization directions of vertically traveling shear waves.

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References


APPENDIX A: NMO ellipse in a layer with weak lateral velocity variation

Here, we obtain a first-order correction for the influence of weak lateral velocity variation (for brevity, we will call it lateral heterogeneity, or LH) on the NMO ellipse in a single horizontal anisotropic layer. The general 3-D NMO equation (2) parameterized by the matrix $W$ [equation (3)] accounts for both anisotropy and lateral heterogeneity, unless deviations from hyperbolic moveout make the very notion of normal-moveout velocity meaningless. Our goal is to find the relationship between the matrix $W^{\text{het}}$, responsible for the NMO ellipse in an anisotropic LH layer, and the matrix $W^{\text{hom}}$ for a reference homogeneous medium.

Assuming that lateral velocity variation is weak, and following the approach developed by Grechka (1998) for vertical transverse isotropy, we apply first-order perturbation theory to express the matrix $W^{\text{het}}$ as the sum of $W^{\text{hom}}$ and small quantities related to LH. We restrict ourselves to anisotropic models with a horizontal symmetry plane (i.e., the medium can be transversely isotropic, orthorhombic, or monoclinic), which imposes additional constraints on the character of lateral velocity variation. While each elastic constant can vary laterally in a different fashion, we assume that these spatial variations do not destroy the symmetry of the phase- and group-velocity surfaces with respect to the horizontal plane.

Using equation (3) for a common midpoint (CMP) located at $y(y_1, y_2) = 0$ (Figure A1), we find

$$ W_{ij}^{\text{het}} = \tau_{ij}^{\text{het}} \frac{\partial^2 \tau_{ij}^{\text{het}}}{\partial x_i \partial x_j} \bigg|_{x=0} , \quad (i, j = 1, 2) \quad (A1) $$

and

$$ W_{ij}^{\text{hom}} = \tau_{ij}^{\text{hom}} \frac{\partial^2 \tau_{ij}^{\text{hom}}}{\partial x_i \partial x_j} \bigg|_{x=0} , \quad (i, j = 1, 2) \quad (A2) $$

where $\tau_{ij}^{\text{het}}$ and $\tau_{ij}^{\text{hom}}$ are the one-way traveltimes from the zero-offset reflection point to the surface location $x$; $\tau_{ij}^{\text{het}}(x = 0)$ and $\tau_{ij}^{\text{hom}}(x = 0)$ are zero-offset traveltimes. $\tau_{ij}^{\text{hom}}$ is obtained for a reference homogeneous, anisotropic layer with the parameters taken at the CMP. One-way traveltimes from the zero-offset reflection point appear in equations (A1) and (A2) because reflection-point dispersions have no influence on normal-moveout velocity, even if the medium is anisotropic and heterogeneous (Hubral and Krey, 1980; Grechka and Tsvankin, 1996).

Since the layer has a horizontal symmetry plane, in the absence of lateral heterogeneity the zero-offset reflection point $R$ coincides with the projection of the CMP onto the boundary and has the coordinates $\{3, 0, z_3\}$.
The traveltime $\tau^{\text{hom}}$ in the homogeneous layer is given by

$$
\tau^{\text{hom}}(x_1, x_2) = \tau^{\text{hom}}(h, \alpha) = \frac{\sqrt{x_1^2 + x_2^2 + x_3^2}}{g(\alpha, \theta)},
$$

where $x_1, x_2$ are the receiver coordinates, $h = \sqrt{x_1^2 + x_2^2}$ is half the source-receiver offset, and $g(\alpha, \theta)$ is the group velocity as a function of the azimuth of the CMP line $\alpha = \tan^{-1}(x_2/x_1)$ and the polar angle $\theta = \tan^{-1}(h/x_3)$.

In general, lateral velocity variation leads to a shift of the zero-offset reflection point from $R$ to a new location $R_c$ (Figure A1). Nonetheless, since LH is assumed to be weak, the traveltimes perturbations due to lateral velocity variation can be computed along the unperturbed ray propagating in the homogeneous model (Grechka and McMechan, 1997). Hence, we obtain the traveltime $\tau^{\text{het}}$ as an integral along the non-specular raypath originated at point $R$:

$$
\tau^{\text{het}}(x_1, x_2) = \sqrt{\frac{h^2 + x_3^2}{h}} \int_0^h \frac{d\xi}{g(\alpha, \theta, y_1(\xi), y_2(\xi))},
$$

where the group velocity $g$ now depends on the lateral coordinates along the ray 

$$y_1 = \xi \cos \alpha \quad \text{and} \quad y_2 = \xi \sin \alpha; \quad \text{(A5)}$$

$\xi$ is the horizontal displacement along the ray. Assuming that the lateral variation in group velocity is sufficiently smooth, it can be expanded in a double Taylor series in the vicinity of the common midpoint $y_1 = y_2 = 0$:

$$g(\alpha, \theta, \xi) = g(\alpha, \theta, y_1, y_2) = g_0 \left[ 1 + \frac{1}{2} g_{ij} y_i y_j + \frac{1}{8} g_{ijk} y_i y_j y_k + \cdots \right],
$$

where

$$g_0 \equiv g(\alpha, \theta, 0, 0), \quad g_{ij} \equiv \frac{\partial g(\alpha, \theta, y_1, y_2)}{\partial y_i \partial y_j} \bigg|_{y_1 = y_2 = 0}, \quad \text{and} \quad g_{ijk} \equiv \frac{\partial^3 g(\alpha, \theta, y_1, y_2)}{\partial y_i \partial y_j \partial y_k} \bigg|_{y_1 = y_2 = 0} \quad \text{A9}
$$

Since the lateral velocity variation is weak, all terms involving $y_i$ in equation (A7) are small compared to unity, and in the linear approximation

$$
-\frac{1}{2g_0} \sum_{i,j=1}^2 g_{ij} y_i y_j - \cdots
$$

Substituting equations (A5), (A7) – (A10) into (A4) and evaluating the integral yields

$$
\tau^{\text{het}}(x_1, x_2) = \sqrt{\frac{h^2 + x_3^2}{g_0}} \left[ 1 - \frac{h}{2g_0} \left( g_{11} \cos^2 \alpha + g_{22} \sin^2 \alpha \right) \right.
$$

$$
- \frac{h^2}{6g_0} \left( g_{11} \cos^2 \alpha + 2g_{12} \sin \alpha \cos \alpha \right.
$$

$$
+ g_{22} \sin^2 \alpha \right) - \cdots \bigg]. \quad \text{(A11)}
$$

From equation (A3) it is clear that the term in front of the brackets in equation (A11) is the traveltime $\tau^{\text{hom}}(x_1, x_2)$. Using the relations $x_1 = h \cos \alpha$ and $x_2 = h \sin \alpha$ (Figure A1), equation (A11) can be rewritten in the form

$$
\tau^{\text{het}}(x_1, x_2) = \tau^{\text{hom}}(x_1, x_2) \left[ 1 - \frac{1}{2g_0} \left( g_{11} x_1 + g_{22} x_2 \right) \right.
$$

$$
- \frac{1}{6g_0} \left( g_{11} x_1^2 + 2g_{12} x_1 x_2 + g_{22} x_2^2 \right) - \cdots \bigg]. \quad \text{(A12)}
$$

Equation (A12) expresses the contribution of the lateral velocity variation to the one-way traveltime in terms of the spatial derivatives of the group-velocity function. Note that since we ignored the influence of LH on reflection-point dispersal, the zero-offset ray in the LH layer remains vertical, and

$$
\tau_0^{\text{het}} = \tau_0^{\text{hom}}. \quad \text{(A13)}
$$

To obtain the matrix $W^{\text{het}}$ defined by equation (A11), we need to find the second-order partial derivatives of the traveltime $\tau^{\text{het}}$ [equation (A12)] with respect to the coordinates $(x_1, x_2)$ at zero offset $x = 0$:

$$
\frac{\partial^2 \tau^{\text{het}}}{\partial x_i \partial x_j} \bigg|_{x=0} = \frac{\partial^2 \tau^{\text{het}}}{\partial x_i \partial x_j} \bigg|_{x=0} = \frac{g_{ij}^{\text{hom}}}{g_0} - \frac{g_{ij}^{\text{hom}}}{g_0} - \frac{g_{ij}^{\text{hom}}}{g_0} - \frac{g_{ij}^{\text{hom}}}{g_0}
$$

$$
+ \left[ \frac{\partial}{\partial x_i} \left( \frac{g_{ij}^{\text{hom}}}{g_0} \right) + \frac{\partial}{\partial x_j} \left( \frac{g_{ij}^{\text{hom}}}{g_0} \right) \right] \bigg|_{x=0} \bigg]. \quad \text{(A14)}
$$

The second and third terms on the right-hand side of equation (A14) depend on the horizontal components of the slowness vector $p$ in the homogeneous layer:

$$
\frac{\partial \tau^{\text{hom}}}{\partial x_i} = p_i. \quad \text{(A15)}
$$
At zero offset, \( p_1(x = 0) = p_2(x = 0) = 0 \) because the slowness vector of the zero-offset ray is vertical [it has to be orthogonal to the (horizontal) reflector].

Next, let us show that the terms \( \frac{\partial \theta}{\partial x_1} (\frac{\partial}{\partial x_1}) \) and \( \frac{\partial \theta}{\partial x_2} (\frac{\partial}{\partial x_2}) \) in equation (A14) vanish as well. Indeed, \( g_0 \) is an even function of \( x_1 \) and \( x_2 \) because the group velocity in a medium with a horizontal symmetry plane is symmetric with respect to vertical in any vertical plane. The derivatives of group velocity with respect to \( y_i \) (\( g_{11} \) and \( g_{22} \)) also have to be even functions of \( x_1 \) and \( x_2 \); otherwise, the symmetry of group velocity with respect to the horizontal plane will not be preserved away from the CMP location. Therefore, the derivatives of \( (\frac{\partial}{\partial x_1}) \) and \( (\frac{\partial}{\partial x_2}) \) with respect to \( x_1 \) and \( x_2 \) go to zero at \( x = 0 \).

Thus, equation (A14) reduces to

\[
\frac{\partial^2 \tau_{\text{het}}}{\partial x_i \partial x_j} \bigg|_{x=0} = \frac{g_{i,j}^{\text{hom}}}{3g_0} \bigg|_{x=0}.
\] (A16)

The group velocity \( g_0 \) and its spatial derivatives \( g_{i,j} \) in equation (A16) are evaluated at the zero offset \( x = 0 \), i.e., for the vertical ray emanating from the zero-offset reflection point. Since in a medium with a horizontal symmetry plane the vertical phase and group velocities are equal to each other, we can replace \( g_0 \) and \( g_{i,j} \) with the vertical phase velocity \( V_0 \) and its derivatives with respect to \( y_i \). Finally, multiplying equation (A16) by \( \tau_0 \) by \( \tau_0 \) [see equation (A13)], we obtain the following relationship between the matrices \( W^{\text{hom}} \) and \( W^{\text{het}} \):

\[
W_{ij}^{\text{het}} = W_{ij}^{\text{hom}} - \frac{\tau_0^2}{3V_0} \frac{\partial^2 V_0}{\partial y_i \partial y_j} \bigg|_{y=0}, \quad (i, j = 1, 2).
\] (A17)

Thus, in the linear approximation employed here, the influence of lateral velocity variation on the NMO ellipse is proportional to the curvature of the vertical-velocity surface \( V_0(y) \) at the CMP location.

**APPENDIX B: NMO ellipse in a two-layered model with weak lateral velocity variation**

Here, we extend the derivation from Appendix A to a model that contains two horizontal anisotropic layers with lateral velocity variation. In addition to the assumptions used in Appendix A (the horizontal plane is a plane of symmetry, the lateral heterogeneity is weak), we consider only weakly anisotropic media. The weak-anisotropy assumption makes it possible not only to simplify the derivation, but also to obtain the final expression for the NMO ellipse in terms of quantities that can be measured from reflection seismic data – the NMO velocities and zero-offset traveltimes for reflections from the two interfaces of our model (Figure B1).

In the model without lateral velocity variation, the one-way traveltime \( \tau_0 \) from the zero-offset reflection point to the surface is given by

\[
\tau^0(h) = \tau_1^{\text{hom}} + \tau_2^{\text{hom}} = \frac{\sqrt{(h^{(1)})^2 + (x_3^{(1)})^2}}{g^{(1)}} + \frac{\sqrt{(h^{(2)})^2 + (x_3^{(2)})^2}}{g^{(2)}},
\] (B1)

where \( g^{(1)} \) and \( g^{(2)} \) are the group velocities along the ray (Figure B1), and \( h = h^{(1)} + h^{(2)} \) is half the source-receiver offset. For convenience, we denote

\[
h^{(2)} = kh \quad \text{and} \quad h^{(1)} = (1-k)h.
\] (B2)

Despite the presence of the horizontal symmetry plane in each layer, the influence of depth-varying azimuthal anisotropy forces the ray to deviate from the vertical incidence plane shown in Figure B1. This deviation, however, produces traveltime distortions quadratic in the anisotropy coefficients (Grechka and Tsvankin, 1996). Since we restrict ourselves to the linear weak-anisotropy approximation, the traveltimes in both homogeneous and L/H models will be evaluated along the rays confined to the vertical incidence plane.

The one-way traveltime \( \tau^{\text{het}} \) in the model with lateral velocity variation can be found by analogy with equation (A4):
where $\xi$ is the horizontal displacement along the ray. Expanding the interval group velocities $g^{(1)}$ and $g^{(2)}$ near the CMP in a double Taylor series in the horizontal coordinates [see equation (A7)] and evaluating the integrals in equation (B3), we express $\tau^\text{het}$ through the traveltimes $\tau^\text{hom}_1$ and $\tau^\text{hom}_2$ in the laterally homogeneous model:

$$
\tau^\text{het}(x) = \tau^\text{hom}_1(x) \left[ 1 - \frac{1 + k^2}{6g_0^{(1)}} \left( g^{(1)}_{1,1} x_1 + g^{(1)}_{1,2} x_2 \right) \right] - \frac{1 + k^2}{6g_0^{(2)}} \left( g^{(2)}_{1,1} x_1 + g^{(2)}_{1,2} x_2 \right) + \tau^\text{hom}_2(x) \left[ 1 - \frac{k^2}{3g_0^{(2)}} \left( g^{(2)}_{1,1} x_1 + g^{(2)}_{1,2} x_2 \right) \right],
$$

where $x_1 = h \cos \alpha$, $x_2 = h \sin \alpha$ ($\alpha$ is the azimuth of the CMP line), and $g_0^{(1,2)}$, $g^{(1,2)}_i$, and $g^{(1,2)}_{ij}$ are defined in the same way as $g_0$, $g_i$, and $g_{ij}$ in Appendix A.

Next, we express the parameter $k$ [equation (B2)] in terms of the velocities and traveltimes. Snell’s law at the transmission point $R_1$ (Figure B1) can be written as

$$
\sin \theta_1 = \frac{g^{(1)}}{g^{(2)}},
$$

where, in the weak anisotropy limit, we have replaced the phase velocities in both layers by the group velocities. Multiplying the numerator and denominator on both sides of equation (B5) by the corresponding ray lengths in each layer $l^{(1,2)} = \tau^\text{hom}_1 g^{(1,2)}$ yields (see Figure B1)

$$
\frac{h^{(1)}}{\tau^\text{hom}_1 g^{(1,2)^2}} = \frac{h^{(2)}}{\tau^\text{hom}_2 g^{(2,2)^2}}.
$$

From equation (B6) and the definition of $k$ [equation (B2)] it follows that

$$
k = \frac{\tau^\text{hom}_2 g^{(1,2)^2}}{\tau^\text{hom}_1 g^{(1,2)^2} + \tau^\text{hom}_2 g^{(2,2)^2}}.
$$

Note that the dependence of $k$ on $x$ in equation (B4) for small incidence angles can be neglected. Indeed, replacing the sines with tangents in equation (B5), we find

$$
\frac{1 - k}{k} = \frac{g^{(1,2)}}{g^{(2,2)}}.
$$

Although the ratio of the group velocities $g^{(1,2)}/g^{(2)}$ (and $k$) does change with $x$, the offset-dependent terms are entirely due to the anisotropy and lateral velocity variation. Hence, these terms can be ignored in equation (B4), where they are multiplied with small quantities related to lateral variation in group velocity.

To obtain the matrix $W^\text{het}$, it is necessary to evaluate the second-order partial derivatives of equation (B4) with respect to $x_1$ and $x_2$ at the CMP location ($x = 0$). Since the interfaces are horizontal and the horizontal plane is a plane of symmetry, $\partial^2 \tau^\text{hom}_1/\partial x_1$ and $\partial^2 \tau^\text{hom}_2/\partial x_i$ vanish at $x = 0$ (see Appendix A). Taking this into account and ignoring the dependence of $k$ on $x$ (see above), we obtain

$$
\frac{\partial^2 \tau^\text{het}}{\partial x_i \partial x_j} \bigg|_{x=0} = \frac{\partial^2}{\partial x_i \partial x_j} \left( \tau^\text{hom}_1 + \tau^\text{hom}_2 \right) \bigg|_{x=0} = \frac{1 + k^2}{3g_0^{(2)}} \left( \tau^\text{hom}_1 g^{(1,2)}_{ij} + \tau^\text{hom}_2 g^{(2,2)}_{ij} \right)_{x=0}.
$$

Multiplying equation (B10) by the zero-offset travelt ime

$$
\tau_0 \equiv \tau^\text{het} = \tau^\text{hom}_1 + \tau^\text{hom}_2 = \tau_{0,1} + \tau_{0,2}
$$

and using the definitions of the matrices $W^\text{het}$ and $W^\text{hom}$ [equations (A1) and (A2)] yields

$$
W^\text{het}_{ij} = W^\text{hom}_{ij} - \tau_0 \left[ \frac{1 + k^2}{3} \frac{\tau_{0,1} g^{(1,2)}_{ij}}{g_0^{(2)}} + \frac{k^2 \tau_{0,2} g^{(1,2)}_{ij}}{g_0^{(2)}} \right] \bigg|_{x=0}, \quad (i, j = 1, 2).
$$

Since both reflectors in our model are horizontal (see Figure B1), the second-order partial derivatives of the vertical velocities $g^{(1,2)}_{ij}$ and $g^{(2,2)}_{ij}$ in equation (B11) can be expressed in terms of derivatives of the vertical traveltimes. Differentiating equations $\tau_{0,1} g^{(1,2)}_{ij} = \text{const}$, ($\ell = 1, 2$) twice with respect to spatial coordinates $y_i$ gives in the linear approximation [see equation (11)]

$$
\tau_{0,1,ij} = -\tau_{0,ij} g_0^{(1,2)},
$$

where $\tau_{0,ij} \equiv \partial^2 \tau_{0,ij}/\partial y_i \partial y_j$. Substituting equations (B12) into equations (B11) yields

$$
W^\text{het}_{ij} = W^\text{hom}_{ij} \left[ \left( 1 + k^2 \right) \tau_{0,1,ij} / g_0^{(1,2)} + k^2 \tau_{0,2,ij} \right] \bigg|_{x=0}, \quad (i, j = 1, 2).
$$
The derivative $\tau_{02,ij}$ can be expressed in terms of $\tau_0$ and $\tau_{01}$ using equation (B10):

$$\tau_{02,ij} = \tau_{0,ij} - \tau_{01,ij}, \quad (i, j = 1, 2).$$  \hfill (B14)

Substituting equation (B14) into (B13), we find the matrix $W^{\text{het}}$ as

$$W^{\text{het}}_{ij} = W^{\text{hom}}_{ij} + \frac{\tau_0}{3} \left[ k^2 \tau_{0,ij} + (1 + k) \tau_{01,ij} \right],$$  \hfill (B15)

$(i, j = 1, 2)$.

The value of $k$ at zero offset is given, by equation (B7), in terms of vertical group velocities and traveltimes. In the weak anisotropy approximation, the vertical velocities $g_0^{(1,2)}$ can be further replaced with the interval normal-moveout velocities $V_{\text{cir}}^{(1,2)}$ [equation (15)] because $g_0^{(1,2)}$ contribute only to the already small terms that involve lateral velocity variation. Then equation (B7) becomes

$$k = \frac{\tau_{01} \left( V_{\text{cir}}^{(1)} \right)^2}{\tau_{01} \left( V_{\text{cir}}^{(1)} \right)^2 + \tau_{02} \left( V_{\text{cir}}^{(2)} \right)^2}. \hfill (B16)$$

The interval NMO velocity in the second layer $V_{\text{cir}}^{(2)}$ cannot be obtained directly from reflection data.

To express the interval NMO velocity $V_{\text{cir}}^{(2)}$ in the second layer through the NMO velocities for reflections from the bottom of the first ($V_{\text{cir}}^{(1)}$) and second ($V_{\text{cir}}$) layer, we use the conventional Dix (1955) formula:

$$\tau_0 V_{\text{cir}}^2 = \tau_{01} \left( V_{\text{cir}}^{(1)} \right)^2 + \tau_{02} \left( V_{\text{cir}}^{(2)} \right)^2. \hfill (B17)$$

In general, normal-moveout velocity in azimuthally anisotropic media should be obtained by averaging the interval NMO ellipses rather than the NMO velocities at a particular azimuth (Tsvankin et al., 1997). Nonetheless, Tsvankin et al. (1997) also show that the conventional Dix equation (B17) provides a linear approximation (in the anisotropic coefficients) to the exact “generalized” Dix-type averaging of the NMO ellipses.

Substituting equation (B17) into equation (B16) and using relation (B10) allows us to express $k$ in terms of quantities that can be obtained from reflection data:

$$k = 1 - \frac{\tau_{01} V_{\text{cir}}^2}{\tau_0 V_{\text{cir}}^2}. \hfill (B18)$$

Hence, the final result is given by equation (B15) with $k$ defined in equation (B18).
Dip moveout of converted waves and parameter estimation in transversely isotropic media

Ilya Tsvankin and Vladimir Grechka

ABSTRACT
For transverse isotropy with a vertical symmetry axis (VTI media), \( P \)-wave reflection data alone are insufficient for building velocity models in depth. Here, we show that all parameters of VTI media responsible for \( P - SV \) propagation (the \( P \)-wave and \( S \)-wave vertical velocities \( V_{P0} \) and \( V_{S0} \) and the anisotropic parameters \( \epsilon \) and \( \delta \)) can be obtained by combining \( P \)-wave traveltimes with the moveout of the converted \( PS \)-wave from a horizontal and dipping reflector. Using converted modes (rather than pure \( S \)-waves) makes the method more practical by avoiding expensive shear-wave excitation.

The inversion algorithm is based on a new analytic description of the dip moveout of \( PS \)-waves developed for symmetry planes of anisotropic media (and for any vertical plane in models with \textit{weak} azimuthal anisotropy). The common-midpoint (CMP) traveltime-offset relationship, derived in a parametric form and represented through the components of the slowness vector of the \( P \) and \( S \)-waves, makes it possible to compute the moveout curve of the \( PS \)-wave without two-point ray tracing. This formalism also yields closed-form solutions for moveout attributes, such as the coordinates \((x_{\text{min}}, t_{\text{min}})\) of the traveltine minimum and the normal-moveout (NMO) velocity defined at the apex of the moveout curve by analogy with pure modes. If reflector dip exceeds 40-50°, and CMP traveltimes of the \( PS \)-wave does not have a minimum, a convenient attribute is the slope of the moveout curve (apparent slowness) at zero offset. We found a simple representation of the moveout slope in CMP geometry by proving that it is always (even in inhomogeneous media) determined by the difference between the ray parameters at the source and receiver locations.

Analytic results were incorporated into a parameter-estimation technique for VTI media (and symmetry planes of orthorhombic media), which operates with reflection moveout of \( P \)- and \( PS \)-waves from a horizontal and dipping reflector. Application of the weak-anisotropy approximation allowed us to simplify the dependence of the \( PS \)-wave moveout attributes on the anisotropic parameters and helped to design the inversion procedure. The NMO velocities of \( P \) and \( PS \)-waves from horizontal events and the ratio of the corresponding zero-offset traveltimes provide three equations for the four unknown medium parameters. The remaining parameter is found from an overdetermined system of equations that includes the \( P \)-wave NMO velocity and moveout attributes of the \( PS \)-wave for a dipping event. Numerical analysis shows that the \( PS \)-wave dip-moveout signature plays a crucial role in obtaining accurate estimates of the anisotropic parameters. The joint inversion of \( P \) and \( PS \) data provides the necessary information not only for \( P \)-wave depth imaging in VTI media, but also for the processing of \( PS \) data.

Introduction
Recent advances in the development and application of anisotropic processing algorithms (e.g., Alkhalifah et al., 1996; Anderson and Tsvankin, 1997) were made possible by new approaches to the inversion of surface seismic data for the anisotropic parameters. Alkhalifah and
Tsvankin (1999) showed that P-wave* reflection move-out and all time-processing steps [NMO and dip-moveout (DMO) corrections, prestack and poststack time migration] in VTI media depend on just two parameters – the normal-moveout velocity from a horizontal reflector \( V_{nmo,P}(0) \) and the "anellipticity" coefficient \( \eta \). In terms of Thomsen's (1986) parameters, \( V_{nmo,P}(0) \) and \( \eta \) are given by

\[
V_{nmo,P}(0) = V_P \sqrt{1 + 2\delta},
\]

\[
\eta = \frac{\epsilon - \delta}{1 + 2\delta},
\]

where \( V_P \) is the P-wave vertical velocity, and \( \epsilon \) and \( \delta \) are the anisotropic parameters responsible for the velocities of P- and SV-waves. [\( V_P, \epsilon \) and \( \delta \) are sufficient to determine all kinematic signatures of P-waves (Tsvankin, 1996), while SV-wave kinematics also depends on the shear-wave vertical velocity \( V_S \).] Both \( V_{nmo,P}(0) \) and \( \eta \) can be found from surface P-wave data using either the dip dependence of NMO velocity (Alkhalifah and Tsvankin, 1996) or nonhyperbolic (long-spread) move-out of horizontal events (Alkhalifah, 1997; Grechka and Tsvankin, 1997). Grechka and Tsvankin (1996) proved that it is possible to obtain \( V_{nmo,P}(0) \) and \( \eta \) from a single dipping event by performing a 3-D moveout analysis and inverting the azimuthal dependence of NMO velocity.

P-wave depth processing (such as prestack depth migration), however, requires knowledge of the vertical velocity \( V_P \) that cannot be obtained from P-wave reflection traveltimes alone. Only if the symmetry axis of TI media is tilted by at least 30-40° from vertical, azimuthally dependent P-wave NMO velocity from two or more reflectors with different dips and/or azimuths can be inverted for all parameters which control P-wave kinematics (Grechka and Tsvankin, 1998). Thus, to resolve the vertical velocity and the anisotropic parameters of VTI media, it is necessary to supplement P-wave traveltimes with additional data. Since the SV-wave velocity also depends on the anisotropic parameters \( \epsilon \) and \( \delta \), a natural option is to include reflection moveout of SV-waves into the inversion procedure. Tsvankin and Thomsen (1995) suggested combining long-spread (nonhyperbolic) moveout of P- and SV-waves from horizontal reflectors to obtain all four parameters, but this approach encounters practical problems stemming from the difficulties in acquiring and processing of long-spread shear data.

Alternatively, input data may include dip-dependent P-wave moveout (e.g., NMO velocities for two different dips), yielding the parameters \( V_{nmo,P}(0) \) and \( \eta \), and the NMO velocity of the SV-wave from a horizontal reflector:

\[
V_{nmo,SV} = V_S \sqrt{1 + 2\sigma},
\]

\[
\sigma \equiv \left( \frac{V_P}{V_S} \right)^2 (\epsilon - \delta).
\]

If pure shear waves are not excited, the SV-wave NMO velocity can be determined from the NMO velocities of the P- and converted PS-waves (Seriff and Siriram, 1991):

\[
t_{PS0} V_{nmo,PS}^2 = t_{P0} V_{nmo,P}^2 + t_{S0} V_{nmo,SV}^2,
\]

where \( t_{P0} \) and \( t_{S0} \) are the vertical traveltimes of the \( P \) and \( S \)-waves, and \( t_{PS0} \equiv t_{P0} + t_{S0} \). Also, if either \( SV \)-or \( PSV \)-waves are available, the ratio of the vertical velocities can be found from the vertical traveltimes:

\[
\frac{V_P}{V_S} = \frac{t_{S0}}{t_{P0}}.
\]

In principle, equations (1), (2), (3), and (6) are sufficient to recover all four unknown parameters \( (V_P, V_S, \epsilon \) and \( \delta ) \). Unfortunately, this inversion procedure turns out to be unstable, with realistic small errors in the input data propagating with considerable amplification into the inverted vertical velocities, \( \epsilon \), and \( \delta \) (Grechka and Tsvankin, 1998). This instability is caused by the form of the dependence of SV-wave NMO velocity on the anisotropic parameters [equations (3) and (4)]. After obtaining \( \eta \approx \epsilon - \delta \) from P-wave data and \( V_P/V_S \) from the vertical traveltimes, equation (3) can be used to find the S-wave vertical velocity. However, the multiplier \( (V_P/V_S)^2 \) translates small errors in \( \epsilon - \delta \) into substantially larger errors in \( \sigma \) and \( V_S \). For a typical \( V_P/V_S = 2 \), a relatively insignificant error of 0.03 in \( \epsilon - \delta \) will cause a distortion of 0.12 in \( \sigma \) and an error of about 12% in \( V_S \) and, consequently, in \( V_P \).

Here, we suggest a more stable parameter-estimation algorithm based on including dip-dependent reflection traveltimes of mode-converted PS-waves in the inversion procedure. Previous work on reflection moveout of converted waves was mostly restricted to isotropic media (e.g., Tassmer and Behle, 1988; Alfaraj, 1993). Equation (5) for NMO velocity of PSV-waves in horizontally layered VTI media was first given by Seriff and Siriram (1991). Tsvankin and Thomsen (1994) presented an analytic expression for the quartic moveout term of PSV conversions for vertical transverse isotropy and used it to describe nonhyperbolic (long-spread) reflection moveout. Anderson (1996) developed a TZO (transformation to zero offset) algorithm for vertical transverse isotropy that produces a zero-offset P-wave section from FMS data.

* For brevity, the qualifiers in "quasi-P-wave" and "quasi-S-wave" will be omitted.
Reflection-point dispersal of PSV-waves in a VTI layer was discussed by Rommel (1997), who noticed a significant influence of anisotropy on the positions of sources and receivers corresponding to the same reflection point. To avoid reflection-point smear during stacking, the data should be resorted into common-reflection-point gathers, but this operation cannot be carried out without knowledge of the medium parameters. Grechka et al. (1997) showed that the azimuthal variation of NMO velocity of converted waves in horizontally layered anisotropic media with a horizontal symmetry plane always has an elliptical form [the result previously proved by Grechka and Tsvankin (1996) for pure modes]. They also generalized relationship (5) between the NMO velocities of pure and converted waves to azimuthally anisotropic media and combined NMO and vertical velocities of P and PSV-waves to obtain the parameters of a horizontal orthorhombic layer.

We begin by giving a general analytic description of reflection moveout of converted waves in a symmetry plane of a homogeneous anisotropic layer. This formalism leads to closed-form expressions for the moveout curve and its attributes (NMO velocity near the traveltime minimum, the shift of the traveltime minimum from zero offset etc.) in terms of the horizontal and vertical slowness components of the P and S-waves. For vertical transverse isotropy, we employ the weak-anisotropy approximation to simplify the exact equations and explain the relationship between the moveout attributes and medium parameters. Then we perform joint inversion of the P- and PS-wave moveout from a horizontal and dipping reflector and show that the new method yields stable estimates of the vertical velocities and anisotropic parameters of VTI media. Although the inversion algorithm is developed for vertical transverse isotropy, it remains fully valid in the vertical symmetry planes of orthorhombic media.

**Dip moveout of converted waves in symmetry planes of anisotropic media**

The main difference between reflection moveout of converted and pure modes in CMP geometry is that mode conversion makes the moveout curve asymmetric with respect to zero offset. Only in the special case of horizontal reflectors and a medium with a horizontal symmetry plane, converted-wave reflection traveltime is an even function of the source-receiver offset (Grechka et al., 1997). The asymmetry of the converted-wave moveout
can be further enhanced by angular velocity variations in anisotropic media. Hence, in general the moveout of \(PS\)-waves cannot be described by the conventional traveltime series \(T(x^2)\) that contains only even powers of the offset \(x\).

Figure 1 shows typical traveltime curves of the \(PSV\)-wave computed for a common-midpoint (CMP) gather in the dip plane of a reflector beneath a VTI layer. The moveout becomes increasingly asymmetric with dip, and the traveltime minimum is recorded at “positive” offsets corresponding to the \(P\)-wave leg located \(downdip\) from the reflection point. For dips beyond 40°, the minimum moves to large offsets exceeding twice the CMP-reflector distance and then disappears altogether.

This behavior of \(PS\) moveout suggests using different sets of moveout attributes for mild and steep dips. If reflector dip is moderate and the \(PS\)-traveltime has a minimum on a CMP gather, it is convenient to introduce such moveout parameters, similar to those for pure modes, as the minimum traveltime \(t_{\text{min}}\), the source-receiver offset \(x_{\text{min}} = x(t_{\text{min}})\), and the normal-moveout velocity \(V_{\text{NMO}}\) responsible for traveltimes near the apex (minimum) of the moveout curve. For steeper dips, a parameter that controls traveltimes at moderate source-receiver offsets is the slope of the moveout curve at \(x = 0\). Below we give concise expressions for these moveout attributes and reflection traveltime as a whole for symmetry planes of an anisotropic layer.

**Parametric representation of \(PS\) traveltime**

Our goal here is to develop an analytic treatment of reflection moveout of converted waves in a homogeneous anisotropic layer overlying a dipping reflector. To make the problem two-dimensional, the incidence plane is assumed to coincide with both the dip plane of the reflector and a symmetry plane of the medium. [The same assumption was made by Tsankin (1995) in his derivation of the 2-D NMO equation for pure modes.] Although our formalism is not exact outside the vertical symmetry planes, it should still provide good accuracy for models with weak azimuthal anisotropy.

In the adopted “2-D” reflection model, the phase-velocity vectors and rays of reflected waves on the dip line do not deviate from the incidence plane. Also, the polarization vector of one of the split shear modes is perpendicular to the dip (incidence) plane, and this \(SH\)-wave is completely decoupled from the \(P\)- and \(SV\)-arrivals. Therefore, a \(P\)- or \(SV\)-wave incident upon the interface generates a single converted mode (\(PSV\) or \(SVP\)); for brevity, hereafter we denote these waves simply \(PS\) and \(SP\).

As shown in Appendix A, the traveltime and source-
tem with a vertical axis $z_3$ (Figure 2), equations (9)–(11) can be concisely expressed through the vertical and horizontal slowness components (Appendix A):

$$N = q_F - p_F q'_F + q_S - p_S q'_S,$$

$$D = 1 + \frac{1}{2} \tan \phi (q'_F + q'_S),$$

$$N_x = q'_F - q'_S,$$

where $p_F$ and $p_S$ are the horizontal components of the slowness vector for the $P$- and $S$-waves (respectively), $q_F$ and $q_S$ are the vertical slownesses, and $q'_F \equiv dq_F/dp_F$, $q'_S \equiv dq_S/dp_S$. The $x_3$-axis in equations (12)–(14) is directed upward, the $x_1$-axis – up dip, the $P$-wave leg is located down dip from the reflection point, and the group-velocity vectors of both waves are assumed to point towards the surface (Figure 2).

The derivative $dq/dp$ for both waves can be obtained in a straightforward way by implicit differentiation of the Christoffel equation (Tvanink et al., 1997). Substitution of equations (12)–(14) into equations (7) and (8) yields the offset and the corresponding traveltime of the converted wave for the CMP location with a given $z_{CMP}$. Although this approach involves solving the Christoffel equation for a range of slownesses $p_{sers}$, it does not require time-consuming two-point ray tracing for each source-receiver offset.

It should be emphasized that equations (12)–(14) are valid for dip moveout of converted waves in a symmetry plane of an anisotropic medium (e.g., the model can be orthorhombic). The analytic representation of dip moveout developed in this section is used below to obtain the NMO velocity and other attributes of the moveout curve.

Attributes of the $PS$ moveout function
The moveout attributes conventionally used in the traveltime inversion of pure-mode reflections include the normal-moveout (NMO) velocity and, sometimes, the higher-order moveout terms responsible for nonhyperbolic moveout. Due to the asymmetric shape of the common-midpoint $PS$ moveout curve with respect to zero offset, the attribute largely responsible for small-offset reflection traveltime is the slope of the moveout curve at $z = 0$. If reflector dip is mild and the $PS$ travelt ime has a minimum at moderate offsets, suitable attributes are NMO velocity and the coordinates (offset, traveltime) of the moveout minimum.

Slope of the moveout curve and position of the traveltime minimum
In Appendix D, we show that the apparent slowness (slope) of any moveout curve recorded in CMP geometry is determined by the difference between the horizontal slownesses (ray parameters) of the incident and reflected ray measured at the source and receiver locations. This representation of moveout slope is valid in any inhomogeneous anisotropic medium if the rays in the CMP gather do not deviate from the incidence plane (i.e., the incidence plane is supposed to be a plane of mirror symmetry). The derivation in Appendix D can be easily modified for data acquired in common-shot or common-receiver gathers. The slope of reflection moveout in a shot gather, for instance, is simply equal to the ray parameter of the reflected ray at the receiver location. (This result also follows from ray theory because the wavefield is excited by a fixed point source, the gradient of the traveltime at any point is equal to the slowness vector.)

For converted $PS$-waves, the slope of the $t(x)$ curve is given by

$$\frac{dt}{dx} = \frac{1}{2} (p_S - p_F),$$

with the horizontal slownesses $p_F$ and $p_S$ measured at the source and receiver locations. Equation (15) not only provides a simple expression for the slope itself, it also helps to obtain concise solutions for NMO velocity and other attributes of the moveout minimum.

If the medium above the reflector is horizontally homogeneous (as is the case with the single-layer model considered here), both $p_F$ and $p_S$ remain constant between the reflector and the surface. To find the moveout slope at zero offset, we determine $p_F$ and $p_S$ from the condition $q'_F = q'_S$, which ensures that the group-velocity vectors of the $P$- and $S$-wave are parallel to each other.

Equation (15) can also be used to find the ray parameter $p_{sers}^{\text{min}}$ corresponding to the minimum of the traveltime curve. Since the derivative $dt/dx$ vanishes at the traveltime minimum,

$$p_F(p_{sers}^{\text{min}}) = p_S(p_{sers}^{\text{min}}).$$

Note that in the special case of a horizontal reflector ($\phi = 0$), equation (16) is satisfied if the slowness vectors of the incident and reflected waves are vertical ($p_F = p_S = p_{sers} = 0$). Therefore, the minimum of the converted-wave traveltime from a horizontal reflector always corresponds to the vertical slowness vector, but the incident and reflected rays are not necessarily vertical, unless the medium has a horizontal symmetry plane. This means that in general the traveltime minimum of the converted wave from a horizontal reflector is located at a non-zero offset $x_{sers} \neq 0$, although the slowness vectors of the corresponding $P$ and $S$-waves are vertical.

Equation (16) also confirms the well-known fact that for a pure-mode reflection and arbitrary reflector dip
\( p_{\text{int}}^{\min} = 0 \). Indeed, if \( p_{\text{int}} = 0 \), the slowness vectors of the incident and reflected waves are orthogonal to the interface and parallel to each other, so in the absence of mode conversion, \( p_P = p_S \). As a result, for pure modes equation (16) is always satisfied at \( p_{\text{int}} = p_{\text{int}}^{\min} = 0 \), and the minimum traveltime is recorded at zero offset.

Using Snell’s law and equation (16), we obtain the following relationship between the slowness components corresponding to the traveltime minimum [equation (B17)]:

\[
2p_P(p_{\text{int}}^{\min}) = [(q_P + q_S) \tan \phi]_{p_{\text{int}}^{\min}}.
\]

The vertical slownesses \( q_P \) and \( q_S \) can be found as functions of \( p_P = p_S \) from the Christoffel equation. Therefore, equation (B17) can be solved in a straightforward way for the horizontal slowness \( p_P(p_{\text{int}}^{\min}) = p_S(p_{\text{int}}^{\min}) \) needed to evaluate the NMO velocity and other attributes associated with the traveltime minimum.

In Appendix B we also give an explicit solution of equation (17) for isotropic media and demonstrate that the traveltime minimum exists only if

\[
\tan \phi \leq \frac{2\gamma}{\gamma^2 - 1},
\]

where \( \gamma \equiv V_P/V_S \), and \( V_P \) and \( V_S \) are the velocities of the \( P \) and \( S \)-waves, respectively. For a typical \( \gamma = 2 \), the moveout curve has a minimum for reflector dips up to 53°. Equation (18) is not exact if the medium is anisotropic, but it still provides a good approximation for small and moderate values of the anisotropic coefficients.

**NMO velocity**

Although the CMP traveltime of converted waves from a dipping reflector is not an even function of source-receiver offset, the moveout curve near the traveltime minimum (if it does exist) can still be described by normal-moveout velocity defined in the same way as that for pure modes:

\[
V_{\text{nmo}}^2 = \left\{ \frac{1}{2} \frac{d^2(t^2)}{dz^2} \right|_{z_{\text{min}}} \right\}^{-1}.
\]

Expressing both the traveltime and source-receiver offset through the slowness projection \( p_{\text{int}} \) and using equation (15) for the moveout slope, we obtain NMO velocity in the following form (Appendix B):

\[
V_{\text{nmo},PS}^2 = \frac{4(q_P q_S A_S^2 + q_P^2 A_P^2)}{(A_P + A_S)^2 [p_P(q_P + q_S) - (q_P + q_S)]},
\]

where

\[
A_P = 1 + q_P^2 \tan \phi, \quad A_S = 1 + q_S^2 \tan \phi;
\]

the derivatives are evaluated at the slowness projection \( p_{\text{int}}^{\min} \), which is determined from equation (17).

For pure (nonconverted) modes, at the moveout minimum \( q_P = q_S \), \( q_P^2 = q_S^2 \), \( q_P^2 = q_S^2 \), and equation (20) reduces to the 2-D NMO equation of Tsvankin (1995) rewritten through the ray parameter (horizontal slowness) by Cohen (1998):

\[
V_{\text{nmo,pure}}^2 = \frac{qq''}{pq' - q} \bigg|_{p_{\text{int}} = 0} = V(\phi) \sqrt{\frac{1}{\cos \phi} \frac{d^2V}{d\phi^2} \bigg|_{\phi = \phi}}.
\]

where \( q' = dq/d\phi, \) and \( q'' \equiv d^2q/d\phi^2, \) and \( V(\theta) \) is phase velocity as a function of phase angle with vertical. It should be mentioned that reflection-point dispersal in a CMP gather, properly treated in our approach, was not accounted for by Tsvankin (1995). The identical result of the two derivations indicates, in agreement with Hubral and Krey (1980), that reflection-point dispersal has no influence on NMO velocity for pure-mode reflections.

For a horizontal reflector (\( \phi = 0 \)), the slowness \( p_P \) vanishes and the parameters \( A_P = A_S = 1 \) [equation (21)]. Hence, equation (20) becomes simply

\[
V_{\text{nmo,PS}}^2(\phi = 0) = - \frac{q_P^2 + q_S^2}{q_P + q_S} \bigg|_{p_{\text{int}}^{\min,0}}.
\]

For media with a horizontal symmetry plane, equation (23) is fully equivalent to the VTI relationship (5). Indeed, the zero-offset ray in this case is vertical, and the ratio of the zero-offset traveltimes of the \( P \)- and \( S \)-waves is equal to the ratio of the vertical slownesses \( q_P \) and \( q_S \):

\[
\frac{t_{P0}}{t_{S0}} = \frac{q_P}{q_S}.
\]

Substitution of equations (24) and (22) (for \( p = 0 \)) into equation (23) immediately yields equation (5).

**Offset of the traveltime minimum**

Another potentially useful attribute of \( PS \) moveout is the offset \( x_{\text{min}} \) of the traveltime minimum (for pure modes, \( x_{\text{min}} \) always equals zero). Since \( x_{\text{min}} \) contains the generally unknown reflector depth \( z_{\text{CMP}} \), it is convenient to normalize it by the minimum traveltime \( t_{\text{min}} \). Using equations (7), (8), (12)–(14) and taking into account that at the traveltime minimum \( p_P = p_S \) [see equation (16)], we find

\[
x_{\text{min}} = \frac{q_P - q_S}{q_P + q_S - p_P(q_P + q_S)} \bigg|_{p_{\text{int}}^{\min}}.
\]

By recording reflection moveout of a converted mode for a range of CMP locations in the dip plane of the reflector, we can also obtain the derivative of \( t_{\text{min}} \) with respect to the CMP coordinate \( y_{\text{CMP}} = z_{\text{CMP}} \tan \phi \). In the pure-mode case, the spatial derivative of the mini-
mum (zero-offset) traveltime determines the slopes of reflections on the zero-offset (stacked) section and is equal to the ray parameter (horizontal slowness) of the zero-offset ray. For converted modes, \( \frac{d t_{\text{min}}}{d y_{\text{CMP}}} \) does not have such a simple interpretation, but it can still provide useful information about the medium parameters. From equations (12)–(14) it follows that

\[
\frac{d t_{\text{min}}}{d y_{\text{CMP}}} = \tan \phi \frac{N}{D} = \tan \phi \frac{q_p + q_s - p_p(q_p + q_s)}{1 + \frac{1}{2} \tan \phi (q_p + q_s)} \bigg|_{p_{\text{int}}}.
\]

(26)

The spatial derivative \( dx_{\text{min}}/dy_{\text{CMP}} \) can be expressed as a combination of \( x_{\text{min}}/t_{\text{min}} \) [equation (25)] and \( dt_{\text{min}}/dy_{\text{CMP}} \) [equation (26)] and, therefore, does not provide an independent equation for the medium parameters.

Weak-anisotropy approximation for VTI media

Analytic developments in the two previous sections are completely general and can be used in a symmetry plane of any anisotropic medium with arbitrary strength of velocity anisotropy. Here, we apply the weak-anisotropy approximation to obtain simple relationships between the moveout attributes and parameters of transversely isotropic media with a vertical symmetry axis (VTI media). These results can also be applied in the vertical symmetry planes of orthorhombic media (assuming that one of the symmetry planes is horizontal) by replacing Thomsen parameters with the equivalent notation introduced in Tsvankin (1997).

In VTI models each vertical plane is a plane of mirror symmetry, and our 2-D formalism is valid for any azimuthal orientation of the reflector. For weakly anisotropic media with small Thomsen’s (1986) parameters \( \epsilon \) and \( \delta \), CMP traveltime and source-receiver offset of the PS-wave can be derived as explicit functions of the projection of the slowness vector on the reflector \( p_{\text{int}} \) [Appendix C, equations (C11), (C13), (C14), and (C15), combined with equations (7)–(11)]. The results of Appendix C make it possible to generate reflection moveout of converted waves for weakly anisotropic VTI media without doing ray tracing and even solving the Christoffel equation.

Despite the explicit form of the weak-anisotropy approximations for traveltime and offset, they are rather lengthy and do not provide an easy insight into the influence of anisotropy on PS reflection moveout. Therefore, to simplify the expressions for NMO velocity and other attributes, we assume that not only the anisotropic coefficients, but also reflector dip is relatively small, and all terms containing the cubic and higher powers of \( \sin \phi \) can be neglected.

For inversion purposes, it is convenient to replace reflector dip \( \phi \) with the ray parameter (horizontal slowness) \( p_{\text{po}} \) of the pure P-wave reflection recorded at zero offset (Alkhalifah and Tsvankin, 1995). Unlike reflector dip, \( p_{\text{po}} \) can be found from surface data by measuring reflection slopes on zero-offset (stacked) P-wave sections. In the “mild-dip” approximation employed here,

\[
p_{\text{po}} \approx \frac{\sin \phi}{V_{\text{po}}}.
\]

(27)

The slowness \( p_{\text{po}}^{\text{min}} \) corresponding to the traveltime minimum, which for arbitrary strength of anisotropy should be obtained from equation (17), in weakly VTI media is given by (Appendix C)

\[
p_{\text{po}}^{\text{min}} = \frac{\sin \phi}{2} \left( \frac{1}{V_{50}} - \frac{1}{V_{\text{po}}} \right) = \frac{p_{\text{po}}}{2} (\gamma - 1),
\]

(28)

where \( \gamma \equiv V_{\text{po}}/V_{50} \). Clearly, for mild dips and in the linear approximation with respect to \( \epsilon \) and \( \delta \), anisotropy has no influence on the value of \( p_{\text{po}}^{\text{min}} \).

The weak-anisotropy approximation for converted-wave NMO velocity is derived in Appendix C as

\[
V_{\text{nmo,PS}}^{-2}(p_{\text{po}}) = V_{\text{nmo,PS}}^{-2}(0) - \frac{p_{\text{po}}^2}{8\gamma} \left[ 3\gamma^4 - 2\gamma^3 \\
+ 6\gamma^2 - 2\gamma + 3 \right] \\
- \frac{p_{\text{po}}^2 (\gamma - 1)}{2\gamma (\gamma + 1)} \left[ 6\sigma (\gamma + 1)^2 \\
- (\sigma - \delta) (3\gamma^2 - 2\gamma - 3) \right].
\]

(29)

The parameter \( \sigma \) was introduced in equation (4), and \( V_{\text{nmo,PS}}^{-2}(0) \) is the NMO velocity from a horizontal reflector:

\[
V_{\text{nmo,PS}}^{-2}(0) = \frac{1}{V_{\text{po}}V_{50}} \left[ 1 - \frac{2(\sigma + \delta \gamma)}{1 + \gamma} \right],
\]

(30)

which coincides with the weak-anisotropy approximation of the exact equation (5) for \( V_{\text{nmo,PS}}^{-2}(0) \).

To estimate the contribution of the anisotropic parameters to the dip dependence of NMO velocity, we rewrite equations (30) and (31) for a typical velocity ratio \( \gamma = 2 \):

\[
V_{\text{nmo,PS}}^{-2}(p_{\text{po}}, \gamma = 2) = \frac{1}{V_{\text{po}}V_{50}} \left[ 1 - 1.3 (\delta + 0.5\sigma) \right] \\
- 3.4p_{\text{po}}^2 - p_{\text{po}}^2 (2.7\gamma + 1.8\delta).
\]

(31)

Thus, the anisotropic dip-dependent term \( p_{\text{po}}^2 (2.7\gamma + 1.8\delta) \) provides an equation for \( \sigma \) and \( \delta \) with comparable weights for both parameters. Equation (31) also shows that for positive values of \( \sigma \), commonly observed in VTI formations, anisotropy amplifies the increase in the NMO velocity with dip (usually \( \sigma > \delta \)). However,
for a typical $\sigma = 0.4 - 0.5$ the magnitude of the anisotropic dip-dependent term can reach only 35-40% of the isotropic one ($3.4p_{p0}^2$), and the dip-dependence of NMO velocity as a whole is not highly sensitive to the anisotropic parameters. Our numerical tests show that the contribution of the anisotropic parameters to the exact NMO velocity is even somewhat smaller than predicted by equation (31). Therefore, we supplement the PS-wave NMO velocity in the anisotropic inversion with the normalized offset of the traveltimes minimum ($x_{\text{min}}/t_{\text{min}}$), which proved to be more sensitive to the parameters $\sigma$ and $\delta$ (see below).

The accuracy of the weak-anisotropy approximation (30) is illustrated by Figure 3. Since we retained just the leading term in $\phi$ and $p_{p0}$, the approximation deviates from the exact solution with increasing dip. Despite this deterioration in accuracy, our approximations are sufficient for understanding the behavior of moveout attributes in the most important regime of moderate dips ($\phi < 35-40^\circ$). For steeper dips, the traveltimes minimum either does not exist at all or corresponds to unusually large source-receiver offsets seldom acquired in practice. Comparison of Figures 3a and 3b also shows that the error is higher for more "anelliptical" models with larger values of $\sigma$.

The linearized approximations for the normalized offset of the traveltimes minimum ($x_{\text{min}}/t_{\text{min}}$) and the spatial derivative of the minimum traveltime ($dt_{\text{min}}/dy_{\text{CMP}}$) have the following form (Appendix C):

$$x_{\text{min}} = \frac{p_{p0}V_{p0}}{2\gamma} \left[ (\gamma - 1) + 2(\delta\gamma - \sigma) \right],$$

$$\frac{dt_{\text{min}}}{dy_{\text{CMP}}} = \frac{p_{p0}}{2(1 + \gamma)} \left[ (1 - \gamma^2) + 4\gamma(\sigma - \delta) \right].$$

The weak-anisotropy approximation (32) for $x_{\text{min}}/t_{\text{min}}$ is less accurate than that for NMO velocity [equation (30)] because $x_{\text{min}}$ is already linear in $p_{p0}$, and we had to drop the terms quadratic in $p_{p0}$ (or in $\sin \phi$) from equation (C19) for $t_{\text{min}}$. Still, the main value of equation (32) and other approximations in this section is in providing analytic insight into the behavior of various moveout attributes. The form of equation (32) suggests that the normalized offset $x_{\text{min}}/t_{\text{min}}$ is quite sensitive to the parameters $\delta$ and $\sigma$ and should be included in the inversion procedure. In contrast, equation (33) for $dt_{\text{min}}/dy_{\text{CMP}}$ is purely isotropic and gives only redundant information about the ratio of the vertical velocities that can be determined in a conventional way using the vertical traveltimes [equation (6)].

The slope of the traveltime curve at zero offset in the weak-anisotropy approximation was obtained in Appendix C [equation (C28)] as

$$\frac{dt}{dx}\bigg|_{x=0} = \frac{p_{p0}}{2(1 + \gamma)} \left[ (1 - \gamma^2) + 4\gamma(\sigma - \delta) \right].$$

For such typical values as $\gamma = 2$ and $\sigma - \delta = 0.5$, the anisotropic term in equation (34) is about 30% greater than the isotropic one. Hence, we can expect $(dt/dx)|_{x=0}$ to provide reliable information about the anisotropic parameters $\delta$ and $\sigma$ (or $\delta$ and $\epsilon$).

On the whole, analytic approximations presented above indicate that dip moveout of the converted PS-wave can be efficiently used in anisotropic parameter estimation. This conclusion is supported below by nu-
merical inversion based on the exact equations for NMO velocity and other moveout attributes.

**Parameter estimation in VTI media using dip moveout of PS-waves**

As suggested by the form of the weak-anisotropy approximations, the addition of the dip moveout of PS-waves to P-wave data can help to stabilize parameter estimation in VTI media. Thus, the data used in the inversion for \( V_{P0}, V_{S0}, \varepsilon, \delta \) include the moveout of \( P \) and PS-waves from a horizontal reflector and on the dip line of a dipping reflector. Also, we assume that the \( P \)-wave ray parameter for the dipping event \((p_{P0})\) was determined from the slope of the \( P \)-wave reflection on the zero-offset (stacked) section. The vertical traveltimes and NMO velocities of the \( P \) and PS-waves from a horizontal reflector allow us to obtain the \( V_{P0}/V_{S0} \) ratio and the NMO velocity of the \( SV \)-wave [see equation (5)].

By using the \( P \)-wave NMO velocity for a dipping event [equation (22)], we include an equation for the anisotropic parameter \( \eta \) since \( V_{nmo,P}(p_{P0}) = f(V_{nmo,P}(0), \eta) \). Although this information [equations (1), (2), (3), and (6)] is sufficient for determination of all four unknowns, the solution of this inverse problem, as discussed in the introduction, suffers from instability; this is further confirmed by a numerical test below. The dip-moveout attributes of the PS-wave allow us to build an overdetermined system of equations needed to obtain more accurate estimates of the vertical velocities and anisotropic parameters.

An important practical issue is how to determine the attributes associated with the PS-wave traveltimes minimum \((t_{min}, x_{min})\) from moveout data. Since the first derivative of the PS-traveltimes curve \((dt/dx)\) goes to zero at \( x_{min} \), we suggest to approximate the PS moveout with a hyperbola centered at the traveltimes minimum:

\[
t^2(x) = t^2_{min} + \frac{(x - x_{min})^2}{V^2_{nmo,PS}}. \tag{35}
\]

If the traveltimes have not been picked, the best-fit hyperbola can be found from semblance velocity analysis along moveout curves described by equation (35).

A typical example illustrating the application of a shifted hyperbola to the recovery of the moveout attributes is shown in Figure 4. Note that the exact PS-wave traveltimes \( t(x) \) (solid) are generally asymmetric with respect to \( x_{min} \) due to the presence of a term cubic in \((x - x_{min})\), which is not included in equation (35). This, however, does not prevent the hyperbola (35) (dashed) from giving the correct position of the moveout apex \((x_{min}, t_{min})\) and an accurate value of the NMO velocity. The errors in the estimates of \( V_{nmo} \) and \( x_{min}/t_{min} \) [compared to the exact values given by equations (20) and (25)] are only 1.1% and 0.05%, respectively. The high accuracy achieved for the model from Figure 4 was ensured by approximately equal range of offsets on each side of the apex of the moveout curve, which mitigates the influence of the cubic moveout term. If the travel-time minimum is substantially shifted with respect to zero offset, it may be necessary either to mute out a certain range of offsets (making the fitting interval more symmetric with respect to \( x_{min} \)) or add the cubic term in \( x - x_{min} \) to the moveout equation. To obtain the slope of the moveout curve at \( x = 0 \), we approximate the travel-times at small source-receiver offsets with a straight line or a quadratic polynomial, depending on the moveout curvature.

Our inversion algorithm is organized in the following way. Using the relationship between the NMO velocities for horizontal events [equation (5)], we determine the \( SV \)-wave NMO velocity [equation (3)] from \( P \) and \( PS \) data. Then, for a given value of \( \delta \), we find the other three parameters [see equations (1), (3), (4), (6)]:

\[
V_{P0} = \frac{V_{nmo,P}}{\sqrt{1 + 2\varepsilon}}, \tag{36}
\]

\[
V_{S0} = \frac{V_{P0}}{\gamma}, \tag{37}
\]

\[
\sigma = \frac{1}{2} \left[ \left( \frac{V_{nmo,SV}}{V_{S0}} \right)^2 - 1 \right], \tag{38}
\]

\[
\epsilon = \frac{\sigma}{\gamma^2} + \delta. \tag{39}
\]

The last step is to use the \( P \)-wave NMO velocity \( V_{nmo,P}(p_{P0}) \) and PS-wave moveout attributes for the
Figure 5. Parameters ε, δ, $V_{P0}$, and $V_{S0}$ determined by inverting $P$ and $PS$ moveout data from a horizontal and dipping reflector. The input data were distorted by random noise with a variance of 0.5% for γ, 1.5% for the zero-dip NMO velocities and 2% for the moveout attributes of the dipping event. (a) and (b) Inversion without dip-moveout attributes of the $PS$-wave; the dip $\phi = 30^\circ$. (c) and (d) Inversion including the dip-moveout attributes of the traveltime minimum of the $PS$-wave [equation (40)]; the dip $\phi = 30^\circ$. (e) and (f) Inversion including the slope of the $PS$-moveout curve [equation (41)]; the dip $\phi = 50^\circ$. The actual values are: $V_{P0} = 2.0$ km/s, $V_{S0} = 1$ km/s, $\epsilon = 0.2$, $\delta = 0.1$. The solid line on plot (a) indicates $\epsilon$'s and $\delta$'s corresponding to the correct value of $\eta$ ($\eta = 0.083$).
same dipping reflector to invert for the remaining unknown parameter \( \delta \). If the PS traveltime does have a minimum on the CMP gather, \( \delta \) is found by minimizing an objective function that contains the PS moveout attributes \( V_{nmo,PS}(P) \) and \( z_{min}/t_{min} \):

\[
\mathcal{F}_{P,SV}^{(1)} = \left[ \frac{V_{nmo,PS}(P) - V_{mean}^{nmo,PS}(P)}{V_{mean}^{nmo,PS}(P)} \right]^2 + \left[ \frac{V_{nmo,PS}(P) - V_{mean}^{nmo,PS}(P)}{V_{mean}^{nmo,PS}(P)} \right]^2 + \left[ \frac{(z_{min}/t_{min}) - (z_{min}/t_{min})^{mean}}{(z_{min}/t_{min})^{mean}} \right]^2.
\]

Here the superscript "meas" denotes the values measured from the data, while the quantities without the subscript are computed from the exact equations (20), (22) and (25). Essentially, the objective function (40) represents an overdetermined system of three nonlinear equations for the single unknown parameter \( \delta \).

For PS traveltime curves without a minimum, the objective function contains a different PS moveout attribute – the slope of the moveout curve at \( x = 0 \) [equation (16)]:

\[
\mathcal{F}_{P,SV}^{(2)} = \left[ \frac{V_{nmo,PS}(P) - V_{mean}^{nmo,PS}(P)}{V_{mean}^{nmo,PS}(P)} \right]^2 + \left[ \frac{(dt/dx)_{x=0} - (dt/dx)_{x=0}^{mean}}{(dt/dx)_{x=0}^{mean}} \right]^2.
\]

In principle, \( (dt/dx)_{x=0} \) can be used in the inversion for mild dips as well. However, if the traveltime minimum is close to zero offset, the slope of the moveout curve at \( x = 0 \) is accurately described by the best-fit shifted hyperbola and, therefore, becomes a redundant attribute.

A numerical example of the joint inversion of \( P \) and \( PS \) data based on equations (36)–(41) is displayed in Figure 5. All input parameters were computed from the exact equations and distorted by Gaussian noise with variances simulating realistic errors in data measurements. To generate the top row of plots (Figures 5a,b), we removed the terms involving the moveout attributes of the PS-wave from the objective function (40). As expected, the parameter \( \eta \) can be accurately determined from the \( P \)-wave NMO velocity of the dipping event, and the \( \epsilon \) and \( \delta \) points in Figure 5a are close to the line corresponding to the correct value of \( \eta \). However, the inversion results for \( V_P \), \( V_S \), \( \epsilon \), and \( \delta \) exhibit significant scatter indicative of high sensitivity to errors in the input data. The standard deviations in all four parameters are too significant for this algorithm to be used in practice (8.6% for \( V_P \) and \( V_S \), 13 for \( \epsilon \) and 0.09 for \( \delta \)).

Minimization using the full objective function (40) (including the NMO velocity of the PS-wave and the normalized offset \( z_{min}/t_{min} \)) leads to a dramatic reduction in the scatter for all medium parameters, with standard deviations of only 3.4% for \( V_P \) and \( V_S \), 0.65 for \( \epsilon \) and 0.03 for \( \delta \) (Figures 5c,d). Clearly, the dip-moveout attributes of the PS-wave allowed us to overcome the problem of error amplification in the transition from \( \eta \) to the vertical velocities and anisotropic coefficients.

Figures 5a–d correspond to a relatively mild reflector dip of 30°. As the dip reaches 50°, the CMP traveltime of the PS-wave no longer has a minimum, and we need to use the second form of the objective function [equation (41)] that includes the slope of the PS moveout curve at zero offset (Figures 5e,f). Since the moveout attributes of both \( P \) and \( PS \)-waves become more sensitive to the anisotropic parameters with increasing dip, the inversion results for the \( \phi = 50^\circ \) are even better than those for \( \phi = 30^\circ \) (the standard deviations are 1.2% for \( V_P \) and \( V_S \), and 0.01 for \( \epsilon \) and \( \delta \)).

**Discussion and conclusions**

Determination of the vertical velocity and the anisotropic parameters of VTI media from surface data is impossible without combining reflection moveout of \( P \)-waves with S-wave traveltimes. Since shear waves are seldom excited in exploration surveys, a more practical option is to supplement \( P \)-wave moveout with converted-wave data. Analysis of the kinematic inverse problem shows that it is necessary to include \( PS \) (PSV) traveltimes not just from a horizontal reflector, but also from at least one dipping interface.

Moveout of \( PS \)-waves generally is asymmetric with respect to zero offset, with the position of the traveltime minimum strongly dependent on reflector dip. For relatively mild dips up to 30–40°, the minimum usually corresponds to moderate offsets and can be recorded in a conventional-length CMP gather. In this case, \( PS \) traveltimes can be used to recover moveout attributes associated with the traveltime minimum \( t_{min}(z_{min}) \), such as the normal-moveout velocity (defined by analogy with pure modes) and the ratio \( z_{min}/t_{min} \). We show that these attributes can be obtained from reflection data by approximating \( PS \)-moveout with a shifted hyperbola centered at the offset \( z_{min} \). If reflector dip exceeds 40–50°, the traveltime minimum either does not exist at all or cannot be captured on conventional spreads. For these traveltime functions, monotonically decreasing with offset, a natural attribute is the slope of the moveout curve at the CMP location.

To apply the \( PS \)-wave moveout attributes in anisotropic parameter estimation, we developed an analytic treatment of dip moveout of converted waves valid in a vertical symmetry plane of an anisotropic layer with arbitrary strength of anisotropy (e.g., the model
can be orthorhombic). Parametric representation of the PS traveltime and CMP offset in terms of the slowness projection on the reflector yields a concise description for reflection moveout involving the vertical and horizontal slowness components of the P- and S-waves. Although the computation of the source-receiver offset and corresponding traveltime involves solving the Christoffel equation in the coordinate system associated with the reflector, our expression can be used to generate the CMP moveout curve without time-consuming two-point ray tracing.

We also proved that the slope of any moveout curve in CMP geometry is always equal to one half of the difference between the ray parameters (horizontal slownesses) evaluated for the incident and reflected ray at the source and receiver locations. This simple result is valid not just for a single layer, but for arbitrary inhomogeneous anisotropic media provided the rays do not deviate from the incidence plane. Combined with the parametric traveltime-offset relationships, the description of moveout slope in terms of the horizontal slownesses of the P and S-waves made it possible to derive closed-form expressions for all moveout attributes described above.

These analytic developments provide a basis for a joint inversion of P and PS data in VTI media. The weak-anisotropy approximation allowed us to find explicit expressions for the traveltime and offset of the PS-wave and study the dependence of the moveout attributes on the anisotropic parameters. The attributes proved to be mostly sensitive to the anisotropy are the normalized offset $x_{\text{min}}/t_{\text{min}}$ and the slope of the t(x) curve at zero offset, while the contribution of anisotropy to the dip-dependence of the PS-wave NMO velocity is noticeably smaller.

Our inversion algorithm is designed to recover the medium parameters (the P- and S-wave vertical velocities $V_{P0}$ and $V_{S0}$ and the anisotropic coefficients $\epsilon$ and $\delta$) using the NMO velocities and vertical traveltimes of the P and PS-waves from a horizontal reflector, P-wave NMO velocity from a dipping reflector, and PS moveout attributes for the same dipping event. Although the number of equations is sufficient to obtain all unknowns even without the dip-moveout attributes of the PS-wave, such an inversion procedure is rather unstable. While the parameter $\eta \approx \epsilon - \delta$ is well-constrained by the dip dependence of P-wave moveout, small errors in $\eta$ propagate with considerable amplification into the vertical velocities and anisotropic parameters. The addition of the PS moveout attributes to the input data leads to a dramatic improvement in the stability of the inversion procedure.

It is interesting that the disappearance of the PS traveltime minimum at steep dips does not impair the stability of the parameter estimation. On the contrary, the accuracy in all inverted parameters increases with dip due to the higher sensitivity of the dip-moveout attributes to the anisotropy.

We showed that combining P and PS reflection traveltimes in moveout inversion yields all parameters of VTI media needed for depth imaging of P, SV, or PSV-waves. Therefore, one of the most important applications of our method is in building velocity models for P-wave prestack or poststack depth migration. Processing of converted waves in VTI media is also impossible without knowledge of both vertical velocities and the parameters $\epsilon$ and $\delta$. Although our inversion has to be performed on common-midpoint gathers, reflection-point dispersal makes it necessary to resort the data into common-reflection-point (CRP) gathers prior to stacking. This operation was proved to be sensitive to the anisotropic coefficients of VTI media and vertical velocities (Rommel, 1997). An alternative way to generate a zero-offset section for converted waves is the transformation to zero offset (TZO), which also requires an accurate VTI model (Anderson, 1998). In addition, our analytic expressions for the converted-wave reflection moveout can be used to extend dip-moveout processing algorithms to PS-modes in anisotropic media.

Due to the kinematic equivalence between the symmetry planes of orthorhombic and VTI media, our results remain valid for CMP reflections in both vertical symmetry planes of models with orthorhombic symmetry (Tsvankin, 1997). The only change required in the weak-anisotropy approximations and the inversion algorithm as a whole is the replacement of $\epsilon$, $\delta$, and the shear-wave vertical velocity $V_{S0}$ with the appropriate set of parameters introduced by Tsvankin (1997).

It is likely that the accuracy of the moveout inversion can be further increased by treating azimuthally varying PS traveltimes in addition to the moveout in the dip plane of the reflector. This methodology can be based on the extension of the 3-D NMO equation, which was originally developed by Grechka and Tsvankin (1996) for pure modes and later generalized by Grechka et al. (1997) for converted waves in horizontally layered media. In sequel publications, we will also discuss the inversion of the PS-moveout attributes for layered anisotropic media.

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**APPENDIX A:** Converted-wave moveout from dipping reflectors

Let us consider a P-S-wave recorded in CMP geometry in the dip direction of a plane reflector. It is assumed that the incidence plane also represents a symmetry plane of the medium, so both rays and the corresponding phase-velocity vectors of the reflected waves are confined to the incidence plane; also, the P-wave generates only one (PSV) conversion. Without losing generality, we assume that the P-ray is located downdip from the reflection point (Figure 2). Then the reflection traveltime can be written as

\[ t = t_P + t_S = \frac{x_r}{g_P \cos \theta^P} + \frac{x_r}{g_S \cos \theta^S} , \]  

(A1)

where \( \theta^P \) and \( \theta^S \) are the group angles with vertical for the P and S segments of the reflected ray (Figure 2), \( g_P \) and \( g_S \) are the corresponding group velocities, and \( x_r \) is the depth of the reflection point. The source-receiver offset in terms of the group angles is given by

\[ x = x_P + x_S = x_r \left( \tan \theta^P + \tan \theta^S \right) . \]  

(A2)

The angle \( \theta^S \) in equation (A1) and below is considered negative if the S-ray is tilted toward the CMP from the reflection point. Introducing the reflector depth beneath the common midpoint \( (z_{\text{CMP}}) \) instead of \( x_r \) yields

\[ z_{\text{CMP}} = x_r \left( 1 + \frac{1}{2} \tan \phi \left( \tan \theta^P - \tan \theta^S \right) \right) , \]  

(A3)

\( \phi \) is reflector dip. Substituting \( z_{\text{CMP}} \) [equation (A3)] into equations (A1) and (A2), we find

\[ t = \frac{N}{D} \]  

(A4)

and
\[ z = z_{\text{CMP}} \frac{N_z}{D}, \]  
(A5)

where

\[ N = \frac{1}{g_P \cos \theta_P^P} + \frac{1}{g_S \cos \theta_S^S}, \]  
(A6)

\[ D = 1 + \frac{1}{2} \tan \phi \left( \tan \theta_P^P - \tan \theta_S^S \right), \]  
(A7)

\[ N_z = \tan \theta_P^P + \tan \theta_S^S. \]  
(A8)

To satisfy Snell's law, the \( P \) and \( S \)-waves should have the same projection of the slowness vector (ray parameter) on the interface at the reflection point. Hence, this projection (\( p_{\text{int}} \); the subscript stands for the interface) can be conveniently used to build a parametric representation of the converted-wave CMP traveltimes. Hereafter, we assume that \( p_{\text{int}} \) is the "updip" projection that corresponds to the upcoming \( S \)-wave and downward \( P \)-wave.

If the medium is isotropic, the group angles \( \theta_P^P \) and \( \theta_S^S \) are equal to the corresponding phase angles and can be easily expressed through \( p_{\text{int}} \) (taken to be non-negative), reflector dip \( \phi \), and the velocities of the \( P \)- and \( S \)-waves (\( V_P \) and \( V_S \)):

\[ \sin \theta_P^P = p_{\text{int}} V_P \cos \phi + \sqrt{1 - p_{\text{int}}^2 V_P^2} \sin \phi, \]  
(A9)

\[ \cos \theta_P^P = \sqrt{1 - p_{\text{int}}^2 V_P^2} \cos \phi - p_{\text{int}} V_P \sin \phi, \]  
(A10)

\[ \sin \theta_S^S = p_{\text{int}} V_S \cos \phi - \sqrt{1 - p_{\text{int}}^2 V_S^2} \sin \phi, \]  
(A11)

\[ \cos \theta_S^S = \sqrt{1 - p_{\text{int}}^2 V_S^2} \cos \phi + p_{\text{int}} V_S \sin \phi. \]  
(A12)

Substitution of equations (A9)–(A12) into equations (A6)–(A8) and then (A4) and (A5) leads to explicit expressions for the reflection traveltimes and source-receiver offset in CMP geometry in terms of the slowness \( p_{\text{int}} \).

For anisotropic media, the transition from the slowness (ray parameter) \( p_{\text{int}} \) to the group angles of the reflected waves involves solving the Christoffel equation for the slowness component orthogonal to the reflector. For \( P - SV \)-waves in a symmetry plane of an anisotropic medium, the Christoffel equation for an unknown slowness component is quartic (it becomes sextic outside the symmetry planes).

Since we may have several reflectors with different dips in the same medium, it is more convenient to express the traveltimes curve through the slowness components in the unrotated coordinate system associated with the earth surface. Also, note that the group angles in equations (A1)–(A5) are defined with respect to the vertical axis rather than the reflector normal. Introducing the projections of the group-velocity vectors of \( P \)- and \( S \)-waves on the vertical (\( z_3 \), subscript "3") and horizontal (\( z_1 \), subscript "1") axes, we obtain the following equivalent form of equations (A6)–(A8):

\[ N = \frac{1}{g_P^3} + \frac{1}{g_S^3}, \]  
(A13)

\[ D = 1 - \frac{1}{2} \tan \phi \left( \frac{g_P^1}{g_P^3} + \frac{g_S^1}{g_S^3} \right), \]  
(A14)

\[ N_z = -\frac{g_P^1}{g_P^3} + \frac{g_S^1}{g_S^3}. \]  
(A15)

Here the \( z_3 \)-axis is directed upward, the \( z_1 \)-axis – updip, and both group-velocity vectors are assumed to point from the reflector towards the surface. The group-velocity components can be related to the slowness vector in the same (unrotated) coordinate system in the following way (Tsvankin et al., 1997):

\[ g_3 = \frac{1}{q - pq}, \]  
(A16)

and

\[ g_1 = -g_3 q', \]  
(A17)

where \( q \) and \( p \) are the vertical and horizontal slowness components, respectively, and \( q' \equiv dq/dp \). Equations (A16) and (A17) allow us to rewrite equations (A13)–(A15) as

\[ N = qP - pqP + gS - psqS, \]  
(A18)
\[ D = 1 + \frac{1}{2} \tan \phi (q_P + q_S), \quad (A19) \]
\[ N_x = q_P - q_S. \quad (A20) \]

Here \( p_P \) and \( p_S \) are the horizontal components of the slowness vector for the \( P \)- and \( S \)-waves, and \( q_P \equiv dq_P/dp_P \), \( q_S \equiv dq_S/dp_S \).

To generate a CMP gather of the converted \( PS \)-wave, we first need to obtain \( p_P \) and \( p_S \) as functions of the projection of the slowness vector on the reflector \( (p_{int}) \) by solving the Christoffel equation for the slowness vectors of the \( P \)- and \( S \)-waves in the rotated coordinate system with one of the axes parallel to the reflecting interface. Then we can find \( p_P \) and \( p_S \) and use them in equations \((A18)-(A20), (A4), \) and \((A5)\) to obtain the moveout curve of the converted mode.

**APPENDIX B: NMO velocity for converted-wave moveout**

If reflection moveout of a converted wave in a CMP gather does have a minimum \( t_{min} = t(x_{min}) \), the traveltime near \( t_{min} \) can be described by normal-moveout velocity \( V_{nmo} \) introduced by analogy with pure modes. To find an analytic expression for \( V_{nmo} \), we expand the squared CMP traveltime \( t^2(x) \) into a Taylor series near the traveltime minimum:

\[ t^2(x) = t^2_{min} + \left( \frac{d^2(t^2)}{dx^2} \right)_{x=x_{min}} (x-x_{min})^2 + \cdots \quad (B1) \]

The first derivative \( d(t^2)/dx \) at \( x = x_{min} \) is equal to zero, while the second derivative yields the NMO velocity that governs the traveltime for small \( (x - x_{min}) \):

\[ V_{nmo}^2 = \left\{ \frac{1}{2} \frac{d^2(t^2)}{dx^2} \right\}_{x=x_{min}}^{-1} = \left\{ \frac{d}{dx} \left( \frac{dt}{dx} \right) \right\}_{x=x_{min}}^{-1}. \quad (B2) \]

Using the results of Appendix D, the moveout slope \( dt/dx \) can be expressed through the difference between the horizontal slownesses of the \( P \)- and \( S \)-waves:

\[ \frac{dt}{dx} = \frac{1}{2} (p_S - p_P). \quad (B3) \]

Considering both \( dt/dx \) and \( x \) as functions of the projection of the slowness vector on the interface \( (p_{int}) \), we can rewrite equation \((B2)\) as

\[ V_{nmo,PS}^2 = \left\{ \frac{t}{2} \frac{d(p_S - p_P)/dp_{int}}{dx/dp_{int}} \right\}_{p_{int}=\min}^{-1}, \quad (B4) \]

where \( p_{int}=\min \) corresponds to the traveltime minimum.

To evaluate the derivatives in equation \((B4)\), \( p_{int} \) should be represented through the slownesses of the \( P \)- and \( S \)-waves using Snell's law:

\[ p_{int} = -(p_P \cos \phi + q_P \sin \phi) = p_S \cos \phi + q_S \sin \phi. \quad (B5) \]

The \( P \)-wave in equations \((A13)-(A15)\) is assumed to travel upward from the reflector, which explains the minus sign in front of the \( P \)-wave term in equation \((B5)\). Differentiating equation \((B5)\), we obtain

\[ \frac{dp_P}{dp_{int}} = -(\cos \phi + q_P \sin \phi)^{-1}, \quad (B6) \]
\[ \frac{dp_S}{dp_{int}} = (\cos \phi + q_S \sin \phi)^{-1}, \quad (B7) \]

where, as in equations \((A18)-(A20)\), \( q_P \equiv dq_P/dp_P \) and \( q_S \equiv dq_S/dp_S \). Hence,

\[ \frac{d(p_S - p_P)}{dp_{int}} = \frac{1}{\cos \phi} \left[ \frac{1}{1 + q_P \tan \phi} + \frac{1}{1 + q_S \tan \phi} \right]. \quad (B8) \]

The derivative \( dx/dp_{int} \) in equation \((B4)\) can be represented through \( D \) and \( N_x \) [equations \((A19)\) and \((A20)\)] as

\[ \frac{dx}{dp_{int}} = z_{CMP} \frac{d(N_x/D)}{dp_{int}} = z_{CMP} (\frac{dN_x}{dp_{int}}) D - (\frac{dD}{dp_{int}}) N_x \frac{1}{D^2}. \quad (B9) \]
Equations (B6) and (B7) allow us to express the derivatives with respect to \( p_{\text{int}} \) through those with respect to \( p_P \) or \( p_S \); for instance,

\[
\frac{dp_P}{dp_{\text{int}}} = -q'_P (\cos \phi + q'_p \sin \phi)^{-1}.
\]  
(B10)

Equation (B10) and an analogous expression for \( q_S \) allow us to obtain the numerator in equation (B9) in the following form:

\[
\frac{dN_x}{dp_{\text{int}}} D - \frac{dD}{dp_{\text{int}}} N_z = -q'_P (1 + q'_p \tan \phi) \frac{q'_P}{\cos \phi + q'_P \sin \phi} - q'_P (1 + q'_P \tan \phi) \frac{q'_S}{\cos \phi + q'_S \sin \phi}.
\]  
(B11)

Note the symmetry in equation (B11) with respect to the subscripts "P" and "S": the second term can be obtained by interchanging these subscripts in the first term. The expression for \( D \), also needed in equation (B9), was derived previously [equation (A19)]:

\[
D = 1 + \frac{1}{2} \tan \phi (q'_P + q'_S).
\]  
(B12)

Equations (B11) and (B12) are sufficient for obtaining the derivative \( dx/dp_{\text{int}} \) from equation (B9).

NMO velocity, as given by equation (B4), also depends on the minimum traveltime that can be found from equations (A4), (A18), and (A19):

\[
t_{\text{min}} = t_{\text{min}}(p_{\text{int}}) = x_{\text{CMP}} \frac{q_P - p_P q'_p + q_S - p_S q'_S}{1 + \frac{1}{2} \tan \phi (q'_P + q'_S)} + t_{\text{min}}(p_{\text{int}}).
\]  
(B13)

Substituting equations (B8), (B9) and (B13) into equation (B4) and taking into account that at the traveltime minimum \( p_P = p_S \) [see equation (B16) below] yields the final expression for NMO velocity as a function of the horizontal slownesses of the P- and S-waves:

\[
V_{nmo,PS}^2 = \frac{4 (q'^2_P A^2_P + q'^2_S A^2_P)}{(A_P + A_S)^2 \left( p_P (q'_P + q'_S) - (q_P + q_S) \right)} + t_{\text{min}}(p_{\text{int}}),
\]  
(B14)

where

\[
A_P = 1 + q'_P \tan \phi, \quad A_S = 1 + q'_S \tan \phi.
\]  
(B15)

To obtain the ray parameter \( p_{\text{int}}^{\text{min}} \), we use equation (B3) for the slope of the moveout curve. Since at the traveltime minimum the slope goes to zero,

\[
p_P(p_{\text{int}}^{\text{min}}) = p_S(p_{\text{int}}^{\text{min}}).
\]  
(B16)

Substituting \( p_P = p_S \) into Snell's law [equation (B5)] and dividing both sides by \( \cos \phi \) gives the following equation for the slownesses corresponding to the traveltime minimum:

\[
2p_P(p_{\text{int}}^{\text{min}}) = [(q_P + q_S) \tan \phi]_{p_{\text{int}}^{\text{min}}},
\]  
(B17)

where the vertical slownesses \( q_P \) and \( q_S \) are related to \( p_P = p_S \) through the Christoffel equation.

For isotropic media, the vertical slownesses are given simply by

\[
q_P = \sqrt{\frac{1}{V_P^2} - p^2_P},
\]  
(B18)

and

\[
q_S = \sqrt{\frac{1}{V_S^2} - p^2_S}.
\]  
(B19)

In this case, it is possible to derive an explicit expression for \( p_P(p_{\text{int}}^{\text{min}}) = p_S(p_{\text{int}}^{\text{min}}) \) by solving equation (B17) with \( q_P \) and \( q_S \) from equations (B18) and (B19):

\[
p_P(p_{\text{int}}^{\text{min}}) = \frac{\sin \phi}{2 V_P} \sqrt{1 + \gamma^2 + S},
\]  
(B20)

where \( \gamma \equiv V_P/V_S \) and

\[
S = \sqrt{4\gamma^2 - \tan^2 \phi (\gamma^2 - 1)^2}.
\]  
(B21)
This solution, however, exists only if the expression under the radical in equation (B21) is nonnegative, or
\[
\tan \phi \leq \frac{2\gamma}{\gamma^2 - 1}. \tag{B22}
\]

**Appendix C: Weak-anisotropy approximation for converted-wave moveout in VTI media**

Here we derive weak-anisotropy approximations for the traveltime-offset relationship and moveout attributes of PS-waves in VTI media by carrying out linearization in Thomsen’s parameters \(\epsilon\) and \(\delta\). For small \(|\epsilon| \ll 1\) and \(|\delta| \ll 1\), phase velocities of \(P-\) and \((SV)-\)waves can be well-approximated by the following linearized expressions (Thomsen, 1986):

\[V_P(\tilde{\phi}) = V_{P0} \left[ 1 + \delta \sin^2 \tilde{\phi} + (\epsilon - \delta) \sin^4 \tilde{\phi} \right] \tag{C1}\]

and

\[V_{SV}(\tilde{\phi}) = V_{SO} \left[ 1 + \sigma \sin^2 \tilde{\phi} \cos^2 \tilde{\phi} \right], \tag{C2}\]

where \(\tilde{\phi}\) is the phase angle with the symmetry axis, and

\[\sigma \equiv \left( \frac{V_{P0}}{V_{SO}} \right)^2 (\epsilon - \delta). \tag{C3}\]

Our first goal is to express the group velocity and group angle for both \(P-\) and \((SV)-\)waves (Figure 2) through the slowness \(p_{int}\). Let us denote the angle between the phase-velocity (slowness) vector of the downgoing \(P\)-wave and the reflector normal (pointing upward) by \(\tilde{\phi}_P\) (Figure C1). Then the phase angle \(\tilde{\phi}\) with the symmetry axis is equal to \(\tilde{\phi}_P - \phi\), and equation (C1) can be written as

\[V_P(\tilde{\phi}_P) = V_{P0} \left[ 1 + \delta \sin^2 (\tilde{\phi}_P - \phi) + (\epsilon - \delta) \sin^4 (\tilde{\phi}_P - \phi) \right]. \tag{C4}\]

In the linearized weak-anisotropy approximation, group velocity (at the group angle) is equal to phase velocity. Also, we can use the isotropic relationship between the phase angle \(\tilde{\phi}_P\) and \(p_{int}\) because \(\tilde{\phi}_P\) is contained only in terms already linear in the anisotropic parameters. Hence, the group velocity of the \(P\)-wave is given by

\[g_P(p_{int}) = V_{P0} \left[ 1 + \delta \sin^2 (\tilde{\phi}_{int} - \phi) + (\epsilon - \delta) \sin^4 (\tilde{\phi}_{int} - \phi) \right], \tag{C5}\]

where
\[ \sin \tilde{\theta}_{i,P} = p_{\text{int}} V_{P0} ; \quad \cos \tilde{\theta}_{i,P} = -\sqrt{1 - (p_{\text{int}} V_{P0})^2}. \]  

(C6)

Next, we need to find the P-wave group angle \( \theta^g_P \) [equation (A1)] as a function of \( p_{\text{int}} \). For the phase angle \( \tilde{\theta}_P \) we have

\[ \sin \tilde{\theta}_P = p_{\text{int}} V_P = p_{\text{int}} V_{P0} (1 + \alpha_{\text{anis},P}) \]  

(C7)

and

\[ \cos \tilde{\theta}_P = -\sqrt{1 - (p_{\text{int}} V_{P0})^2} \frac{1 - (p_{\text{int}} V_{P0})^2}{1 - (p_{\text{int}} V_{P0})^2} \alpha_{\text{anis},P}, \]  

(C8)

where

\[ \alpha_{\text{anis},P} \equiv \delta \sin^2(\tilde{\theta}_{i,P} - \phi) + (\epsilon - \delta) \sin^4(\tilde{\theta}_{i,P} - \phi). \]  

(C9)

Using the weak-anisotropy relationship between the group and phase angles in TI media (Thomsen, 1986) and taking into account that the P-wave propagates upward from the reflection point yields

\[ \tan \theta^g_P = -\tan(\tilde{\theta}_P - \phi) \left[ 1 + 2\delta + 4(\epsilon - \delta) \sin^2(\tilde{\theta}_{i,P} - \phi) \right]. \]  

(C10)

Thus, the weak-anisotropy approximation makes it possible to find explicit expressions for group velocity and group angle in terms of the slowness projection \( p_{\text{int}} \). Substituting \( \sin \tilde{\theta}_P \) and \( \cos \tilde{\theta}_P \) from equations (C7) and (C8) into equation (C10) and further linearizing in the anisotropic parameters, we obtain

\[ \tan \theta^g_P = -\tan(\tilde{\theta}_{i,P} - \phi) \left[ 1 + 2\delta + 4(\epsilon - \delta) \sin^2(\tilde{\theta}_{i,P} - \phi) + \alpha_{\text{anis},P} \tan \tilde{\theta}_{i,P} \right]. \]  

(C11)

From equation (C10) it also follows that

\[ \frac{1}{\cos \theta^g_P} = \frac{1}{\cos(\tilde{\theta}_P - \phi)} \left\{ 1 + \sin^2(\tilde{\theta}_{i,P} - \phi) [2\delta + 4(\epsilon - \delta) \sin^2(\tilde{\theta}_{i,P} - \phi)] \right\}. \]  

(C12)

Combining equations (C5) and (C12) gives the term \( 1/g_P \cos \theta^g_P \) needed to find the traveltime along the P-wave leg:

\[ \frac{1}{g_P \cos \theta^g_P} = \frac{1}{V_{P0} \cos(\tilde{\theta}_{i,P} - \phi)} \left[ 1 + 2(\epsilon - \delta) \sin^4(\tilde{\theta}_{i} - \phi) + \alpha_{\text{anis},P} \cos \tilde{\theta}_{i,P} \cos(\tilde{\theta}_{i,P} - \phi) \right]. \]  

(C13)

Similar algebraic transformations yield the corresponding expressions for the S-wave leg:

\[ \tan \theta^g_S = \tan(\tilde{\theta}_{i,S} - \phi) \left[ 1 + 2\sigma - 4\sigma \sin^2(\tilde{\theta}_{i,S} - \phi) + \alpha_{\text{anis},S} \frac{\tan \tilde{\theta}_{i,S}}{\sin(\tilde{\theta}_{i,S} - \phi) \cos(\tilde{\theta}_{i,S} - \phi)} \right], \]  

(C14)

\[ \frac{1}{g_S \cos \theta^g_S} = \frac{1}{V_{S0} \cos(\tilde{\theta}_{i,S} - \phi)} \left[ 1 - 2\sigma \sin^4(\tilde{\theta}_{i,S} - \phi) + \alpha_{\text{anis},S} \frac{\cos \tilde{\theta}_{i,S}}{\cos \tilde{\theta}_{i,S} \cos(\tilde{\theta}_{i,S} - \phi)} \right], \]  

(C15)

where

\[ \alpha_{\text{anis},S} \equiv \sigma \sin^2(\tilde{\theta}_{i,S} - \phi) \cos^2(\tilde{\theta}_{i,S} - \phi) \]  

(C16)

and

\[ \sin \tilde{\theta}_{i,S} = p_{\text{int}} V_{S0} ; \quad \cos \tilde{\theta}_{i,S} = \sqrt{1 - (p_{\text{int}} V_{S0})^2}. \]  

(C17)

Note that the terms involving anisotropic coefficients in equations (C14) and (C15) can be obtained from the anisotropic terms in the corresponding P-wave equations (C11) and (C13) by making the following substitutions: \( V_{P0} \) should be replaced with \( V_{S0} \), \( \delta \) with \( \sigma \), and \( \epsilon \) set to zero. Equations (C11), (C13), (C14), and (C15) are sufficient for obtaining the traveltime and offset for the P-wave leg using the general relationships (A4)–(A8).

Prior to obtaining the attributes associated with the traveltime minimum, it is necessary to determine the corresponding slowness projection \( p_{\text{int}}^{\text{min}} \). To derive concise approximations for NMO velocity and other moveout attributes, hereafter we drop all terms with the cubic and higher powers of \( \sin \phi \) (i.e., the dip is assumed to be relatively mild). Linearizing equation (B17) in the anisotropic parameters and keeping terms up to quadratic in \( \sin \phi \) yields
\[ p_{\text{int}} = \frac{\sin \phi}{2} \left( \frac{1}{V_{S0}} - \frac{1}{V_{P0}} \right) \]  

Equation (C18) is fully equivalent to the isotropic result (B20), which can be shown by recalculating \( p_{\text{int}} \) into the corresponding horizontal slowness \( p_{P0} \) of \( \phi_{\text{int}} \).

Next, we substitute \( p_{\text{int}} \) from equation (C18) into equations (C11), (C13), (C14), and (C15), carry out expansion in \( \sin \phi \) and truncate the series after the \( \sin^2 \phi \)-term. The results allows us to obtain the minimum traveltime \( t_{\text{min}} \) and the corresponding offset \( x_{\text{min}} \) from the general equations (A4)–(A8) using symbolic software Mathematica:

\[ t_{\text{min}} = \text{CMP} \left( \frac{V_{P0} + V_{S0}}{V_{P0} V_{S0}} \right) \left( 1 - \frac{\sin^2 \phi (V_{P0} + V_{S0})}{8 V_{P0} V_{S0}} + \frac{\sin^2 \phi (V_{P0} + V_{S0})}{4 V_{S0}^2} [V_{P0} (\delta - \epsilon) - V_{S0} (1 + 2 \delta)] \right), \]  

\[ x_{\text{min}} = \text{CMP} \left( \frac{\sin \phi}{2 V_{P0} V_{S0}^2} \right) \left( V_{P0} + V_{S0} \right) \left[ 2 V_{P0}^2 (\delta - \epsilon) + V_{P0} V_{S0} (1 + 2 \delta) - V_{S0}^2 \right]. \]  

Next, we introduce the ray parameter (horizontal slowness) \( p_{P0} \) of the pure P-wave reflection recorded at zero offset:

\[ p_{P0} = \frac{\sin \phi}{V_{P}(\phi)} \approx \frac{\sin \phi}{V_{P}}. \]  

Equation (C21) has a purely "isotropic" form because the anisotropic terms in \( p_{P0} \) are multiplied with high powers of \( \sin \phi \) neglected in our approximation.

Combining equations (C19), (C20), and (C21) and dropping cubic and higher-order terms in \( p_{P0} \), we find

\[ \frac{x_{\text{min}}}{t_{\text{min}}} = \frac{p_{P0} V_{P0}^2}{2 \gamma} [(\gamma - 1) + 2(\delta \gamma - \sigma)], \]  

where \( \gamma \equiv V_{P0}/V_{S0} \).

Likewise, the weak-anisotropy, mild-dip approximation for the spatial derivative of the minimum traveltime (C19) is given by

\[ \frac{dt_{\text{min}}}{dy_{\text{CMP}}} = \tan \phi \frac{dt_{\text{min}}}{dx_{\text{CMP}}} = p_{P0} (1 + \gamma). \]  

To obtain the weak-anisotropy approximation for normal-moveout velocity, we linearize the exact equation (B14) in the anisotropic parameters. Expanding the linearized version of equation (B14) in \( \sin \phi \) up to the quadratic term and substituting \( p_{\text{int}} \) from equation (C18) leads to the following form of \( V_{\text{nmo,PS}} \) (this result was also obtained in Mathematica):

\[ V_{\text{nmo,PS}}^{-2} = \frac{1}{V_{P0} V_{S0}} - \frac{\sin^2 \phi}{8 V_{P0} V_{S0}^3} \left[ 3 V_{P0}^4 - 2 V_{P0}^2 V_{S0} + 6 V_{P0} V_{S0}^3 - 2 V_{P0} V_{S0}^3 + 3 V_{S0}^4 \right] \]

\[ + \frac{\epsilon^2}{2 V_{P0} V_{S0}^4 (V_{P0} + V_{S0})} \left[ -4 V_{P0} V_{S0}^3 + \sin^2 \phi \left( 3 V_{P0}^4 - 11 V_{P0}^2 V_{S0} - V_{P0} V_{S0}^3 + 3 V_{P0} V_{S0}^3 + 6 V_{S0}^2 \right) \right] \]

\[ + \frac{\delta}{2 V_{P0} V_{S0}^4 (V_{P0} + V_{S0})} \left[ -4 V_{P0} V_{S0}^3 + \sin^2 \phi \left( 3 V_{P0}^4 - 8 V_{P0}^2 V_{S0} - 6 V_{P0} V_{S0}^3 - 8 V_{P0} V_{S0}^3 + 3 V_{S0}^4 \right) \right]. \]

Introducing \( \gamma \) and \( p_{P0} \), replacing \( \epsilon \) with \( \sigma \), and making further simplifications in equation (C-24) yields the final expression for NMO velocity:

\[ V_{\text{nmo,PS}}^{-2} (p_{P0}) = V_{\text{nmo,PS}}^{-2} (0) - \frac{p_{P0}^2}{8 \gamma} (3 \gamma^4 - 2 \gamma^2 + 6 \gamma^2 - 2 \gamma + 3) \]

\[ - \frac{p_{P0} (\gamma - 1)}{2 \gamma (\gamma + 1)} \left[ 6 \sigma (\gamma + 1)^2 - (\sigma - \delta) \gamma (3 \gamma^2 - 2 \gamma + 3) \right], \]

where

\[ V_{\text{nmo,PS}}^{-2} (0) = \frac{1}{V_{P0} V_{S0}} \left[ 1 - \frac{2(\sigma + \delta \gamma)}{1 + \gamma} \right]. \]

The approximate slope of the moveout curve at zero offset can be found by linearizing the exact equation (B3) in \( \epsilon \) and \( \delta \). The value of \( p_{\text{int}} \) corresponding to \( x = 0 \) satisfies the condition
which follows from equation (A17), if we take into account that at \( x = 0 \) the group-velocity vectors of \( P \)- and \( S \)-waves are parallel to each other. The mild-dip, weak-anisotropy approximation for the moveout slope at \( x = 0 \) is given by

\[
\frac{dt}{dx} \bigg|_{x=0} = \frac{p_{R_0}}{2(1 + \gamma)} \left[ (1 - \gamma^2) + 4\gamma (\sigma - \delta) \right].
\]

(C28)

APPENDIX D: General expression for the slope of reflection moveout

The goal of this appendix is to prove that the apparent slowness (slope) of the CMP moveout curve for any converted or pure reflection is determined by the difference between the ray parameters (slownesses) corresponding to the two legs of the reflected ray. As before, we assume a 2-D model of wave propagation, with the group velocities of the reflected waves confined to the incidence plane. The medium above the reflector, however, is no longer restricted to a single homogeneous layer and may be arbitrary inhomogeneous and anisotropic (with the incidence plane still being a plane of symmetry).

Suppose the reflected wave represents a converted \( PS \) mode recorded in CMP geometry (Figure D1). The slope of the moveout curve at any offset \( x_0 \) is given by

\[
\frac{dt}{dx} \bigg|_{x_0} = \frac{d(t_P + t_S)}{dx} \bigg|_{x_0},
\]

(D1)

where \( t_P \) is the travelt ime along segment \( S_1O_1 \) and \( t_S \) corresponds to \( O_1R_1 \) (Figure D1). To relate the moveout slope to the ray parameters of the \( P \)- and \( S \)-waves, it is convenient to add and subtract from \( dt/dx \) the slope of the reflection traveltime \( t^R \) along a non-specular raypath \( S_1O_1R_1 \). Expressing \( t^R \) through the sum of the \( P \) and \( S \) traveltimes \( t^P_R \) and \( t^S_R \), we rewrite equation (D1) in the form

\[
\frac{dt}{dx} \bigg|_{x_0} = \frac{d(t^P_R + t^S_R)}{dx} \bigg|_{x_0} + \frac{d(t_P + t_S - t^P_R - t^S_R)}{dx} \bigg|_{x_0}.
\]

(D2)

Since the \( P \)- and \( S \)-legs of the non-specular raypath originate from the fixed reflection point \( O \), the slope of the corresponding moveout curve can be expressed as

\[
\frac{d(t^P_R + t^S_R)}{dx} \bigg|_{x_0} = \frac{d(t^P_R + t^S_R)}{d(2h)} \bigg|_{h_0} = \frac{1}{2} [ -p_P(-h_0) + p_S(h_0)] ,
\]

(D3)

where \( h = x/2, h_0 = x_0/2 \), \( p_P \) and \( p_S \) are the horizontal slownesses (ray parameters) evaluated at the surface for the \( P \)- and \( S \)-rays \( OS \) and \( OR \), and the horizontal coordinate axis runs up dip.

To prove that the remaining (second) term in the right-hand side of equation (D2) is equal to zero, we consider all possible non-specular reflections with the source-receiver offset \( x = S_1R_1 \) (Figure D1). The traveltimes of these arrivals can be expanded into a Taylor series in the distance \( l = OO_1 \) between the specular \( (O_1) \) and non-specular
reflection point. According to Fermat’s principle, the minimum traveltime corresponds to the specular reflection, so the term linear in $l$ in this expansion should vanish. Hence, dropping the cubic and higher-order terms in $l$, we obtain the following relationship between the nonspecular ($t_{ns} = t_{P}^{ns} + t_{S}^{ns}$) and specular ($t = t_{P} + t_{S}$) traveltimes:

\[
t_{ns} = t + \frac{1}{2} \frac{d^{2}t_{ns}}{dl^{2}} \bigg|_{l=0} l^{2} + \ldots .
\]

(D4)

Therefore, the difference between the traveltimes along the raypaths $S_{1}O_{1}R_{1}$ and $S_{1}OR_{1}$ (Figure D1) becomes

\[
t - t_{ns} = A(x)l^{2}(x),
\]

(D5)

where

\[
A(x) = -\frac{1}{2} \frac{d^{2}t_{ns}}{dl^{2}} \bigg|_{l=0} .
\]

(D6)

Differentiating equation (D5) with respect to $x$ at $x = x_{0}$ yields

\[
\frac{d(t - t_{ns})}{dx} \bigg|_{x_{0}} = \frac{dA(x)}{dx} l^{2}(x) \bigg|_{l=0} + A(x) 2l \frac{dl}{dx} \bigg|_{l=0} = 0 .
\]

(D7)

Therefore, the slope of the moveout curve can be determined from the non-specular traveltime [equation (D3)]:

\[
\frac{dt}{dx} \bigg|_{x_{0}} = \frac{1}{2} [-P_{F}(-h_{0}) + P_{S}(h_{0})] .
\]

(D8)
AVO analysis in finely layered azimuthally anisotropic media

Vladimir Grechka

ABSTRACT

Over the years, amplitude-variation-with-offset (AVO) analysis has proved its usefulness in exploration of oil and gas reservoirs. However, the model conventionally used to interpret AVO anomalies – a single isolated interface between two isotropic half-spaces – is often too simplistic. Here, I examine what can be obtained from AVO responses for significantly more complicated reservoir model – a stack of plane azimuthally anisotropic layers, the model which can be used to simulate AVO signatures over finely layered fractured reservoirs.

I describe a processing technique which takes seismic data in frequency – slowness ($\omega - p$) domain, properly corrects it for slant wave propagation in finely layered medium, and produces instantaneous AVO intercept and azimuthally varying AVO gradient as functions of vertical travelt ime. Ideally, obtained AVO intercept and gradient are those which would be recorded in the case of isolated interfaces and in the absence of interference between closely spaced reflections. I use azimuthally dependent AVO gradient to obtain instantaneous AVO azimuth, which corresponds to the direction of the greatest AVO gradient. In fractured reservoirs, AVO azimuth is related to the orientation of vertical cracks, therefore, fracture characterization is one of potential applications of the described technique.

I perform a numerical study to examine stability of obtained AVO azimuth with respect to errors in velocity model of the reservoir, inaccuracies in wavelet estimation, and random noise in the data. The results of numerical tests indicate that AVO azimuth is reasonable stable, and suggest the possibility of detecting principal directions of azimuthal anisotropy in layers which are thinner than half dominant seismic wavelength.

Key words: AVO analysis, azimuthally anisotropic media, fine layers

Introduction

Amplitude-variation-with-offset (AVO) analysis has been used for years to estimate reservoir lithology and fluid content. Since work of Ostrander and Gassaway (1983) and Ostrander (1984), who first demonstrated potential of AVO as an indicator of hydrocarbons, many studies have been undertaken to improve reliability of AVO analysis and overcome its numerous complications and pitfalls. An extensive list of inherent problems in measuring and interpreting offset-dependent reflectivity is given by Castagna (1993). Some of the factors causing amplitude variations such as spherical spreading, source-receiver directivity, and influence of free surface can be accounted for and approximately compensated during processing, the others require revision of conventional reflectivity model – an isolated interface between two sufficiently thick isotropic homogeneous layers. In this paper, I examine the combination of two factors which have the first-order influence on measured AVO signatures: fine layering and anisotropy within the reservoir. I study $P$-wave AVO response and construct corresponding processing algorithm for a model that consists of a stack of thin (as compare to the wavelength) azimuthally-anisotropic plane layers. The presence of fine layering in the model enables one to simulate and properly account for various tuning and interference phenomena. Permit-
ting azimuthal anisotropy in the layers allows to model AVO response for naturally-fractured reservoirs.

Inability of reflection coefficient from a single interface to describe real seismic reflections, which usually contain superposition of events from closely spaced interfaces, was recognized in the very beginning of applying AVO technology (e.g., Sherwood et al., 1983; Ostrander, 1984) Interference between reflections from adjacent interfaces causes the classical tuning phenomenon (Widess, 1973) and becomes increasingly complicated when considered as a function of offset or an incidence angle (Swan, 1988). As a result, an apparent AVO response is observed and the single-interface AVO signatures (i.e., AVO intercept and AVO gradient which would be recorded if interference does not exist) become obscured. Wapenaar et al. (1995) related this apparent AVO behavior to the fact that waves illuminating layered medium have different apparent vertical wavelengths \( \lambda_v \) at different incidence angles. Wapenaar et al. (1995) suggested considering the wave frequency \( \omega \) as a function of the incidence angle or the horizontal slowness \( p \) and change \( \omega \) in such a way to make \( \lambda_v \) independent on \( p \). The result was a technique which takes the data in \( t_0 \rightarrow p \) (to be the vertical time) domain, corrects it for apparent AVO phenomena due to fine layering, and reconstructs true reflectivity of acoustic media. The same idea was applied to laterally varying acoustic and elastic isotropic media by van Geloven and Wapenaar (1996, 1997). I express this idea in terms of slowness dependent plane-wave time delays \( t_p(p) \) and generalize the technique to azimuthally anisotropic media.

Anisotropy as well as fine layering is known to have a strong effect on seismic reflectivity. Numerical results of Wright (1984, 1987) indicate that transverse isotropy of overlying shales severely influences AVO gradient and must be accounted for in AVO analysis. Thomsen (1993), assuming weak anisotropy and small contrast in elastic properties across an interface, showed analytically that transverse isotropy with the vertical symmetry axis (VTI model) has the first-order influence on angular dependence of P-wave reflection amplitude. Exact formulae for reflection coefficients in VTI media were derived by Graebner (1992) who used the horizontal slowness \( p \) rather than the incidence angle as independent parameter. I will follow the same idea here. Having slowness \( p \) as an argument is especially attractive because the slowness is preserved during wave propagation in plane-layered media to be examined.

The influence of anisotropy on P-wave AVO signatures is even more pronounced for azimuthally anisotropic media. As was first numerically demonstrated by Mallick and Frazer (1991), AVO gradient becomes azimuthally dependent in this case. Since azimuthal anisotropy is often associated with aligned vertical cracks, the result of Mallick and Frazer (1991) suggests a possibility for fracture detection using measurements of P-wave AVO gradient in several azimuths. The relation between azimuthal variations in P-wave amplitudes and anisotropic coefficients, which can be derived from parameters of cracks, became apparent after studies done by Rüger and Tsvankin (1995) and Rüger (1996, 1997) who obtained analytical weak-anisotropy, small-contrast approximations for P-wave reflection coefficients in two important seismic models of vertical fractures – transversely isotropic media with a horizontal symmetry axis (HTI model) and orthorhombic media. They explicitly showed which combinations of anisotropic coefficients influence AVO signatures and, therefore, can be estimated from AVO inversion. The results of Rüger and Tsvankin (1995) and Rüger (1996, 1997) were extended to arbitrary anisotropy by Vavrčuk and Psencík (1997) and Zillmer et al. (1997). Sayers and Rickett (1997) numerically examined azimuthal variation in AVO response from the top and the base of a layer of fractured gas sand. Still, all the above-mentioned studies concentrate on reflections from isolated interfaces and do not take into account fine layering.

From exploration standpoint, there is a great demand in characterizing fractured reservoirs, and multi-azimuth P-wave AVO studies have been performed in a number of exploration areas. Lefevre (1994) applied AVO-AVAZ (amplitude versus offset and amplitude versus azimuth) analysis in the Paris basin, France and found that P waves, calibrated by shear-wave data, can detect fractures in the subsurface. Lynn et al. (1996) and Lynch et al. (1997) successfully used multi-azimuth P-wave AVO analysis in the Wind River Basin, Wyoming and at the Rulison Field, Picance Basin, Colorado to determine fracture orientation within production intervals. Mallick et al. (1996) described a fracture detection method (the fractogram technique) applied to a 3-D P-wave data set over a gas field in central Wyoming. Although the method of Mallick et al. (1996) aims to produce principal directions of AVO gradient as functions of vertical time, influence of fine layering within reservoir was not removed from final fractogram displays.

Here, I develop a technique for azimuthal AVO analysis designed to reconstruct true AVO intercept and a quadratic form responsible for azimuthally varying AVO gradient as functions of the vertical time (or the depth, if the vertical velocity is known) in finely layered azimuthally anisotropic media. The word "true" in the previous sentence means that ideally I estimate series of spikes for AVO intercept and azimuthally dependent
AVO gradient which would be recorded if all interfaces are isolated from each other and interference of closely spaced reflections does not exist. In reality, the reconstructed AVO signatures are influenced by the shape and the frequency content of the seismic wavelet, thus, wavelet estimation becomes important for deconvolving and obtaining correct reflectivities. Then, the estimated AVO gradients can be interpreted in terms of differences of elastic parameters at each given vertical time using, for instance, linearized approximations derived by Vavrytuk and Pšenčík (1997) or nonlinear inversion as suggested by Neves and de Hoop (1997).

I begin with introducing an exploding reflector-type model for a finely layered azimuthally anisotropic reservoir. This model contains plane-wave reflection coefficients of primaries. I demonstrate that the introduced model properly accounts for the influence of fine layering on AVO signatures and can be used to obtain the true AVO intercept and gradient. Then, I perform numerical study to show how errors in estimated wavelet, uncertainties in velocity model, and random noise distort the extracted AVO signatures.

**Reflectivity model for layered reservoir**

I model layered reservoir as a stack of plane azimuthally anisotropic layers between two homogeneous halfspaces \( x_3 \leq 0 \) and \( x_3 \geq z \) (Figure 1) and concentrate on wavefield \( U \) reflected from the reservoir and recorded at its top \( x_3 = 0^* \). It is convenient to examine the wavefield \( U \) in frequency-slowness domain, i.e., \( U \equiv U(\omega, p_1, p_2) \equiv U(\omega, p) \) because both temporary frequency \( \omega \) and the horizontal components \( p_1 \) and \( p_2 \) of the slowness vector are preserved during propagation of a plane wave within the reservoir.

Conventional convolutional model (e.g., Yilmaz, 1991)

\[
U(\omega, p) = S(\omega) R(\omega, p)
\]

in \( \omega - p \) domain can be used to relate the wavefield \( U(\omega, p) \) to the source spectrum \( S(\omega) \) and the generalized P-wave reflection coefficient \( R(\omega, p) \). Note that the source spectrum \( S \) is independent of \( p \) if the size of the source is smaller than the dominant wavelength (e.g., Buland et al., 1996), which is usually the case in practice. Assuming normal incidence (i.e., \( p = 0 \)), one can deconvolve the wavefield \( U(\omega, 0) \) and estimate the reflection coefficient \( R(\omega, 0) \) and the source spectrum \( S(\omega) \).

Conventional assumptions under which deconvolution is being applied are discussed in Yilmaz (1991). The spectra \( R(\omega, 0) \) and \( S(\omega) \) can be Fourier-transformed into time domain to obtain the reflectivity series \( r(t_0, 0) \) and the wavelet \( s(t_0) \) as functions of the vertical time \( t_0 \).

After the source wavelet has been estimated using conventional processing, equation (1) explicitly gives the generalized reflection coefficient \( R(\omega, p) \) for any slowness \( p \). The initial objective is to express \( R(\omega, p) \) in terms of individual P-wave reflection coefficients \( r(t_0, p) \) from all interfaces within the reservoir. The obtained coefficients \( r(t_0, p) \) then can be smoothed by fitting surfaces \( G(p) \) at each vertical time \( t_0 \). The values \( G(0) = r(t_0, 0) \) are AVO intercepts and the curvatures of \( G(p) \) at \( p = 0 \) are AVO gradients in slowness domain.

**A single-layer reservoir model**

Let us begin with the simplest reservoir model that does not contain intermediate layers. Since, there is only one plane wave reflected from the reservoir base at depth \( z \) (Figure 1), the reflectivity series in time domain contains a single spike \( r(t_0, p) \) which can be related to the reflection response \( R(\omega, p) \) as

\[
R(\omega, p) = e^{2i\omega t_p(t_0, p)} r(t_0, p),
\]

The term \( r(t_0, p) \) in this equation represents the spike while traveltime \( t_p \equiv t_p(t_0, p) \) describes slowness-dependent time delay; the factor \( e^{2i\omega t_p} \) is the exponential stands to account for two-way propagation time. The relation between two time delays, \( t_p \) and \( t_0 \), is given by:

\[
t_p = z \frac{\cos \theta(p)}{V(p)} = t_0 \frac{v_0 q(p)}{q(p)},
\]

\* This wavefield can be obtained as a result of true-amplitude migration applied to the wavefield recorded at the earth surface.
where $\theta(p)$ is the propagation angle (Figure 1), $V(p)$ is the phase velocity in direction $\theta(p)$, $v_0$ is the vertical group velocity, and $q(p)$ is the vertical component of the slowness vector.

**Multi-layer reservoir model**

Now, I generalize equation (2) and construct reflection response from a multi-layer reservoir. The simplest way of doing that is to sum up reflections from all interfaces at their appropriate time delays:

$$R(\omega, p) = \sum_k e^{2i\omega t_\ell(t_{0,k}, p)} r(t_{0,k}, p), \quad (4)$$

where

$$t_{0,k} = \sum_{\ell \leq k} \Delta t_{\ell, k} \quad (5)$$

and

$$t_\ell(t_{0,k}, p) = \sum_{\ell \leq k} \Delta t_{\ell, k} v_{0,\ell} q_\ell(p). \quad (6)$$

In equations (5) and (6), $\Delta t_{\ell, k}$ is the interval one-way vertical traveltime in the $\ell$th layer, $v_{0,\ell}$ is the interval vertical group velocity, and $q_\ell(p)$ is the interval vertical slowness.

Equation (4) represents a linear "exploding reflector-type" reflectivity model which includes primaries only. This model accounts neither for internal multiples, nor for various converted waves which both present in true reflection response. Also, equation (4) does not compensate for propagation effects through interfaces at vertical times $t_{0,\ell}$ for $\ell < k$. Nevertheless, the simplistic reflectivity model (4) has one important feature: it is linear with respect to unknown reflection coefficients $r(t_{0,k}, p)$ which can be found from system of equations

$$R_j = E_{jk} r_k, \quad (7)$$

where

$$R_j \equiv R(\omega_j, p), \quad E_{jk} \equiv e^{2i\omega_j t_\ell(t_{0,k}, p)}, \quad (8)$$

and $r_k \equiv r(t_{0,k}, p)$.

Note, that the time delays $t_{0,k}$ for the reflection coefficients $r_k$ are not unknowns; $t_{0,k}$ should be specified before solving equations (7). If there is no reflector at time $t_{0,k}$ corresponding reflection coefficient is supposed to be zero.

Ideally, equations (7) should be solved independently for each value of slowness to produce individual reflection coefficients $r_k$ as functions of $p$. It is possible if the number of frequencies $\omega_j$ in reflection response $R(\omega_j, p)$ is greater than the number of unknowns $r_k$, and system (7) is overdetermined. This constraints the maximum number of reflection coefficients that can be resolved. Another constraint — on temporary resolution or on the minimum value of $\Delta t_{0,\ell}$ — is determined by spectral bandwidth of the response $R$. I will discuss these and other numerical issues of obtaining $r_k$ in the following sections; here, I examine some options for making the reflectivity model more accurate.

The reflectivity model (4) may be improved by including some of the factors — multiples, conversions, and propagation effects — which were initially ignored. There is not much freedom, though, because I would not like to increase the number of unknowns as those additional model parameters may introduce some trade-offs with the primary unknowns — the individual $P$-wave reflection coefficients $r_k$ — and degrade their resolution. Below, I discuss some possibilities and include the factors which can be expressed in terms of $r_k$ and, therefore, do not add new unknowns.

**Multiples**

Including multiples into the reflectivity model (4) is not difficult. However, it makes the model nonlinear with respect to $r_k$. For example, amplitude of the first-order multiple is proportional to $r_{k_1} r_{k_2} r_{k_3}$, where indices $k_i$ indicate the interfaces where the multiple has reflections. Nonlinearity of the model is an undesirable feature because it complicates calculating reflection coefficients and may lead to non-uniqueness of the solution. On the other hand, one may believe that the influence of multiples within the reservoir on reflection response $R(\omega, p)$ is not that significant. For example, reflection coefficient of about $r = 0.2$ indicates a fairly strong interface. There may be only a few such strong interfaces within the reservoir. Therefore, amplitude of the first-order multiple, reflected from these interfaces, can expected to be approximately 4% of amplitudes of corresponding primaries. Given the accuracy of measuring seismic amplitudes, an error of 4% can be ignored. Thus, I choose not to make reflectivity model (4) more complicated to improve its accuracy by just a few percent. Numerical examples presented below support this choice.

**Converted waves**

The problem with including various converted waves into the reflectivity model is that their reflection coefficients depend on parameter combinations of contacting layers which are different from those that control $P$-wave reflection coefficients $r_k$. This can be explicitly observed from linearized weak-anisotropy small-contrast approximations for reflection coefficients given by Rüger (1996). Exact reflection coefficients are expected to be even
more complicated. As a consequence, converted-mode reflection coefficients cannot be expressed in terms of $r_k$. Therefore, to account for converted waves explicitly, some unknown parameters, which influence converted-mode AVO signatures but do not present in P-wave ones, should be added to the reflectivity model (4). This, again, complicates the reflectivity model and may reduce stability in estimating $r_k$.

There is another consideration, which allows one to believe that conversions can be often approximately subtracted from composite reservoir response even without explicit estimating of corresponding converted-mode reflection coefficients. Let us concentrate on a common model of fractured reservoir which can be characterized by the presence of vertical cracks. All anisotropic media (HTI, orthorhombic, or monoclinic), which are used to simulate seismic response from a vertically fractured reservoir, have a horizontal symmetry plane. In those media, converted-wave reflection coefficients equal zero at normal incidence (at $p = 0$) and depend linearly on slowness $p$ for small $p$ (or small offsets) as opposed to pure-mode reflection coefficients which are quadratic functions of $p$ when $p$ is small. This provides us with a simple tool for separating converted-mode reflections from pure-mode ones even if reflectivity model (4) is used.

Let us suppose that equations (7) have already being solved and $r_k = r_k(p)$ are found. At small $p$, reflection coefficients can be approximated by double a Taylor series

$$ r_k(p) = r_k(p_1, p_2) \approx \sum_{m,n=0}^{m+n \leq N} \frac{1}{(m+n)!} g_k^{(m,n)} p_1^m p_2^n, \quad (9) $$

where $g_k^{(0,0)}$ is the normal-incidence reflection coefficient (AVO intercept), the coefficients $g_k^{(m,n)}$ with $m+n = 1$ describe amplitudes of converted waves arriving at the vertical time $2t_{0,k}$, and $g_k^{(m,n)}$ with $m+n = 2$ are the elements of a symmetric quadratic form which represents azimuthally dependent AVO gradient in the slowness domain. Criteria for choosing $N$ will be discussed in the next section.

Thus, by fitting reflection coefficients $r_k(p)$ with equation (9), I may hope to account for some influence of converted waves. I will verify this by performing numerical tests.

**Propagation effects**

Since I consider the reflection response $R(\omega, p)$ for each plane wave (specified by slowness $p$) separately, the spherical spreading, which exists in time-offset domain, has already been removed by 3-D Fourier transformation to $\omega - p$ domain. Another factor that can be accounted for is transmission effects.

At normal incidence ($p = 0$), transmission coefficient through $k$th interface is $1 - r_k$. Since wave transmits through each interface twice on the way down and up and the reflection coefficients of down- and upgoing waves are opposite, amplitude reduction for two-way propagation is $1 - r_k^2$. Therefore, due to transmission, reflection response $R$ contains

$$ \tilde{r}_k = r_k \prod_{\ell < k} [1 - r_\ell^2] \quad (10) $$

instead of just $r_k$. Thus, I can replace $r_k$ by $\tilde{r}_k$ in equation (7) and, after $\tilde{r}_k$ are found, use equation (10) to obtain actual $r_k$. Although this equation is nonlinear with respect to $r_k$, it has a triangular-type structure which becomes apparent if equation (10) is presented in the form:

$$ \tilde{r}_0 = r_0, \quad \tilde{r}_1 = r_1 [1 - r_0^2], \quad \tilde{r}_2 = r_2 [1 - r_1^2] [1 - r_0^2], \ldots, \quad (11) $$

where $r_0$ is reflection coefficient from the top of reservoir.

Thus, equation (10) is easy to solve. Some difficulties appear if one wants to correct transmission effects for non-normal incidence, i.e., for $p \neq 0$. The problem is that reflection coefficient $r_k(p)$ for downgoing wave approaching interface from the top is no longer opposite to that for upgoing wave which approaches the same interface from the bottom (e.g., Castagna, 1993). Therefore, strictly speaking, reflection coefficient is asymmetric, and equation (10) becomes incorrect. On the other hand, all known linearized weak-anisotropy small-contrast approximations show that reflection coefficients from two sides of the same interface are opposite to each other. This fact indicates that asymmetry in reflection coefficient is usually small and probably can be ignored for practical purposes. The numerical results support that and I will apply equation (10) to correct for transmission effects within reservoir for both normal and non-normal incidence.

Thus, the reflectivity model for layered reservoir, which will be used to extract individual reflection coefficients $r_k(p)$, is given by

$$ R(\omega, p) = \sum_k e^{2i\omega t_k(p_0, \omega, p)} \tilde{r}_k(t_{0,k}, p), \quad (12) $$

or, in matrix form,

$$ R_j = E_{jk} \tilde{r}_k, \quad (13) $$

where $\tilde{r}_k \equiv \tilde{r}_k(p) = \tilde{r}(t_{0,k}, p)$, and $R_j$ and $E_{jk}$ are defined by equations (8). Equation (10) relates reflection coefficients $r_k$ to $\tilde{r}_k$. 
Extracting AVO intercept and gradient

Once equations (13) and (10) are solved, I obtain individual reflection coefficients \( r(p) \) as functions of the vertical time \( t_0 \). Then, using equation (9), I estimate AVO intercept

\[
A(t_0) \equiv G^{(0,0)}(t_0)
\]

and symmetric matrix

\[
B(t_0) \equiv \begin{pmatrix} B_{11}(t_0) & B_{12}(t_0) \\ B_{12}(t_0) & B_{22}(t_0) \end{pmatrix}
\]

\[
\equiv \begin{pmatrix} G^{(2,0)}(t_0) & G^{(1,1)}(t_0) \\ G^{(1,1)}(t_0) & G^{(0,2)}(t_0) \end{pmatrix}
\]

which describes azimuthally varying AVO gradient in the slowness domain. Quantities \( A \) and \( B \) represent AVO intercept and gradient at interface which corresponds to one-wave vertical traveltime \( t_0 \). They are an ideal input for AVO inversion schemes and can be interpreted in exactly the same way as AVO intercept and gradient at a single isolated reflector.

In reality, however, recorded reservoir response \( R(\omega, p) \) may contain some portion of the source wavelet \( \tilde{s}(t_0) \) which was not properly deconvolved. This portion of the wavelet can be put into equation (12) explicitly:

\[
\hat{S}(\omega) R(\omega, p) = \sum_k e^{2i\omega t_p(t_0, k, p)} [\tilde{s}(t_0) \ast \tilde{r}(t_0, p)]_k,
\]

where \( \hat{S}(\omega) \) is the portion the source spectrum which was not removed, and \( \ast \) denotes convolution. From equations (16), I can find the convolved reflection coefficients \( \tilde{s}(t_0) \ast \tilde{r}(t_0) \). Then, solving equations (10) with the left-hand sides \( \tilde{s}(t_0) \ast \tilde{r}(t_0) \), yields the result which is approximately for small \( \tilde{r}(t_0) \) the convolution \( \tilde{s}(t_0) \ast \tilde{r}(t_0) \). This approximation becomes exact if reflection coefficients \( \tilde{r}(t_0) \) are so small that the influence of transmission effects can be ignored. Finally, I obtain \( A(t_0) \) and \( B(t_0) \) which are also approximately convolutions of true AVO intercept and gradient with \( \tilde{s}(t_0) \).

Even when the source wavelet is correctly estimated, obtained \( \tilde{r}(t_0) \) may be still inaccurate because of errors in the anisotropic velocity model used to compute \( t_p(t_0, p) \). As equation (6) indicates, correct calculating \( t_p(t_0, p) \) requires knowing not only interval vertical velocities \( v_0 \) but also interval anisotropic coefficients which control the vertical slowness \( q(p) \). Such detailed information apparently cannot be extracted from seismic data. Therefore, in practice, velocity model used to compute \( t_p(t_0, p) \) in equations (16) will always be erroneous. This means that reflections from different interfaces within reservoir are going to be mispositioned which, in turn, will lead to errors in \( \tilde{r}(t_0, p) \) and, consequently, to errors in AVO intercept \( A(t_0) \) and gradient \( B(t_0) \). Fortunately, for relatively thin reservoirs, errors in \( t_p(t_0, p) \) due to inaccuracy in anisotropic velocity model are small even if some average isotropic velocity model is used. As I demonstrate below, these errors do not lead to noticeable distortions in \( A(t_0) \) and \( B(t_0) \), thus, in fact, detailed velocity information within reservoir is not needed for AVO analysis.

Numerical examples

Here, I demonstrate capabilities of the developed technique and clarify various computational issues. I apply the technique to two reservoir models. For all examples presented, I generate reflection response \( R(\omega, p) \) using anisotropic reflectivity code. Thus, the responses I examine contain all possible multiples and converted waves. I use equations (12) and (10) to find \( r_k(p) \); then, applying equations (9), (14), and (15), obtain AVO intercept \( A(t_0) \) and gradient \( B(t_0) \).

Model 1

The first model contains two orthorhombic layers between two isotropic half-spaces (Table 1). This model was intentionally designed in such a way that azimuthal variations of \( P \)-wave reflection coefficients from its interfaces have all theoretically possible shapes (for small slowness \( p \)): a saddle, a maximum, and a minimum (Figure 2). Note that the range of \( p \) values is smaller in Figure 2a compared to that in Figure 2b and 2c. I reduced the range of \( p \) in Figure 2a to show more explicitly the shape of reflection-coefficient surface at small \( p \) or small offsets. Corresponding surface for conventional offsets (when incidence angle reaches 30°) is shown in Figure 3. Clearly, the order of the surface in Figure 3 is higher than quadratic. To accommodate those high-order variations in reflection coefficient, I use \( N = 4 \) in equation (9) but analyze only AVO intercept and gradient†.

An important quantity which can be extracted from azimuthal AVO analysis is the azimuth \( \beta \) of greater principal value of AVO gradient (I will call \( \beta \) the AVO azimuth). It equals to the azimuth of eigenvector corresponding to the greater eigenvalue of matrix \( B \) [equation (15)]:

\[
\beta = \frac{1}{2} \tan^{-1} \frac{2B_{12}}{B_{11} - B_{22}}.
\]

The principal values of AVO gradient (eigenvalues of matrix \( B \)) are given by

\[
b_{1,3} = \frac{1}{2} \left[ B_{11} + B_{22} \pm \sqrt{(B_{11} - B_{22})^2 + 4B_{12}^2} \right].
\]

I will be mostly interested in reconstructing \( \beta(t_0) \)

† This is similar to using stacking velocity obtained from nonhyperbolic velocity analysis of long-spread reflection moveout.
Table 1. Reservoir model 1: two orthorhombic layers between isotropic half-spaces. Density 2.5 g/cm³ is the same in all layers and half-spaces. Azimuths of \([\varepsilon_1, \varepsilon_3]\) symmetry planes of the layers are measured counter-clockwise from the positive direction of axis \(p_2 = 0\).

Figure 2. P-wave reflection coefficients at three interfaces in model 1. Maximum incidence angle is 15° for the first interface (a) and 30° for the second (b) and the third (c) interfaces.

Figure 3. P-wave reflection coefficient at the first interface in model 1. Maximum incidence angle is 30°.

since it relates to the symmetry directions of contacting anisotropic layers (assuming that the layers have orthorhombic or HTI symmetry) and, therefore, indicates the azimuth of vertical fractures, which is of great importance for seismic exploration. If only one of contacting layers is azimuthally anisotropic, fracture orientation in this layer is parallel or orthogonal to the azimuth \(\beta\). If both layers are azimuthally anisotropic, \(\beta\) incorporates influence of anisotropy of both.

Figure 4 demonstrates the results of applying the procedure to reflection response in model 1. I put reflection from the top (the first interface) at time \(t_0 = 0\)

Figure 4. Normalized AVO attributes as functions of vertical time estimated for model 1: (a) AVO intercept \(A\), (b) principal values of AVO gradient [solid – \(b_1\) and dashed – \(b_2\)] obtained from equation (18), and (c) AVO azimuth \(\beta\) (in degrees) given by equation (17).

and used correct anisotropic velocity model given in Table 1 to calculate \(t_p(t_0, p)\) [see equations (12) and (6)]. The reflection response was convolved with a bell-shaped wavelet,

\[
s(t_0) = \exp(-\nu^2 t_0^2),
\]

(19)
\( \nu = 100 \text{ s}^{-1} \); the side lobes of wavelets in Figure 4 appeared because of spline-interpolation between time samples \( \tau = 4 \text{ ms} \).

Figure 4 presents almost ideal result:

- Normal incidence amplitudes for all there reflections are equal (Figure 4a) as it is supposed to be (correct normal-incidence reflection coefficients, which can be calculated based on Table 1, are \( \pm 0.067 \)). This indicates that the correction for transmission effects [equation (10)] is sufficiently accurate, and ignoring multiples and converted waves for this model is justified.

- The signs of the principal values of AVO gradient shown in Figure 4b are in exact accordance with the shapes of reflection-coefficient surfaces (Figure 2). Reflection at \( t_0 = 0 \) has a saddle-type shape (Figure 2a); correspondingly, its \( b_1 \) and \( b_2 \) values [equation (18)] are of opposite sign. Reflection from the second interface at \( t_0 = 24 \text{ ms} \) has a maximum at normal incidence and both its \( b_1 \) and \( b_2 \) are negative. Similarly, for reflection from the third interface at time \( t_0 = 48 \text{ ms} \), which has the minimum at normal incidence, both principal values of AVO gradient are positive.

- The AVO azimuths \( \beta \) (Figure 4c), whose correct values are \( 60^\circ, -14^\circ, \) and \(-50^\circ), were reconstructed with error less than \( 1^\circ \). This error can be attributed to the influence of ignored multiples and converted waves. The azimuths \( \beta \) for the first and third reflections correspond to the \( x_2, x_3 \)- and \( x_1, x_3 \)-symmetry planes in the first and second orthorhombic layers (compare Figure 4c and Table 1).

Figure 5 demonstrates a more realistic example. It shows the results of applying the AVO procedure to the same reflection response as that in Figure 4 with the only difference: an incorrect isotropic model with constant velocity \( v_0 = 4 \text{ km/s} \) corresponding to one in the lower half-space (Table 1) was used to calculate traveltimes \( t_p \). The traveltimes were computed from equation (6) which, under the assumptions of constant velocity and isotropy, yields

\[
t_p(t_0, p) = t_0 \sqrt{1 - v_0^2 (p_1^2 + p_2^2)}.
\] (20)

Since intentionally incorrect velocity model was used, the matrix \( e^{2i\omega t_p(t_0, p)} \) in equations (12) is erroneous, which leads to incorrect values of \( r(t_0, p) \). Moreover, due to incorrect \( t_p \), reflections at \( t_0 = 24 \text{ ms} \) and \( t_0 = 48 \text{ ms} \) are mispositioned which produces additional errors in \( A(t_0) \) and \( B(t_0) \). However, Figure 5 demonstrates that obtained AVO signatures are very close to the correct ones shown in Figure 4. Some differences can be observed between \( b_2 \) values for reflection at \( t_0 = 48 \). Also, there are errors up to \( 5^\circ \) in estimated AVO azimuths. To explain such good results, one should take
Table 2. Reservoir model 2: Barinas field, Venezuela. The model contains isotropic layers and half-spaces and one HTI layer. HTI anisotropy can be viewed as a special case of orthorhombic anisotropy with the following constraints on elastic constants:

\[ c_{12} = c_{13}, c_{22} = c_{33}, c_{23} = c_{32} = -2c_{44}, \text{ and } c_{55} = c_{66}. \]

Azimuth \( \alpha \) (in degrees) is given for the symmetry-axis plane of the HTI layer with respect to the positive direction of axis \( p_2 = 0 \).

Figure 8. Normalized AVO signatures as functions of vertical time estimated for model 2: (a) AVO intercept \( A \), (b) principal values of AVO gradient [solid - \( b_1 \) and dashed - \( b_2 \)], and (c) AVO azimuth \( \beta \) (in degrees).

The previous two examples were somewhat artificial because frequency of the bell-shaped wavelet was high enough so there was no interference between reflections from different interfaces. A more realistic example is shown in Figure 6. I have chosen the same wavelet (19) but this time with \( \nu = 7 \ s^{-1} \). The same incorrect isotropic velocity model with \( v_0 = 4 \ km/s \) is used again. Figure 6a can be viewed as a low-pass filtered version of Figure 5a. A single reflection event is observed this time. It is important that the curve \( \beta(t_0) \) shown in Figure 6c can also be thought as a low-resolution version of corresponding curve in Figure 5c. Thus, useful information about azimuthal anisotropy can be obtained even without isolating individual reflections.

The last example for model 1 is shown in Figure 7. This time, the Ricker wavelet with dominant frequency 20 Hz was convolved with the reservoir reflection response. Assuming that I am not aware of that, I, again, use an incorrect isotropic velocity model with \( v_0 = 4 \ km/s \) to obtain \( \hat{\alpha}(t_0) \) from equation (16). Comparing Figure 7a and 7b with Figure 5a and 5c shows that both AVO intercept and the principal values of AVO gradient are close to convolutions of signatures shown in Figure 5 with Ricker wavelet. The wavelet shape has its imprint on AVO azimuth as well: the curve \( \beta(t_0) \) (Figure 7c) has a spurious lobe at time \( t_0 = 55 \ ms \). Unfortunately, such false AVO azimuths cannot be avoided because the algorithm interprets wavelet lobes as real reflections. Therefore, Figure 7 emphasizes the necessity of wavelet estimation and deconvolution for AVO processing in fine layered reservoirs.

Model 2

For the second test I chose the model of carbonate fractured reservoir in the Barinas field, Venezuela (Table 2). This model was initially built by Michelen and al. (1994) and Ata et al. (1994) to describe splitting of converted
waves and, then, refined by Perez and Gibson (1996) to explain P-wave AVO signatures recorded along different azimuths. I assumed HTI anisotropy in P member (layer 5 in Table 2) and selected its parameters in such a way to produce a saddle-type AVO responses observed by Perez et al. (1998).

Figure 8 demonstrates the results of the first test for model 2. I again put reflection from the top (interface Paguey – Gobernador, see Table 2) at time $t_0 = 0$, convolve the reflection response with bell-shaped wavelet (19) [$\nu = 100$ s$^{-1}$], and use incorrect isotropic constant-velocity ($v_0 = 4.1$ km/s) model to obtain AVO intercept and gradient. Figure 8a shows three strong reflections: from interfaces Paguey – Gobernador ($t_0 = 0$), O member – P member ($t_0 = 96$ ms), and Aquardiente – Basement ($t_0 = 146$ ms). There are also several weaker reflections at $t_0$ from approximately 70 to 120 ms and some multiples observed at $t_0 > 150$ ms. Figure 8a clearly illustrates interference of all these events. However, even in the presence of interference, principal values of AVO gradient shown in Figure 8b indicate that there is no azimuthal anisotropy (i.e., $b_1(t_0) = b_2(t_0)$) in more shallow horizons than P member. At $t_0 > 96$ ms, after the reflection from interface O member – P member arrives, the difference between $b_1$ and $b_2$ becomes visible showing the presence of azimuthal anisotropy. The difference between $b_1$ and $b_2$ extends beyond P member, which can be attributed to combination of errors due to wrong velocity used for AVO analysis with some influence of ignored multiples and converted waves. Computing AVO azimuth $\beta(t_0)$, I set $\beta(t_0)$ to 0 when the difference $b_1(t_0) - b_2(t_0)$ or AVO intercept $At(t_0)$ is less than 25% of their maximum values. Figure 8c shows that the azimuth of the symmetry-axis plane ($-70^\circ$) within P member is correctly recovered.

P-wave seismic data recorded over Barinas field have much lower frequency than that in Figure 8 (Perez and Gibson, 1996; Perez et al., 1998). To simulate the expected data, I convolved reflection response from the reservoir with Ricker wavelet which has dominant frequency 25 Hz. Again, I used incorrect velocity model with $v_0 = 4.1$ km/s to extract AVO attributes. The results of applying AVO analysis are shown in Figure 9. Despite obvious interference of various reflection events, which is seen at normal incidence (Figure 9a), a reasonable estimation of AVO azimuth $\beta$ is obtained (Figure 9c). Although the curve $\beta(t_0)$ contains small false

\[ \text{Figure 9. The same as Figure 8 but reflection response from reservoir was convolved with Ricker wavelet. Dots in (c) represent AVO azimuths estimated from "conventional" azimuthal AVO analysis.} \]

\[ \text{Figure 10. The same as Figure 9 but Gaussian noise was added to reflection response convolved with Ricker wavelet.} \]

\[ \text{Figure 11. NMO-corrected seismic traces recorded for incidence angles $\theta = 0^\circ$ (a), 10$^\circ$ (b), 20$^\circ$ (c), and 30$^\circ$ (d). Azimuth of incident wave is 45$^\circ$.} \]

\[ \dagger \text{The reservoir, filled with crude oil, includes O member, P member, S shale, and Aquardiente. Seismic measurements of fractures within O and P members are supported by borehole data (M.Perez, personal communication).} \]
lobes at \( t_0 \) of about 70 and 130 ms (those correspond to the lobes of Ricker wavelet), it shows almost exact value \( \beta = -72^\circ \) within P member. This result looks especially encouraging noting that time thickness of P member (16 ms) is less than half dominant period (20 ms) of the wavelet. Thus, the presented numerical example indicates the potential of AVO analysis in detecting azimuthal anisotropy in very thin layers.

The results shown in Figure 9 can be distorted by adding some random noise to the reflection response. The errors in AVO azimuths become visible when the variance of Gaussian noise reaches 30% of the maximum reflection amplitude. Figure 10 illustrates what happens in this case. Since AVO intercept is a better determined quantity than is AVO gradient, the values of \( b_1(t_0) \) (Figure 10b) have greater distortions as compare to that in \( A(t_0) \) (Figure 10a). The errors in AVO gradient produce spurious AVO azimuth \( \beta \) at \( t_0 = 0 \) and corrupt AVO azimuth in the vicinity of \( t_0 = 100 \) ms (compare Figure 9c with Figure 10c). Even thought this example combines all errors and inaccuracies I have examined before, the obtained curve \( \beta(t_0) \) still shows where azimuthal anisotropy can be expected and thus provides useful information for interpretation.

It is of interest to compare estimations of AVO azimuth \( \beta \) obtained from "conventional" azimuthal AVO analysis with the presented results. To simulate "conventional" AVO analysis, I transformed the reflection response \( R(\omega, p) \) into time-offset domain and corrected obtained traces for normal moveout (NMO). Figure 11 shows some of those NMO-corrected traces and illustrates the change in reflection response as a function of incidence angle \( \theta \) in upper half-space. Then, I picked amplitudes of peaks and troughs in time interval from 50 to 160 ms and used those amplitudes to find AVO azimuths [applying equation (9) with \( N = 4 \) and equations (15) and (17)] corresponding to each peak and trough. The results (dots) are shown in Figure 9c. Although AVO azimuth \( \beta \) at \( t_0 = 100 \) ms was determined correctly, the overall result is difficult to interpret due to significant scatter in \( \beta \) values. The scatter is caused by interference of various reflection events whose dependence on offset (or incidence angle) and azimuth cannot be properly accounted for in time-offset domain.

Discussion and conclusions
I have developed a method for AVO analysis in finely layered azimuthally anisotropic media. The method takes seismic reflection data in \( \omega - p \) domain and reconstructs true AVO intercept \( A \) and azimuthally varying AVO gradient \( B \) which would be recorded in the case of isolated interfaces and in the absence of interference between closely spaced reflection events. The obtained AVO attributes, instantaneous AVO intercept \( A(t_0) \) and gradient \( B(t_0) \), are functions of the vertical time \( t_0 \) and represent the required input for various AVO inversion schemes because they can be interpreted in exactly the same way as AVO intercept and gradient at a single isolated reflector at time \( t_0 \). The technique I have proposed operates in slowness domain\(^\S\) because the horizontal slowness \( p = (p_1, p_2) \), which is preserved during wave propagation in horizontally layered media, is a natural parameter for reflection coefficients from interfaces located at different vertical times \( t_0 \). Otherwise, if I parameterize AVO attributes as functions of offset \( X \) or the incidence angle \( \theta \), I would have to recalculate \( X \) or \( \theta \) for each \( t_0 \) since they change in vertically varying media. Although this can be done for any given set of the model parameters, the whole AVO procedure becomes more complicated and probably less stable because the required model parameters are unknown.

The key idea of developed AVO procedure [sketched in Figure 1 and mathematically expressed by equation (4)] is to place each prospective primary plane-wave reflection, specified by the horizontal slowness \( p \), at its proper vertical time \( t_0 \). The idea I applied here is fully equivalent to the concept initially proposed by Wapenaar et al. (1995) that one has to make the apparent vertical wavelength \( \lambda_\nu \) independent of slowness \( p \) to properly account for layering. In this regard the developed procedure can be viewed as an extension of technique of Wapenaar et al. (1995) to azimuthally anisotropic media.

To extract individual reflection coefficients \( r(t_0, p) \), I used the "exploding reflector-type" reflectivity model, which contains primaries and approximately accounts for transmission effects. At first glance, the reflectivity model may seem too simplistic because it completely ignores multiples and converted waves. However, the numerical examples I have presented indicate that the simplifications made are adequate in the sense that they allowed reconstructing true AVO signatures in the absence of any other errors. I also have shown that the method produces good results even when correct anisotropic velocity within reservoir is replaced by some average isotropic constant-velocity model. This is a very important feature of the algorithm because it allows one to perform AVO analysis without knowing the fine-scale reservoir velocity model. This characteristic of the technique, in fact, could be expected because I used information

\(^\S\) The estimated AVO gradient \( B(p) \) is also in the slowness domain. Although there is a relation between \( B(p) \) and conventional AVO gradient \( B(X) \) measured as a function of the offset \( X = (x_1, x_2) \), generally \( B(p) \neq B(X) \).
about velocity just to calculate traveltimes of prospective reflection arrivals. It is known that traveltimes are influenced by only relatively long-period velocity variations and almost insensitive to velocity structure on a fine scale.

One of the quantities produced by the procedure is AVO azimuth $\beta(t_0)$ which corresponds to instantaneous direction of larger principal value of AVO gradient $B(t_0)$. AVO azimuth is of great importance for seismic exploration in fractured reservoirs because it indicates the azimuth of vertical cracks at a given vertical time $t_0$. I have shown that $\beta(t_0)$ is a reasonable well determined quantity so one can expect to obtain it from real data. I have found that errors in $\beta$ are mostly associated with wrong estimation of seismic wavelet. Generally, the wavelet is supposed to be deconvolved before calculating AVO azimuth; otherwise spurious values of $\beta(t_0)$ are produced.

I compared estimated values of $\beta$ with those obtained by “conventional” azimuthal AVO analysis (i.e., when one looks at variation with offset and azimuth of reflection amplitude corresponding to a certain peak or trough on seismic traces). I found that “conventional” estimates of AVO azimuths are less robust and characterized by greater scatter. This scatter can be explained by offset- and azimuth-dependent interference, which is not properly accounted and compensated for in the time-offset domain.

I have tested the technique on seismic model developed for the Barinas field, Venezuela. This model contains several layers which are thinner than half dominant wavelength observed in seismic data. Although I have not examined the limits of spatial resolution of azimuthal AVO analysis in detail, the presented results suggest that it might be possible to detect the presence and principal directions of azimuthal anisotropy in layers with thickness less than half dominant wavelength.

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Reflection coefficients for weak anisotropic media:
A synthetic example of anisotropic $P$-wave processing for a model from the Gulf of Mexico

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ABSTRACT

Transverse isotropy with a vertical symmetry axis (VTI media) is the most common anisotropic model observed in most sedimentary basins. Here, we apply $P$-wave processing algorithms developed for VTI media to a 2-D synthetic data set generated by a finite difference code. The model, typical for the Gulf of Mexico, has a moderate structural complexity and includes a salt body and a dipping fault plane.

Using the Alkhalifah-Tsvankin dip-moveout (DMO) inversion method, we obtain the anisotropic coefficient $\eta$ responsible for the dip dependence of $P$-wave NMO velocity in VTI media. In combination with the normal-moveout (NMO) velocity from a horizontal reflector $[V_{nmo}(0)]$, $\eta$ is sufficient for performing all $P$-wave time-processing steps (NMO, DMO, prestack and poststack time migration). The NMO (stacking) velocities needed to determine $V_{nmo}(0)$ and $\eta$ are picked from conventional semblance velocity panels for reflections from sub-horizontal interfaces, the dipping fault plane and the flank of the salt body. To mitigate the instability in the interval parameter estimation, the time-dependent $V_{nmo}(0)$ and $\eta$ functions are approximated by Chebyshev polynomials with the coefficients found by “global” fitting of all velocity picks.

We perform prestack depth migration for the reconstructed anisotropic model and two isotropic models with different choices of the velocity field. The anisotropic migration result has a good overall quality, but reflectors are mispositioned in depth because the vertical velocity cannot be obtained from surface data. To build an anisotropic model for depth migration, $P$-wave data have to be combined with well information (e.g., well logs, check shots) or reflection moveout of shear or converted waves.

The isotropic migrated section with the NMO velocity $V_{nmo}(0)$ substituted for the isotropic velocity also has the wrong depth scale and is somewhat inferior to the anisotropic result in the focusing of dipping events. Still, the image distortions are not significant because the parameter $\eta$, which controls NMO velocity for dipping reflectors, is rather small (the average value of $\eta$ is about 0.05). In contrast, the isotropic section migrated with the vertical velocity $V_0$ has a poor quality (although the depth of the subhorizontal reflectors is correct) due to the fact that in VTI media $V_0$ can be used to stack neither dipping nor horizontal events. The difference between $V_0$ and the zero-dip stacking velocity $V_{nmo}(0)$ is determined by the anisotropic coefficient $\delta$, which is greater than $\eta$ in our model (on average, $\delta \approx 0.1$).

Introduction

TI model with a vertical symmetry axis was shown to give accurate predictions for the kinematics of wave propagation in most sedimentary basins. Extending seismic processing even to this simplest anisotropic model, however, requires estimating anisotropic parameters from surface (preferably, $P$-wave) seismic data. Alkhalifah and Tsvankin (1995) showed that $P$-wave ve-
locity analysis in vertically inhomogeneous VTI media can yield a single anisotropic parameter ($\eta$) in addition to the normal-moveout (NMO) velocity for horizontal events $V_{nmo}(0)$. In terms of Thomsen’s (1986) parameters $\epsilon$ and $\delta$ and the $P$-wave vertical velocity $V_0$, $V_{nmo}(0)$ and $\eta$ are expressed as

$$V_{nmo}(0) = V_0 \sqrt{1 + 2\delta}$$  \hspace{1cm} (1)

and

$$\eta = \frac{\epsilon - \delta}{1 + 2\delta}. \hspace{1cm} (2)$$

Obtained as functions of the vertical traveltine $\tau$, the parameters $V_{nmo}(0)$ and $\eta$ control all time-processing steps (NMO, DMC, time migration) in VTI media.

The interval values $V_{nmo, int}(0, \tau)$ can be found using conventional Dix (1955) differentiation of NMO (stacking) velocities from horizontal (or subhorizontal) interfaces. To obtain the interval $\eta_{int}(\tau)$, Alkhalifah and Tsvankin (1995) and Alkhalifah (1997) suggested to use the Dix-type differentiation of NMO velocities for dipping events. This procedure, however, is known to produce unreasonably strong variations in the interval $\eta$ values (Alkhalifah and Rampton, 1997). Here, we stabilize the inversion by interval $\eta$ by representing the $\eta_{int}(\tau)$ curve as a superposition of Chebyshev polynomials (Grechka et al., 1996). This allows us to take advantage of the redundancy in the available velocity picks and estimate only those (smooth) components of $\eta_{int}(\tau)$, which are necessary to fit the NMO velocity to a given degree of accuracy.

Alkhalifah-Tsvankin parameter-estimation methodology has been successfully used to perform anisotropic imaging in such exploration areas as offshore Africa (Alkhalifah et al., 1996) and Trinidad (Alkhalifah and Rampton, 1997), where massive shale formations are characterized by significant anisotropy. In both areas, accounting for vertical transverse isotropy leads to dramatic improvements in the imaging of dipping reflectors (fault planes) and helps to remove the distortions caused by nonhyperbolic moveout in the stacking of subhorizontal events. Similar benefits can be expected from VTI processing in the Gulf of Mexico, where widespread misities in time-to-depth conversion provide evidence of non-negligible anisotropy.

Here, we apply the anisotropic processing sequence to a 2-D synthetic data set generated by an anisotropic finite-difference code. The model used in our synthetic test was fashioned after a typical cross-section from the Gulf of Mexico (J. Leveille and F. Qin, pers. comm.) and has a number of VTI layers. Although the structural complexity of the model is moderate, it includes a salt dome surrounded by sedimentary layers and a relatively steep fault plane (Figure 1). The anisotropic parameters can be considered as “best-guess” values that may well understate the magnitude of anisotropy in many areas in the Gulf of Mexico.

After the parameter-estimation step, we perform prestack depth migration of the data by means of a 45° finite-difference scheme (Han, 1998). First, we use the correct anisotropic model to produce a section that serves as a benchmark for comparison with other results. To simulate the output of a conventional processing sequence, we carry out isotropic migration with two different choices of the velocity function and discuss the distortions caused by the influence of anisotropy. Finally, we interpolate (and extrapolate, if necessary) the results of the anisotropic parameter estimation and perform depth migration with this approximate anisotropic model. Although the model is weakly anisotropic, the VTI images have a superior quality, especially in the focusing and positioning of the fault plane.

**Parameter estimation**

**Methodology**

Suppose a dipping reflector is embedded in a vertically inhomogeneous transversely isotropic medium with a vertical symmetry axis. The effective normal-moveout velocity and one-way zero-offset traveltie for such a model are given by (Appendix A)

$$V_{nmo, eff}^2(p, \tau) = -\frac{1}{t(p, \tau)} \int_0^\tau V_0(\xi) q''(\xi) \, d\xi$$  \hspace{1cm} (3)

and

$$t(p, \tau) = \int_0^\tau V_0(\xi) [q(\xi) - pq'(\xi)] \, d\xi.$$  \hspace{1cm} (4)

In equations (3) and (4), $p$ is the horizontal component of the slowness vector (the ray parameter) of the zero-offset ray, the integration variable $\xi$ has the meaning of the one-way vertical traveltie ($\tau$ is the one-way vertical traveltie from the surface to the zero-offset reflection point), and $t(p, \tau)$ is the one-way traveltie along the zero-offset ray. The vertical slowness component $q \equiv q(p)$ and its derivatives $q' \equiv dq/dp$ and $q'' \equiv d^2q/dp^2$ can be obtained in an explicit form from the Christoffel equation.

A key result of Alkhalifah and Tsvankin (1995) is that both $V_{nmo, eff}(p, \tau)$ and $t(p, \tau)$ depend on only two combinations of interval parameters of VTI media – the zero-dip NMO velocity $V_{nmo, int}(0, \tau)$ and the parameter $\eta_{int}(\tau)$. Therefore, the measurements of the effective NMO velocity $V_{nmo, eff}(p, \tau)$ for two different dips (or for two values of $p$) can be inverted for $V_{nmo, int}(0, \tau)$ and $\eta_{int}(\tau)$.

In most cases, we can use horizontal events to determine $V_{nmo, eff}(p = 0)$ as a function of the vertical trav-
Anisotropic processing of synthetic data

\[ V_{nmo, eff}^2(0, \tau) = \frac{1}{\tau} \int_0^\tau V_{nmo, int}^2(0, \xi) \, d\xi. \]  

(5)

Obtaining \( V_{nmo, int}(0, \tau) \) from equation (5) essentially amounts to differentiating the effective velocities \( V_{nmo, eff}(0, \tau) \), which inevitably leads to amplification of errors in velocity picking. To mitigate this instability, equation (5) can be solved by the technique described in Grechka et al. (1996). This approach is based on approximating the velocity picks by Chebyshev polynomials and finding the interval velocity \( V_{nmo, int}(0, \tau) \) in the Chebyshev domain. The desired smoothness of the solution and the degree to which errors in the effective velocities propagate into the interval values can be regulated by choosing the appropriate number of polynomials.

Once \( V_{nmo, int}(0, \tau) \) has been obtained, the interval parameter \( \eta \) can be found from the NMO velocity and zero-offset traveltime of dipping events [equations (3) and (4)]. The input data include the triplets of the horizontal slowness \( p \) (reflection slopes on zero-offset sections), corresponding zero-offset traveltime \( t \), and the effective NMO velocity \( V_{nmo, eff} \). These triplets can be picked from the zero-offset time sections generated by scanning the stacking velocity or from the semblance velocity panels at a number of adjacent common-midpoint (CMP) locations. The time-varying function \( \eta_{int}(\tau) \) is represented as a sum of Chebyshev polynomials and reconstructed from the triplets \( \{t, p, V_{nmo, eff}\} \) using equations (3) and (4) in the following way. For a trial solution \( \eta_{int}(\tau) \) (specified at each iteration) and the zero-offset traveltime \( t(p, \tau) \) of a particular velocity pick, we find the corresponding vertical time \( \tau \) using equation (4). Then we calculate the velocity \( V_{nmo, eff}(p, \tau) \) from equation (3) and find the difference between the computed and actual value. Then we update \( \eta_{int}(\tau) \) to minimize the difference between the picked and computed values of \( V_{nmo, eff}(p, \tau) \) for all velocity picks.

**Data processing**

A complete anisotropic processing sequence was applied to a 2-D data set computed by finite differences for the VTI model shown in Figure 1. The section contains a salt body and a fault plane, which produce dipping events needed to estimate the parameter \( \eta \). Judging by the values of the parameters \( \epsilon \) and \( \delta \), some of the layers may be considered strongly anisotropic (the parameter \( \epsilon \) reaches 0.28, see Figure 1b). P-wave reflection moveout, however, is controlled by the parameter \( \eta \) [equation (2)], which is relatively small throughout the model, with a maximum value of 0.09 and an average of about 0.05 (Figure 1d).

| Model size: | 21,945 m × 9,144 m |
| Number of shots: | 361 |
| Shot spacing: | 61 m |
| Number of receivers per shot: | 550 |
| Receiver spacing: | 12.2 m |
| Dominant frequency: | 20 Hz |
| Recording time: | 8 s |
| Sample rate: | 4 ms |
| Aperture for modeling: | −3,048 m to +6,096 m |

Table 1. Parameters used in the finite-difference modeling.

The function \( \eta_{int}(\tau) \) was obtained as follows:

- Common-shot gathers were resorted into common-midpoint (CMP) gathers. Since the inversion algorithm needs moveout of dipping events to estimate \( \eta \), we used only those CMP gathers which contain reflections from the top of the salt body (CMP locations from 4.9 to 6.7 km) and from the fault plane (CMP locations from 11.0 to 16.8 km). Thus, only about 30% of the data (Table 1) was actually included in the anisotropic parameter estimation.

- Conventional semblance analysis was used to obtain \( V_{nmo, eff}(0, \tau) \) from subhorizontal events and the triplets \( \{t, p, V_{nmo, eff}\} \) from the reflections from the right flank of the salt body and the fault plane. The ray parameter (horizontal slowness) \( p \) for dipping events was determined using the lateral time shift of the corresponding semblance velocity maxima.

- The algorithm based on equations (3)–(5) was applied to invert the zero-dip velocities \( V_{nmo, eff}(0, \tau) \) and the triplets \( \{t, p, V_{nmo, eff}\} \) for the interval values \( V_{nmo, int}(0, \tau) \) and \( \eta_{int}(\tau) \).

There are two main sources of distortions in the estimation of the interval values of \( V_{nmo} \) and \( \eta \): erroneous model assumptions and errors in velocity picking. The modeling errors stem from the assumption that the medium above each dipping reflector consists of horizontal homogeneous layers. In our model, most “subhorizontal” interfaces have dips \( \phi \) of up to 15°. Ignoring the dips in evaluating \( V_{nmo}(0) \) leads to velocity errors (estimated from the isotropic relation \( V_{nmo}/\cos \phi \)) that can reach 3.5%.

Uncertainty in the velocity picking for dipping events may give rise to errors of similar magnitude. Figure 2 displays a typical seismogram and a semblance panel for the CMP gather with the midpoint at 15.0 km. The reflection from the salt body is recorded at the zero-offset time 3.85 s. Apparently, this reflection has a much higher stacking (NMO) velocity than the events at zero-offset times of 3.67 and 4.05 s, which represent the reflections from subhorizontal interfaces near the fault (Figure 2b). Partly due to the interference with
Figure 1. Parameters of the VTI model: (a) $P$-wave vertical velocity $V_p$; (b) and (c) Thomsen's (1986) anisotropic parameters $\epsilon$ and $\delta$; (d) the Alkhalifah-Tsvankin parameter $\eta$. 
the event at 3.67 s, the fault-plane reflection produces a relatively broad semblance maximum (Figure 2a) covering approximately 0.15 km/s along the velocity axis. As a result, we can expect the uncertainty in velocity picking of about 3–4%. Naturally, this error propagates into the interval values of $V_{\text{nmo}}$ and $\eta$ with amplification, thus causing instability in the straightforward Dix-type differentiation. Application of Chebyshev polynomials, however, amounts to a smoothing operation that helps to stabilize the inversion procedure and eliminate spurious points in the interval curves.

**Parameter-estimation results**

The time-dependent curves $V_{\text{nmo, int}}$ and $\eta_{\text{int}}$ obtained for the left portion of the model are shown in Figure 3. The zero-dip NMO velocity was determined by semblance velocity analysis of subhorizontal events, while $\eta$ was in-
Figure 5. Prestack depth migration for the correct anisotropic model.

Figure 6. Migration using an anisotropic model obtained by interpolation and extrapolation of the inverted functions $\eta_{\text{int}}$ [Figure 3 and Figure 4]. The vertical and NMO velocity are assumed to be equal to each other ($\delta = 0$).

verted from the NMO velocities and zero-offset traveltimes of reflection events generated by the right flank of the salt body (Figure 1). Due to the regularization (smoothing) properties of our inversion algorithm, the curve $\eta_{\text{int}}(\tau)$ represents an accurate but smoothed version of the actual discontinuous $\eta$ function (dashed line in Figure 3b).

Figure 4 displays the inversion results for the right side of the model, obtained using reflections from the dipping fault plane. In this case, we were unable to reconstruct the low-$\eta$ layer at a vertical time of 3 s (Figure 4b), which is too thin to produce a noticeable change in the effective stacking (NMO) velocity (for the level of velocity errors described above). Apart from this problem, our algorithm adequately reconstructed the general character of the interval function $\eta_{\text{int}}(\tau)$. 
**Depth migration**

While the parameters $V_{nmo}(0)$ and $\eta$ are sufficient for time-domain processing including prestack and poststack time migration, depth imaging requires knowledge of the vertical velocity and Thomsen parameters $\epsilon$ and $\delta$. To resolve the vertical velocity, $P$-wave reflection moveout has to be combined with other data, such as reflection traveltimes of shear or converted waves; this joint inversion was not attempted on our synthetic data. Nonetheless, we carried out prestack depth migration to evaluate image distortions caused by replacing the correct anisotropic velocity field with the following models:

1. Anisotropic model with the inverted $\eta$ and the "best-guess" vertical velocity equal to the zero-dip NMO velocity.

2. Purely isotropic model with the velocity equal to either zero-dip NMO velocity or actual vertical velocity.

We used the migration algorithm of Han (1998) based on a 45° finite-difference scheme. Figure 5 shows the section obtained by prestack depth migration of the data using the correct VTI model from Figure 1. Both the salt body in the left part of the model and the fault plane are imaged reasonably well. The left part of the bottom of the salt body, however, is almost invisible because of the limited recording aperture (see Table 1). Also, some of the multiple arrivals, such as those inside the salt, were only partially attenuated on the migrated section.

The migrated image in Figure 6 was obtained for an anisotropic velocity model based on the parameter-estimation results described above. Since the time-dependent $\eta$ function was determined only at two spatial locations, the $\eta$ section was build by interpolating and extrapolating the curves from Figures 3b and 4b. This smoothed and rather crudely interpolated version of the actual $\eta$ field, however, did not degrade the quality of image, except for a slight deterioration in the focusing of the fault plane. Clearly, fine details of the $\eta$ section do not have much influence on the migration results. Also, the magnitude of $\eta$ in the model was so small (on average, $\eta \approx 0.05$, Figure 1d) that high accuracy in restoring $\eta$ was unnecessary.

To specify the vertical velocity for the anisotropic migration, we assumed that the parameter $\delta = 0$ [i.e., $V_0 = V_{nmo}(0)$], which leads to the incorrect depth scale for the whole image. It is clear from equation (1) that the percentage depth error in Figure 6 should be close to the average value of $\delta$ above the reflector. Indeed, the depth of subhorizontal reflectors in the lower right part of Figure 6 is overstated by about 7%, while the average $\delta$ in this part of model is about 0.1 (Figure 1c). Except for the depth error, the image in Figure 6 is quite close to the benchmark section from Figure 5. For more structurally complicated models, however, it may be necessary to know all three relevant parameters ($V_0$, $\epsilon$ and $\delta$) to avoid distortions in the time image.

Figure 7 can be considered as an ideal output of the conventional isotropic processing sequence. The isotropic velocity model used to generate Figure 7 is based on the correct NMO velocity $V_{nmo}(0)$ that may be obtained
by error-free semblance velocity analysis. Comparing the
lower right portions in Figures 7 and Figure 5, we
notice that the subhorizontal reflectors are mispositioned
in depth, which was also the case in Figure 6. The overall
quality of the image, however, is comparable to that in
Figures 5 and 6. For example, there are no conflicting
dips in the vicinity of the fault, which could be indicative
of the presence of anisotropy. The continuity and
crispness of the fault-plane reflection is somewhat infe-
rior than in Figure 6, but the difference is not dramatic.
The good quality of this isotropic image is explained by
the small values of the $\eta$ coefficient that controls the dip-
development of NMO velocity in VTI media. Ignoring $\eta$
in our model cannot cause substantial distortions in the
stacking of dipping events, while the correct $V_{nmo}(0)$ en-
sures that the horizontal reflectors are well focused.

Another option in choosing the “isotropic” migration velocity is illustrated by Figure 8. This time, the
data were migrated using the correct vertical velocity
$V_0$, which in some cases may be obtained from check
shots and well legs (but not from surface $P$-wave data).
As expected, all subhorizontal reflectors in Figure 8 are
correctly positioned in depth. However, the quality of
the image is considerably lower than in Figures 5–7. The
difference between $V_0$ and the correct zero-dip NMO ve-
cocity $V_{nmo}(0)$ causes missetting of the subhorizontal
events throughout the model. Since the wrong values of
$V_{nmo}(0)$ also distort the dip-dependence of NMO veloc-
ity, the reflections from the fault plane and the flank of
the salt dome are poorly focused and positioned. Also no-
tice the intersecting reflectors with different dips marked
by arrows in Figure 8.

Discussion and conclusions
We applied anisotropic velocity analysis and depth mi-
gration to a synthetic data set generated by finite dif-
fences for a VTI model typical for the Gulf of Mexico.
Combining NMO (stacking) velocities for subhorizontal
and dipping events, we obtained the zero-dip NMO ve-
cocity $V_{nmo}(0)$ and the Alkhalifah-Tsvankin coefficient $\eta$
(the two parameters needed for time imaging) as func-
tions of vertical time. The inversion algorithm, based on
the representation of both parameters through Cheby-
shev polynomials, allowed us to reduce the instability in
the Dix-type differentiation and reconstruct the smooth
components of the vertical variation in $V_{nmo}(0)$ and $\eta$.
Parameter estimation was carried out separately over
two ranges of CMP locations, with subsequent interpola-
tion and extrapolation of the two $\eta$ curves for the whole
model.

Depth migration of the reconstructed anisotropic ve-
cocity model produces a high-quality image quite close
to that generated for the exact anisotropic velocity field.
Inaccuracies in $\eta$ estimation do not cause visible distor-
tions in the focusing and positioning of reflection events,
in part because the magnitude of $\eta$ in the model is rather
small (the average $\eta \approx 0.05$). However, since the ver-
tical velocity cannot be determined from surface $P$-wave
data, the image has the wrong depth scale (we assumed
that the vertical and NMO velocities are equal to each other).

The same mispositioning of subhorizontal events is observed on the depth section generated for a purely isotropic model with the velocity equal to the correct zero-dip NMO velocity $V_{\text{nom}}(0)$. Due to the small values of $\eta$ in our model, the overall quality of the image is comparable to that of the anisotropic section, except for some degradation in the focusing and continuity of the fault-plane reflection. Distortions in the isotropic image based on the correct zero-dip stacking velocity become significant only for models with average $\eta$ values reaching or exceeding 0.1 (Alkhalifah et al., 1996; Alkhalifah, 1997).

Since one needs to use the vertical velocity $V_0$ to image horizontal events at the correct depth, we also attempted to migrate the data with the isotropic velocity model based on $V_0$. Although the depth of reflectors in this case is indeed accurate, the quality of the image is unacceptable because in anisotropic media the vertical velocity is inappropriate for stacking reflection events of any dip.

Thus, application of isotropic depth migration in anisotropic media leads to inferior image quality and/or inaccurate positions of reflectors in depth. In models with moderate structural complexity, anisotropic migration with the correct inverted parameters $V_{\text{nom}}(0)$ and $\eta$ provides good focusing and positioning of reflection events, but may have the wrong depth scale. To obtain the vertical velocity and avoid depth errors in migration, $P$-wave reflection traveltimes should be supplemented by shear data or borehole information, such as check shots or well logs.

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APPENDIX A: NMO velocity and zero-offset traveltime in vertically inhomogeneous VTI media

We consider reflection moveout in the dip plane of a reflector overlaid by vertically inhomogeneous TI media with a vertical symmetry axis. The goal of this appendix is to express both NMO velocity and zero-offset traveltime through the interval values of the horizontal and vertical components of the slowness vector.

An exact expression for the dip-line normal-moveout velocity in a symmetry plane of an anisotropic layer was given by Tsvankin (1995) and rewritten in terms of the components of the slowness vector by Cohen (1998):

$$v_{\text{nom}}^2(p) = \frac{q''}{p q' - q}$$  \hspace{1cm} (A1)

where $p$ and $q \equiv q(p)$ are the horizontal and the vector components of the slowness vector, respectively, $q' = dq/dp$, and $q'' = d^2q/dp^2$; all quantities are evaluated for the zero-offset ray. The slowness vector can be obtained analytically by solving Christoffel equation,
which reduces to a cubic equation for the squared slowness in a given phase (slowness) direction. The derivatives \( q' \) and \( q'' \) can be found in an explicit form by differentiating the Christoffel equation. Equation (A1) is a special case of a more general expression obtained by Tsvankin et al. (1997) for azimuthally varying NMO velocity in an arbitrary anisotropic layer above a dipping reflector.

The one-way \( P \)-wave traveltime \( t(p) \) along an oblique ray to the reflection point in a homogeneous medium is given by (Grechka and Tsvankin, 1997)

\[
t(p) = \tau V_0 (q - pq'),
\]

where \( \tau \) is the one-way vertical traveltime from the surface to the reflection point and \( V_0 \) is the \( P \)-wave vertical velocity.

For vertically inhomogeneous media, the differential of the traveltime along a ray arc can be written as

\[
dt(p) = V_0 (q - pq') \, d\tau.
\]

(A3)

In the integral form, equation (A3) yields

\[
t(p, \tau) = \int_0^\tau V_0(\xi) [q(\xi) - pq'(\xi)] \, d\xi.
\]

(A4)

According to Snell's law, the horizontal slowness \( p \) in equation (A4) does not change along the ray.

To obtain NMO velocity in vertically inhomogeneous media, we apply the Dix-type averaging (Alkhalfah and Tsvankin, 1995) to equation (A1):

\[
V_{\text{nmo,eff}}^2(p, t(p, \tau)) = \frac{1}{t(p, \tau)} \int_0^t \frac{q''(\nu)}{pq'(\nu) - q(\nu)} \, d\nu,
\]

(A5)

where integration is performed along the (generally oblique) zero-offset ray. To make equation (A5) compatible with equation (A4), it is convenient to use the vertical traveltime as the integration variable. Taking equation (A3) into account, we represent equation (A5) as

\[
V_{\text{nmo,eff}}^2(p, \tau) = -\frac{1}{t(p, \tau)} \int_0^\tau V_0(\xi) q''(\xi) \, d\xi.
\]

(A6)

Equations (A4) and (A6) express the NMO velocity and zero-offset traveltime in vertically inhomogeneous VT1 media in a form convenient for moveout inversion.
Extrapolation of seismic data to small offsets

Carlos E. Theodoro and Ken Larner

ABSTRACT

Recorded data starting from offset-zero are required for surface-related multiple-attenuation methods that use the data themselves to develop the multiple prediction operator. Traces close to the source are not usually recorded in the field so they have to be extrapolated from the recorded field data. Multi-valued moveout (e.g., reflections at about the same recording time, but with different moveout) is a problem for extrapolation methods presently in use in industry. We present an offset-continuation method, based on Hale’s dip-moveout approach for constant-velocity background, for extrapolating data from one offset to a smaller offset, and compare the quality of predicted traces with that of those predicted using a conventional approach. Tests with synthetic data show better prediction of smaller-offsets when compared with predictions by the conventional method. Although results of the method presented here are substantially better than that of the conventional approach, they may still not be good enough for predicting the data needed for multiple-attenuation methods over areas of complex geology.

Introduction

A difficulty for modern methods of surface-related multiple attenuation (Carvalho et al., 1992; Verschuur et al., 1992; Dragset and MacKay, 1993; and Borselen et al., 1996) is that these methods require a continuous spatial sampling of the wavefield over a large range of offsets, starting with zero-offset. Other requirements of these methods are an accurate estimate of the source wavelet and, since multiples are predicted from primaries present in the data, full physical consistency between primary and multiple events. The absence of data recorded at offsets close to the source, which typically are not recorded in the field, results in an error in the prediction of the multiples; because this error propagates along the record during the process of multiple prediction and subtraction, strong residuals will be present in the data after the multiple-suppression process. A way of minimizing the problem is to extrapolate data from the shorter recorded offsets in order to model the missing close-to-the-source traces.

To extrapolate the short-offset traces in marine seismic data, two procedures are usually followed in the industry. In the first approach the smallest-offset recorded trace, or a stack of a few traces with small offsets, is simply repeated, usually in the common-midpoint (CMP) domain, after normal-moveout (NMO) correction, to represent the missing offsets of interest; then inverse NMO correction is applied based on those offsets. In the second approach, also implemented in the CMP domain, a split-spread record is first simulated by the symmetric repetition of traces from one side of the source to the other, obeying the offset distribution. The data are then NMO corrected, the missing traces are laterally interpolated, and finally inverse NMO correction is applied as mentioned above. We use results of this procedure, split-spread simulation and interpolation (SSSI), to compare with those of the offset-continuation (OC) extrapolation presented here.

Problems may occur with these conventional approaches: (a) for reflections at about the same recording time, but with different moveout (e.g., reflections from interfaces with different dip occurring at nearly the same recording time), proper NMO correction for one of these events will leave residual NMO for the other event that will degrade the extrapolation; (b) random noise may introduce spurious events and incorrect interpolated amplitudes, posing a problem for interpolation methods; and (c) where the water bottom is shallow, the information recorded at early times, even close to the source, may be post-critical, so amplitude and phase corrections are necessary for the extrapolation. The method for extrapolation of near-offset data proposed here addresses primarily the first, and, to some extent, the second of these problems.
Kabir and Verschuur (1995) presented a method for extrapolation of the traces close to the source based on the assumption of parabolic residual moveout of seismic events, after a partial NMO correction in CMP gathers. They use a band-limited parabolic Radon transform of the data to extrapolate data to smaller offsets and thus restore missing offsets. Several iterations of forward and inverse transform are required to replace the missing traces in the original gather with reconstructed ones. Where moveout is multi-valued (the first problem mentioned above), however, one curvature for the Radon parabolic transform must be chosen from among the two or more multi-valued moveout events, resulting in good extrapolation for the event with chosen curvature, but poor extrapolation for the other events.

Dip-moveout (DMO) is a technique for correction of distortions in CMP data due to the dip of reflectors (Judson et al., 1978). It reduces the scattering of reflection points over dipping interfaces, thus yielding proper common-reflection-point gathers. If we could apply DMO before the multiple-attenuation process, we would overcome the problem of multi-valued moveout. Unfortunately, in applying DMO to the data, the spatial and temporal relationship between primaries and multiples is distorted, thus harming the result of the multiple-attenuation process.

Dereckowski (1986) lists ten different purposes for DMO, and, among those, one consists of interpolation of dead traces. A zero-amplitude trace present in a common-offset gather will be interpolated after DMO, due to the partial migration of events from neighboring traces.

Bolondi et al. (1982) defined OC as a continuous process of gradual change of offset, all the way to zero-offset. The process involves solution of a partial differential equation proposed by Dereckowski and Rocca (1981) by means of a pure phase-shift operator relating different common-offset sections. Salvador and Savelli (1982) presented a solution by an approximate numerical method in the space-time domain, and Bolondi et al. (1982) solved this equation in the wavenumber-time domain. Being based on the small-offset, small-dip approximation, Bolondi’s equation fails for large ratio of offset to depth and for steep reflectors. Fomel (1995a, 1995b) introduced a revised version of the OC differential equation and showed that, under the assumption of constant propagation speed, it provides the correct kinematics and correct leading-order amplitude of the continued wavefield for any offset and reflector dip. Fomel and Bleistein (1996) then used the Kirchhoff integral as a link between the wave-equation theory and the kinematically-derived OC equation, and proved that the OC correctly transforms common-offset seismic data modeled by the Kirchhoff integral approximation. Finally, data mapping (Bleistein et al., 1998) is a general procedure for transforming data recorded from a given source/receiver configuration and prescribed model to another configuration and model. True-amplitude OC is one of several possibilities offered by data mapping.

DMO can be thought as a special case of OC, where the output consists of zero-offset data. Here we are using the term “offset continuation” as the prediction of the wavefield at half-offset \( h_{\text{output}} \) from the wavefield at half-offset \( h_{\text{input}} \), with \( h_{\text{input}} > h_{\text{output}} \). To extrapolate to obtain information close to the source, we follow the formulation of Hale (1984) in the frequency domain, making the assumption that input offsets as well as output offsets are small. In Hale’s approach, the OC extrapolation is given by the multiplication of phase and amplitude terms by a function depending on the difference in offsets between input and output data, and on the wavenumber, temporal frequency, and recording time of the sample being processed.

In the next section, we describe the method and state its assumptions. Two synthetic datasets are used in the following section to compare the quality of OC extrapolation and conventional SSSI. The first set, composed of plane reflectors with different dips, shows the influence of residual moveout and random noise on the quality of the extrapolated results for a model with constant velocity or velocity varying with depth \([v(z)]\). The second set, a synthetic line from the SEG/EAGE salt-model, shows results for a more realistic model composed of curved and truncated reflectors, diffractions, and a time- and space-varying velocity field. Before concluding, we discuss results and review implications of assumptions.

The Method

For reflections from the dipping base of a homogeneous medium, the dip-dependent relationship between nonzero-offset recording time \( t \) and zero-offset time \( t_0 \) is given by

\[
t^2 = t_0^2 + (2h)^2 \cos^2 \theta/v^2,
\]

where \( h \) is the half-offset, \( v \) is the velocity, and \( \theta \) is the reflector dip.

Following Hale (1984), equation (1) can be expanded as

\[
t^2 = t_0^2 + (2h)^2/v^2 - (2h)^2 \sin^2 \theta/v^2.
\]

Considering the NMO corrected time \( t_n \),

\[
t_n^2 = t^2 - (2h)^2/v^2,
\]
equation (2) can be rewritten as

\[ t_n^2 = t_0^2 - (2h)^2 \sin^2 \theta / \omega^2. \tag{3} \]

Recall that

\[ 2 \sin \theta / v = \Delta t / \Delta x = k / \omega, \]

where \( \Delta x \) is the offset, and \( k \) and \( \omega \) are the spatial and temporal frequencies, respectively. So equation (3) can be rewritten as

\[ t_n^2 = t_0^2 - h^2 k_x^2 / \omega^2. \]

Consider two half-offsets, a recorded one, \( h_I \), and a desired one, \( h_O \) (where the subscripts I and O refer to input and output, respectively), so

\[ t_{n,I}^2 = t_0^2 - h_I^2 k_x^2 / \omega_I^2, \]

and

\[ t_{n,O}^2 = t_0^2 - h_O^2 k_x^2 / \omega_O^2, \]

giving the relationship between \( t_{n,I} \) and \( t_{n,O} \),

\[ t_{n,O}^2 = t_{n,I}^2 + (h_I^2 k_x^2 / \omega_I^2) - (h_O^2 k_x^2 / \omega_O^2). \tag{4} \]

Let us assume that

\[ k_I / \omega_I \approx k_O / \omega_O \approx k / \omega, \tag{5} \]

which means that the slopes in the NMO-corrected common-offset sections, for the offsets of interest, are approximately the same. The slopes would be equal if the NMO correction were perfect, since reflection events would then appear horizontal in any CMP gather. That is, they would all have the same corrected reflection time, independent of offset. Common-offset data after such perfect NMO correction would show the same slope, independent of the offset. Obviously the assumption expressed in equation (5) will be violated where NMO is multi-valued. We discuss this situation below, and, in Example 1, we will see that even for a strong violation of this assumption, the result for OC extrapolation is significantly better than that for the SSI method.

Under assumption (5), equation (4) reduces to

\[ t_{n,O} = A t_{n,I}, \tag{6} \]

where

\[ A = \sqrt{1 + (h_I^2 / h_O^2) k^2 / (\omega^2 t_{n,I}^2)}. \]

Since the extrapolation is from larger to smaller offsets, \( h_I \approx h_O \) making \( A \) real.

We want to compute the wavefield \( p_0(t_{n,O}, x, h_O) \), given the NMO-corrected recorded data \( p_I(t_{n,I}, x, h_I) \). Under the small-offset approximation one can assume

\[ p_0(t_{n,O}, x, h_O) = P_0(t_{n,O}, x, h_I), \]

which means that recorded data at position \( x \), after NMO correction, at time \( t_{n,O} \) and half-offset \( h_O \) are similar to NMO-corrected data at the same spatial position, same time \( t_{n,O} \), and half-offset \( h_I \). In conventional extrapolation to fill in missing small offsets, this is exactly the requirement used. Then the double Fourier transform of \( p_0(t_{n,O}, x, h_O) \) can be written as

\[ P_0(\omega_{n,O}, k, h_O) = \int dt_{n,O} e^{i \omega_{n,O} t_{n,O}} \int dxe^{-ikx} p_0(t_{n,O}, x, h_I). \tag{7} \]

From equation (6), recalling that \( A \) also is a function of \( t_{n,I} \), one gets

\[ dt_{n,O} = A^{-1} dt_{n,I}. \tag{8} \]

Changing variables, from \( t_{n,O} \) to \( t_{n,I} \) in equation (7), using equations (6) and (8), and observing that

\[ p_0(A t_{n,I}, x, h_I) = p_I(t_{n,I}, x, h_I), \]

which is exactly the DMO correction, one gets

\[ P_0(\omega_{n,O}, k, h_O) = \int dt_{n,I} A^{-1} e^{i \omega_{n,O} t_{n,I} A} \int dxe^{-ikx} p_I(t_{n,I}, x, h_I). \tag{9} \]

Now, to get \( p(t_{n,O}, x, h_O) \) is just a matter of double inverse Fourier transformation of equation (9),

\[ p_0(t_{n,O}, x, h_O) = \frac{1}{(2\pi)^3} \int d\omega e^{-i \omega_{n,O} t_{n,O}} \int dke^{ikx} P_0(\omega_{n,O}, k, h_O). \tag{10} \]

The number of computations for the OC approach is basically that of two double Fourier transformations (one direct and one inverse transform) and \( 2n \) multiplications (where \( n \) is the number of time samples per trace), \( n \) multiplications for the amplitude term and another \( n \) for the phase term.

Application to Synthetic Data

Following are tests on synthetic data for models of reflectors with different dips, with additive random noise, and with and without surface-related multiples. As we shall see, except for the simple case of noiseless data from plane reflectors, the methodology presented here yielded better prediction of small-offset data than did
the conventional SSSI. In that simple case, both results are equivalent.

The advantage of testing with synthetic data is that we can compare the extrapolated offset with data directly modeled for the same offset. Since field data usually are not recorded close to the source, extrapolation using field data does not offer the possibility of such comparison. In future tests we will use field data for comparison of results for attenuation of surface-related multiples using small-offset data predicted by conventional SSSI versus the OC extrapolation.

Example 1

The model for the first dataset has two horizontal reflectors and two plane reflectors, with dip of 45 and 75 degrees. A Ricker pulse with 25-Hz peak-frequency was used as the source wavelet. The purpose of this test is to assess the results of extrapolation in the presence of residual moveout. For this model the velocity field is constant, equal to 2000 m/s. Use of this velocity for NMO correction for both conventional SSSI and OC extrapolation causes residual moveout for the dipping events. To make the test more interesting, we added to the signal temporally bandlimited random noise, with Gaussian amplitude distribution, over the frequency range from 8 to 60 Hz. The noise is uncorrelated spatially, and has a root-mean-square amplitude that varies along the line. The noise-contaminated, zero-offset section (generated with the SUSYNLV and SUADDNOISE programs, within the Seismic Unix system) is shown in Figure 1. The data consist of 161 CMP gathers, with 12.5 m between midpoints, each gather with a trace every 25 m, from offset 0 to 800 m. Offsets from 0 to 175 m, which were excluded from the data that were used for the extrapolation, were generated only for comparison with the extrapolated data.

For the conventional SSSI method, we used as input NMO-corrected data over the offset range 200 to 800 m. With these data, we simulated split-spread data for use in interpolating offsets from 0 to 175 m. To assess the quality of the extrapolation we will look at difference sections - the difference between the result of the extrapolation for a given offset and a noiseless version of the modeled synthetic data for that same offset. Figure 2 shows the difference section, for the 25-m common-offset dataset, displayed at the same amplitude scaling as that in Figure 1. As expected, in Figure 2 the horizontal events are well predicted (i.e., extrapolated) because the SSSI process should predict well those events that are enhanced by the NMO correction. The dipping events, in contrast, are poorly extrapolated, leaving a strong residual.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Zero-offset data for the first model, contaminated with random noise having spatially varying amplitude.}
\end{figure}

The difference section for the OC extrapolation of the 25-m offset is shown in Figure 3. The input data used for the extrapolation were common-offset at an offset of 200 m, again contaminated with noise having the same amplitude variation with CMP as before. Except for the steepest event at early time, all reflections have been well extrapolated, independent of dip. The residual signal present in Figure 3 is a mixed result of edge-effect, violation of the assumption of small-offset, and erroneous amplitude prediction for the shallow, dipping extrapolated data.

In presence of dip, using \( v \) for the NMO correction yields reflections with residual moveout in CMP gathers, violating the assumption of equation (5). Departure from this assumption is a function of dip, recording time, and difference between input and output offsets; the steeper the reflector, the earlier the time, or the bigger the difference in offsets, the weaker is the assumption of equation (5). As illustrated in Figure 3, however, even for steep features (75 degrees) and a difference of 175 m between
For the OC extrapolation we modified the factor that multiplies $A^{-1}$ in expressions (9) and (11) to

$$1 + 2(h_1^2 - h_0^2)[k/(\omega t)]^2.$$  \hspace{1cm} (12)

To illustrate the influence of this factor, instead of using the noisy data, we used a noiseless version of Example 1. Figure 4 shows the difference section for the prediction of common-offset 25-m data from common-offset 200-m data without this amplitude correction factor, and Figure 5 shows OC extrapolation of the same data as in Figure 4, but computed using the amplitude-correction term. Figure 5 shows a somewhat better prediction of the dipping events, with most improvement for the steep event at early recording time.

Going one step further, consider again the structural model of Example 1, but instead of constant velocity, let velocity vary with depth. Since OC extrapolation is based on the assumption of constant velocity, we wish to assess any shortcomings that result from ignoring velocity variation in the processing. Let the velocity function be...
\[ v(z) = 2000 + 0.000z, \]

where \( z \) is in m, and \( v \) is in m/s. The zero-offset section for this variation of the model in Example 1 is shown in Figure 6, and the difference section for the prediction of common-offset 25 m from common-offset 200 m is shown in Figure 7. The prediction of the shorter offset is as good as that for the model with constant velocity (Figure 3), despite the fact that the OC extrapolation used was based on the assumption of constant velocity. Thus, we do not expect typical vertical variation in velocity to pose a problem for the OC method.

**Example 2**

Data from the 3D SEG/EAGE salt-model offer a more realistic situation for comparison of the actions of the SSSI and OC in extrapolating small offsets. For this model, velocity varies with both depth and lateral position, and reflectors are curved and truncated. Profile A-A', extracted from the 3D SEG/EAGE salt-model (Aminzadeh et al., 1996), is composed of 325 shot records, each one with 176 traces, with offsets ranging from 0 to 14000 ft (80 ft between traces). The source spacing is 160 ft; therefore, common-offset sections have 160 ft between traces. We processed two sets of data, one without free-surface multiples and another with free-surface multiples (i.e., events with at least one bounce off the free-surface). Figures 8 and 9 display the zero-offset sections for the datasets. For both sets, the velocity field used for NMO correction was computed by conventional velocity analysis from the multiple-free data at each 200 CMP, starting at CMP 400. The velocity function used for the OC extrapolation was that for CMP gather 600, and we used a logarithmic stretching of the time axis to accommodate the time-varying velocity function.

The difference between the modeled 160-ft offset data and the conventional SSSI extrapolation to 160-ft offset obtained using offsets ranging from 800 to 6560 ft (only a few which are live for early portions of the data), is displayed in Figure 10. With the exception of the
most complex zone of diffractions (CMP 900 to 1300), simple NMO correction and interpolation has predicted the zero-offset data well. The most shallow reflection is poorly predicted because stretch muting after NMO correction removed this event before doing the extrapolation. The shortcoming of the SSSI in predicting the diffractions in the central part of the profile results from departures of the NMO velocity used for the dynamic correction of the data from the true stacking velocities for the various dipping features. Where amplitudes in the difference section are small, we can expect the extrapolated data to be valid input to multiple-suppression methods. This seems to be the case in regions away from the complex central portion of the profile, but over the central portion, because primaries are less well predicted, multiple-suppression methods can be expected to suffer.

Figure 11 shows the difference section for OC extrapolation to 160-ft offset, using just offset 800 ft as input to the process. For this extrapolation as well, the shallow horizontal event is again poorly predicted. This is a serious problem for multiple prediction and subtraction; no method in use in industry nowadays can extrapolate smaller offsets from post-critical recorded data. This remains an open subject for future research. As was the case for the SSSI result, away from the complex central portion of the profile, the prediction of the zero-offset trace is good, even for the shallow (.3 s) sub-horizontal event between CMPs 400 and 600. Over the complex zone the OC extrapolation is substantially better than that of the simple SSSI, although it exhibits observable residuals. While most apexes of diffractions are well predicted, the apex of the diffraction close to CMP 1000 and time 1.6 s shows some residual. We are not yet sure about the reason for the shortcoming in the prediction for this event. The same complex zone also shows residual from aliased events. Recall that the spatial interval between traces in these common-offset sections is 160 ft, which is larger than the spatial interval between traces typically encountered in field data. Modern field data usually are sampled more densely, implying less severe
spatial aliasing and likely better prediction of smaller offset than that found here.

Figure 12 shows the difference section, this time for the data containing surface-related multiples, resulting from conventional SSSI extrapolation of 160-ft offset using as input data offsets ranging from 640 to 6500 ft. The small-offset prediction here is much poorer than that for the data in which multiples were absent (Figure 10), especially in the complex central portion of the profile, but even to some extent to the left of that region. Also, the sub-horizontal primary at about 0.3 s, between CMPs 400 and 600, is less well predicted than before; now a multiple of the first event interferes with that primary event. Clearly, the pattern of multiples in the central region is not as all well predicted, suggesting that suppression of second- and higher-order surface multiples will suffer.

Figure 13 displays the difference section for OC extrapolations to 160-ft offset using just offset 800 ft as input to the process. Comparing the result here with that in Figure 12, throughout the section we see improvement of the small-offset prediction by the method of offset continuation. Recall that both Figures 10 and 11 showed a good prediction of primary events in the time window from .3 to 2 s, exception to the complex area. Comparing the same time window in Figures 12 and 13, OC extrapolation exhibits weaker residual, meaning a better extrapolation of free-surface multiple contaminated data. The central zone, nevertheless, still displays significant shortcomings as a prediction of small-offset data. The presence of multiples on the input data here has compromised the small-offset prediction of some primary events, such as the two diffraction apexes between CMPs 1200 and 1400, close to 1.6 s, which were well predicted by OC for the multiple-free data (Figure 11), but not for the SSSI (Figure 10). The rationale for using offset continuation is that it would be a better predictor of primaries (of all dips) and multiples in areas of low structural complexity than is the SSSI method. Offset continuation, however, may not be a sufficiently good predictor of multiples in complex areas of this model.
Discussion and Conclusion

Multi-dimensional techniques for attenuating surface-related multiples that use the data themselves in the prediction of multiples require a continuous spatial sampling of the wavefield starting with zero-offset data. Here, we presented a method for extrapolation, by offset continuation (OC), of traces ranging from offset zero to the nearest offset recorded in the field. The method is based in Hale’s (1984) dip-moveout formulation and is limited to data obeying the small-offset approximation.

OC extrapolation, like both the method using parabolic Radon transform and the conventional method in use in industry (split-spread simulation of CMP gathers and interpolation of the missing offsets), does not correct for reflection amplitude and phase variation with offset (or incidence angle) between input and output data. While including the amplitude correction term given by Bleistein et al. (1997) in the OC extrapolation improved the amplitude prediction, mostly for shallow dipping events, like any other method in use nowadays, OC extrapolation cannot properly extrapolate smaller-offset data from post-critical recorded data.

Although the larger difference between the results of the SSSI and OC extrapolations is in the prediction of the primaries, we also observed improvement in the extrapolation of multiple events to shorter offsets in areas with low structural complexity. Good prediction of the smaller offsets by SSSI requires perfect NMO correction (using $v/\cos\theta$, where $v$ is the velocity and $\theta$ is the dip). OC extrapolation also requires a good approximation of velocity for NMO correction (here NMO is corrected using $v$, only), but that requirement is less stringent. Since primaries are better NMO-corrected than are multiples, OC results in better extrapolation of primaries than of multiples. However, as seen for the salt-model data with multiples, OC still yields a better extrapolation of multiples than does the SSSI method in areas of low structural complexity.

OC extrapolation, like migration, spreads amplitudes from one trace to its surrounding traces, and since random noise does not have lateral consistency it tends
to attenuate. Also, it is possible to perform as many extrapolations as desired (or as are consistent with the small-offset approximation) to the target offset, using data with different common-offsets as input, and then stack the results of each extrapolation at each CMP.

Compared to other methods of short-offset extrapolation, this is a relatively low-cost option. Extrapolation by SSSI is of course even less costly since the interpolation is simply done laterally over the input traces. Parabolic Radon requires several iterations with direct and inverse transforms, whereas OC extrapolation is given by one direct and one inverse 2D Fourier transform. Fomel's (1995a, 1995b) true-amplitude extrapolation requires the computationally intensive solution of an integral equation.

Tests using synthetic data showed better extrapolation for the OC method as compared with the conventional SSSI. For data from areas with low structural complexity, whether contaminated or not by random noise, and whether velocity is constant or depth-dependent, OC extrapolation results in better prediction of events (primaries, multiples, and diffractions) than does SSSI. Only for noiseless data from horizontal reflectors are both results equivalent. For data from areas of complex subsurface structure, with curved and truncated reflectors, diffractions, and velocity varying both vertically and laterally, although the result of the extrapolation is better for the OC extrapolation than for the conventional SSSI, the error in the prediction is substantial and will likely degrade the quality of any subsequent multiple-attenuation process.

How much OC helps or fails to help for multiple suppression must await application of multiple suppression to field data having short offsets predicted by conventional SSSI and OC extrapolation.

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Extrapolation of seismic data to small offsets

References


Geophys.


Transient elastic plane-wave diffraction by a semi-infinite fluid-filled crack in an isotropic, dispersive solid

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ABSTRACT

The canonical problem of the diffraction of a transient elastic plane wave by a semi-infinite fluid-filled crack in an unbounded, homogeneous, isotropic, dispersive solid is considered. The dispersion properties of the solid (and its associated loss mechanism) are accounted for by general, global relaxation functions that meet the causality criterion and encompass the influence of frictional forces and a Maxwell-type viscoelasticity that can be used to describe the temporal dispersion of seismic waves. The problem is solved by a combination of the Wiener-Hopf and modified Cagniard techniques, together with an appropriate correspondence principle that interrelates the transient elastic wave motion in a solid with global relaxation functions and the one in a perfectly elastic solid. Several examples are discussed and numerical results are presented.

Introduction

The detection of cracks in a solid is an issue of technical importance in the non-destructive evaluation of mechanical and civil-engineering structures as well as in the evaluation of oil reservoirs in fossil energy production. In all of these cases, position, extent, orientation and nature of the crack (fluid-filled or not) are to be inferred from scattered elastic wavefield measurements carried out at locations that are often remote from the location of the crack. In such a remote sensing setup, canonical scattering geometries are needed as test cases in feasibility studies that serve a number of purposes. As far as the forward problem is concerned, we mention the evaluation of accuracy and effectiveness of low- and high-frequency asymptotic approximations. With regard to the inverse problem, design and performance of inversion and imaging algorithms, and their associated acquisition of data can be analyzed. In the category of canonical problems, the ones that admit closed-form analytic solutions have the advantage that their answers can be constructed with arbitrary numerical accuracy, while a sensitivity analysis as to the influence of the different parameters can readily be carried out. The model of the present investigation also includes a class of dispersion mechanisms through which wave attenuation and dispersion in the host material can be quantitatively analyzed, while in accordance with the Kramers-Kronig causality relations.

Another area where canonical problems provide a useful tool of analysis is the field of controlled laboratory experiments. In fact, our study was started in direct relationship with a rock physics experiment where the elastic wavefield generated by an ultrasonic transducer and scattered by a fluid-filled crack present in a piece of uniform rock was measured. Here, the model data were used to calibrate the wavefield transmitted by the transducer and to further analyze the wavefield constituents that showed up in the measured data. In this respect, an additional advantage of having the calculated data in an analytic closed form is that each particular wave phenomenon showing up in the scattered wavefield (reflected body waves, transmitted body waves, intercepted body waves, diffracted body waves, surface waves, compressional to shear converted head waves) has a one-to-one relationship with a specific singularity in the analytic answer, a feature that is not available in purely computationally generated model data.

The diffraction of a plane elastic wave by a semi-infinite crack in an unbounded homogeneous, isotropic solid offers such a canonical problem. The problem is analyzed in the two-dimensional approximation where the wave motion takes place in the plane perpendicular to the edge of the crack. The corresponding elastic wave motion then decomposes into a 'horizontally polarized' shear-wave (SH-wave) constituent (with an out-of-plane particle velocity of the wave motion) and a combined compressional-wave (P-wave) and 'vertically polarized' shear-wave (SV-wave) constituent (with an in-plane particle velocity) (Achenbach, 1973).
The particle velocity and the dynamic stress of these wave constituents are determined with the aid of a proper combination of the modified Cagniard method for analyzing transient waves in layered media (Achenbach, 1973; Aki & Richards, 1980; Miklowitz, 1978; De Hoop, 1958; de Hoop, 1960; De Hoop & van der Ijden, 1983; De Hoop & van der Ijden, 1984; De Hoop & van der Ijden, 1985; De Hoop, 1988b; De Hoop, 1988a; de Hoop, 1990) and the Wiener-Hopf or factorization method for analyzing 'semi-infinite' diffraction problems (Noble, 1958; Weinstein (Vayshileyn), 1969). This combination has previously been applied to the analysis of the diffraction of plane elastic waves by a semi-infinite perfectly rigid plane baffle and a semi-infinite plane traction-free crack in a homogeneous, isotropic, perfectly elastic solid (Achenbach, 1973; De Hoop, 1958). Since in the laboratory experiment with regard to which the research initiated, the rock material presumably showed dispersion in its elastodynamic response, we have included in the analysis global elastodynamic relaxation mechanisms of the type that admit the construction of the solution for the dispersive case from the one pertaining to the perfectly elastic counterpart by the application of an appropriate correspondence principle (Chao & Achenbach, 1964; de Hoop, 1995b).

Mathematically, the solution also provides the expressions for the edge diffraction coefficients that are needed for the construction of the asymptotic ray solution to two-dimensional time-domain diffraction problems associated with cracks of the type under consideration (Achenbach et al., 1982).

Position in three-dimensional configuration space \( \mathbb{R}^3 \) is specified by the coordinates \( (x_1, x_2, x_3) \) with respect to an orthogonal Cartesian reference frame with the origin \( O \) and the three mutually perpendicular base vectors \( \{ \hat{e}_1, \hat{e}_2, \hat{e}_3 \} \) of unit length each. The subscript notation for Cartesian vectors and tensors is used and the summation convention applies. The position vector is also indicated as \( \mathbf{r} \). The time coordinate is \( t \). Partial differentiation with respect to \( x_m \) is denoted by \( \partial_m \); \( \partial_t \) is a reserved symbol for differentiation with respect to time. The semi-infinite fluid-filled crack is present at \( \{ x \in \mathbb{R}^2; x_1 > 0, x_3 = 0 \} \). The wave propagation takes place in the \( \{ x_1, x_3 \} \)-plane; the wave motion has components both parallel to the edge of the crack and in the plane perpendicular to it.

### Formulation of the problem

In an unbounded, homogeneous, isotropic, elastic solid with volume density of mass

\[
\rho_k = \rho \delta_{kr},
\]

where \( \delta_{kr} \) is the Kronecker tensor: \( \delta_{rr} = 1 \) for \( k = r \) and \( \delta_{kr} = 0 \) for \( k \neq r \), compliance (de Hoop, 1995a, pp.320)

\[
S_{ijpq} = \Lambda \delta_{ij} \delta_{pq} + M (\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp}),
\]

with

\[
\Lambda = -\frac{\lambda}{3\lambda + 2\mu}, \quad M = \frac{1}{4\mu},
\]

where \( \lambda \) and \( \mu \) are the Lamé coefficients, global inertia relaxation function \( \phi = \phi(t) \) and global compliance relaxation function \( \psi = \psi(t) \), the particle velocity \( v_r = v_r(x, t) \) and the dynamic stress \( \tau_{pq} = \tau_{pq}(x, t) \) satisfy the first-order coupled elastic wave equations (de Hoop, 1995a, pp.327)

\[
-\Delta_{ijpq}^\pm \partial_m \tau_{pq} + \rho_k \partial_t (\phi \ast v_r) = f_k,
\]

\[
\Delta_{ijpq} \partial_m v_r - S_{ijpq} \partial_t (\psi \ast \tau_{pq}) = h_{ij},
\]

where

\[
\Delta_{ijpq}^\pm = \frac{1}{2} (\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp})
\]

is the symmetrical unit tensor of rank four, \( f_k = f_k(x, t) \) is the volume source density of force, \( h_{ij} = h_{ij}(x, t) \) is the volume source density of deformation rate and \( \ast \) denotes time convolution. The relaxation functions satisfy the causality condition \( \phi(t) = 0 \) for \( t < 0 \) and \( \psi(t) = 0 \) for \( t < 0 \).

The elastic properties of the fluid-filled crack are characterized by the boundary conditions of the continuity type

\[
\lim_{x_3 \to 10} v_3 = \lim_{x_3 \to 0} v_3 \quad \text{for} \quad 0 < x_1 < \infty, \quad -\infty < x_2 < \infty,
\]

\[
\lim_{x_3 \to 10} \tau_{33} = \lim_{x_3 \to 0} \tau_{33} \quad \text{for} \quad 0 < x_1 < \infty, \quad -\infty < x_2 < \infty,
\]

that require the continuity of the normal component of the particle velocity and the normal component of the traction across the
crack, and the explicit boundary conditions

\[ \lim_{x_1 \to 0} \tau_{13} = \lim_{x_1 \to 0} \tau_{12} = 0 \text{ for } 0 < x_1 < \infty, \quad -\infty < x_2 < \infty, \]
\[ \lim_{x_1 \to 0} \tau_{23} = \lim_{x_1 \to 0} \tau_{21} = 0 \text{ for } 0 < x_1 < \infty, \quad -\infty < x_2 < \infty, \]

(9) (10)

that require the vanishing of the tangential components of the traction upon approaching the crack on either side. Further, all components of the particle velocity and the stress are continuous across the unruptured part of the plane of the crack, i.e., the half-plane \( \{ x \in \mathbb{R}^3; x_1 < 0, x_3 = 0 \} \).

The total wave motion is the superposition of the incident wave (to be indicated by the superscript \(^1\)), which we shall take to be either a plane SH wave, a plane P wave, or a plane SV wave, all propagating in the \( \{ x_1, x_3 \} \)-plane perpendicular to the edge of the crack, and the scattered wave (to be indicated by the superscript \(^s\)). In view of the uniformity of the medium as well as the crack in the \( x_2 \)-direction, and the isotropy of the solid, the wave diffraction problem is then two-dimensional in nature and independent of \( x_2 \), and the scattered waves, too, decouple into an SH wave for which \( \mathbf{v} = v_2 \mathbf{i}_2 \) and a combined P/SV wave for which \( \mathbf{v} = v_1 \mathbf{i}_1 + v_3 \mathbf{i}_3 \).

First, the diffraction problem will be solved for a perfectly elastic (i.e. instantaneously reacting) solid, for which \( \phi(t) = \delta(t) \) and \( \psi(t) = \delta(t) \), where \( \delta(t) \) is the Dirac delta distribution. The solution for the solid with relaxation is subsequently determined by invoking the correspondence principle for a medium with global relaxation functions (de Hoop, 1995b).

**Spectral representations for the scattered wavefield constituents**

The incident plane wave is assumed to hit the edge of the crack at the instant \( t = 0 \). Under this condition, the scattered wavefield vanishes prior to \( t = 0 \). The first step towards solving the problem through the combined Wiener-Hopf and modified Cagniard techniques then consists of carrying out a Laplace transformation with respect to time according to

\[ \{ \hat{u}_r, \hat{\tau}_{pq} \}(x, s) = \int_0^\infty \exp(-st) \{ u_r, \tau_{pq} \}(x, t) dt. \]

(11)

The time Laplace-transform parameter (or complex frequency) \( s \) is taken to be real and positive, which, in view of Lerch's theorem (Widder, 1946), is a sufficient condition for the uniqueness of the relationship between a causal time function and its time Laplace transform. Next, the spectral representations of the scattered wavefield quantities are introduced according to

\[ \{ \hat{u}_r, \hat{\tau}_{pq} \}(x, x_3, s) = \frac{1}{2\pi i} \int_{\mathcal{L}} \exp(-spx_1) \{ \hat{u}_r, \hat{\tau}_{pq} \}(p, x_3, s) dp, \]

(12)

where \( i \) is the imaginary unit and \( p \) is the complex slowness parameter in the \( x_1 \)-direction. The (infinite) path of integration \( \mathcal{L} \) in the complex \( p \)-plane remains to be determined, but a necessary condition for the convergence of the integral is that \( \mathcal{L} \) should be parallel to the imaginary \( p \)-axis as \( |p| \to \infty \). The domain to the left of \( \mathcal{L} \) will be denoted by \( \mathcal{D}^{-} \), the domain to the right of \( \mathcal{L} \) will be denoted by \( \mathcal{D}^{+} \).

**SH wave**

The spectral representation to be used for the particle velocity of the scattered SH wave is

\[ \hat{s}_2^2 = \hat{s}_2^{2,\text{SH}} = \hat{q}(s) \begin{cases} \hat{A}^{S\text{H}}(p) \exp(s \gamma_S x_3) & \text{for } x_3 < 0, \\ \hat{A}^{S\text{H}}(p) \exp(-s \gamma_S x_3) & \text{for } x_3 > 0, \end{cases} \]

(13)

Here, it has been taken into account that the scattered waves should travel away from the crack, with the shear wave speed \( c_s = (\mu/\rho)^{1/2} \), which entails the vertical S-wave slowness

\[ \gamma_S = (c_s^2 - p^2)^{1/2} \quad \text{with } \Re(\gamma_S) \geq 0 \text{ for all } p \in \mathbb{C}, \]

(14)

while \( \hat{q} = \hat{q}(s) \) is the time Laplace transform of the particle-velocity pulse shape (signature) \( q = q(t) \) of the incident SH wave. The condition on the definition of the square root in the expression for \( \gamma_S \) is at this point only needed for \( p \in \mathbb{C} \); the extension of the condition to the entire complex \( p \)-plane is needed in the later application of the modified Cagniard method and implies that branch cuts are introduced along \( \{ p \in \mathbb{C}; 1/c_s < |\Re(p)| < \infty, \Im(p) = 0 \} \). Using the relation

\[ s \hat{s}_3^2 = \mu \partial_3 \hat{s}_2^2 \]

(15)
that follows from Equations (2), (3) and (5), and invoking the continuity of $\hat{\tau}_{33}$ across the entire plane $\{x \in \mathbb{R}^3, x_3 = 0\}$, it follows that
\[ -\hat{A}^{SH\dagger}_3 = \hat{A}^{SH}_l = \bar{A}_\perp. \] (16)

With this, the expression for $\hat{v}_3$ becomes
\[ \hat{v}_3^{SH} = \mp \hat{q}(s) \hat{A}_l(p) \exp(-s\gamma_3|x_3|) \quad \text{for } x_3 \leq 0. \] (17)

How to determine $\bar{A}_\perp = \bar{A}_l(p)$ from the remaining boundary conditions, will be discussed in Section .

P/SV wave

The spectral representations to be used for the particle velocity of the scattered $P$ and $SV$ waves are
\[ \hat{v}_1^{P\dagger} = \hat{q}(s) \left\{ \begin{array}{ll}
p\hat{A}_P^\dagger(p) \exp(s\gamma_P x_3) & \text{for } x_3 < 0, \\
p\hat{A}_P(p) \exp(-s\gamma_P x_3) & \text{for } x_3 > 0,
\end{array} \right. \] (18)
\[ \hat{v}_3^{P\dagger} = \hat{q}(s) \left\{ \begin{array}{ll}
-\gamma_P p\hat{A}_P^\dagger(p) \exp(s\gamma_P x_3) & \text{for } x_3 < 0, \\
\gamma_P p\hat{A}_P^\dagger(p) \exp(-s\gamma_P x_3) & \text{for } x_3 > 0,
\end{array} \right. \] (19)

and
\[ \hat{v}_1^{SV\dagger} = \hat{q}(s) \left\{ \begin{array}{ll}
gamma_S p\hat{A}_S^\dagger(p) \exp(s\gamma_S x_3) & \text{for } x_3 < 0, \\
-\gamma_S p\hat{A}_S(p) \exp(-s\gamma_S x_3) & \text{for } x_3 > 0,
\end{array} \right. \] (20)
\[ \hat{v}_3^{SV\dagger} = \hat{q}(s) \left\{ \begin{array}{ll}
p\hat{A}_S^\dagger(p) \exp(s\gamma_S x_3) & \text{for } x_3 < 0, \\
p\hat{A}_S(p) \exp(-s\gamma_S x_3) & \text{for } x_3 > 0.
\end{array} \right. \] (21)

Here, it has been taken into account that the scattered waves should travel away from the crack, that the $P$-wave constituent is to be cullifree and travels with the compressional wave speed $c_P = [(\lambda + 2\mu)/\rho]^{1/2}$, which entails the vertical $P$-wave slowness
\[ \gamma_P = (c_P^2 - p^2)^{1/2} \quad \text{with } \text{Re}(\gamma_P) \geq 0 \text{ for all } p \in \mathbb{C}, \] (22)

that the $SV$-wave constituent is to be divergencefree and travels with the shear wave speed $c_S$, while $\hat{q} = \hat{q}(s)$ is the time Laplace transform of the particle-velocity pulse shape (signature) $q = q(t)$ of the incident $P$ or $SV$ wave. The condition on the definition of the square root in the expression for $\gamma_P$ is at this point only needed for $p \in \mathcal{L}$; the extension of the condition to the entire complex $p$-plane is needed in the later application of the modified Cagniard method and implies that branch cuts are introduced along $\{p \in \mathbb{C}; 1/c_P < |\text{Re}(p)| < \infty, \text{Im}(p) = 0\}$. The total scattered wave in this case consists of the superposition of the $P$- and $SV$-wave constituents, i.e.
\[ \hat{v}_1^{\dagger} = \hat{v}_1^{P\dagger} + \hat{v}_1^{SV\dagger}, \] (23)
\[ \hat{v}_3^{\dagger} = \hat{v}_3^{P\dagger} + \hat{v}_3^{SV\dagger}. \] (24)

Using the relations
\[ s\hat{\tau}_{13}^{\dagger} = \mu(\hat{\tau}_3^{\dagger} + \partial_3 \hat{v}_1^{\dagger}), \] (25)
\[ s\hat{\tau}_{33}^{\dagger} = \lambda(\hat{\tau}_3^{\dagger} + \partial_3 \hat{v}_3^{\dagger}) + 2\mu \partial_3 \hat{v}_3^{\dagger}, \] (26)

that follow from Equations (2), (3) and (5), and invoking the continuity of $\hat{\tau}_3^{\dagger}, \hat{\tau}_{13}^{\dagger}$ and $\hat{\tau}_{33}^{\dagger}$ across the entire plane $\{x \in \mathbb{R}^3, x_3 = 0\}$, it follows that
\[ -\hat{A}^{P\dagger}_3 = \hat{A}_P^{P\dagger}_l = p\gamma_S \hat{A}_l, \] (27)
\[ \hat{A}^{SV\dagger}_3 = \hat{A}_S^{SV\dagger}_l = (p^2 - \frac{1}{2}c_S^{-2}) \hat{A}_l. \] (28)

With this, the expressions for $\hat{v}_1^{P\dagger}, \hat{v}_3^{P\dagger}, \hat{v}_1^{SV\dagger}$ and $\hat{v}_3^{SV\dagger}$ become
\[ \hat{v}_1^{P\dagger} = \mp\hat{q}(s)p^2\gamma_S \hat{A}_3(p) \exp(-s\gamma_P|x_3|) \quad \text{for } x_3 \leq 0, \] (29)
\[ \hat{v}_3^{P\dagger} = \hat{q}(s)p\gamma_P\gamma_S \hat{A}_l(p) \exp(-s\gamma_P|x_3|) \quad \text{for } x_3 \leq 0, \] (30)
\[ \hat{v}_1^{SV\dagger} = \pm\hat{q}(s)g(p^2 - \frac{1}{2}c_S^{-2}) \hat{A}_l(p) \exp(-s\gamma_S|x_3|) \quad \text{for } x_3 \leq 0, \] (31)
\[ \hat{v}_3^{SV\dagger} = \hat{q}(s)p(p^2 - \frac{1}{2}c_S^{-2}) \hat{A}_l(p) \exp(-s\gamma_S|x_3|) \quad \text{for } x_3 \leq 0. \] (32)
How to determine $\tilde{A}_{\parallel} = \tilde{A}_{\parallel}(p)$ from the remaining boundary conditions, will be discussed in Section 3.

**Determination of the scattered SH-wave spectral amplitude**

In this section, the SH-wave spectral amplitude $\tilde{A}_{\perp} = \tilde{A}_{\perp}(p)$ will be determined with the aid of the Wiener-Hopf method ('factorization method'). Using Equation (17), the condition that $\tilde{v}_2^{\text{s SH}}$ should be continuous across the half-plane $\{ x \in \mathbb{R}^3; x_1 < 0, x_3 = 0 \}$ leads to

$$\tilde{v}_2^{\text{s SH}} = \frac{1}{2\pi i} \int_{C} \tilde{A}_{\perp}(p) \exp(-spx_1)dp \quad \text{for} \quad x_1 < 0.$$  \hspace{1cm} (33)

A sufficient condition for this relation to be satisfied is

$$\tilde{A}_{\perp}(p) = F^{-}\left( p \right) \quad \text{for} \quad p \in \mathcal{L},$$  \hspace{1cm} (34)

where $F^{-} = F^{-}(p)$ is some function of $p$ that is regular in $\mathcal{D}^{-}$ and has the property $F^{-}\left( p \right) = o(1)$ as $|p| \to \infty$ in $\mathcal{D}^{-}$, uniformly in $\text{arg}(p)$. The proof follows by supplementing $\mathcal{L}$ with a semi-circular arc at infinity in $\mathcal{D}^{-}$ (whose contribution vanishes by virtue of Jordan's lemma) and applying Cauchy's theorem to the resulting closed contour. Finally, the explicit boundary condition

$$\tilde{r}_{33}^s = -\tilde{r}_{23}^s \quad \text{for} \quad x_1 > 0, \quad x_3 = 0$$  \hspace{1cm} (35)

is invoked. With the incident plane SH wave specified by

$$v_i = q[t - \kappa_{31}^s x_1 - \gamma_s (p_0^s) x_3],$$  \hspace{1cm} (36)

with $0 \leq p_0^s \leq c_s^{-1}$ in view of the condition of causality on the scattered wave, the boundary condition (35) leads, with the use of Equation (15) to

$$\frac{1}{2\pi i} \int_{C} \gamma_s(p) \tilde{A}_{\perp}(p) \exp(-spx_1)dp = -\gamma_s(p_0^s) \exp(-sp_0^s x_1) \quad \text{for} \quad x_1 > 0.$$  \hspace{1cm} (37)

A sufficient condition for this relation to be satisfied is

$$\gamma_s(p) \tilde{A}_{\perp}(p) = \frac{\gamma_s(p_0^s)}{p - p_0^s} + H^+(p) \quad \text{for} \quad p \in \mathcal{L},$$  \hspace{1cm} (38)

where $H^+(p) = H^+(p)$ is some function of $p$ that is regular in $\mathcal{D}^+$ and has the property $H^+(p) = o(1)$ as $|p| \to \infty$ in $\mathcal{D}^+$, uniformly in $\text{arg}(p)$, provided that $\mathcal{L}$ is chosen such that $p_0^s \in \mathcal{D}^+$. The proof follows by supplementing $\mathcal{L}$ with a semi-circular arc at infinity in $\mathcal{D}^+$ (whose contribution vanishes by virtue of Jordan's lemma) and applying the theorem of residues to the resulting closed contour. Combining Equations (34) and (38), we arrive at

$$\gamma_s(p) F^{-}(p) = \frac{\gamma_s(p_0^s)}{p - p_0^s} + H^+(p) \quad \text{for} \quad p \in \mathcal{L}.$$  \hspace{1cm} (39)

Equation (39) belongs to the class of Wiener-Hopf relations (Wiener & Hopf, 1931; Titchmarsh, 1948; Baker & Copson, 1950; Noble, 1958; Achenbach, 1973) that are amenable to a solution with the aid of the factorization method (Weinstein (Vaynshteyn), 1969). The procedure goes as follows.

First, observe that ('factorization by inspection')

$$\gamma_s(p) = \gamma_s^+(p) \gamma_s^-(p) \quad \text{for} \quad p \in \mathcal{L},$$  \hspace{1cm} (40)

where

$$\gamma_s^-(p) = (c_s^{-1} - p)^{1/2} \quad \text{with} \quad \text{Re}(\gamma_s^-) \geq 0$$  \hspace{1cm} (41)

is regular in $\mathcal{D}^-$ and

$$\gamma_s^+(p) = (c_s^{-1} + p)^{1/2} \quad \text{with} \quad \text{Re}(\gamma_s^+) \geq 0$$  \hspace{1cm} (42)

is regular in $\mathcal{D}^+$. With this, Equation (39) is cast into the form

$$\gamma_s^-(p) F^{-}(p) = \frac{1}{\gamma_s^+(p_0^s)} \frac{\gamma_s(p_0^s)}{p - p_0^s} \left[ \frac{1}{\gamma_s^-(p)} - \frac{1}{\gamma_s^+(p_0^s)} \right] \frac{\gamma_s(p_0^s)}{p - p_0^s} + \frac{H^+(p)}{\gamma_s^-(p)} \quad \text{for} \quad p \in \mathcal{L}.$$  \hspace{1cm} (43)

By taking $\mathcal{L}$ such that the branch point $p = c_s^{-1}$ of $\gamma_s^-$ and the simple pole $p = p_0^s$ of $\left( p - p_0^s \right)^{-1}$ are located in $\mathcal{D}^+$ and the branch
point $p = -c_3^{-1}$ of $\gamma S$ is located in $D^-$, the left-hand side of this equation is regular in $D^-$ and the right-hand side is regular in $D^+$. According to Liouville’s first theorem (Titchmarsh, 1950, pp.85), the left-hand side and the right-hand side of Equation (43) then are representations of one and the same entire function that is regular in the entire $p$-plane. Since the left-hand side is of order $o(p^{1/2})$ as $|p| \to \infty$ in $D^−$ and the right-hand side is of order $o(1)$ as $|p| \to \infty$ in $D^+$, this entire function is, by virtue of Liouville’s second theorem (Titchmarsh, 1950, pp.85), identically equal to zero. Consequently, by putting the left-hand side equal to zero, we obtain

$$
F = \frac{1}{\gamma S(p) \gamma S(p_0)} \gamma S(p_0^\circ) \quad \text{for } p \in \{D^− \cup L\}.
$$

(44)

In view of Equation (34), the expression for $\tilde{A}_\perp$ has herewith been determined and the spectral representation of the scattered $SH$ wave is known.

To arrive at the corresponding space-time expression for the scattered $SH$ wave, the analytic continuation of $\tilde{A}_\perp$ into $D^+$ is needed. To this end, the definitions of $\gamma S$ and $\gamma S^\circ$ as given in Equations (41) and (42) are taken to hold in the entire cut $p$-plane. In accordance with the condition put on the square roots, the branch cut of $\gamma S(p)$ is along $\{p \in \mathbb{C}; c_3^{-1} < \text{Re}(p) < \infty, \text{Im}(p) = 0\}$ and the branch cut of $\gamma S^\circ(p)$ along $\{p \in \mathbb{C}; -\infty < \text{Re}(p) < -c_3^{-1}, \text{Im}(p) = 0\}$. With this procedure, the expression for $\tilde{A}_\perp$ to be used in the transformation back to the space-time domain becomes

$$
\tilde{A}_\perp = \frac{\gamma S(p_0^\circ)}{\gamma S(p)} \frac{1}{p - p_0^\circ}
$$

in the entire $p$-plane cut along $\{p \in \mathbb{C}; c_3^{-1} < \text{Re}(p) < \infty, \text{Im}(p) = 0\}$.

### Determination of the scattered $P/SV$-wave spectral amplitude

In this section, the $P/SV$-wave spectral amplitude $\tilde{A}_\parallel = \tilde{A}_\parallel(p)$ will be determined with the aid of the Wiener-Hopf method (‘factorization method’). Using Equations (23), (29) and (31), the condition that $\tilde{\sigma}^\parallel$ should be continuous across the half-plane $\{x \in \mathbb{R}^2; x_1 < 0, x_3 = 0\}$ leads to

$$
0 = \frac{1}{2\pi i} \int \gamma S(p) \gamma S(p_0^\circ) \tilde{A}_\parallel(p) \exp(-spx_1)dp \quad \text{for } x_1 < 0.
$$

(46)

A sufficient condition for this relation to be satisfied is

$$
\tilde{A}_\parallel = \frac{2c_3}{\gamma S(p)} F^−(p) \quad \text{for } p \in L,
$$

(47)

where $F^−(p)$ is some function of $p$ that is regular in $D^-$ and has the property $\lim_{|p| \to \infty} F^−(p) = o(1)$ as $|p| \to \infty$ in $D^−$, uniformly in $\arg(p)$. The proof follows by supplementing $L$ with a semi-circular arc at infinity in $D^−$ (whose contribution vanishes by virtue of Jordan’s lemma) and applying Cauchy’s theorem to the resulting closed contour. Finally, the explicit boundary condition

$$
\tilde{r}_1 = -\tilde{r}_3 \quad \text{for } x_1 > 0, x_3 = 0
$$

(48)

is invoked. From here on, the cases of an incident plane $P$ wave and an incident plane $SV$ wave have to be discussed separately.

### Incident plane $P$ wave

Let the incident plane $P$ wave be specified by

$$
\begin{align*}
\tilde{v}_1^P &= cp_0^P q[t - p_0^\circ x_1 - \gamma P(p_0^\circ)x_3], \\
\tilde{v}_3^P &= cp_0^P \gamma P(p_0^\circ) q[t - p_0^\circ x_1 - \gamma P(p_0^\circ)x_3],
\end{align*}
$$

(49)

(50)

with $0 \leq p_0^P \leq c_3^{-1}$ in view of the condition of causality of the scattered wave. Then, the boundary condition (48) leads, with the use of Equation (25) and denoting $\tilde{A}_\parallel$ for this case by $\tilde{A}_\parallel^P$, to

$$
\begin{align*}
0 &= \frac{1}{2\pi i} \int \gamma S(p)\gamma S(p_0^\circ) \tilde{A}_\parallel^P(p) \exp(-spx_1)dp \\
 &= \frac{1}{2\pi i} \int \gamma S(p)\gamma S(p_0^\circ) \tilde{A}_\parallel^P(p) \exp(-spx_1)dp.
\end{align*}
$$

(51)
In this expression, we recognize
\[ \Delta_{R}(p) = \frac{p^2}{2} \gamma_{P}(p) \gamma_{S}(p) + (p^2 - \frac{1}{2} c_{S}^2) \gamma_{S}(p), \] (52)
the Rayleigh determinant whose zeros are \( p = \pm c_{R}^{-1} \) where \( c_{R} \) is the wavespeed of Rayleigh surface waves along a traction-free boundary of a semi-infinite perfectly elastic solid.

A sufficient condition for relation (51) to be satisfied is
\[ \Delta_{R}(p) A_{R}^{2}(p) = \frac{c_{PP} p^2}{p - p_{0}^R} \gamma_{P}(p_{0}^R) + H^{+}(p) \quad \text{for } p \in \mathcal{L}, \] (53)
where \( H^{+}(p) \) is a function of \( p \) that is regular in \( \mathcal{D}^{+} \) and has the property \( H^{+}(p) = o(1) \) as \( |p| \to \infty \) in \( \mathcal{D}^{+} \), uniformly in \( \arg(p) \), provided that \( \mathcal{L} \) is chosen such that \( p_{0}^R \in \mathcal{D}^{+} \). The proof follows by supplementing \( \mathcal{L} \) with a semi-circular arc at infinity in \( \mathcal{D}^{+} \) (whose contribution vanishes by virtue of Jordan’s lemma) and applying the theorem of residues to the resulting closed contour. Combining Equations (47) and (53), we arrive at
\[ \Delta_{R}(p) \frac{2 c_{S}^{-2}}{\gamma_{S}(p)} F^{-}(p) = \frac{c_{PP} p^2}{p - p_{0}^R} \gamma_{P}(p_{0}^R) + H^{+}(p) \quad \text{for } p \in \mathcal{L}. \] (54)

Equation (54) belongs to the class of Wiener-Hopf relations and is amenable to a solution with the aid of the factorization method. For the factorization of the 'kernel function' (i.e. the function multiplying \( F^{-}(p) \)) we now have, however, to rely on the Plemelj formulas (Noble, 1958; Sparenberg, 1958), since a factorization by inspection (as could be carried out for the case of SH-wave scattering) is not known. The application of the Plemelj formulas requires a kernel function that approaches the value one as \( |p| \to \infty \) along \( \mathcal{L} \). To meet this condition, Equation (54) is rewritten as
\[ \left( 1 - \frac{c_{S}^{-2}}{c_{P}^{-2}} \right) K(p) \frac{1}{c_{R}^{-2} - p^2} \gamma_{S}(p) \left( 1 - \frac{1}{c_{R}^{-2} - p^2} \right) F^{-}(p) = \frac{c_{PP} p^2}{p - p_{0}^R} \gamma_{P}(p_{0}^R) + H^{+}(p) \quad \text{for } p \in \mathcal{L}, \] (55)
in which
\[ K(p) = \frac{2}{c_{S}^{-2} - c_{P}^{-2} c_{R}^{-2} - p^2} \Delta_{R}(p) \quad \text{for } p \in \mathcal{L}. \] (56)

A Taylor expansion around \( p = \infty \) shows that indeed, \( K(p) = 1 + O(p^{-2}) \) as \( |p| \to \infty \) along \( \mathcal{L} \). As Equation (56) shows, \( K(p) \) can be continued analytically away from \( \mathcal{L} \) into the entire \( p \)-plane cut along the common branch cuts of \( \gamma_{P}(p) \gamma_{S}(p) \), i.e. along \( \{ p \in \mathbb{C}; c_{P}^{-1} < |\text{Re}(p)| < c_{S}^{-1}, \text{Im}(p) = 0 \} \). For this function, the asymptotic relation \( K(p) = 1 + O(p^{-2}) \) as \( |p| \to \infty \) holds uniformly in \( \arg(p) \). These properties facilitate the factorization procedure carried out with the aid of the Plemelj formulas (see Appendix A). Let
\[ K(p) = K^{-}(p) K^{+}(p) \quad \text{for } p \in \mathcal{L}, \] (57)
where \( K^{-}(p) \) as given by Equations (A12) and (A19) is regular in \( \mathcal{D}^{-} \) and \( K^{+}(p) \) as given by Equations (A14) and (A20) is regular in \( \mathcal{D}^{+} \). With the aid of Equations (40)-(42) and (57), Equation (56) is cast into the form
\[
\begin{align*}
(1 - \frac{c_{S}^{-2}}{c_{P}^{-2}}) & \quad K^{-}(p) \left( 1 - \frac{1}{c_{R}^{-2} - p^2} \right) F^{-}(p) - c_{PP} p^2 \gamma_{P}(p_{0}^R) \frac{\gamma_{S}^{+}(p_{0}^R)}{K^{+}(p_{0})} \frac{1}{c_{R}^{-1} + p_{0}^R} \frac{1}{p - p_{0}^R} \\
&= c_{PP} p^2 \gamma_{P}(p_{0}^R) \left[ \frac{\gamma_{S}^{+}(p)}{K^{+}(p)} \frac{1}{c_{R}^{-1} + p} - \frac{\gamma_{S}^{+}(p_{0}^R)}{K^{+}(p_{0})} \frac{1}{c_{R}^{-1} + p_{0}^R} \right] \frac{1}{p - p_{0}^R} \\
&\quad + \frac{\gamma_{S}^{+}(p)}{K^{+}(p)} \frac{1}{c_{R}^{-1} + p} H^{+}(p) \quad \text{for } p \in \mathcal{L}.
\end{align*}
\] (58)

By taking \( \mathcal{L} \) such that the branch points \( p = c_{R}^{-1} \) and \( p = c_{S}^{-1} \) of \( K^{-}(p) \), the branch point \( p = c_{S}^{-1} \) of \( \gamma_{S}(p) \) as well as the simple pole \( p = c_{R}^{-1} \) of \( (c_{R}^{-1} - p)^{-1} \) and the simple pole \( p = p_{0}^R \) of \( (p - p_{0}^R)^{-1} \) are all located in \( \mathcal{D}^{+} \) and the branch points \( p = -c_{P}^{-1} \) and \( p = -c_{S}^{-1} \) of \( K^{+}(p) \), the branch point \( p = -c_{S}^{-1} \) of \( \gamma_{S}^{+}(p) \) as well as the simple pole \( p = -c_{R}^{-1} \) of \( (c_{R}^{-1} + p)^{-1} \) are all located in \( \mathcal{D}^{-} \), the left-hand side of this equation is regular in \( \mathcal{D}^{-} \) and the right-hand side is regular in \( \mathcal{D}^{+} \). According to Liouville’s first theorem (Titchmarsh, 1950, pp.85), the left-hand side and the right-hand side of Equation (58) are then representations of one and the same entire function that is regular in the entire \( p \)-plane. Since the left-hand side is of order \( O(p^{1/2}) \) as \( |p| \to \infty \) in \( \mathcal{D}^{-} \) and the right-hand side is of order \( O(p^{-1/2}) \) as \( |p| \to \infty \) in \( \mathcal{D}^{+} \), this entire function is, by virtue of Liouville’s second theorem (Titchmarsh, 1950, pp.85), identically equal to zero. Consequently, by putting the left-hand side equal to zero, we obtain
\[ F^{-} = \frac{c_{PP} p^2 \gamma_{P}(p_{0}^R)}{1 - c_{S}^{-2}/c_{P}^{-2}} \frac{\gamma_{S}^{+}(p)}{K^{-}(p)(c_{R}^{-1} - p)} \frac{1}{K^{+}(p_{0})} \frac{1}{c_{R}^{-1} + p_{0}^R} \frac{1}{p - p_{0}^R} \quad \text{for } p \in \{ \mathcal{D}^{-} \cup \mathcal{L} \}. \] (59)
In view of Equation (47), the expression for \( \hat{A}_\parallel^p \) then becomes

\[
\hat{A}_\parallel^p = \frac{2c_s^2c_p^2p^6\gamma(p_0^6)}{1 - c_s^2/c_p^2} \frac{\gamma_\Sigma(p)}{\gamma_\Sigma(p)K^-(p)(c_R^{-1} - p)K^+(p_0^6) c_R^{-1} + p_0^6} \frac{1}{p - p_0^6} \quad \text{for } p \in \{D^- \cup \mathcal{L}\}.
\] (60)

Substitution of this result into Equations (29)-(32) yields the spectral amplitudes of the scattered \( P \) and \( SV \) waves.

To arrive at the space-time expressions for the particle velocities of the scattered \( P \) and \( SV \) waves, the analytic continuation of \( F^- \) into \( D^+ \) is needed. To this end, the definitions of \( K^- \) and \( K^+ \) as given in Appendix A are taken to hold in the entire cut \( p \)-plane.

**Incident plane \( SV \) wave**

Let the incident plane \( SV \) wave be specified by

\[
v_i^{-SV} = -c_s\gamma_\Sigma(p_0^6)q[t - p_0^6x_1 - \gamma_\Sigma(p_0^6)x_3],
\] (61)

\[
v_i^{+SV} = c_s p_0^6q[t - p_0^6x_1 - \gamma_\Sigma(p_0^6)x_3],
\] (62)

with \( 0 \leq p_0^6 \leq c_s^{-1} \) in view of the condition of causality of the scattered wave. Then, the boundary condition (48) leads, with the use of Equation (25) and denoting \( \hat{A}_\parallel^p \) by \( \hat{A}_\parallel^p \), to

\[
\frac{1}{2\pi i} \int_{\mathcal{L}} \Delta_R(p) \hat{A}_\parallel^p(p) \exp(-spx_1) dp = -2cs[(p_0^6)^2 - \frac{1}{2}c_s^{-2}] \exp(-sp_0^6x_1) \quad \text{for } x_1 > 0.
\] (63)

A sufficient condition for this relation to be satisfied is

\[
\Delta_R(p) \hat{A}_\parallel^p(p) = \frac{2cs[(p_0^6)^2 - \frac{1}{2}c_s^{-2}]}{p - p_0^6} + H^+(p) \quad \text{for } p \in \mathcal{L},
\] (64)

where \( H^+(p) \) is some function of \( p \) that is regular in \( D^+ \) and has the property \( H^+(p) = o(1) \) as \( |p| \to \infty \) in \( D^+ \), uniformly in \( \arg(p) \), provided that \( \mathcal{L} \) is chosen such that \( p_0^6 \in D^+ \). The proof follows by supplementing \( \mathcal{L} \) with a semi-circular arc at infinity in \( D^+ \) (whose contribution vanishes by virtue of Jordan’s lemma) and applying the theorem of residues to the resulting closed contour. Combining Equations (47) and (64), we arrive at

\[
\Delta_R(p) \frac{2cs[(p_0^6)^2 - \frac{1}{2}c_s^{-2}]}{p - p_0^6} + H^+(p) \quad \text{for } p \in \mathcal{L}.
\] (65)

Equation (65) belongs to the class of Wiener-Hopf relations and is amenable to a solution with the aid of the factorization method. Proceeding as in the case of an incident \( P \) wave, Equation (65) is rewritten as

\[
\left(1 - c_s^2/c_p^2\right) \frac{K(p)}{\gamma_\Sigma(p)} \left(1 - \frac{1}{c_R^{-1} - p}\right) F^-(p) = \frac{2cs[(p_0^6)^2 - \frac{1}{2}c_s^{-2}]}{p - p_0^6} + H^+(p) \quad \text{for } p \in \mathcal{L},
\] (66)

in which \( K = K(p) \) is again given by Equation (56). With the aid of Equations (40)-(42) and Equation (57), Equation (66) is cast into the form

\[
\left(1 - c_s^2/c_p^2\right) \frac{K^-(p)}{\gamma_\Sigma(p)} \left(1 - \frac{1}{c_R^{-1} - p}\right) F^-(p) = 2cs[(p_0^6)^2 - \frac{1}{2}c_s^{-2}] \frac{\gamma_\Sigma^+(p_0^6)}{K^+(p_0^6) c_R^{-1} + p_0^6} \frac{1}{p - p_0^6} + \gamma_\Sigma^+(p_0^6) \frac{1}{K^+(p_0^6) c_R^{-1} + p_0^6} \frac{1}{p - p_0^6} \frac{1}{p - p_0^6}
\]

\[
+ \frac{\gamma_\Sigma^+(p_0^6)}{K^+(p_0^6) c_R^{-1} + p_0^6} \frac{1}{p - p_0^6} \quad \text{for } p \in \mathcal{L}.
\] (67)

By taking \( \mathcal{L} \) such that the branch points \( p = c_R^{-1} \) and \( p = c_s^{-1} \) of \( K^-(p) \), the branch point \( p = c_s^{-1} \) of \( \gamma_\Sigma(p) \) as well as the simple pole \( p = c_R^{-1} \) of \( (c_R^{-1} - p)^{-1} \) and the simple pole \( p = p_0^6 \) of \( (p - p_0^6)^{-1} \) are all located in \( D^+ \) and the branch points \( p = -c_R^{-1} \) and \( p = -c_s^{-1} \) of \( K^+(p) \), the branch point \( p = -c_s^{-1} \) of \( \gamma_\Sigma(p) \) as well as the simple pole \( p = -(c_R^{-1} + p)^{-1} \) are all located in \( D^- \), the left-hand side of this equation is regular in \( D^- \) and the right-hand side is regular in \( D^+ \). According to Liouville’s first theorem (Titchmarsh, 1950, pp.85), the left-hand side and the right-hand side of Equation (67) are then representations of one and the same entire function that is regular in the entire \( p \)-plane. Since the left-hand side is of order \( o(p^{1/2}) \) as \( |p| \to \infty \) in \( D^- \) and the
right-hand side is of order $o(p^{-1/2})$ as $|p| \to \infty$ in $D^+$, this entire function is, by virtue of Liouville's second theorem (Titchmarsh, 1950, pp.85,) identically equal to zero. Consequently, by putting the left-hand side equal to zero, we obtain

$$F^- = \frac{2c_S\left[(p_0^S)^2 - \frac{1}{2}c_S^{-2}\right]}{1 - c_S^2/c_p^2} \frac{\gamma_S(p)}{K^-(p)(c_R^{-1} - p)} \frac{\gamma_S^+(p_0^S)}{K^+(p_0^S) c_R^{-1} + p_0^S p - p_0^S}$$

for $p \in \{D^- \cup L\}$. \hspace{1cm} (68)

In view of Equation (47), the expression for $\tilde{A}_S$ then becomes

$$\tilde{A}_S = \frac{4c_S^2\left[(p_0^S)^2 - \frac{1}{2}c_S^{-2}\right]}{1 - c_S^2/c_p^2} \frac{\gamma_S(p)}{K^-(p)(c_R^{-1} - p)} \frac{\gamma_S^+(p_0^S)}{K^+(p_0^S) c_R^{-1} + p_0^S p - p_0^S}$$

for $p \in \{D^- \cup L\}$. \hspace{1cm} (69)

Substitution of this result into Equations (29)-(32) yields the spectral amplitudes of the scattered $P$ and $SV$ waves.

To arrive at the space-time expressions for the particle velocities of the scattered $P$ and $SV$ waves, the analytic continuation of $F^-$ into $D^+$ is needed. To this end, the definitions of $K^-$ and $K^+$ as given in Appendix A are taken to hold in the entire cut $p$-plane.

**Time-domain expression for the scattered $SH$ wave**

The transformation of the spectral-domain expressions to the space-time domain is carried out with the aid of the modified Cagniard method. Starting from Equations (12) and (17), it follows that the expression for the time Laplace transform of the particle velocity of the scattered $SH$ wave is given by

$$\tilde{v}_2^{SH} = \mp q(s) \frac{1}{2\pi i} \int_L \tilde{A}_L(p) \exp\{-s[pz_1 + \gamma_S(p)|x_3|]\} dp \quad \text{for} \quad x_3 \leq 0,$$ \hspace{1cm} (70)

in which $\tilde{A}_L$ is given by Equation (45). In the representation, the integrand can be continued analytically into the complex $p$-plane away from the path of integration $L$. In this process, we meet the singularities of the integrand, viz. the branch points at $p = -c_S^{-1}$ and $p = c_S^{-1}$, and the simple pole at $p = p_0^S$. In the modified Cagniard method, the integration in the right-hand side of Equation (70) is replaced with one along a path where the exponential function is of the form $\exp(-r\tau)$, where $\tau$ is real and positive, upon which $r$ is introduced as the variable of integration. Along such a path, we have

$$pz_1 + \gamma_S(p)|x_3| = r.$$ \hspace{1cm} (71)

Candidates are: the part of the real $p$-axis in between the branch points $p = -c_S^{-1}$ and $p = c_S^{-1}$ and the hyperbolic path $\{p \in \mathbb{C}; p = p_S\} \cup \{p \in \mathbb{C}; p = p_0^S\}$, with

$$p_S = p_S(x_1, |x_3|, \tau) = \frac{x_1}{r} \tau + \frac{1}{2} |x_3| (r^2 - T_S^2)^{1/2} \quad \text{for} \quad T_S < \tau < \infty,$$ \hspace{1cm} (72)

in which

$$r = (x_1^2 + x_3^2)^{1/2}$$ \hspace{1cm} (73)

is the distance from the edge of the crack to the point of observation and

$$T_S = r/c_S$$ \hspace{1cm} (74)

is the $SH$-wave travel time from the edge of the crack to the point of observation, while $^*$ denotes complex conjugate. Further, the modified Cagniard path must originate from $L$ through a continuous deformation without passing singularities of the integrand in order that Cauchy's theorem can be applied. To this end, circular arcs at infinity are to join $L$ and the modified Cagniard path. The behavior of $\tilde{A}_L(p)$ as $|p| \to \infty$ ensures that the contribution from the joining circular arcs at infinity vanishes in view of Jordan’s lemma. Hence, the integration along $L$ can be replaced by an integration along the hyperbolic path $\{p \in \mathbb{C}; p = p_S\} \cup \{p \in \mathbb{C}; p = p_0^S\}$ in the region where its point of intersection with the real $p$-axis lies to the left of the pole $p = p_0^S$ in the expression for $\tilde{A}_L$, i.e., for $x_1/r < c_S p_0^S$, whereas in the range $x_1/r \geq c_S p_0^S$ the contribution from the pole $p = p_0^S$ must be taken into account. Taking the contributions from $\{p \in \mathbb{C}; p = p_S\}$ and $\{p \in \mathbb{C}; p = p_0^S\}$ together, applying Schwarz's reflection principle of complex function theory, and introducing $\tau$ as the variable of integration, we thus arrive at

$$\tilde{v}_2^{SH} = \tilde{q}(s) G_2^{SH}(x_1, x_3, \phi),$$ \hspace{1cm} (75)
with
\[ \ddot{G}_2^{\text{SH}} = \ddot{G}_2^{\text{g:SH}} + \ddot{G}_2^{\text{d:SH}}, \]

(76)
in which
\[ \ddot{G}_2^{\text{g:SH}} = \pm \{0, \frac{1}{2}, 1\} \exp\{-s[p_0^2 x_1 + \gamma s(p_0^2) x_3]\} \]

for \( \{x_1/r < c_s p_0^2, x_1/r = c_s p_0^2, x_1/r > c_s p_0^2\} \), \( x_3 \leq 0 \),

(77)
will be identified as the 'geometrical' contribution to \( \ddot{G}_2^{\text{SH}} \) (in the sense of geometrical ray tracing via the crack) and
\[ \ddot{G}_2^{\text{d:SH}} = \mp \frac{1}{\pi} \int_{T_0}^{\infty} \exp(-s \tau) \text{Im} \left[ \tilde{A}_\perp(p\tau) \frac{\partial \text{Im} \left[ \tilde{A}(p) \right]}{\partial \tau} \right] \tau \text{d}\tau \text{ for } x_3 \leq 0, \]

(78)
will be identified as the 'diffraction' contribution to \( \ddot{G}_2^{\text{SH}} \) (associated with the edge of the crack). In Equation (78),
\[ \frac{\partial \text{Im} \left[ \tilde{A}(p\tau) \right]}{\partial \tau} = \frac{x_1}{\tau^2} + \frac{1}{r^2} \frac{\tau}{(\tau^2 - \frac{T_0^2}{r^2})^{1/2}} \text{ for } T_0 < \tau < \infty, \]

(79)
while for \( x_1/r = c_s p_0^2 \) the lower limit \( T_0 \) of the integration has to be approached from above. In view of Lerch's theorem on the uniqueness of the one-sided Laplace transform, the time-domain values corresponding to Equations (75)-(78) are obtained as
\[ v_2^{\text{SH}} = q(t) * \dot{G}_2^{\text{SH}}(x_1, x_3, t), \]

(80)
with
\[ G_2^{\text{SH}} = G_2^{\text{g:SH}} + G_2^{\text{d:SH}}, \]

(81)
in which
\[ G_2^{\text{g:SH}} = \pm \{0, \frac{1}{2}, 1\} \delta[t - p_0^2 x_1 - \gamma s(p_0^2) x_3]\]

for \( \{x_1/r < c_s p_0^2, x_1/r = c_s p_0^2, x_1/r > c_s p_0^2\} \), \( x_3 \leq 0 \),

(82)
is the 'geometrical' contribution to \( G_2^{\text{SH}} \) and
\[ \ddot{G}_2^{\text{d:SH}} = \mp \frac{1}{\pi} \text{Im} \left[ \tilde{A}_\perp(p\tau) \frac{\partial \text{Im} \left[ \tilde{A}(p\tau) \right]}{\partial \tau} \right] \text{H}(t - T_0) \text{ for } x_3 \leq 0, \]

(83)
where \( \text{H}(t) \) denotes the Heavside unit step function \( \{H(t) = \{0, 1/2, 1\} \text{ for } \{t < 0, t = 0, t > 0\}\} \), is the diffraction contribution to \( G_2^{\text{SH}} \) and, again, for \( x_1/r = c_s p_0^2 \) the value \( T_0 \) has to be approached from above.

Equation (80) indicates that the scattered wave motion is the time convolution of the signature of the incident wave and a space-time Green's function given by Equations (81)-(83). The lower sign in Equation (82) leads to the annihilation of the incident plane wave in the 'geometrical shadow region' \( \{x \in \mathbb{R}^3; x_1/r > c_s p_0^2, x_3 > 0\} \); the upper sign yields the geometrically reflected plane wave in the 'geometrically illuminated region' \( \{x \in \mathbb{R}^3; x_1/r > c_s p_0^2, x_3 < 0\} \). The right-hand side of Equation (83) yields the cylindrical diffraction wave originating at the edge of the crack. The latter's expression is the same as for the diffraction of a plane SH wave by a traction-free crack, which, in terms of the polar coordinates in the \( \{x_1, x_3\} \)-plane can be found in A.T. de Hoop (1958) and in Achenbach (1973, pp.378).

Scattered wave in the plane of the crack

In the plane of the crack, the particle velocity of the wave motion takes a particularly simple form. Substituting \( x_3 = 0 \) in Equation (70), we have
\[ \ddot{v}_2^{\text{SH}}(x_1, \mp, s) = \mp q(s) \frac{1}{2 x_1} \int L \tilde{A}_\perp(p) \exp(-spx_1) dp. \]

(84)
The cases \( x_1 < 0 \) and \( x_1 > 0 \) require different treatments and will be discussed separately below.
Diffraction by fluid-filled crack

Unruptured part \((x_1 < 0)\)

For the unruptured part of the plane of the crack we have \(x_1 < 0\). For this case, the path of integration \(\mathcal{L}\) is supplemented with a semi-circular arc at infinity in \(D^-\). Since \(\tilde{A}_1\) as given by Equation (45) has no singularities in \(D^-\) and satisfies the conditions for the application of Jordan’s lemma, it follows that

\[ \tilde{v}_2^{\text{SH}}(x_1, \mp 0, s) = 0 \quad \text{for} \quad x_1 < 0. \]  

Consequently,

\[ \tilde{v}_2^{\text{SH}}(x_1, \mp 0, t) = 0 \quad \text{for} \quad x_1 < 0, \]  

as it should be in view of the continuity of \(v_2^{\text{SH}}\) across the unruptured part of the plane of the crack in conjunction with the occurrence of the \(\mp\) on the right-hand side of Equation (84).

Ruptured part \((x_1 > 0)\)

For the ruptured part of the plane of the crack we have \(x_1 > 0\). For this case, the path of integration \(\mathcal{L}\) is supplemented with a semi-circular arc at infinity in \(D^+\). Since Jordan’s lemma applies, we replace \(\mathcal{L}\) by a loop around the branch cut of \(\gamma_0^\infty\). Along this loop, we introduce \(\tau = px_1\) as the variable of integration, after which the application of Lerch’s theorem leads to

\[ \tilde{v}_2^{\text{SH}}(x_1, \mp 0, t) = Q(t) \int G^{\text{SH}}_2(x_1, \mp 0, t), \]  

with

\[ G^{\text{SH}}_2(x_1, \mp 0, t) = G^{\text{SH}}_2(x_1, \mp 0, t) + G^{\text{SH}}_2(x_1, \mp 0, t), \]  

in which

\[ G^{\text{SH}}_2(x_1, \mp 0, t) = \mp \delta(t - p_0^t x_1) \quad \text{for} \quad x_1 > 0 \]  

and

\[ G^{\text{SH}}_2(x_1, \mp 0, t) = \mp \frac{1}{\pi x_1} \Im \left[ \mathcal{A}_{\perp}(p) \right]_{|p=\pm x_1} \mathcal{H}(t - x_1/c_S) \quad \text{for} \quad x_1 > 0. \]

Time-domain expression for the scattered P/SV wave (incident P wave)

The transformation of the spectral domain expressions for the P/SV wave back to the time domain with the aid of the modified Cagniard method runs, in principle, along the same lines as the one for the SH wave, but it is that now the spectral domain expressions are more complicated and show more singularities in the complex \(p\)-plane. The modified Cagniard path is again dictated by the argument of the exponential functions involved. In view of this, the transformations for the P-wave part and the SV-wave part have to be carried out separately, after which the two results are to be added. Also, the two cases of an incident P wave and an incident SV wave have to be considered separately, because of their difference in possible location of the corresponding poles in the complex \(p\)-plane. The present section deals with the case of an incident P wave; Section deals with the case of an incident SV wave.

Scattered P wave (incident P wave)

Starting from Equations (12), (29) and (30), it follows that the expressions for the time

Laplace transform of the particle velocity components of the scattered \(p\) wave due to an incident \(p\) wave are given by

\[ \tilde{v}_1^{\text{P}} = \mp \bar{q}(s) \frac{1}{2 \pi i} \int_{\mathcal{L}} p^2 \gamma_S(p) \mathcal{A}_{\parallel}^g(p) \exp(-s[p x_1 + \gamma_P(p)]x_3] \) \, dp \quad \text{for} \quad x_3 \leq 0, \]  

\[ \tilde{v}_3^{\text{P}} = \bar{q}(s) \frac{1}{2 \pi i} \int_{\mathcal{L}} p \gamma_P(p) [\gamma_S(p) \mathcal{A}_{\parallel}^g(p) \exp(-s[p x_1 + \gamma_P(p)]x_3)] \) \, dp \quad \text{for} \quad x_3 \leq 0, \]  

in which \(\mathcal{A}_{\parallel}^g\) is given by Equation (60). In this representation, the integrand can be continued analytically into the complex \(p\)-plane away from the path of integration \(\mathcal{L}\). In this process, we meet the singularities of the integrand, viz. the branch points \(p = -c_p^1\) and \(p = c_p^{-1}\), the branch points \(p = -c_S^1\) and \(p = c_S^{-1}\), the simple pole at \(p = c_R^{-1}\) and the simple pole at \(p = p_0^t\). Now (de Hoop, 1995a, pp.333), \(c_S \leq (\sqrt{3}/2)c_p\) and \(c_R \leq c_S\), which implies that the poles \(p = \pm c_R^{-1}\) are always more remote from the origin in
the complex \( p \)-plane than the most remote branch points \( p = \pm c \rho^{-1} \). Further, for a uniform plane incident wave that hits the crack at the instant \( t = 0 \), we have \( 0 \leq p_0^p \leq c_0 \rho^{-1} \).

The modified Cagniard path associated with Equations (91) and (92) must satisfy the equation

\[
p z_1 + \gamma_\rho(p) |z_1| = \tau,
\]

where \( \tau \) is real and positive. Candidates are: the part of the real \( p \)-axis in between the branch points \( p = -c \rho^{-1} \) and \( p = c \rho^{-1} \) of \( \gamma_\rho \) and the hyperbolic path \( \{ p \in C; p = p_0^p \} \cup \{ p \in C; p = p_0^\rho \} \), with

\[
p_0^p = p_0^p(x_1, x_3, \tau) = \frac{x_1}{\tau^2} \tau + \frac{1}{\tau^2} \frac{|x_3|}{\tau^2} (\tau^2 - T_P^2)^{1/2} \quad \text{for} \quad T_P < \tau < \infty,
\]

in which \( \tau \) is given by Equation (73) and

\[
T_P = \tau/c_\rho
\]

is the \( P \)-wave travel time from the edge of the crack to the point of observation. Further, the modified Cagniard path must originate from \( L \) through a continuous deformation without passing singularities of the integrand in order that Cauchy's theorem can be applied. To this end, circular arcs at infinity are to join \( L \) and the modified Cagniard path. In view of the behavior of \( \bar{A}_\rho \) at infinity, viz. \( \bar{A}_\rho = O(p^{-5/2}) \) as \( |p| \to \infty \), the contribution from the joining circular arcs at infinity vanishes in view of Jordan's lemma as long as \( |z_1| > 0 \). Under this condition, the integration along \( L \) can be replaced by an integration along the hyperbolic path \( \{ p \in C; p = p_0^p \} \cup \{ p \in C; p = p_0^\rho \} \) in the region of space where its point of intersection with the real \( p \)-axis lies to the left of the pole \( p = p_0^p \) in the expression for \( \bar{A}_\rho \), i.e., for \( x_1/c_\rho p_0^\rho \leq x_1/r < c_\rho p_0^p \), whereas in the range \( x_1/r > c_\rho p_0^p \) the contribution from the pole \( p = p_0^\rho \) must be taken into account. Taking the contributions from \( \{ p \in C; p = p_0^p \} \) and \( \{ p \in C; p = p_0^\rho \} \) together, applying Schwarz's reflection principle of complex function theory and introducing \( \tau \) as the variable of integration, we thus arrive at

\[
\bar{G}^p_{1,3} = \bar{G}_{1,3}(x_1, x_3, s, \tau) \quad \text{(96)}
\]

with

\[
\bar{G}^p_{1,3} = \bar{G}^p_{1,3} + \bar{G}^d_{1,3}
\]

in which

\[
\bar{G}^p_{1,3} = \pm \left\{ \begin{array}{ll}
(1, \frac{1}{2}, 1) & (p_0^\rho)^3 \gamma_\rho(p_0^\rho) R^p(p_0^\rho) \exp\left\{-s[p_0^p x_1 + \gamma_\rho(p_0^\rho)|x_3|]\right\}
\end{array} \right.
\]

\[
\bar{G}^d_{1,3} = \left\{ \begin{array}{ll}
-\left(1, \frac{1}{2}, 1\right) & p_0^\rho \gamma_\rho(p_0^\rho) \gamma_\rho(p_0^\rho) R^p(p_0^\rho) \exp\left\{-s[p_0^p x_1 + \gamma_\rho(p_0^\rho)|x_3|]\right\}
\end{array} \right.
\]

for \( x_1/r < c_\rho p_0^p \), \( x_1/r = c_\rho p_0^p \), \( x_1/r > c_\rho p_0^p \) \( x_3 \leq 0 \),

with

\[
R^p(p_0^\rho) = \text{Res}_{p=p_0^\rho} \bar{A}_\rho(p) = \frac{c_\rho p_0^\rho \gamma_\rho(p_0^\rho)}{\Delta_R(p_0^\rho)}
\]

will be identified as the 'geometrical' contribution to \( \bar{G}^p_{1,3} \) (in the sense of geometrical ray tracing via the crack) and

\[
\bar{G}^d_{1,3} = \mp \frac{1}{\pi} \int_{T_P}^{\infty} \exp(-s\tau) \text{Im} \left[ p_0^\rho \gamma_\rho(p_0^\rho) \bar{A}_\rho(p) \frac{\partial p_0^p}{\partial \tau} \right] d\tau \quad \text{for} \quad x_3 \leq 0
\]

\[
\bar{G}^d_{1,3} = \frac{1}{\pi} \int_{T_P}^{\infty} \exp(-s\tau) \text{Im} \left[ p_0^\rho \gamma_\rho(p_0^\rho) \bar{A}_\rho(p) \frac{\partial p_0^p}{\partial \tau} \right] d\tau \quad \text{for} \quad x_3 \leq 0
\]

will be identified as the 'diffraction' contribution to \( \bar{G}^p_{1,3} \) (associated with the edge of the crack). In these expressions

\[
\frac{\partial p_0^p}{\partial \tau} = \frac{x_1}{\tau^2} + \frac{|x_3|}{\tau^2} \frac{\tau^2 - T_P^2}{(\tau^2 - T_P^2)^{1/2}} \quad \text{for} \quad T_P < \tau < \infty,
\]

while for \( x_1/r = c_\rho p_0^p \) the lower limit of integration \( T_P \) has to be approached from above. In view of Lerch's theorem on the uniqueness of the one-sided Laplace transform, the time-domain expressions corresponding to Equations (96)-(99) and (101)-(102) are obtained as

\[
v_{1,3} = v_{1,3} = G_{1,3}^p(x_1, x_3, t)
\]

with

\[
G_{1,3}^p = \bar{G}_{1,3} + \bar{G}^d_{1,3}
\]
in which
\begin{align}
G_{1}^{\text{d},p} &= \pm \{ 0, \frac{1}{2}, 1 \} (p_0^P)^2 \gamma_S(p_0^P) R^P(p_0^P) \delta [ t - p_0^P x_1 - \gamma_P(p_0^P) | x_3 ] \\
G_{3}^{\text{d},p} &= \{-0, \frac{1}{2}, 1 \} p_0^P \gamma_P(p_0^P) \gamma_S(p_0^P) R^P(p_0^P) \delta [ t - p_0^P x_1 - \gamma_P(p_0^P) | x_3 ]
\end{align}
\quad (106, 107)
for \( x_1/t < c_P p_0^P \), \( x_1/t = c_P p_0^P \), \( x_1/t > c_P p_0^P \), \( x_3 \leq 0 \), and
\begin{align}
G_{1}^{\text{d},p} &= \pm \frac{1}{\pi} \text{Im} \left[ p_0^P \gamma_S(p_0^P) \hat{A}_1^P(p_0^P) \frac{\partial}{\partial r} \right] H(t - T_P) \quad \text{for} \quad x_3 \leq 0, \\
G_{3}^{\text{d},p} &= \frac{1}{\pi} \text{Im} \left[ p_0^P \gamma_P(p_0^P) \gamma_S(p_0^P) \hat{A}_1^P(p_0^P) \frac{\partial}{\partial r} \right] H(t - T_P) \quad \text{for} \quad x_3 \leq 0,
\end{align}
\quad (108, 109)
while for \( x_1/t = c_P p_0^P \), in the time convolution, the lower limit of integration \( T_P \) has to be approached from above.

**Scattered SV wave (incident \( P \) wave)**

Starting from Equations (12), (31) and (32), it follows that the expressions for the time Laplace transform of the particle velocity components of the scattered SV wave due to an incident \( P \) wave are given by
\begin{align}
\ddot{u}_1^{\text{SV}} &= \frac{\pm q(s)}{2 \pi i} \int_{\mathcal{L}} \gamma_S(p) \left( p^2 - \frac{1}{2} c_S^2 \right) \hat{A}_1^P(p) \exp \{-s[p x_1 + \gamma_S(p) | x_3]\} \, dp \quad \text{for} \quad x_3 \leq 0, \\
\ddot{u}_3^{\text{SV}} &= \frac{q(s)}{2 \pi i} \int_{\mathcal{L}} p \left( p^2 - \frac{1}{2} c_S^2 \right) \hat{A}_3^P(p) \exp \{-s[p x_1 + \gamma_S(p) | x_3]\} \, dp \quad \text{for} \quad x_3 \leq 0,
\end{align}
\quad (110, 111)
in which \( \hat{A}_1^P \) is given by Equation (60).

The modified Cagniard path associated with Equations (110) and (111) must satisfy the equation
\[ p x_1 + \gamma_S(p) | x_3 | = r, \]
\quad (112)
where \( r \) is real and positive. Candidates are: the part of the real \( p \)-axis in between the branch points \( p = -c_S^{-1} \) and \( p = c_S^{-1} \) of \( \gamma_S \) and the hyperbolic path \( \{ p \in \mathbb{C}; p = p_S \} \cup \{ p \in \mathbb{C}; p = p_S^\ast \} \), with
\[ p_S = p_S(x_1, | x_3 |, \tau) = \frac{x_1}{\tau^2} r + i \frac{| x_3 |}{\tau^2} (\tau^2 - T_S^2)^{1/2} \quad \text{for} \quad T_S < r < \infty, \]
\quad (113)
in which \( r \) is given by Equation (73) and
\[ T_S = r/c_S \]
\quad (114)
is the \( S \)-wave travel time from the edge of the crack to the point of observation. Further, the modified Cagniard path must originate from \( \mathcal{L} \) through a continuous deformation without passing singularities of the integrand in order that Cauchy's theorem can be applied. To this end, circular arcs at infinity are to join \( \mathcal{L} \) and the modified Cagniard path. In view of the behavior of \( \hat{A}_1^P \) at infinity, viz. \( \hat{A}_1^P \sim O(p^{-5/2}) \) as \( |p| \to \infty \), the contribution from the joining circular arcs at infinity vanishes in view of Jordan's lemma as long as \( | x_3 | > 0 \). Under this condition, the integration along \( \mathcal{L} \) can be replaced by an integration along the hyperbolic path \( \{ p \in \mathbb{C}; p = p_S \} \cup \{ p \in \mathbb{C}; p = p_S^\ast \} \) in the region of space where its point of intersection with the real \( p \)-axis lies to the left of the pole \( p = p_0^P \) as well as to the left of the branch point \( p = c_P^{-1} \), which singularities occur in the expression for \( \hat{A}_1^P \) (the latter through \( K^-(p) \)). As a consequence, in the region of space where \( x_1/r > c_S p_0^P \) the contribution from the pole \( p = p_0^P \) must be taken into account, while in the region of space where \( x_1/r > c_S/c_P \) the deformation of the path of integration involves a loop integral around the branch cut associated with \( K^-(p) \), joining the points of intersection of the hyperbolic part of Equation (112) with the real \( p \)-axis on either side of the branch cut. The contribution from this loop integral is associated with the shear head wave and is parametrized by \( \{ p \in \mathbb{C}; p = p_H \} \cup \{ p \in \mathbb{C}; p = p_H^\ast \} \), with
\[ p_H = p_H(x_1, | x_3 |, \tau) = \frac{x_1}{\tau^2} r - i \frac{| x_3 |}{\tau^2} (T_S^2 - \tau^2)^{1/2} + i 0 \quad \text{for} \quad x_1/r > c_S/c_P, \quad T_H < \tau < T_S, \]
\quad (115)
with
\[ T_H = x_1/c_P + (c_S^{-2} - c_P^{-2})^{1/2} | x_3 |. \]
\quad (116)
Taking the contributions from \( \{ p \in \mathbb{C}; p = p_S \} \) and \( \{ p \in \mathbb{C}; p = p_H \} \) as well as from \( \{ p \in \mathbb{C}; p = p_H^* \} \) together, applying Schwarz's reflection principle of complex function theory, and introducing \( \tau \) as the variable of integration, we thus arrive at

\[
\psi_{1,3}^{SV} = \tilde{q}(s) \hat{G}_{1,3}^{SV}(x_1, x_3, s),
\]

(117)

with

\[
\hat{G}_{1,3}^{SV} = \hat{G}_{1,3}^{SV} + \hat{G}_{1,3}^{H} + \hat{G}_{1,3}^{dSV},
\]

(118)

where

\[
\hat{G}_{1,3}^{SV} = \pm \{ 0, \frac{1}{2}, 1 \} \gamma_S(p_S^0)(p_S^0)^2 - \frac{1}{2} c_S^{-2} \} R^p(p_S^0) \exp \{- \text{sl}[p_S^0 x_1 + \gamma_S(p_S^0) x_3]\}
\]

(119)

\[
\hat{G}_{2,3}^{SV} = - \{ 0, \frac{1}{2}, 1 \} p_S^0 (p_S^0)^2 - \frac{1}{2} c_S^{-2} \} R^p(p_S^0) \exp \{- \text{sl}[p_S^0 x_1 + \gamma_S(p_S^0) x_3]\}
\]

(120)

for \( x_1/r < c_S p_S^0, x_1/r > c_S p_S^0, x_3 \leq 0 \),

where \( R^p(p_S^0) \) is given by Equation (100), will be identified as the 'geometrical' contribution to \( \hat{G}_{1,3}^{SV} \) (in the sense of geometrical ray tracing via the crack),

\[
\hat{G}_{1,3}^{H} = \pm \{ 0, 1 \} \frac{1}{\pi} \int_{T_H}^{T_S} \exp(-\text{s\textit{l}}r) \text{Im} \left[ \gamma_S(p_H)(p_H^2 - \frac{1}{2} c_S^{-2}) \hat{A}_0^p(p_H) \frac{\partial p_H}{\partial r} \right] \text{d}r
\]

(121)

\[
\hat{G}_{2,3}^{H} = \{ 0, 1 \} \frac{1}{\pi} \int_{T_H}^{T_S} \exp(-\text{s\textit{l}}r) \text{Im} \left[ p_H(p_H^2 - \frac{1}{2} c_S^{-2}) \hat{A}_0^p(p_H) \frac{\partial p_H}{\partial r} \right] \text{d}r
\]

(122)

for \( x_1/r < c_S p_S^0, x_1/r > c_S p_S^0, x_3 \leq 0 \),

will be identified as the head-wave contribution to \( \hat{G}_{1,3}^{SV} \) (associated with the edge of the crack) and

\[
\hat{G}_{1,3}^{dSV} = \pm \frac{1}{\pi} \int_{T_S}^{\infty} \exp(-\text{s\textit{l}}r) \text{Im} \left[ \gamma_S(p_S)(p_S^2 - \frac{1}{2} c_S^{-2}) \hat{A}_0^p(p_S) \frac{\partial p_S}{\partial r} \right] \text{d}r \quad \text{for } x_3 \leq 0,
\]

(123)

\[
\hat{G}_{2,3}^{dSV} = \frac{1}{\pi} \int_{T_S}^{\infty} \exp(-\text{s\textit{l}}r) \text{Im} \left[ p_S(p_S^2 - \frac{1}{2} c_S^{-2}) \hat{A}_0^p(p_S) \frac{\partial p_S}{\partial r} \right] \text{d}r \quad \text{for } x_3 \leq 0,
\]

(124)

will be identified as the 'diffraction' contribution to \( \hat{G}_{1,3}^{SV} \) (associated with the edge of the crack). In these expressions,

\[
\frac{\partial p_H}{\partial r} = \frac{x_1}{r^2} + \frac{x_3}{r^2} \frac{\tau}{(T_S^2 - \tau^2)^{1/2}} \quad \text{for } x_1/r > c_S p_S, \quad T_H < \tau < T_S
\]

(125)

and

\[
\frac{\partial p_S}{\partial r} = \frac{x_1}{r^2} + i \frac{x_3}{r^2} \frac{\tau}{(T_S^2 - \tau^2)^{1/2}} \quad \text{for } T_S < \tau < \infty,
\]

(126)

while for \( x_1/r = c_S p_S^0 \) the lower limit of integration \( T_S \) has to be approached from above. In view of Lerch's theorem on the uniqueness of the one-sided Laplace transform, the time-domain expressions corresponding to Equations (117)-(124) are obtained as

\[
\psi_{1,3}^{SV} = q(t) \ast G_{1,3}^{SV}(x_1, x_3, t),
\]

(127)

with

\[
G_{1,3}^{SV} = G_{1,3}^{SV} + G_{1,3}^{H} + G_{1,3}^{dSV},
\]

(128)

in which the geometrical part is given by

\[
G_{1,3}^{SV} = \pm \{ 0, \frac{1}{2}, 1 \} \gamma_S(p_S^0)(p_S^0)^2 - \frac{1}{2} c_S^{-2} \} R^p(p_S^0) \delta[t - p_S^0 x_1 + \gamma_S(p_S^0) x_3]
\]

(129)

\[
G_{2,3}^{SV} = - \{ 0, \frac{1}{2}, 1 \} p_S^0 (p_S^0)^2 - \frac{1}{2} c_S^{-2} \} R^p(p_S^0) \delta[t - p_S^0 x_1 + \gamma_S(p_S^0) x_3]
\]

(130)

for \( x_1/r < c_S p_S^0, x_1/r = c_S p_S^0, x_1/r > c_S p_S^0, x_3 \leq 0 \),
the head-wave part by

$$G_{1}^{H} = \pm (0,1) \frac{1}{\pi} \text{Im} \left[ \gamma_5 (p_H) (p_H - \frac{1}{2} c_s^{-2}) \tilde{A}_n^{P} (p_H) \frac{\partial H}{\partial r} \right] \left[ H(\tau - T_H) - H(\tau - T_S) \right]$$

$$(131)$$

$$G_{3}^{H} = \{0,1\} \frac{1}{\pi} \text{Im} \left[ p_H (p_H - \frac{1}{2} c_s^{-2}) \tilde{A}_n^{P} (p_H) \frac{\partial H}{\partial r} \right] \left[ H(\tau - T_H) - H(\tau - T_S) \right]$$

$$(132)$$

and the diffracted part by

$$G_{1}^{SV} = \pm \frac{1}{\pi} \text{Im} \left[ \gamma_5 (p_S) (p_S - \frac{1}{2} c_s^{-2}) \tilde{A}_n^{P} (p_S) \frac{\partial p_S}{\partial r} \right] H(\tau - T_S) \quad \text{for} \ x_3 \leq 0,$$

$$(133)$$

$$G_{3}^{SV} = \frac{1}{\pi} \text{Im} \left[ p_S (p_S - \frac{1}{2} c_s^{-2}) \tilde{A}_n^{P} (p_S) \frac{\partial p_S}{\partial r} \right] H(\tau - T_S) \quad \text{for} \ x_3 \leq 0.$$

$$(134)$$

Scattered wave in the plane of the crack

In the plane of the crack, we have $x_3 = 0$ and hence Jordan’s lemma fails to apply for the scattered $P$ and $SV$ waves separately. Upon taking the $P$ and $SV$ contributions together, we can, however, follow the standard procedures. Using Equations (23), (24) and (29)-(32), and substituting $x_3 = 0$, we obtain

$$v_{1}^{P;SV}(p, \mp 0, s) = \mp \frac{1}{2} q(s) \gamma_5 (p) c_s^{-2} \tilde{A}_n^{P} (p),$$

$$(135)$$

$$v_{3}^{P;SV}(p, \mp 0, s) = \tilde{q}(s) [p \gamma_5 (p) \gamma_5 (p) + p(p^2 - \frac{1}{2} c_s^{-2})] \tilde{A}_n^{P} (p),$$

$$(136)$$

in which $\tilde{A}_n^{P}$ is given by Equation (60). Using these expressions in Equation (12), the behavior of $\tilde{A}_n^{P} (p)$ as $|p| \to \infty$ ensures that Jordan’s lemma is indeed applicable. The cases $x_1 < 0$ and $x_1 > 0$ require different treatments and will be discussed separately below.

Unruptured part ($x_1 < 0$)

For the unruptured part of the plane of the crack we have $x_1 < 0$. For this case, the path of integration $\mathcal{L}$ is supplemented with a semi-circular arc at infinity in $\mathcal{D}^-$. Since the right-hand side of Equation (135) is free from singularities in $\mathcal{D}^-$, it then follows that

$$v_{1}^{P;SV}(x_1, \mp 0, s) = 0 \quad \text{for} \ x_1 < 0.$$  

$$(137)$$

Consequently,

$$v_{1}^{P;SV}(x_1, \mp 0, t) = 0 \quad \text{for} \ x_1 < 0,$$

$$(138)$$

as it should be in view of the continuity of $v_{1}^{P;SV}$ across the unruptured part of the plane of the crack in conjunction with the occurrence of the $\mp$ on the right-hand side of Equation (135). As far as $v_{3}^{P;SV}$ is concerned, the integration along $\mathcal{L}$ is replaced by one along a loop around the branch cut $\{p \in C; \ -\infty < \text{Re}(p) < -1/c_p, \text{Im}(p) = 0\}$ of the function on the right-hand side of Equation (136). Taking the contributions from the lower and upper parts of this loop together, applying Schwarz’s reflection principle of complex function theory, introducing $\tau = px_1$ as the variable of integration, and applying Lerch’s theorem, we arrive at

$$v_{3}^{P;SV}(x_1, \mp 0, t) = q(t) \frac{1}{x_1} \text{Im} \left[ p \gamma_5 (p) \gamma_5 (p) + p(p^2 - \frac{1}{2} c_s^{-2}) \tilde{A}_n^{P} (p) \right]_{p = x_1 + i 0} \text{H}(t + x_1/c_p)$$

$$(139)$$

with

$$G_{3}^{P;SV}(x_1, \mp 0, t) = G_{3}^{d;SV}(x_1, \mp 0, t)$$

$$(140)$$

consisting of a diffracted part only, given by

$$G_{3}^{d;SV} = \frac{1}{\pi x_1} \text{Im} \left[ p \gamma_5 (p) \gamma_5 (p) + p(p^2 - \frac{1}{2} c_s^{-2}) \tilde{A}_n^{P} (p) \right]_{p = x_1 + i 0} \text{H}(t + x_1/c_p)$$

$$(141)$$
Ruptured part \((x_1 > 0)\)

For the ruptured part of the plane of the crack we have \(x_1 > 0\). For this case, we replace \(\mathcal{L}\) by a loop around the branch cut \(\{p \in \mathbb{C}; c_1^{-1} < \text{Re}(p) < \infty, \text{Im}(p) = 0\}\) of the right-hand sides of Equations (135)-(136) and take into account the residue of the simple pole at \(p = c_1^{-1}\). Along the loop, we introduce \(\tau = px_1\) as the variable of integration, take the contributions from the lower and upper parts together, apply Schwarz's reflection principle of complex function theory, and use Lerch's theorem. Via this procedure we arrive at

\[
v^{S,P}_{1,3}(x_1, \tau = 0, t) = q(t) \ast G^{S,P}_{1,3}(x_1, \tau = 0, t), \quad (142)
\]

with

\[
G^{S,P}_{1,3}(x_1, \tau = 0, t) = G^{S,P}_{1,3}(x_1, \tau = 0, t) + G^{d,P}_{1,3}(x_1, \tau = 0, t), \quad (143)
\]

in which the geometrical part is given by

\[
G^{S,P}_{1}(x_1, \tau = 0, t) = \pm \frac{1}{2 \pi x_1 c_1} \gamma_S(p) \delta(t - p x_1) \quad \text{for} \quad x_1 > 0, \quad (144)
\]

\[
G^{S,P}_{3}(x_1, \tau = 0, t) = -\frac{1}{2 \pi x_1} \gamma_S(p) \gamma_S(p) \delta(t - p x_1) \quad \text{for} \quad x_1 > 0, \quad (145)
\]

and the diffracted part by

\[
G^{d,P}_{1}(x_1, \tau = 0, t) = \pm \frac{1}{2 \pi x_1} \gamma_S(p) \delta(t - x_1/c_1) \quad \text{for} \quad x_1 > 0, \quad (146)
\]

where the value \(t = x_1/c_1\) has to be approached from either side, and

\[
G^{d,P}_{3}(x_1, \tau = 0, t) = \frac{1}{2 \pi x_1} \left[ \frac{1}{\gamma(p)} \gamma_S(p) \left| \gamma_S(p) \right| \right] H(t - x_1/c_1) \\
-\left[ c_R^{-2} \gamma_S(c_R^{-1}) \gamma_S(c_R^{-1}) + c_R^{-1} (c_R^{-2} - \frac{1}{2} c_R^{-2}) \right] \text{Res}_{p = c_R} \delta(p) \quad \text{for} \quad x_1 > 0. \quad (147)
\]

Time-domain expression for the scattered \(P/SV\) wave (incident \(SV\) wave)

In this section, the case of an incident \(SV\) wave is dealt with. The procedure to be followed is the same as in Section for the incident \(P\) wave as long as the direction of incidence is precritical, i.e., as long as the line of intersection of the wave front of the incident wave with the plane of the crack travels along the crack with a speed that is greater than \(c_1\). For postcritical incidence, i.e., if this latter speed is less than \(c_1\) (though always greater than or equal to \(c_S\)), special features show up due to the fact that the simple pole that is indicative for the direction of incidence is now located on the branch cut associated with the diffracted \(SV\) wave. As a consequence, the residue theorem, that yields the geometrical contribution to the scattered wave motion can no longer be applied in its simple form.

Scattered \(P\) wave (incident \(SV\) wave)

Starting from Equations (12), (29) and (30), it follows that the expressions for the time Laplace transform of the particle-velocity components of the scattered \(P\) wave due to an incident \(SV\) wave are given by

\[
v^{S,P}_{1} = -\dot{q}(t) \ast \frac{1}{2 \pi i} \int_{\mathcal{L}} p^2 \gamma_S(p) \left| \gamma_S(p) \right| \exp(-s[p x_1 + \gamma_S(p)|x_3]|) dp \quad \text{for} \quad x_3 \leq 0, \quad (148)
\]

\[
v^{S,P}_{3} = q(t) \ast \frac{1}{2 \pi i} \int_{\mathcal{L}} p \gamma_S(p) \left| \gamma_S(p) \right| \exp(-s[p x_1 + \gamma_S(p)|x_3]|) dp \quad \text{for} \quad x_3 \leq 0, \quad (149)
\]

in which \(\tilde{A}^S\) is given by Equation (69). Proceeding as in Section we arrive at

\[
v^{S,P}_{1,3} = q(t) \ast G^{S,P}_{1,3}(x_1, x_3, t), \quad (150)
\]

with

\[
G^{S,P}_{1,3} = G^{S,P}_{1,3} + G^{d,P}_{1,3}, \quad (151)
\]
\[
G_{11}^{\text{P}} = \pm \{0, \frac{1}{2}, 1\} \gamma_S(p_0^S) R^S(p_0^S) \delta[t - p_0^S x_1 - \gamma_F(p_0^S)|x_3|] \\
G_{33}^{\text{P}} = \{0, \frac{1}{2}, 1\} p_0^S \gamma_F(p_0^S) \gamma_S(p_0^S) R^S(p_0^S) \delta[t - p_0^S x_1 - \gamma_F(p_0^S)|x_3|] \\
\text{for } \{x_1/r < c_F p_0^S, x_1/r = c_F p_0^S, x_1/r > c_F p_0^S\}, x_3 \leq 0,
\]

with \[
R^S(p_0^S) = \text{Res}_{p = p_0^S} \tilde{A}_h^S(p) = \frac{2cs([p_0^S]^2 - \frac{1}{2}c_s^{-2})}{\Delta_R(p_0^S)},
\]

is the 'geometrical' contribution to \(\hat{G}_{1,3}^{\text{P}}\) (in the sense of geometrical ray tracing via the crack) and

\[
G_{11}^{\text{dP}} = \mp \frac{1}{\pi} \text{Im} \left[ p_0^S \gamma_S(p_0^S) \tilde{A}_h^S(p_0^S) \frac{\partial \phi_{\text{dP}}}{\partial \tau} \right] \text{ for } x_3 \leq 0,
\]

\[
G_{33}^{\text{dP}} = \frac{1}{\pi} \text{Im} \left[ p_0^S \gamma_F(p_0^S) \gamma_S(p_0^S) \tilde{A}_h^S(p_0^S) \frac{\partial \phi_{\text{dP}}}{\partial \tau} \right] \text{ for } x_3 \leq 0,
\]

is the 'diffraction' contribution to \(\hat{G}_{1,3}^{\text{P}}\) (associated with the edge of the crack).

**Scattered SV wave (incident SV wave)**

Starting from Equations (12), (31) and (32), it follows that the expressions for the time Laplace transform of the particle velocity components of the scattered SV wave due to an incident SV wave are given by

\[
\hat{v}_1^{\text{SV}} = \pm \hat{q}(s) \frac{1}{2\pi i} \int_{\mathcal{L}} \gamma_S(p) (p^2 - \frac{1}{2}c_s^2) \tilde{A}_h^S(p) \exp\{-s[p x_1 + \gamma_S(p)|x_3|]\} dp \text{ for } x_3 \leq 0,
\]

\[
\hat{v}_3^{\text{SV}} = \hat{q}(s) \frac{1}{2\pi i} \int_{\mathcal{L}} p(p^2 - \frac{1}{2}c_s^2) \tilde{A}_h^S(p) \exp\{-s[p x_1 + \gamma_S(p)|x_3|]\} dp \text{ for } x_3 \leq 0,
\]

in which \(\tilde{A}_h^S\) is given by Equation (69). Proceeding as in Section 5, we arrive at

\[
v_{1,3}^{\text{SV}} = q(t) \ast G_{1,3}^{\text{SV}}(x_1, x_3, t),
\]

with

\[
G_{1,3}^{\text{SV}} = G_{1,3}^{\text{P}} + G_{1,3}^{\text{dP}} + G_{1,3}^{\text{dSV}},
\]

in which the geometrical part is given by

\[
G_{11}^{\text{SV}} = \mp \{0, \frac{1}{2}, 1\} \gamma_S(p_0^S) [(p_0^S)^2 - \frac{1}{2}c_s^2] R^S(p_0^S) \delta[t - p_0^S x_1 - \gamma_S(p_0^S)|x_3|]
\]

\[
G_{33}^{\text{SV}} = -\{0, \frac{1}{2}, 1\} p_0^S [(p_0^S)^2 - \frac{1}{2}c_s^2] R^S(p_0^S) \delta[t - p_0^S x_1 - \gamma_S(p_0^S)|x_3|]
\]

\[
\text{for } \{x_1/r < c_F p_0^S, x_1/r = c_F p_0^S, x_1/r > c_F p_0^S\}, x_3 \leq 0,
\]

with \[
R^S(p_0^S) = \text{Res}_{p = p_0^S} \tilde{A}_h^S(p) = \frac{2cs([p_0^S]^2 - \frac{1}{2}c_s^{-2})}{\Delta_R(p_0^S)},
\]

for \(0 \leq p_0^S < c_F^{-1}\) (i.e. for precritical incidence, for which case \(\gamma_F(p_0^S)\) is real-valued) and

\[
R^S(p_0^S) = \text{Re} \left[ \text{Res}_{p = p_0^S + i0} \tilde{A}_h^S(p) \right] = \text{Re} \left[ \frac{2cs([p_0^S]^2 - \frac{1}{2}c_s^{-2})}{\Delta_R(p_0^S)} \right],
\]

for \(c_F^{-1} \leq p_0^S < c_s^{-1}\) (i.e. postcritical incidence for which case \(\gamma_F(p_0^S) + i0\) is negative imaginary), the head-wave part by

\[
G_{11}^{\text{dH}} = \pm \{0, 1\} \frac{1}{\pi} \text{Im} \left[ \gamma_S(p_0^S) [(p_0^S)^2 - \frac{1}{2}c_s^2] \tilde{A}_h^S(p_0^S) \frac{\partial \phi_{\text{dH}}}{\partial \tau} \right] [H(\tau - T_H) - H(\tau - T_S)]
\]

\[
G_{33}^{\text{dH}} = \{0, 1\} \frac{1}{\pi} \text{Im} \left[ p_0^S [(p_0^S)^2 - \frac{1}{2}c_s^2] \tilde{A}_h^S(p_0^S) \frac{\partial \phi_{\text{dH}}}{\partial \tau} \right] [H(\tau - T_H) - H(\tau - T_S)]
\]
for \( x_1/r < c_s/c_p, x_1/r > c_s/c_p \), \( x_3 \leq 0 \),

and the diffracted part by

\[
G_d^{4,SV}_1 = \pm \frac{1}{\pi} \text{im} \left\{ \frac{1}{\gamma_S(p_S)(p_S^2 - \frac{1}{2}c_s^2)} \frac{\partial}{\partial \tau} \right\} H(\tau - T_S) \text{ for } x_3 \leq 0,
\]

\[
G_d^{4,SV}_3 = \frac{1}{\pi} \text{im} \left\{ \frac{1}{p_S(p_S^2 - \frac{1}{2}c_s^2)A_{||}(p_S)} \frac{\partial}{\partial \tau} \right\} H(\tau - T_S) \text{ for } x_3 \leq 0.
\]

(167)

(168)

As far as the contribution from the pole \( p_0^S \), and hence the expression for \( R^S(p_0^S) \) is concerned, we have to distinguish between the cases of precritical incidence (for which \( 0 \leq p_0^S < c_p^{-1} \)) and postcritical incidence (for which \( c_p^{-1} \leq p_0^S \leq c_s^{-1} \)). In the case of precritical incidence the pole \( p_0^S \) is situated to the left of the branch point \( c_p^{-1} \), and \( R^S(p_0^S) \) is given by Equation (161) follows from a straightforward application of the theorem of residues. In the case of postcritical incidence the pole \( p_0^S \) is situated on the branch cut joining \( c_p^{-1} \), and two semi-circular arcs on either side of the branch cut have to be introduced to avoid the pole. These semi-circular arcs symmetrically break up the loop associated with the head-wave contribution. Taking together their contributions, we arrive at Equation (164) as the corresponding expression for \( R^S(p_0^S) \).

**Scattered wave in the plane of the crack**

In the plane of the crack, we have \( x_3 = 0 \) and hence Jordan’s lemma fails to apply for the scattered \( P \) and \( SV \) waves separately. Upon taking the \( P \) and \( SV \) contributions together, we can, however, follow the standard procedures. Using Equations (23), (24) and (29)-(32), and substituting \( x_3 = 0 \), we obtain

\[
\tilde{v}_1^{P,SV}(p, \mp 0, s) = \mp \frac{i}{2} \hat{q}(s) \gamma_S(p) c_s^{-2} \tilde{A}_0^S(p),
\]

\[
\tilde{v}_2^{P,SV}(p, \mp 0, s) = \hat{q}(s)p \gamma_S(p) c_s^{-2} \tilde{A}_\perp^S(p),
\]

(169)

(170)

in which \( \tilde{A}_0^S(p) \) is given by Equation (69). Using these expressions in Equation (12), the behavior of \( \tilde{A}_0^S(p) \) as \( |p| \to \infty \) ensures that Jordan’s lemma is indeed applicable. The cases \( x_1 < 0 \) and \( x_1 > 0 \) require different treatments; their results will be presented separately below.

**Unruptured part \((x_1 < 0)\)**

Proceeding as in Section we arrive for the unruptured part of the plane of the crack at

\[
v_1^{P,SV}(x_1, \mp 0, t) = G^{P,SV}_1(x_1, \mp 0, t),
\]

as it should be in view of the continuity of \( v_1^{P,SV}(x_1, \mp 0, t) \) across the unruptured part of the plane of the crack in conjunction with the occurrence of the \( \mp \) on the right-hand side of Equation (169). Furthermore,

\[
v_2^{P,SV}(x_1, \mp 0, t) = \frac{1}{2} \hat{q}(t) G^{P,SV}_2(x_1, \mp 0, t),
\]

(172)

with

\[
G^{P,SV}_2(x_1, \mp 0, t) = G^{d,P,SV}_2(x_1, \mp 0, t)
\]

(173)

consisting of a diffracted part only, given by

\[
G^{d,P,SV}_2 = \frac{1}{\pi i} \text{im} \left\{ \frac{1}{p \gamma_S(p) \gamma_S(p) + p(p^2 - \frac{1}{2}c_s^2)A_\perp^S(p)} \right\}|_{p=t/x_1+i0} H(t + x_1/c_p)
\]

(174)

for \( x_1 < 0 \).

**Ruptured part \((x_1 > 0)\)**

Proceeding as in Section we arrive for the ruptured part of the plane of the crack at

\[
v_1^{P,SV}(x_1, \mp 0, t) = \frac{1}{2} \hat{q}(t) G^{P,SV}_1(x_1, \mp 0, t),
\]

(175)

with

\[
G^{P,SV}_1(x_1, \mp 0, t) = G^{P,SV}_1(x_1, \mp 0, t) + G^{d,P,SV}_1(x_1, \mp 0, t),
\]

(176)
in which the geometrical part is given by

\begin{align}
G_1^{EP/SV}(x_1, \mp 0, t) &= \pm \frac{1}{2} c_S^2 \gamma_S (p_S^2 R_S^S (p_S^2) \delta[t - p_S^0 x_1] \quad \text{for} \quad x_1 > 0, \\
G_2^{EP/SV}(x_1, \mp 0, t) &= \mp \frac{1}{2} c_S^2 \gamma_P (p_S^2 R_S^S (p_S^2) \delta[t - p_S^0 x_1] \quad \text{for} \quad x_1 > 0,
\end{align}

and the diffracted part by

\begin{align}
G_1^{dP/SV}(x_1, \mp 0, t) &= \frac{1}{2 \pi x_1 c_S^2} \text{Im} \left[ \gamma_S (p) \tilde{A}_S^S (p) \right]_{p = t / x_1} H(t - x_1 / c_P) \quad \text{for} \quad x_1 > 0, \\
G_2^{dP/SV}(x_1, \mp 0, t) &= \frac{1}{2 \pi x_1} \text{Im} \left[ [p \gamma_P (p) \gamma_S (p) + p(p^2 - \frac{1}{2} c_S^2)] \tilde{A}_S^S (p) \right]_{p = t / x_1} H(t - x_1 / c_P) \\
&\quad - [c_R^{-1} \gamma_P (c_R^{-1}) \gamma_S (c_R^{-1}) + c_R^{-1} (c_R^{-2} - \frac{1}{2} c_S^2)] \text{Res}_{p = c_R^{-1}} \tilde{A}_S^S (p) \quad \text{for} \quad x_1 > 0.
\end{align}

Global relaxation in the solid

In this section, global relaxation in the solid is included in the analysis. The relevant mechanism is represented through the global relaxation functions \( \phi = \phi(t) \) for the inertial and \( \psi = \psi(t) \) for the compliance as introduced in Section. The consequences of this are most easily investigated in the complex frequency or s-domain. In particular, from the particle velocity elastodynamic wave equation, it is seen that the result of the introduction of the relaxation mechanism is to replace \( s^2 \) in this equation by \( s^2 \tilde{\phi}(s) \tilde{\psi}(s) \).

Since, further, the boundary conditions at the crack are of the homogeneous type, the spectral representations of the scattered field quantities can now profitably taken of the form

\begin{align}
\tilde{w}(p, x_3, s) &= \hat{\phi}(s) \hat{\psi}(p, x_3) \\
\hat{\phi}(p, x_3) &= \hat{\tilde{A}}(p) \exp \{-s[\hat{\phi}(s) \hat{\psi}(s)]^{1/2} \gamma_{PP, s}(p)|x_3| \}, \\
\hat{\psi}(p, x_3) &= \hat{\tilde{A}}(p) \exp \{-s[\hat{\phi}(s) \hat{\psi}(s)]^{1/2} \gamma_{PP, s}(p)|x_3| \},
\end{align}

where \( \gamma_{PP, s}(p) \) is, still, given by either Equation (22) for P waves or Equation (14) for SH and SV waves. The factorization method remains unaltered, whereas the modified Cagniard method now leads to complex-frequency domain expressions of the type

\begin{align}
\hat{g}(p, x_3, s) &= \frac{1}{\pi} \int_{T_{PP, s}} \exp \{-s[\hat{\phi}(s) \hat{\psi}(s)]^{1/2} \} \text{Im} \left[ \hat{\tilde{A}}(p) \frac{\partial \tilde{A}_{PP, s}}{\partial \tau} \right] \text{d} \tau \\
\text{for diffracted body-wave contributions, and} \\
\hat{g}(x_1, x_3, s) &= \frac{1}{\pi} \int_{T_{SH}} \exp \{-s[\hat{\phi}(s) \hat{\psi}(s)]^{1/2} \} \text{Im} \left[ \hat{\tilde{A}}(p) \frac{\partial \tilde{A}_{PH, s}}{\partial \tau} \right] \text{d} \tau \\
\text{for diffracted head-wave contributions.}
\end{align}

Now, for a number of relaxation functions, \( \exp \{-s[\hat{\phi}(s) \hat{\psi}(s)]^{1/2} \} \) admits a representation of the type

\begin{align}
\exp \{-s[\hat{\phi}(s) \hat{\psi}(s)]^{1/2} \} = \int_t^\infty \text{exp}(-st) U(t, r) \text{d}t.
\end{align}

Using Equation (186) in Equation (185) and employing Lerch's theorem on the uniqueness of the time-Laplace transform for real, positive values of the transform parameter, we end up with

\begin{align}
g(x_1, x_3, t) &= \left[ \int_{T_{PP, s}} U(t, r) G(x_1, x_3, r) \text{d}r \right] H(t - T_{PP, s})
\end{align}
for diffracted body-wave contributions, and
\[
g(x_1, x_3, t) = \left[ \int_{T_H}^{\min(t, T_S)} U(t, \tau) G(x_1, x_3, \tau) d\tau \right] \left[ H(t - T_H) - H(t - T_S) \right] \tag{188}
\]
for diffracted head-wave contributions. Here, \(G(x_1, x_3, \tau)\) in all cases is scattered wave pertaining to the lossless case for which \(\phi(t) = \delta(t)\) and \(\psi(t) = \delta(t)\).

'Special case: Maxwell-type visco-elasticity

For the special case \(\phi(t) = \delta(t) + \alpha H(t), \psi(t) = \delta(t) + \beta H(t)\), we have
\[
\hat{\phi}(s) = 1 + \frac{\alpha}{s} \quad \text{for} \quad \text{Re}(s) > 0, \tag{189}
\]
\[
\hat{\psi}(s) = 1 + \frac{\beta}{s} \quad \text{for} \quad \text{Re}(s) > 0. \tag{190}
\]
Here, \(\alpha > 0\) and \(\beta > 0\). This case includes both the influence of frictional forces and a Maxwell-type visco-elasticity (for the latter see Kolsky (1964, pp.107)).

The relation corresponding to Equation (186) becomes (Abramowitz & Stegun, 1965)
\[
\exp[-(s + \alpha)^{1/2}(s + \beta)^{1/2} t] = \int_{-\infty}^{\infty} \exp(-st) U_1(t, \tau) dt, \tag{191}
\]
in which
\[
U_1(t, \tau) = -\partial_t U_0(t, \tau), \tag{192}
\]
with
\[
U_0(t, \tau) = \exp[-(\alpha + \beta)t/2] I_0(\beta - \alpha/2)(t^2 - \tau^2)^{1/2} H(t - \tau), \tag{193}
\]
where \(I_0\) denotes the modified Bessel function of the first kind and order 0. This type of relaxation has been used to generate the numerical results for wave diffraction in a dispersive solid.

Numerical results

In this section, some numerical results will be presented. The source signature of the incident plane wave is taken to be the power-exponential pulse,
\[
q(t) = A \left(\frac{\alpha t}{\nu}\right)^{\nu} \exp(-at + \nu) H(t). \tag{194}
\]
Here, \(A\) is the amplitude of the pulse and the parameters \(\nu\) and \(\alpha\) are related to the pulse rise time \(t_r\) and the pulse time width \(t_w\) via
\[
\nu = \frac{t_r}{\alpha}, \tag{195}
\]
\[
\alpha = \frac{1}{t_w - t_r}. \tag{196}
\]
The pulse rise time and the pulse time width must satisfy the condition \(t_w > t_r > 0\), which entails \(\nu > 0\) and \(\alpha > 0\).

For any component of the particle velocity, the incident-wave source signature has to be convolved with a Green’s function. In the latter, there always occurs a Jacobian of the type \(\partial p_{p,s}/\partial r\) associated with the modified Cagniard method. This Jacobian has an inverse square-root singularity at the arrival time \(T_{P,S}\) of the diffracted body waves, which singularity causes a difficulty in the numerical evaluation of the time-convolution integral. This difficulty can be circumvented by rewriting the integral as a Stieljes-type integral in which \(p_{p,s} = p_{p,s}(x_1, x_3, \tau)\) is incorporated in the differential through \((\partial p_{p,s}/\partial r) d\tau = d[p_{p,s}(.,., \tau)]\).

Another difficulty can be caused by the presence of a factor \(1/T_{p,s}\) in the expression for the Green’s function in case the modified Cagniard path comes close to the pertaining branch point. This factor, too, can be accommodated in the the differential of the Stieljes-type integral through \([1/T_{p,s}(p_{p,s})]d[p_{p,s}(.,., \tau)] = d[-2\gamma_{p,s}(p_{p,s}(.,., \tau))]\).
Diffraction by fluid-filled crack

Let the generic form of the convolution integral under consideration be given by

\[ v^d = \int_{t_{P,S}}^{t} q(t - \tau) \frac{1}{\pi} \text{Im} \left[ \frac{\hat{N}(p P, S)}{\gamma_{P,S}(p P, S)} \frac{\partial N}{\partial r} \right] d\tau H(t - T_{P,S}). \] (197)

With the indicated procedure, this expression is rewritten as

\[ v^d = \text{Im} \left[ \frac{1}{\pi} \int_{t_{P,S}}^{t} q(t - \tau) \hat{N}(p P, S) d[-2 \gamma_{P,S}(p P, S, r)] \right] H(t - T_{P,S}). \] (198)

This Stieltjes-type integral is, finally, evaluated numerically with the aid the quasi-trapezoidal rule given in Appendix B.

**SH-wave scattering**

Here, we illustrate the wavefield in the case of SH-wave scattering. In Figure 1 a snapshot of the incident plane wave is shown. Figure 2 represents a snapshot associated with Equation (77); Figure 3 represents a snapshot associated with Equation (78). In Figure 4 a snapshot of the total field is shown.
Figure 1. SH: snapshot of the incident plane wave.

Figure 3. SH: snapshot of the diffraction contribution.

Figure 2. SH: snapshot of the geometrical contribution.

Figure 4. SH: snapshot of the total field.
References


APPENDIX A: The Plemelj additive decomposition formulas and their application to kernel function factorization

The Plemelj decomposition formulas (Muskheishvili, 1953) deal with the additive decomposition of a function $Q = Q(p)$ of the type

$$Q(p) = Q^-(p) + Q^+(p) \quad \text{for } p \in \mathcal{L},$$

(A1)

where $\mathcal{L}$ is an oriented curve in the complex $p$-plane with the property that it divides the plane into a domain $\mathcal{D}^-$ to the left of $\mathcal{L}$ and a domain $\mathcal{D}^+$ to the right of $\mathcal{L}$. $Q(p)$ is defined and continuous on $\mathcal{L}$, $Q^-(p)$ is regular in $\mathcal{D}^-$ and $Q^+(p)$ is regular in $\mathcal{D}^+$. For our purposes, $\mathcal{L}$ extends to infinity and is the path of integration in the spectral representation of the wave constituents. It will be shown that under certain additional restrictions the decomposition is accomplished by

$$Q^-(p) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{Q(w)}{w - p} \, dw \quad \text{for } p \in \mathcal{D}^-$$

(A2)

and

$$Q^+(p) = -\frac{1}{2\pi i} \int_{\mathcal{L}} \frac{Q(w)}{w - p} \, dw \quad \text{for } p \in \mathcal{D}^+.$$  

(A3)

A sufficient condition for the right-hand sides of Equations (A2) and (A3) to exist is

$$Q(w) = O(w^{-1}) \quad \text{as } |w| \to \infty \text{ along } \mathcal{L}, \text{ with } II > I.$$  

(A4)

Under this condition, $Q^-$ is an analytic function of $p$ that is regular in $\mathcal{D}^-$ and $Q^+$ is an analytic function of $p$ that is regular in $\mathcal{D}^+$. Now for an arbitrary point $p = p_0$ on $\mathcal{L}$ define

$$Q^-(p_0) = \lim_{p \to p_0, \quad p \in \mathcal{D}^-} \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{Q(w)}{w - p} \, dw$$

(A5)

and

$$Q^+(p_0) = -\lim_{p \to p_0, \quad p \in \mathcal{D}^+} \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{Q(w)}{w - p} \, dw.$$  

(A6)

A sufficient condition for the limits on the right-hand sides of Equations (A5) and (A6) to exist is that $Q = Q(w)$ satisfies on $\mathcal{L}$ the Hölder condition

$$|Q(w_1) - Q(w_2)| < A|w_1 - w_2|^\alpha \quad \text{with } A > 0 \text{ and } \alpha > 0$$

(A7)

for all $w_1 \in \mathcal{L}$ and $w_2 \in \mathcal{L}$. Under this condition (Muskheishvili, 1953),

$$Q^-(p_0) = \frac{1}{2} Q(p_0) + \frac{1}{2\pi i} \text{PV} \int_{\mathcal{L}} \frac{Q(w)}{w - p_0} \, dw \quad \text{for } p_0 \in \mathcal{L}$$

(A8)

and

$$Q^+(p_0) = \frac{1}{2} Q(p_0) - \frac{1}{2\pi i} \text{PV} \int_{\mathcal{L}} \frac{Q(w)}{w - p} \, dw, \quad \text{for } p_0 \in \mathcal{L},$$

(A9)

where $\text{PV} \int$ denotes the Cauchy principal value of the relevant integral, and the property

$$Q^-(p_0) + Q^+(p_0) = Q(p_0) \quad \text{for } p_0 \in \mathcal{L}$$

(A10)

obviously holds.

To apply these results to the kernel factorization problem of Section 6, we take

$$Q(p) = \log[K(p)],$$

(A11)

where $K(p)$ is given by Equation (56). Through the way in which we have constructed $K(p)$, in particular its behavior as $|p| \to \infty$, $\log[K(p)]$ satisfies all conditions laid on $Q(p)$ and hence we can take

$$K^-(p) = \exp[Q^-(p)] \quad \text{for } p \in \mathcal{D}^-,$$

(A12)
with
\[ Q^-(p) = \frac{1}{2\pi i} \int_{\mathcal{L}} \log \left[ K(w) \right] \frac{dw}{w - p} \quad \text{for} \ p \in \mathcal{D}^- \]  
(A13)

and
\[ K^+(p) = \exp \left[ Q^+(p) \right] \quad \text{for} \ p \in \mathcal{D}^+, \]  
(A14)

with
\[ Q^+(p) = -\frac{1}{2\pi i} \int_{\mathcal{L}} \log \left[ K(w) \right] \frac{dw}{w - p} \quad \text{for} \ p \in \mathcal{D}^+. \]  
(A15)

In view of Equations (A8) and (A9) we have
\[ Q^-(p_0) = \frac{1}{2} \log[K(p_0)] + \frac{1}{2\pi i} \text{PV} \int_{\mathcal{L}} \log \left[ \frac{K(w)}{w - p_0} \right] \frac{dw}{w - p_0} \quad \text{for} \ p_0 \in \mathcal{L} \]  
(A16)

and
\[ Q^+(p_0) = \frac{1}{2} \log[K(p_0)] - \frac{1}{2\pi i} \text{PV} \int_{\mathcal{L}} \log \left[ \frac{K(w)}{w - p_0} \right] \frac{dw}{w - p_0} \quad \text{for} \ p_0 \in \mathcal{L}, \]  
(A17)

from which it follows that
\[ K^-(p_0)K^+(p_0) = K(p_0) \quad \text{for} \ p_0 \in \mathcal{L}. \]  
(A18)

Since through Equation (56) the kernel function \( K = K(p) \) is defined everywhere in the complex \( p \)-plane cut along the branch cuts \( \{ p \in \mathbb{C}; \infty/p < |\text{Re}(p)| < 1/c_0, \text{Im}(p) = 0 \} \), the right-hand sides of Equations (A13) and (A15) can be transformed into expressions that are more amenable to numerical evaluation. To this end, the path of integration \( \mathcal{L} \) is, in the expression for \( Q^- \), supplemented by a semi-circular arc of arbitrarily large radius in \( \mathcal{D}^+ \) and in the expression for \( Q^+ \) by a semi-circular arc of arbitrarily large radius in \( \mathcal{D}^- \). In view of Jordan’s lemma, the contribution from these circular arcs vanishes in the limit as their radius goes to infinity. Subsequently, Cauchy’s theorem is applied to the domain in between the resulting closed contours and a loop around the branch cut in \( \mathcal{D}^+ \) for \( Q^- \) and a loop around the branch cut in \( \mathcal{D}^- \) for \( Q^+ \). Taking into account that along these branch cuts \( \gamma_p \) is imaginary and \( \gamma_S \) is real, we arrive at the following expressions:
\[ Q^-(p) = \frac{1}{\pi} \int_{1/c_0}^{1/c_p} \arctan \left[ \frac{w^2 - 1/c_p^2}{(w^2 - 1/c_0^2) w - p} \right] \frac{dw}{w - p} \quad \text{for} \ \text{Re}(p) < 1/c_p \]  
(A19)

and
\[ Q^+(p) = \frac{1}{\pi} \int_{1/c_p}^{1/c_0} \arctan \left[ \frac{w^2 - 1/c_0^2}{(w^2 - 1/c_p^2) w + p} \right] \frac{dw}{w + p} \quad \text{for} \ \text{Re}(p) > -1/c_p, \]  
(A20)

where for the last result a change of the variable of integration into its opposite has been carried out. From the expressions it follows that \( Q^-(p) = Q^+(p) \) for all \( p \).

To circumvent possible difficulties in the numerical evaluation of the right-hand sides of Equations (A19) and (A20) due to the occurrence of an algebraic square root behavior near the end points of the integration interval, the variable of integration is replaced by
\[ w = \left[ (1/c_p^2) \cos^2(\psi) + (1/c_0^2) \sin^2(\psi) \right]^{1/2} \quad \text{with} \ 0 < \psi < \pi/2, \]  
(A21)

through which
\[ (w^2 - 1/c_p^2)^{1/2} = (1/c_0^2 - 1/c_p^2)^{1/2} \sin(\psi), \]  
(A22)
\[ (1/c_0^2 - w^2)^{1/2} = (1/c_0^2 - 1/c_p^2)^{1/2} \cos(\psi), \]  
(A23)

while the Jacobian of the transformation is
\[ \frac{\partial w}{\partial \psi} = \frac{(1/c_0^2 - 1/c_p^2) \cos(\psi) \sin(\psi)}{[(1/c_p^2) \cos^2(\psi) + (1/c_0^2) \sin^2(\psi)]^{1/2}}. \]  
(A24)

The resulting expressions for \( K^-(p) \) and \( K^+(p) \) are used in the main text.
APPENDIX B: Numerical evaluation of Stieltjes-type integral

In this appendix, the quasi-trapezoidal rule algorithm for the numerical evaluation of Stieltjes-type integral will be presented. Let the relevant integrals be

\[ I = \int_{t_1}^{t_2} f(t) \, d[g(t)]. \]  

(B1)

With linear interpolation of \( f \) and \( g \) on the interval \([t_1, t_2]\), i.e.,

\[ f(t) \simeq f(t_1) \frac{t - t_1}{t_2 - t_1} + f(t_2) \frac{t_2 - t}{t_2 - t_1}, \]  

(B2)

and

\[ g(t) \simeq g(t_1) \frac{t - t_1}{t_2 - t_1} + g(t_2) \frac{t_2 - t}{t_2 - t_1}, \]  

(B3)

the expression for \( I \) is found to be

\[ I \simeq \frac{1}{3} [f(t_1) + f(t_2)] [g(t_2) - g(t_1)]. \]  

(B4)

This quadrature formula is used in the main text.
3D multi-valued traveltime and amplitude maps

Lan Wang & Norman Bleistein

ABSTRACT

Ray-theoretic modeling requires accurate amplitude as well as phase both for forward modeling and Kirchhoff inversion, among other applications. There are no analytical solutions to the ray equations in realistic earth models, thus, we must use numerical solutions to solve problems of interest. For three dimensional applications, it is a challenge to develop numerical modeling codes that require reasonable cpu time while achieving sufficient amplitude accuracy to be useful in applications.

For the case of linear sloth (slowness squared or inverse wavespeed squared), analytical solutions of the ray equations do exist, leading to a combined numerical analytical technique. In this method, the physical model is decomposed into tetrahedral blocks of sufficiently small size to allow for the linear sloth approximation to be valid in each. Analytical solutions in each tetrahedron are then pieced together to provide global solutions. The ray tracing with this method is relatively fast. However, the wavespeed model generated by this technique is not sufficiently smooth to produce accurate amplitudes, numerically. Recent attempts to further smooth the physical model defeat the advantage of speed of the algorithm because the smoothness conditions across the faces of the tetrahedra generate a coupled system of equations of a size proportional to the number of tetrahedra in the global physical model. This is not practical in 3D. Thus, we conclude that a standard smoothed physical model on a Cartesian grid is likely to lead to a computer code of competitive cpu speed, when amplitude accuracy—dynamics—is of as much concern as traveltime accuracy—kinematics.

In either case, we use a wavefront construction technique, in which the size of triangular plates connecting three nearby rays on the isochron (surface of constant traveltime) is used as an indicator of adequate density of rays. When the criteria for density of rays are violated, data at new points on the wavefront are interpolated into the family of rays and the wavefront construction continues. In this manner, the method does not require excessive density of rays at small traveltimes in order to maintain adequate density of rays at larger traveltimes. The technique allows for multi-pathing (caustics) and for amplitude propagation along each of the branches of the wavefront.

Applications of the modeling technique are shown.

Key words: amplitude, dynamic ray tracing, analytic ray tracing, wavefront construction

Introduction

In this report, we address the problem of accurate and efficient determination of multi-valued 3D maps for amplitude as well as traveltimes or any other ray-related variables throughout the target zone from any shot and receiver position. The current interest in 3D seismic imaging has considerably increased the importance of ray tracing methods in wave field computations. Among seismic modeling methods, ray tracing methods provide a reasonable compromise between ac-
accuracy and computational efficiency. For the computation of traveltimes, various methods have been described. Among those, the finite-differencing (FD) method, i.e., FD-solvers of the eikonal equation, has recently become a popular method for calculation of "first arrival" traveltimes (Vidale, 1988). However, this method suffers from the disadvantages that it is restricted to the computation of first arrivals only and it produces unreliable amplitudes. Both are severe disadvantages for Kicchoff-type algorithms, such as the Bleistein/Cohen inversion (Bleistein et al., 1987; Bleistein et al., 1996), where the calculation of amplitudes is necessary to determine the weighting factor in inhomogeneous media. Furthermore, in complex media, such as near salt domes and in sub-salt regions, later arrival traveltimes should be considered to obtain better image quality. Amplitudes can be used, among other things, to find most energetic arrivals.

Simultaneous computation of traveltimes and amplitudes is possible by dynamic ray tracing (DRT). It provides accurate multiple arrivals, amplitude and phase. Estimation of these ray data can be carried out either by numerical solution of ray tracing equations in general smooth grid-based models or by piecewise analytic solutions for certain simple velocity functions in tetrahedral models. Among the choices of ray tracing procedures, the simplest and fastest solution of the ray tracing system is usually based on its analytic solution, wherever the complexity of the model allows one. This is usually referred to as analytic ray tracing or cell ray tracing. Generally, the whole medium is divided into suitable cells (usually tetrahedra in 3D), in which the velocity can be approximated by simple functions that permit analytic ray solutions. The ray in the whole model is then obtained as a chain of analytically computed segments. The analytic ray tracing is usually performed for models in which either the velocity, \( v(x_i) \), or \( 1/v(x_i) \), or \( 1/v^2(x_i) \), is a linear function of Cartesian coordinates. The simplest analytic solution for inhomogeneous medium is the one for constant gradient of squared slowness, also referred as linear sloth media (Červený, 1987; Meng & Bleistein, 1997). However, this assumption leads to tetrahedral cells with artificial second-order discontinuities at their interfaces. As a consequence, this approach produces unreliable amplitude coefficients across the internal boundaries. Körnig (1995) proposed a method using quadratic sloth. In this approach, the squared slowness and its gradient with respect to spatial variables are continuous across each cell boundary. The analytic solutions for such a velocity function are determined by using Laplace transform. Also, the computation of amplitude can be largely simplified by calculating the ray Jacobian directly from the analytical ray equations. However, the problem of determining the cell constants in quadratic sloth is rather difficult. The model design leads to a huge matrix inversion problem, and is impractical in 3D. Only 2D implementation of the traveltine computation was carried out by Körnig (1995). We have concluded that the analytic approach in tetrahedral cells does not likely offer efficient algorithms in dynamic applications.

Traditionally, numerical DRT is performed by shooting a fan of rays from the source and extrapolating traveltimes and amplitude away from the rays into their nearby regions (Červený, 1987; Virieux & Farra, 1991; Sun & Biondi, 1995). The main disadvantage of the conventional shooting method is the lack of control of ray density in the search fan. Therefore, it is hard to reach a favorable compromise between efficiency and reliability, especially in complex 3D models. It also produces shadow zones in areas of large velocity contrasts. The wavefront construction traveltine computation method (Vinje et al., 1996) offers a solution to this problem by dynamically adding rays as needed. In this method, rays are maintained by a triangular network, and are traced stepwise in traveltine through the model. The wavefronts are then obtained automatically as a by-product of the ray tracing. In this report, the idea of wavefront construction is applied to 3D complex models for estimation of both traveltine and amplitude coefficients. The dynamic interpolation of new rays assures that the wavefront is equipped with sufficient ray density at each computational step. Linear interpolation of traveltine with respect to the simulated wavefronts and linear interpolation of amplitude in terms of tube cross sectional area are performed at grid points that fall into the subvolume formed by every two successive wavefronts. A grid point can be passed by different sequences of wavefronts and, thereby, multi-valued arrivals can be detected and recorded. In this manner, all the grid points in the model are equipped with accurate—perhaps multi-valued—traveltimes and amplitudes.

In the following sections, we first discuss the possibility of applying analytic solutions in tetrahedral models for amplitude estimation. Thereafter, we address some important issues in numerical DRT such as interpolation of new rays and estimation of parameters at grid points. We also propose a smooth gridded model representation for the purpose of computational efficiency. Finally, we show results of applying this method to different velocity models.

**Dynamic ray tracing**

This section is a brief review of the dynamic ray tracing theory, based on Červený (1987), (1995). We begin by introducing two coordinate systems involved in
DRT — the ray-centered coordinate system and the ray parameter coordinate system.

The ray-centered coordinate system, denoted by \((q_1, q_2, q_3)\), is a curvilinear orthogonal coordinate system associated with any selected ray \(\Omega\) (see Fig 1). One coordinate, say \(q_3\), corresponds to any monotonic parameter along the ray, such as the arc length \(s\), the traveltime \(\tau\) or the parameter \(\sigma\), with \(dq = ds/\nu^2\). Here, we take \(q_3 = \tau\), the traveltime of ray \(\Omega\) away from the source. Thus, the traveltime, itself, is one of the coordinate axes in the ray-centered coordinate system. Coordinates \(q_1\) and \(q_2\) form a 2D Cartesian coordinate system in the plane \(\Sigma\) perpendicular to \(\Omega\) at \(q_3 = \tau\), with the origin at \(\Omega\). The vector basis of the ray-centered coordinate system connected with \(\Omega\) is formed at any arbitrary point \(q_3 = \tau\) of ray \(\Omega\) by a right-handed triplet of unit vectors \(\hat{e}_1(\tau), \hat{e}_2(\tau), \hat{e}_3(\tau)\), as shown in Fig 1. Unit vectors \(\hat{e}_3(\tau)\) can also be viewed as polarization vectors for isotropic media. The unit vector \(\hat{e}_3\) determines the direction of the displacement vector of \(P\) waves, which is always linearly polarized. Especially important are unit vectors \(\hat{e}_1, \hat{e}_2\), since they determine the polarization of \(S\) waves, when we are dealing with vector solutions of the elastic wave equation.

The ray parameter coordinates, \((\gamma_1, \gamma_2, \gamma_3)\), are defined as following: \(\gamma_1\) and \(\gamma_2\) are the ray parameters that specify the ray, usually they are either the take-off angles or the slowness vector components at the source; \(\gamma_3\) is any monotonic parameter along the ray, \(s, \tau\) or \(\sigma\). The Jacobian \(J\) of transformation from ray coordinates, \((\gamma_1, \gamma_2, \gamma_3)\), to the general Cartesian coordinates, \((x_1, x_2, x_3)\), is an important factor in computation of the ray amplitude (Bleistein, 1984). The ray amplitude has the following form,

\[
A(x, x_0) = \frac{\text{const}}{\sqrt{|J|}},
\]

with,

\[
J = \det \left[ \frac{\partial(x_1, x_2, x_3)}{\partial(\gamma_1, \gamma_2, \gamma_3)} \right].
\]

Here, \(A(x, x_0)\) is the amplitude at \(x = (x_1, x_2, x_3)\) corresponding to the source \(x_0 = (x_{10}, x_{20}, x_{30})\); \(J\) is the Jacobian; and the constant is determined by the choice of \((\gamma_1, \gamma_2, \gamma_3)\).

The rays are defined as the characteristics of the eikonal equation. That is, by transforming the eikonal equation to the following six ray equation by using the method of characteristic, (Bleistein, 1984)

\[
\frac{dx_i}{d\tau} = v^2 p_i,
\]

\[
\frac{dp_i}{d\tau} = -\frac{1}{v} \frac{\partial v}{\partial x_i}, \quad i = 1, 2, 3.
\]

Here, \(x_i(\tau)\) denotes the coordinates of position along the ray, \(p_i(\tau)\) denotes the components of the slowness vector, \(\tau\) denotes the traveltime along the ray, and \(v(x_i)\) denotes the velocity. System (3) is often referred as the kinematic ray tracing (KRT) system.

Differentiating the KRT system (3) with respect to the ray coordinates \(\gamma_i\) and applying Taylor approximation up to second-order in \(\gamma_i\) will generate the dynamic ray tracing system. The DRT system can be expressed in many forms and in various coordinate systems. The simplest form of the DRT system is obtained in ray-centered coordinates connected with the ray \(\Omega\):

\[
\frac{dQ_i}{d\tau} = v^2 P_i,
\]

\[
\frac{dP_i}{d\tau} = -\frac{1}{v} V Q_i,
\]

where \(Q, P\) and \(V\) are \(2 \times 2\) matrices defined as

\[
Q_{ij} = \frac{\partial q_j}{\partial \gamma_i}, \quad i, j = 1, 2,
\]

\[
P_{ij} = \frac{\partial p_j}{\partial \gamma_i}, \quad i, j = 1, 2,
\]

\[
V_{ij} = \left. \frac{\partial^2 V(q_1, q_2, S)}{\partial q_i \partial q_j} \right|_{q_1=q_2=0} = H_{kl} v_{kl} H_{ij}, \quad i, j = 1, 2,
\]

\[
H_{kl} = i_k \cdot \hat{e}_l.
\]

Here, \(i_k\) are the basis vectors in general Cartesian coordinates \((x_1, x_2, x_3)\) and \(H\) is the transform matrix from \((q_1, q_2, q_3)\) to \((x_1, x_2, x_3)\), its element \(H_{kl}\) represents the
kth Cartesian component of basis vector \( \vec{e}_i \). Elements \( V_{ij} \) of the matrix \( V \) are the second derivatives of velocity \( \dot{v} \) with respect to \( q_i \), and equivalent to the second derivatives with respect to \( x_i \) under transformation \( H \). Therefore, dynamic ray tracing (4) requires continuity of the velocity field up to second derivatives.\(^*\) \( Q \) is a transformation matrix from the ray parameters \( \gamma_1, \gamma_2 \) to the ray-centered coordinate \( q_1, q_2 \). Its determinant measures \( \gamma \) Jacobian (2), and is also called the geometrical spreading factor. \( P \) is a transformation matrix from the ray parameters \( \gamma_1, \gamma_2 \) to the slowness vector component in the ray-centered coordinate system. The quantity \( P \) may also be defined by

\[
P = \frac{KQ}{v}
\]

where \( K \) is the wavefront curvature. The KRT system (3) computes the first derivatives of the traveltime field, while the DRT system (4) relates the second derivatives by the relationship,

\[
M(\tau) = PQ^{-1}.
\]

where \( M \) is a 2x2 matrix of second derivatives of the traveltime field with respect to the ray-centered coordinate \( q_1, q_2 \), \( M_{ij} = \partial^2 \tau / \partial q_i \partial q_j \).

At caustic points, the ray Jacobian (2) or the determinant of matrix \( Q \) vanishes. In 3D structure, there are two kinds of caustic points depending on the range of the matrix. The ray tube may shrink to a caustic surface (envelope of rays) which is perpendicular to the direction of propagation (a caustic point of the first order); or the ray tube may shrink to a point (a caustic point of the second order). In passing through the caustic point of the first order, the ray Jacobian \( J \) changes sign and the argument of \( J^{1/2} \) takes on the phase term \( \pm \pi / 2 \). Similarly, in passing through the caustic point of the second order, the phase term is \( \pm \pi \). The phase shift due to caustics is cumulative. If we pass through several caustic points along the ray, the total phase shift is the sum of the individual phase shifts, this is often referred as the KMAH amplitude.

Analytic ray tracing

It is known that the realistic velocity field of interest is often rather complicated and can hardly allow a general analytic solution of the ray tracing system. However, analytic ray tracing plays an important role in wave field computation. This is due, in part, because the analytic solutions are valuable in the cell approach, in which the whole model is subdivided into a set of tetrahedral cells with simple velocity functions within cells. The models allowing analytic solutions are usually those with either the velocity, or slowness, or squared slowness being a linear function of Cartesian coordinates. As we have mentioned before, this constraint does not provide enough smoothness for amplitude estimation. In this section, we discuss the analytic solutions for quadratic slot media.

The quadratic slot distribution, denoted by \( q \), is defined as a quadratic function in space,

\[
q(x_1, x_2, x_3) = \frac{1}{v^2(x_1, x_2, x_3)}
\]

\[
= A + 2B_i x_i + C_{ij} x_i x_j.
\]

The analytical solutions for this distribution were found by Könnig (1995) using the Laplace transform. In the Laplace domain, the ray coordinates, \( X_i(s) \), are ratios of polynomials of sixth and seventh order, respectively, in \( s \); this is the Laplace variable corresponding to \( \sigma \), the ray tracing integral variable with \( ds = v^2 dr \). The expressions for the ray trajectories, \( x_i(\sigma) \), can be obtained explicitly by inverse Laplace transform of \( X_i(s) \) using partial fraction expansions. Depending on the distribution of eigenvalues of \( C_{ij} \), the solutions of \( x_i(\sigma) \) are generalized into seven different forms. For each case, the ray trajectories are in the general form

\[
x_i(\sigma) = w_{ik} f_k(\sigma), \quad i = 1, 2, 3, \quad k = 1, 2, \ldots, 7.
\]

Here, the \( w_{ik} \)'s are the weighting factors, which are functions of \( x_{i0}, p_{i0}, B_j \) and \( C_{ij} \) with \( x_{i0} \) and \( p_{i0} \) being the initial position and slowness components; \( f_k(\sigma) \) are the basis functions corresponding to the inverse transform of the partial fraction expansions. One of them is unity, the others are either low-order polynomials in \( \sigma \), or trigonometric, or hyperbolic function.

Now we propose an alternative to Könnig's (1995) approach to calculate the amplitudes along rays. Notice that the \( f_k(\sigma) \)'s in (9) are functions depending only on \( \sigma \), and \( w_{ik} \)'s can be expressed in terms of \( x_{i0}, p_{i0} \), and constants \( B_j \) and \( C_{ij} \). Therefore, if we choose \( \gamma_1 = p_{10}, \gamma_2 = p_{20} \) and \( \gamma_3 = \sigma \) in (2), the ray Jacobian \( J \) can be calculated analytically,

\[
J = \det \begin{bmatrix}
\frac{\partial w_{ik}}{\partial p_{i0}} f_k(\sigma) \\
\frac{\partial w_{ik}}{\partial p_{20}} f_k(\sigma) \\
\vdots \\
w_{ik} g_k(\sigma)
\end{bmatrix}
\]

with summation over the repeated index, \( k \), and

\(^*\) It is the numerical sampling across tetrahedral interfaces that causes amplitude instability when the second derivative is not continuous.
\[ j_k(\sigma) = \begin{cases} f_k(\sigma), & i = 1, 2 \\ g_k(\sigma), & i = 3. \end{cases} \] (11)

The coefficients \( j_{ijk} \) can be expressed for all the seven cases in (9). Notice that expression (10) has taken the place of the DRT system (4), and it is less computationally costly than solving the eight integral equations in small time steps.

This approach of making quadratic sloth assumption in tetrahedral models has eliminated the smoothing procedure across the internal interfaces. However, the constants \( A, B_i \) and \( C_{ij} \) in (8) are usually not known in advance from the given physical model. They have to be determined from the discrete model. The assumption of quadratic sloth is equivalent to the continuity of both \( q \) and the gradient of \( q \) across the internal cell faces. Therefore, the 10 constants \( A, B_i \) and \( C_{ij} \) in one tetrahedron cannot be determined by the velocity values at its four apices only, but also depend on the values in the neighboring cells. Such a model design problem for all the tetrahedral cells leads to a huge inverse problem. If the whole model is divided into \( N \) cells, the size of the coupled system of equation is proportional to \( 10N \), making the computation very time consuming. Furthermore, this inverse system is not always solvable, or has solutions in a least squares sense, at best. This is impractical in 3D and considerably limits the applicability of this approach.

From the above discussion, we see that although the assumption of quadratic sloth in tetrahedral models provides accurate amplitudes, this assumption leads to a difficult and inefficient numerical problem for determining the cell constants. And this problem exists in all extensions to quadratic physical models, not particularly in a quadratic sloth model. We conclude that analytic ray tracing in tetrahedral models is not likely to give us an efficient module for dynamics, although it has its applications in traveltime calculations.

Wavefront construction on smooth gridded models

Here and below, we will focus on numerically solving (3) and (4) for smooth gridded models. We will apply the technique of wavefront construction (WFC) to both kinematics and dynamics.

In the wavefront construction method, a relatively sparse number of rays are shot initially. They differ from each other by the two take-off angles, and are extrapolated into the zone of interest by solving (3) and (4) numerically with appropriate initial conditions. Required accuracy of traveltime and amplitude can be approached by various standard numerical procedures, such as Runge-Kutta or predictor-corrector, for example. At any computational step, the wavefront is obtained as a by-product of the ray tracing. The wavefront is represented by triangular plates that connect every three neighboring ray endpoints on the WF. The nearby rays in 3D are then defined and organized by such a triangular mesh consisting of the internal ordering of connecting endpoints in each triangle and adjacent triangle(s) to each of its sides. The processes of checking, interpolating of new rays and estimating of grid point parameters (described below) are all performed within such a triangular network. Rays are added and the original triangle is subdivided into new triangles when certain criteria, restricting the size of the triangular plates, are violated. In this manner, the wavefront always has sufficient ray density without a priori estimation of the number of rays needed, or by imposing an excessive ray density on initiation. For complex 3D velocity models, the wavefront surface may be very complicated, folding in on itself at some parts, for example; however, no tears or holes in the interior of the surface are allowed. In this sense, the wavefront is complete. On the other hand, a grid point can be passed by different sequences of wavefronts; multi-valued arrivals can then be estimated and distinguished by their initial take-off angles.

The most attractive advantage of the WFC method is that it is more efficient than the conventional ray tracing method. In addition, WFC gives better ray coverage, especially in areas of large geometrical spreading where conventional ray tracing may give no arrivals. Furthermore, compared to FD-solvers, the WFC method is not restricted to the calculation of only first arrivals. Amplitude and other ray theoretical quantities are also available. Thus, it meets the requirements for accurate modeling of amplitude as well as phase, a requirement for inversion as opposed to migration.

Ray interpolation

The wavefront construction method is largely dependent on the procedure of interpolation of the wavefront at each step. New ray endpoints must be added along the simulated wavefront and must have the propagation direction that the ray would have had if it had been shoot from the source. This section addresses an algorithm for this procedure.

Rays diverge and the wavefronts expand through the wave-field. When new ray endpoints are needed to keep a certain ray density on the wavefront, the whole triangular network will have to be reorganized. The criterion for this interpolation can be the area of triangles, which must not exceed a pre-specified limit, and/or the angle deviation of the slowness vectors of two adjacent rays, which cannot be too large. New rays are always added
Figure 2. Interpolation of new rays and estimation of grid parameters are performed in a "ray tube". (a) Interpolation of a single new ray. The two ray endpoints and their propagation directions form two straight lines in 3D. An "approximate" center can be defined as the midpoint of the line segment that connects the two straight lines at their points of shortest distance. This approximate center, along with the two ray ends at the old WF, form a fan and a circular curve connecting the two ray end points. The new ray position is then found along the dividing direction from the approximate center, and at its intersection with the circular curve. Then the interpolated ray is traced from the old to the new WF. (b) Simple ray cell with an interior grid point. Ray data are estimated with respect to the two simulated WFs.

Figure 3. 3D wave field of a linear sloth model using WF construction method. The grey part is the shadow zone.

Figure 4. The rays (white) of linear sloth model have a parabolic shape. The grey surface is the caustic surface of the ray equations for this model.

in between the pairs of existing rays in order to meet the criteria of size and shape of triangular plates on the waveform. Fig 2-a illustrates the interpolation of a single new ray. The two ray endpoints and their propagation directions form two straight lines in 3D. An "approximate" center can be defined as the midpoint of the line segment that connects the two straight lines at their points of shortest distance. This approximate center, along with the two ray ends at the old WF, form a fan and a circular curve connecting the two ray end points. The new ray position is then found along the dividing direction from the approximate center, and at its intersection with the circular curve. Other parameters along the new ray are interpolated linearly.

There are other alternative methods of interpolating new rays, such as the parameterization of a wavefront by a third-order polynomial (Vinje et al., 1996). However, they require special treatment in the vicinity of caustic points, since only rays belonging to the same phase must be used to determine quantities of the new ray. Moreover, we do not use the curvature of the wavefront obtained from DRT for the interpolation to keep the problems of KRT and DRT separate. We have found this method very stable.
Grid interpolation

Another interpolation procedure in DRT is the estimation of ray data at the output grid points. This is our final goal of the ray tracing algorithm. We perform the grid interpolation within ray tubes, which are prism-shaped bodies bounded by three rays and the triangles that connect them on the two WFs. First the grid points falling into (or close to) each cell are found. Then, the traveltimes can be estimated at each grid point in a similar way to the interpolation procedure for new rays (see Fig 2-b). The approximate center is determined by the three rays with ray endpoints on any of the two WFs. The distances from each grid point to the approximate center and to the simulated wavefront are calculated. The traveltimes at the grid point is then recorded as \( t_0 + d/v \), with \( t_0 \) the time at the wavefront, \( d \) the distance of the grid point away from the wavefront, and \( v \) the velocity at the grid point.

The above procedure is not suitable for interpolation of amplitude, because the isochrons surface is usually not the iso-amplitude surface. Here, we apply linear interpolation for amplitudes, which is based on the assumption that the amplitudes vary only slowly; otherwise, the validity conditions of the underlying asymptotic theory would be violated. Since the ray Jacobian—the determinant of \( Q \) in (4)—is proportional to the cross sectional area of the ray tube, we interpolate the ray Jacobian linearly with respect to the triangle areas on the two wavefronts.

Another parameter that requires interpolation at every grid point is the initial shooting direction, i.e., the initial take-off angles of the ray that would reach the grid point if the ray actually had been traced. This parameter is stored in order to distinguish between arrivals because two arrivals at a grid point cannot have almost equal take-off angles.

Model representation

The smoothness of the velocity model representation is critical to the calculation of amplitudes. The integration of the DRT system (4) requires continuity of the velocity field up to the second derivatives. Many ray tracing procedures (Farra, 1990) involve a type of spline interpolation for the evaluation of velocities at arbitrary points. Spline interpolation, however, is a time consuming procedure. Here, we define the velocity model on a fine grid (about three or four grid points per shortest significant model wavelength) and pre-calculate its first and second derivatives at all grid points by finite differences of second order. For the evaluation of the velocities and their derivatives at arbitrary points we use linear interpolyation.

Figure 5. Four cross sections of the 3D SEG salt dome velocity model (subsection) at \( x = 1.5km, z = 3.5km, y = 1.5km \), and \( y = 3.5km \).
tion. For the smooth models defined on fine grids, the difference between this linear representation and a spline representation of the model is negligible.

When the considered model contains discontinuous velocities, a smoothing procedure must be applied to guarantee that the velocities vary smoothly. For the sake of computational efficiency, the interface conditions are eliminated here by using proper smoothing and a densely sampled grid model.

Examples

The first example provides a test of the accuracy of this modeling technique. The synthetic model we choose for this test is the one with constant gradient of squared slowness, i.e.,

$$\frac{1}{v^2(x)} = \frac{1}{v_0^2} + B_1x_1 + B_2x_2 + B_3x_3.$$  \hfill (12)

In such a medium, the rays have a parabolic shape; both traveltimes and amplitude field can be expressed exactly by analytical solutions for comparison purposes. Figure 3 shows the 3D wave field with the gradient constants being \((0, 0, -0.2)s^2/km^3\). The relative difference of computed traveltimes and analytic ones is less than 0.1% through the whole interest area, while the differences between computed and analytical amplitudes are no more than 1%. The grey part in Fig 3 is the shadow zone, where no rays are entered. This is due to the negative gradient constant of \(B_3\), which is equivalent to the increasing velocity with depth. In Figure 4, the grey surface is the caustic surface of the ray equations. It is the envelope of the parabolic rays.

The aim of the second example is to prove that our tracing algorithm can operate on a complex velocity model. This example is performed on a 3D SEG salt dome velocity model. This synthetic velocity model contains one complex salt dome with high velocity in the dome and low and slowly varying lateral layers outside the salt dome. Due to the RAM capacity of the computer, we extracted a subsection of the original model, which has part of the salt in the middle. The strong velocity contrast at the salt wall has violated the smoothness requirement of ray tracing, therefore we first smoothed this reference model. Figure 5 shows four cross sections of the smoothed velocity model. The grid size is \(40m \times 40m \times 40m\). Figure 6 shows some wavefronts for this model. The wavefronts expand and become more complex for the later traveltimes. However, by using the wavefront construction technique, all the wavefronts have sufficient ray density. Figure 7 presents cross sections of traveltimes maps of the above smoothed velocity model. The isochrons are spherical-like at the shallow parts, but the shape changes due to the salt in the middle depth.

Conclusions

We have demonstrated that numerically solving the ray equations on a smooth gridded model provides forward modeling that is fast enough for three-dimensional computations. Both the traveltimes and amplitudes proved to be smooth and stable in our examples. It remains to check the numerical accuracy of this technique, as compared to analytical solutions and alternative numerical methods. However, it is already known that the tetrahedral-based approach produces unacceptable amplitudes, due to the difficulty in efficiently obtaining accurate amplitudes across the internal interfaces.

The WFC procedure based on proper interpolation of new rays makes the dynamic ray tracing more efficient and results in a dense and consistent ray coverage throughout the model, even in areas of large geometrical spreading. When accurate amplitudes are required, we believe that this is a competitive method for development of ray theoretic wavefields.

References


Figure 7. Cross sections of traveltime maps of the 3D salt dome model. The source is at $x = 2\text{km}$, $y = 2\text{km}$, $z = 0\text{km}$.


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Parallel implementation of four depth-migration algorithms in a heterogeneous network

Baoniu Han

ABSTRACT

This paper covers the parallel implementation, using PVM (Parallel virtual machine) in a heterogeneous network environment, of four related depth-migration methods that are based on wavefield-extrapolation theory. Use of a master-slave topology, in which a common master code communicates with different slave processes, facilitates future development of new migration code since only the slave code needs to be changed to accommodate different algorithms. Automatic load balancing is incorporated in these codes to take full advantage of the available computing resources in a heterogeneous network. I show comparisons of efficiency tests on both shared and distributed computer architectures for prestack common-shot migration.

Key words: PVM, MPI, Wave equation, Depth migration.

Introduction

Migration algorithms that work in frequency-space and frequency-wavenumber domains parallelize naturally because frequency components are independent of each other, and hence can be processed separately. Here, I present parallel implementation of four different depth-migration methods: phase-shift-plus-interpolation (PSPI), split-step Fourier (SSF), implicit \( \omega - \varsigma \) finite-difference (FD), and Fourier finite-difference (FFD). These implementations required little effort to convert from previous sequential migration code to the parallel code. Most of the initial converting work for all four methods was done within a weekend, though debugging of these codes took another few days.

The PVM library (Geist et al., 1994), as opposed to MPI specification, is the communication library I used, the main reason for this choice being that the PVM library has more flexibility and power for computing within a heterogeneous network than does MPI, although some efficiency is sacrificed to achieve such characteristics. Murillo and Uzcategui (1997) suggested a general strategy on the parallelism of migration algorithms, and the PVM implementation here benefited from suggestions in Gouveia (1996) and Murillo (1996).

The motivation of this research is the need to compare the relative performance of different algorithms for migration applied to large volumes of 2-D poststack and prestack data. This parallel implementation turned out to be highly rewarding in that the complex Marmousi data, for example, can be migrated within half an hour compared to the nine hours required by the sequential codes. Benchmark tests performed on both shared and distributed computer architectures showed an almost linear relationship between the number of processors and speedup factors.

Though four algorithms are now parallelized, a pair of master-slave codes developed to accommodate the different algorithms into a common structure allows the ready generation of new migration code; only the downward-propagation part needs to be modified, facilitating the coding work for other algorithms.

Heterogeneous computing environment

The Center for Wave Phenomena (CWP) at Colorado School of Mines has about thirty workstations connected via an Ethernet network, forming the following heterogeneous computing environment.

1. Different hardware architectures

Among the thirty computers, 15 are PC workstations with Intel Pentium CPUs, their clock frequency ranging
from 90 Hz to 120 Hz. Nine of them are PC workstations with Intel Pentium Pro CPUs having a clock frequency of 200 Hz, and we have a six-processor SGI Power Challenge; the clock frequency of the MIPS-R8000 processor is at 75 Hz.

2. Different software environment
All the PC workstations were installed with the Linux operating system using free GNU development tools from FSF (Free Software Foundation). Although most of them have the latest Linux operating system, Slackware 3.4 with 2.0.30 kernel, some of them are still working with Slackware 3.2 (2.0.27 kernel) or 3.1 (2.0.0 kernel). The reason that we kept some of the old operating system is for the compatibility with older software that can run on only the old Linux system. As to the SGI Power Challenge, its operating system is IRIX 6.1.

3. Different network bandwidth
Most of the workstations are connected via the slow Ethernet network, with peak bandwidth at 10 Mbyte/s. Eight workstations are connected with a fast Ethernet network; it has a peak bandwidth of 100 Mbyte/s. When I test my implementation, I usually run a parallel job using 20 workstations, so my only choice is to cope with the 10 Mbyte/s slow network, which becomes a bottleneck for communication-demanding jobs.

Implementation considerations
Below is some of the key issues I have considered in the implementation of these migration codes.

• PVM versus MPI
To convert the sequential code to parallel code, I needed a communication library to adopt the master-slave topology. Two good choices are PVM (parallel virtual machine) and MPI (message passing interface) specification. Both of them are freely available from the web site of Netlib (http://www.netlib.org), with good documentation and some sample code. In my implementation, I chose PVM as the communication library based on a fair comparison of the above two systems by Geist et al. (1996).

Though generally less efficient than the MPI specification, PVM was designed at the outset to work for a heterogeneous-network computing environment. It has better portability and inter-operability performance than any other existing communication library. Specifically, the code written in PVM can be compiled and run on any hardware architecture and operating system without changing the source code; PVM jobs on different hardware architecture can communicate and exchange data with one another; and PVM supports both C and F77. Also C and Fortran code can exchange data between them even though arrays in C and Fortran have opposite storage sequences; for example, one can code a master program in C and a slave program in Fortran without any ill-effect. As to the MPI specification, though the source code written in MPI can also be ported between different architectures without any modification, it lacks the inter-operability feature: jobs on different platforms cannot exchange data with one another, and codes in C and Fortran cannot communicate with each other. The high efficiency of MPI can be achieved only in a homogeneous, distributed environment, thus sacrificing the flexibility that is necessary for heterogeneous network computing.

Furthermore, PVM contains all the necessary requirements of process control. It can start tasks, keep track of the running processes and stop them. In contrast, in the widely available MPI-1.1 standard, no function has been defined to start a parallel task within a program. Although MPI-2 will accommodate some of the process-control specification, no public-domain MPI-2 implementation has been available to date.

Though all the source codes I have implemented are written in PVM, considering the similarity in the communication functions between PVM and MPI, if desired, they could be converted to utilize the MPI functions without any difficulty.

• A common platform for different algorithms
The four related depth-migration algorithms I have implemented in both poststack and prestack shotgathering domains are PSPI (phase-shift-plus-interpolation), SSF (split-step Fourier), FD (implicit $\omega - \tau$ finite-difference), and FFD (Fourier finite-difference) methods. All of the algorithms were derived from the one-way acoustic wave equation based on downward-continuation theory, and work in the frequency-space domain. These algorithms are naturally parallelized; that is, all the frequency components are independent of one another, and hence can be processed separately. For details of these algorithms, refer to Han (1998), Clearbout (1985), Ristow and Rühl (1994), Stoffa et al. (1990), and Gazdag and Squazzero (1984).

In my implementation, I have adapted a master-slave topology. The master program distributes the frequency components to different slave tasks, and collects and stacks the final image after sending out all the data. Since the only differences among these migration algorithms are in the downward-continuation propagator, I code a common platform for all of them to make the code reusable, consequently saving implementation effort and facilitating future development of other migration algo-
rithms. The master code is identical for all the migration algorithms. In fact, at the same time one can run a master program to communicate to the slave tasks with different downward-continuation propagators. The communication part of the slave code is also the same among the different algorithms; only the core part — the propagator — differs. Changing a parallel code from algorithm A to B can be easily achieved by substituting the new propagator for the old one.

- Automatic load balancing

In migration, the total computational cost is determined by the size of the migration aperture, the frequency range of the input data, the grid spacing, the number of depth steps and the migration algorithm itself. Once the above parameters have been fixed, the total computing cost will be static and unchanged for a given hardware configuration. But as opposed to the static computing expense, the available computing resources in a heterogeneous computing environment can and do change dynamically. Besides the difference in the speed of the processor (for example, the Intel Pentium Pro 200 Hz processor is about twice as fast as the Pentium 90 Hz processor in floating-point computation.), the load of different workstations varies dynamically. Also, typically a parallel job needs to compete for the available resources with those of other users of the parallel network. In such situations, it is necessary to incorporate dynamic load balancing to take care of differences in computer speed and process load.

In parallelizing the four depth-migration algorithms, dynamic load balancing can be easily and readily put into the codes. Instead of dividing the entire data just by the available number of processors \( N \), I multiply \( N \) by a data-dependent factor \( a \) (in my codes, it is usually set to 3 or 5), then partition the data using the new number \( aN \). Smaller-size data sets are sent to different slave processors, and, once they are done, signals are sent back to the master process to request more data. In this way, with faster processors processing more data than slower ones, the data load between different workstations will be balanced. To reduce the network loading, all the intermediate migrated images are kept in the slave processors instead of sending them back to the master program. When all the frequency components have been processed, the master program will collect the partial images from the slave processors and stack them together to get the final image. Hence, the parallel implementation here will be generally computation, as opposed to communication, intensive, as is needed in a low-bandwidth network.

![Figure 1. Marmousi section migrated with a parallelized phase-shift-plus-interpolation algorithm.](image)

**Benchmarking comparison**

The performance measurements, done here on both shared and distributed computer structures, involve prestack migration on the Marmousi data (Versteeg & Grau, 1990).

**Description of testing model and the measuring procedure.**

The Marmousi data have been used extensively in the past decade to test the accuracy of prestack depth-migration algorithms and velocity analysis techniques on complex geological structures. It contains 240, 96-fold shot gathers, each trace with 726 samples and a sampling interval of 4 ms. The velocity profile used for migration is 375 by 369 samples, and the grid spacings are 25 m laterally and 8 m vertically. For the migrated Marmousi section shown in Figure 1, the migrated data were constructed with 60 frequency components uniformly incremented between 15 Hz and 35 Hz.

To test the performance of the parallel codes, instead of migrating the entire data set, I migrated only 15 shot gathers and saved the computing time for each shot gather to a file. After migration, the latter ten of the computing times were averaged to get a mean value. A speedup factor was computed by dividing the sequential time for a shot gather by the averaged parallel time. To test the relationship between the total computing time and the data-block size sent each time to the processors, I ran tests with, respectively, 1, 3, and 5 frequency components sent each time from the master process to the slave tasks.

The above tests have been repeated for all four algorithms with similar speedup results, showing that these implementations are independent of the algorithms.
Performance comparison for the PC and SGI Power Challenge

To see what speedup we can get from these codes, the performance tests were done on two homogeneous platforms: a PC-based network with 8 Pentium Pro workstations and the 6-processor SGI Power Challenge.

The speedups achieved on the two platforms are shown in Figures 2 and 3. The speedup factors increase linearly on both testing platforms, although they still could not catch up with the ideal speedup factors. The difference between the ideal expectation and the test results is due to the time spent on communication through the network.

The communication time takes only a small portion of the total computing time, so the total computing time is almost inversely proportional to the number of processors, resulting in the somewhat linear increase of speedup factors in both figures. Though I tried different data sizes sent each time from the master to slaves, their results almost coincide with one another, again because these implementations are computation-intensive rather than communication-intensive.

In addition to the speedup tests here, I also performed load balancing tests for a heterogeneous network that include five different workstations: Pentium 90, Pentium 120, Pentium Pro 200, SGI Power Challenge, and SGI Indigo 2. For the codes without load balancing, each slave workstation works on the same amount of data. Using the SSF algorithm, for migration of a single shot gather of the Marmousi data, the code without loading balancing took about 68 as compared to the 39 seconds achieved by the code with load balancing. Without load balancing, more than 70% of additional time was wasted in waiting for the slower processors to finish their jobs.

Conclusions

The parallel codes appear to be stable (I ran PVM jobs on 20 workstations for 9 hours with no problem) and give satisfactory speedup factors. Using 60 frequency components (15 – 35 Hz) in a 16-PC heterogeneous network, the migration of the multi-offset Marmousi data took from half an hour for the 650 FD algorithm to two hours for the PSPI algorithm — the most expensive of the four approaches considered here. Later, when more PC workstations are connected via the fast Ethernet network, I will implement the MPI specification in these codes to see what, if any, improvement in efficiency is gained by using MPI.

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The influence of hardware on software optimization

Alberto Villarreal

ABSTRACT

One of the key factors determining computer performance is the degree to which the hardware can take advantage of Instruction Level Parallelism (ILP) that software often exhibits. ILP refers to the parallelism between individual operations in a program, and it is exploited by hardware elements called pipelines that are able to execute multiple instructions concurrently. Research in this area has been reported extensively in the literature; however, much of it contains technical language and is not addressed to the final user. In this document, I describe in simple terms how modern processors are able to deliver high performance, and how ILP in programs can be exploited using pipelining and other hardware elements (cache memories and branch predictors) to increase the delivered performance of a processor.

Key words: software optimization, pipelining, cache memory, branch prediction

Introduction

When running computer programs, one computer frequently will deliver better results than another, even if both computers have processors with the same clock speed, for example, a 75 MHz Pentium processor compared to a 75 MHz MIPS R6000 processor. Also, incrementing the clock speed of a processor may not increment the performance of the program proportionately. In some cases, it may not increment at all. The main reason for this is that processors commonly used in workstations today incorporate supercomputer technology to a certain degree, such as pipelining and other hardware-implemented parallel techniques. Note that here I am not referring to the coarse-grained parallelism between large sets of operations, such as subprograms, which is exploited by multiprocessor systems. Rather, I am referring to a fine-grain one called Instruction level parallelism (ILP) which refers to the parallelism between individual operations that can be exploited by uniprocessor systems.

If the software does not take advantage of ILP processing, the delivered performance will be less than optimal (Franklin, 1993). On the other hand, attention paid to a few key optimization procedures can pay off with important increases in performance. Recent studies have confirmed that a large amount of ILP exists in ordinary numerical programs. This parallelism can be exploited by the hardware (Franklin, 1993), (Wall, 1993). This means that no matter how much parallelism is exploited by coarse-grain parallel processors, a substantial amount of parallelism will still remain to be exploited at the instruction level. Therefore, irrespective of the speedup given by coarse-grain parallelism, ILP processing can increase that speedup substantially.

Hardware is not the only issue involved in software optimization. In the past few years, optimizing compilers have become an essential component of modern high-performance computer systems. Besides generating executable code, these compilers analyze the user's source code and apply various transformations to improve its performance by exploiting ILP present in the software (Bacon et al., 1994). However, these compilers are not always effective when analyzing complex code (Saavedra & Smith, 1992). In particular, some widely used compilers, such as the freely available GNU's gcc and g77, have limited code analysis and code transformation capabilities.

On the other hand, modifying existing codes (or designing new ones) to exploit ILP, independently of the compiler, presents the following advantages:

- The performance of the software will be more pre-
dictable in terms of hardware parameters, such as depth of the pipelines, amount of cache memory available, etc.

- The program will be able to take advantage of hardware optimization mechanisms, such as hardware-based branch prediction and out-of-order execution, irrespective of the compiler being used; Thus, the performance of non-expensive and broadly available computer systems using public domain software, such as Pentium-based systems currently installed at CWP, may be able to increase substantially and to approach that of more expensive systems.

- If an efficient optimizing compiler is available, it will be able to extract the best performance of an already optimized code. This would be achieved without performing extensive and time-consuming analysis and transformations that could affect the floating point accuracy of the program.

- Obtaining the best performance in a uniprocessor system is critical when the code is to take advantage of coarse-grain multiprocessor parallelism. If the program is to be executed in parallel in a heterogeneous environment, where different compilers will generate the code for each architecture present in the distributed system, caution must be taken to insure that the part of the code at each node is running fast in order not to decrease the overall performance of the distributed system.

Understanding something about crucial hardware and software elements built into modern computer systems is extremely useful to increasing the delivered performance of computation-intensive programs.

In this document I describe in simple terms some of the hardware elements present in modern processors, such as pipelines, branch predictors and cache memories, and how those elements can take advantage of the ILP present in software to reduce the execution time. I also discuss how taking these elements into account when coding can often increase the speed of a program to an important fraction of the processor's peak speed.

A few words on modern computer architecture

High performance in modern computer systems is achieved by incrementing the speed when performing arithmetic operations, as well as incrementing the speed of the memory delivering data for those arithmetic operations to be completed. In this section I briefly describe how modern processors improve those two parameters using pipelines, cache memories, and branch prediction mechanisms. These processors are designed to exploit the parallelism that programs exhibit at the instruction level. As an example of this parallelism, let us look at the following code fragment:

\[
\begin{align*}
i_1 &= \text{array}[j]; \\
i_2 &= i_3 \times N; \\
\text{array}[j+1] &= i_3;
\end{align*}
\]

This code fragment consists of three instructions that can be executed concurrently, because they have no data dependencies, i.e., they do not depend on each other's results. The following code, on the other hand, does have data dependencies and cannot be executed in parallel:

\[
\begin{align*}
i_1 &= \text{array}[j]; \\
i_2 &= i_1 \times N; \\
\text{array}[i_2] &= i_3;
\end{align*}
\]

Much of the work in scalar optimization is aimed at eliminating or reducing data dependencies in programs in order to increase the ILP and permit the hardware to efficiently execute the instructions.

Pipelines, together with cache memories and branch prediction systems, are some of the most important components of superscalar processors. A superscalar processor is one that can issue multiple independent instructions in the same cycle. Superscalar processing has been acclaimed as "vector processing for scalar programs," and speedup factors between 4 and 10, depending on the amount of ILP present in the test codes, have been reported when using this technique (Wall, 1993). Descriptions of the superscalar architecture of some common processors can be found in Zaqha (1996), Hunt (1995), and the Pentium Pro and Pentium II pages on Intel's site*

Note, however, that other hardware issues such as networking, hard disk speed, graphical accelerators, etc., also have important influence on the delivered performance of the system. Here I will only talk about some in-processor components that affect performance.

Pipelines

Pipelining is the most common implementation technique used today to increase the performance of processors. Pipelines in modern processors replace sequential arithmetic units present in old hardware. Their purpose is to reduce the number of processor ticks, or clock cycles, needed to execute an individual instruction, by taking advantage of ILP found in everyday programs.

The idea behind pipelining is that if the hardware unit performing the arithmetic operations (the pipeline)

* http://www.intel.com
Sequential mode: 5 ticks/instr.

Pipelined mode: 1 ticks/instr.

**Figure 1.** Flow of instructions through a pipeline. In this 5-stage pipeline, 1 instruction takes 5 ticks to complete, but a new instruction is started every tick, and 1 instruction is completed in each tick.

is segmented in several stages, and each stage executes pieces of an operation concurrently and independently of one another, then it is possible to start another operation just after the previous one completes the first stage of the pipeline, providing operations are of the same type (Dowd, 1993). A pipeline can be thought of as an assembly line where multiple actions (instructions, in this case) are performed concurrently.

Figure (1) shows schematically how a 5-stage floating point pipeline works. Multiplication operations enter the pipeline and at every cycle, the pipeline rolls them over until, after 5 cycles, the first result is popped out. After that, because the pipeline stages are independent of one another, up to five operations can be in the pipeline at a time, and the pipeline will produce a result every clock cycle as long as input operations exist. This is a powerful mechanism: where before it would have taken five ticks for a single result to come out, now the pipeline generates as much as one result every tick.

As an example of how the pipeline affects the peak performance of processors, let us look at a simplified representation of the structure of a typical 75 Mhz processor as shown in Figure (2).

This processor contains 2 pipelined floating point units and 2 pipelined integer units. When working in fully pipelined mode, each of the floating point units can perform up to 1 floating point operation per tick, as described in Figure (1). If both floating point units are working simultaneously, then the peak floating point performance of this processor is 2 flops/tick, which at a clock speed of 75 MHz, corresponds to a peak performance of 150 Mflops.

In general, a pipeline with $S$ stages working in fully pipelined mode can execute $N$ operations in

$$T_p = S + (N - 1)$$

(1)
ticks. And we can observe that as the number of operations increases, the speed of the pipeline approaches the one operation per tick rate.

However, if the pipeline is working in sequential mode, i.e., starting the next operation once the previous one has been fully completed, then the same $N$ operations will be executed in

$$T_s = NS$$

(2)
ticks. From this we see that the ideal pipeline speedup:

$$T_s/T_p$$

(3)
equals the number of stages $S$ when $N$ is large, assuming there is no overhead involved in the pipeline mechanism).
This explains why processors with longer pipelines increment the theoretical (peak) performance. Note, however, that the longer the pipeline, the bigger the penalty when the pipeline fails to work in fully pipelined mode, so another solution to keep improving the performance, without increasing the clock speed or the number of stages, is to add extra pipeline units to the processor, as in the case of the processor shown in Figure (2), which has 2 floating point pipelines and 2 integer pipelines.

Every instruction in a processor, floating point addition, subtraction, and multiplication, as well as integer operations and memory references, can be pipelined. However, in order for this technique to work, the code must be as close as possible to a stream of operations with no branches, i.e., no ifs, go to's, subroutine calls, etc. The reason is that every time a branch is encountered the pipeline will stop the flow of operations, reset, and start again with a new flow of operations, and this will result in a big penalty in processing time.

Branch prediction and speculative execution

Unfortunately, most programs make heavy use of branches, and both the compiler and the hardware must have mechanisms to cope with control transfers in the code.

How does the pipeline know when a branch is coming? There are several mechanisms for this. In some cases an optimizing compiler will try to predict which way it believes a branch is likely to go, and it will generate appropriate object code trying to use as much of the pipeline capabilities as it can. In other cases, hardware mechanisms working at execution time permit the CPU to guess where a branch is going to appear based on when other branches have occurred (Dowd, 1993). In either case, the more we simplify the branches in the code, the more efficient these mechanisms will work, and the more efficient the pipeline will behave.

The worst scenario (unfortunately a very common one when developing complex software) is when every iteration in a loop depends on a branch decision (an if inside a loop). If this scenario cannot be transformed by the compiler or the hardware, the processor will be working in sequential mode at a very low speed. Fortunately, recent research in algorithms show that most of these kinds of inefficient structures can be transformed into others that can be fully, or at least partially, pipelined. For example consider the following commonly used structure, known as invariant test:

```
DO I=1,K
  IF (N .EQ. 0) THEN
    A(I) = A(I) + B(I) * C
  ELSE
    A(I) = 0.
  ENDF
ENDDO
```

"Invariant" in this case means that the result of the test will not depend on the result of the operations in the loop. The value of N will be the same regardless of the values of the variables A, B and C. Therefore, the loop can be rewritten making the test outside the loop and replicating the loop body twice: once for a valid outcome, and once for a false outcome:

```
IF (N .EQ. 0) THEN
  DO I=1,K
    A(I) = A(I) + B(I) * C
  ENDDO
ELSE
  DO I=1,K
    A(I) = 0.
  ENDDO
ENDIF
```

This is one of the simplest transformations in loops, yet the effect on the run time can be important, because now the computations inside the loop do not depend on the if-statement, and are, therefore, much easier to pipeline. Besides, N - 1 instances of the test have been eliminated. Although most optimizing compilers will recognize this structure and will make the appropriate transformations in order for the loop to exhibit ILP and be processed in superscalar mode, it is a good practice to make the transformations manually. Code written in this way will be able to run efficiently independently of the compiler.

Another frequently used structure is the so called Dependent loop conditional:

```
DO I=1,K
  IF (X .LT. A(I)) THEN
    X = X + B(I)
  ELSE
    X = X + C(I)
  ENDF
ENDDO
```

This loop has dependencies between iterations, which means that it is not possible to know which way the branch will go for the next iteration until the current one is done. This is a less trivial case than the previous Invariant Test and the compiler may not be able
to optimize it. The way the hardware can cope with this kind of structures is by trying to estimate which is the most probable result of the test in the if-statement using branch prediction systems. These systems try to predict the way the instruction will branch the next time it is used.

Research has been done on how to increase parallelism on conditionals as the one shown above using branch prediction. Several authors have suggested ways of predicting the direction of conditional branches with hardware and/or software that uses the history of previous branches. Different mechanisms take advantage of different observed patterns in branch behavior. Descriptions and comparisons of different branch prediction mechanisms can be found in Wall (1994), (1993) and McFarling (1993). One of the most recent methods that has been implemented in actual processors is the Dynamic Branch Prediction mechanism, which looks multiple steps ahead in the program and makes a prediction for the direction of the following branches based on the direction the branch went the last times it was executed. More accurate predictions can be made by utilizing more branch history. Prediction accuracy of 90 - 97% has been reported using these systems (Wall, 1994), and they perform well either when each branch is strongly biased in a particular direction, for example, when the condition is almost always false in the loop shown above, or for branches with simple repetitive patterns.

Once the processor has predicted the direction of the upcoming branches, it creates a schedule of instructions to be executed in an optimal way by the pipelines, trying to use its full supercalar power, and executes it "speculatively," storing the results of these instructions as "speculative results" until the final state can be determined, when the branches are resolved. Once this happens, the temporary results can be either copied to the registers, if the prediction of the branch was successful, or discarded and the pipeline flushed, if the branch was incorrectly predicted. This process is called Speculative execution, and if the processor is also capable of scheduling the instructions in a different order than the original order implied by the program, to avoid data dependencies, for example, then the process is called Out-of-order execution. In this case, the instructions will be returned to their proper program order once the branch direction is resolved.

This is a powerful mechanism for exploiting ILP masked by conditional branches. And most modern workstations are based on processors using some implementation of branch prediction and speculative execution (MIPS, Alpha, HP-PA, Power, Pentium Pro, among others).

Regarding the example loops shown above, the compiler may be able to rearrange the conditional by itself (more likely in the first example than in the second one), but if the software is written following basic optimization techniques to increase ILP, then the code will perform well on any machine, even on those that do not have smart compilers. The second example shows how the software can take advantage of hardware-implemented optimization capabilities, such as branch prediction systems and speculative execution techniques. Optimizing these structures via software requires more programmer input and, therefore, the compiler will be rather conservative with these kinds of transformations. For this case, the conditional can be arranged manually in such a way that the condition is either true or false for a large number of consecutive iterations (i.e., making the outcome of the branch more predictable), increasing this way the chances that the processor successfully predicts the outcome of the conditional.

These and other examples are well referenced in the literature, for example in Dowd (1993) and in Bacon (1994).

Cache memory

Almost any modern computer system has two types of system memory: main memory and cache memory. The cache memory is a (usually) small, high speed memory that contains the most recently accessed pieces of main memory (Figure (3)). This high speed memory is necessary in modern systems because the times it takes to bring data into the processor from main memory is long compared to the time it takes to execute an instruction, and this performance gap between processors and memory is widening; for example, in a Pentium Pro-based system, the access time for main memory is about 60 nanoseconds (ns), while a 100 MHz processor can execute most instructions in 1 clock or 10 ns. Therefore, a bottleneck is formed at the input of the processor. A typical access time for cache memory in this systems is about 15 ns. As a result, cache memory improves the efficiency of the system allowing small portions of data in main memory to be accessed about 4 times faster.

Since pipelines are trying to execute an operation every tick and, therefore, loading and storing data every tick, then the software will not be able to exploit pipelining power by using only main memory accesses, even if the software presents a high degree of ILP. For this reason, the efficiency of the memory system must be increased in order to take advantage of superscalar processing. The cache memory improves efficiency using the concept of "data locality" (spatial or temporal). "Locality" refers to a characteristic that programs often ex-
hibit, in which they very often make use of data which are near in memory, both in space and time, to other pieces of data which are going to be used immediately. More specifically, we can say that Temporal Locality implies that, when a data element is referenced in a program, it will be referenced again soon, and Spatial Locality implies that when a data element is referenced, nearby data elements will be referenced soon (Dowd, 1993). Under this scenario, chances are good that the next time the program needs data, these data will be ready in cache memory for fast access, and the CPU will not have to perform the slow process of retrieving it from main memory.

Therefore, a good way to improve the performance of the program is to improve data locality (Saavedra & Smith, 1993), which can be performed in several ways. Again, an optimizing compiler will try to transform the code in order to exploit locality, but this will not always occur, especially in complex code. Manually organizing the loops in the program so that contiguous operations use contiguous pieces of data in memory is the most effective way to improve cache memory use and, therefore, to increase pipelining use.

A common example of cache memory optimization is Blocking memory references (see for example Dowd, 1993). This example is common in geophysical software where large multidimensional arrays must be accessed in memory, and loop transformations are necessary to improve locality. The following loop:

\[
\begin{align*}
&\text{DO } I=1,N \\
&\quad \text{DO } J=1,N \\
&\quad \quad \text{DO } K=1,N \\
&\quad \quad \quad C(I,J)=C(I,J)+A(I,K)*B(K,J) \\
&\quad \text{END DO} \\
&\text{END DO} \\
&\text{END DO} \\
\end{align*}
\]

has a big stride in some matrix (matrix B if FORTRAN is used, or matrix A if C is used), so there will be cache misses, i.e., some data will not be available in cache when the operation is ready to execute in the pipeline, and a slow main memory access will be necessary. This can be avoided transforming the loop to operate in submatrices that can fit in cache:

\[
\begin{align*}
&\text{DO } I=1,N \\
&\quad \text{DO } J=1,N \\
&\quad \quad \text{DO } K=1,N \\
&\quad \quad \quad C(I,J)=C(I,J)+A(I,K)*B(K,J) \\
&\quad \text{END DO} \\
&\text{END DO} \\
&\text{END DO} \\
\end{align*}
\]

where \(\text{NB}\) is the number of submatrices the original matrix is partitioned into. The size of the submatrices must be such that the three submatrices (from A, B and C) fit into cache memory. We can estimate the size of these submatrices considering that if each one in this example is \(n \times n \times n\), and each one contains elements of type REAL*8 (in FORTRAN nomenclature), the biggest \(n\) such that all three submatrices fit completely in cache must be

\[
n < \text{extra} + \sqrt{\frac{\text{cache} - \text{size}}{3S}}
\]

where \(\text{extra}\) is extra space in cache that will be required by other variables and \(\text{cache} - \text{size}\) is the total cache size in bytes.
Remarks

Code transformation for best performance in uniprocessor systems (also called scalar optimization) is an important issue in modern high performance processors. Scalar optimization techniques are aimed basically at uncovering the ILP present in software.

Most of the CPU time consumed by a program developed for a scientific application is spent on loop-based manipulation of arrays, so the analysis of a program can be restricted only to the most time-consuming loops contained in it. Optimizing transformations in loops and conditionals is an active area of current research, and much of this research is oriented toward developing better optimizing compilers. However, some of these transformations and techniques can be applied manually when developing or updating software for high-performance processors. From my experience, in most cases only a few transformations that improve the locality and, hence, the ILP of the code, will cause important improvement in the performance of the program, because the locality will permit pipelines and branch predictors to work together providing superscalar performance.

In summary, exploiting the ILP of software to get maximum efficiency from a processor has several important implications when developing software:

- Often, expensive hardware can be substituted by low-cost workstations when the programs are optimized, because the peak performance of expensive processors is often not much higher than that of moderate-cost processors.
- It makes little sense to compare different algorithms using only the processing time it takes for the code to run, even in the same computer, because different algorithms will present different degrees of ILP. Therefore, besides the number of operations itself, the structure of the algorithm becomes another metric for algorithm comparison. In particular, one characteristic in algorithms called "locality" is one of the main factors affecting the performance. Poor locality in algorithms will decrease almost any opportunity for optimization. I have discussed locality and the way it can take advantage of cache memories, branch predictors, and pipelines in order to obtain maximum performance from software.
- Obtaining optimum performance in scalar mode, i.e., in one processor, is critical before the coarse-grained parallelization is done. If the program is making inefficient use of processor and memory resources, this inefficiency will be multiplied when going parallel and the overall efficiency of the system will decrease.

Extensive surveys of optimizing transformations can be found in the literature, for example, in Bacon et al., (1994).

References


A platform for Kirchhoff data mapping in scalar models of data acquisition

Norm Bleistein & Herman Jaramillo

ABSTRACT
Kirchhoff data mapping (KDM) is a procedure for transforming data from a given input source/receiver configuration and background earth model to data corresponding to a different output source/receiver configuration and background model. The generalization of NMO/DMO, datuming and offset continuation are three examples of KDM applications. This paper describes a "platform" for KDM for scalar wavefields. The word, platform, indicates that no calculations are carried out in this paper that would specialize the derived formula to any one of a list of KDM's that are presented in the text. Platform formulas are presented in 3D and in 2.5D. For the latter, the validity of the platform equation is verified by applying it to a Kirchhoff approximate representation of the upward scattered data from a single reflector and for an arbitrary source/receiver configuration. The KDM formalism is shown to map this Kirchhoff model data in the input source/receiver configuration to Kirchhoff data in the output source/receiver configuration, with one exception. The method does not map the reflection coefficient. Thus, we verify that, asymptotically, the ray theoretical geometrical spreading effects due to propagation and reflection (including reflector curvature) are mapped by this formalism, consistent with the input and output modeling parameters, while the input reflection coefficient is preserved. In this sense, this is a "true amplitude" formalism. As with earlier Kirchhoff inversion, a slight modification of the kernel of KDM provides alternative integral operators for estimating the specular angle, both in the input configuration and in the output configuration, thereby providing a basis for amplitude-versus-angle analysis of the data.


H(x, ti) The Beylkin determinant (2 x 2) that arises in two dimensional inversion. Equation (15).

h(x, ti) The Beylkin determinant (3 x 3) that arises in three dimensional inversion. Equation (9).

k1, ko Curvatures of input and output isochrons, respectively.

ωt, ωo Input, output frequencies, respectively.

nR Unit normal vector to the reflection surface.

R(x, xg) Geometrical optics reflection coefficient at x due to a source at xg.

α1, α2, Σ0x, Σ0y Running parameters along the rays from source or receiver in the input or output configurations, respectively.

T1 Isochron coordinate of fixed input time in 2D: T1 = τ1(x' (γ, ti)).

τ1(x, ti), τ0(x, ξo) Travel time from source to point at depth to receiver in input or output source/receiver configuration, respectively.

Glossary
A(x, xO, ξO) A Green's function amplitude, with source at xO and observation point at x. Other choices, s replaced by g and/or O replaced by I. Equation (2).

α1(x, ξO), α0(x, ξO) Amplitudes of modeling (O) and inversion. Equation (2). (I), operators, each a product of Green's function.

β(x) The reflectivity function, product of a singular function of a surface and a reflection coefficient at a distinguished incidence angle. Equation (7).

c(x) Wavespeed.

γ Variable in 2D coordinate system, varying along the isochron, T1 = τ1(x', (γ, ti)).

ΔT1 Difference of second derivatives of travel time in a stationary phase calculation. Equation (28).

δ(nR) A Dirac delta function with argument being normal distance to a surface; the singular function of the surface.
\(\theta_1, \theta_O\) Half angles between rays from source and receiver to a point at depth in the input or output configuration, respectively.

\(u_i(\xi_1, \omega_1)\) Input data, observed data.

\(u_o(\xi_O, \omega_O)\) Output data, result of data mapping.

\(x'(\gamma, T_f)\) Function defining the coordinate transformation from variables, \(x\) to variables, \((\gamma, T_f)\) in 2D. Equations (10) and (16).

\(x_R(\ell)\) Function defining the reflector in 2.5D as a function of the parameter, \(\ell\).

\(x_s, x_g\) Coordinates of source and geophone in input or output variables, depending on argument, \(\xi_1, \xi_O\) respectively.

\(\xi_1, \xi_O\) Input, output parameters used to define the source/receiver coordinates.

**Introduction**

"Kirchhoff data mapping" (KDM) is a process for transforming data from one prescribed source/receiver configuration and model of the propagating medium to another configuration and model. KDM is a "true amplitude" process in the following sense.

(i) Travel time and point source geometrical spreading effects of the input configuration are transformed to those effects of the output configuration.

(ii) Reflector curvature geometrical spreading effects of the input configuration are transformed to those effects of the output configuration.

However, the reflection coefficient of the input configuration is preserved in the output data. On the other hand, the formalism provides a mechanism for determining both the input and the output geometrical optics incidence angles of the reflection process in these applications, thereby providing a basis for amplitude-versus-angle (AVA) analysis. Both of these transformations are model consistent; that is, they depend on the input and output physical models that are assumed for the processing.

This research is grounded in the process of transformation to zero offset (TZO)—the true amplitude mapping of data from finite offset between source and receiver to zero offset. The constant background specialization of that technique is the Normal MoveOut/Dip MoveOut [NMO/DMO] method, which are standard tools in seismic data processing. The research was initially motivated by a desire to develop a specific technology and computer software for TZO in a depth-dependent earth model. That application will be reported on separately in Jaramillo [1998]. It extends the earlier NMO/DMO process in depth dependent background of Artley and Hale [1992] and Hale and Artley [1993] to a true amplitude process.

It became clear to us that the approach of Tygel, et al. [1997], as well as earlier papers, [Tygel, et al., 1995a, 1995b] had certain advantages over other approaches, including one of our own [Bleistein, et al., 1998]. However, we exploit an approximation of the obliquity factor in the Kirchhoff model that allows us to push the transformation process further than the results of the motivating papers. In a companion paper, Jaramillo and Bleistein [1997], the authors have derived a migration/demigration formalism in the space-time domain, based on the generalized Radon transform inversion technique. In Jaramillo [1998], that work is extended to a data mapping formalism in space-time. There, our obliquity approximation is also exploited. Furthermore, Tygel, et al. [1998], did not mention the possibility of transforming the physical model, as suggested here. However, the necessary machinery to do so is available in their work.

The basic idea of these methods is to cascade an inversion formula with a modeling formula. The combined formula maps a given data set to another. Both of the operators used are integral operators. Thus, the result is an integral over the variables of the input data set—here denoted by \(\xi_I\) and \(\omega_I\)—to produce an estimated model of a reflectivity in the subsurface combined with an integral over the coordinates of that model—here denoted by \(x\)—to produce the output data set—with variables \(\xi_O\) and \(\omega_O\).* The input data set depends only on the variables of the acquisition geometry and time (or frequency)—the variables \(\xi_I\) and \(\omega_I\). The operator is a function of the input and output parameters and the reflectivity model variables, as well; that is, all of the variables introduced above. The idea, then, is to carry out the integration over (earth modeling) variables, \(x\), asymptotically, to obtain a weight that is a function of the input and output variables, only. This weight is then applied to the input data set to produce the output data set. It is this asymptotic analysis that can only be partially carried out in the absence of an explicit KDM implementation; hence, the word, "platform," to characterize this point of departure from which to derive the mapping of specific implementations.

Tygel et al [1998] and Jaramillo and Bleistein [1997] carry out this process in space-time, while a space-frequency derivation is presented here. An advantage of

* In this notation, the source and receiver coordinates are given parametrically as functions of the variables \(\xi_I\) and \(\xi_O\), respectively. By this device, specification of the source/receiver configurations is postponed to specific applications.
space-time methods is that the time domain integral produces a delta function that essentially reduces the dimension of the space-domain integration. The equivalent process, in the approach here, is to decompose the volume integral into an integral along an isochron, a surface of constant travel time (We use isochrons of the input travel time) followed by an integral over that travel time variable. This leads to an \( n-1 \) dimensional (1 or 2) integral followed by a one-dimensional integral. The iterated asymptotic analysis of this approach is easier than the approach of Bleistein, et al [1998], where an \( n \) dimensional stationary phase calculation is carried out. It also lends itself more easily to geometric (ray theoretical) interpretation.

The forward model used here, and in the companion Jaramillo and Bleistein [1997] paper, is a volume integral version of the Kirchhoff approximation. Tygel et al [1998] also start from a Kirchhoff approximation for the modeling of data, but as a surface integral. Cohen, et al. [1986] start from the Born approximation, which is also a volume integral. Thus, the issue of "small perturbations," as opposed to more general variations in medium parameters, arises in the Bleistein, et al [1998] approach. Although this assumption is overcome through further asymptotic analysis of the final result [Bleistein, 1987], it is certainly much cleaner to start from the Kirchhoff approximation. This is a single reflector model, requiring negligible errors in the earth model above the reflector of interest. It also neglects multiple reflections, as do all methods for data mapping (including classic DMO) to date. However, the approach used here does allow for larger jumps in medium parameters across the reflector of interest. Thus, in the absence of multi-pathing, the interpretation of the output in terms of the geometrical optics reflection coefficient is direct and immediate in this approach, while the former approach of Bleistein, et al [1998] only predicts a linear approximation of the reflection coefficient in the output. It is only through a posteriori analysis that the Bleistein, et al [1998] approach confirms an output that is linear in the geometrical optics reflection coefficient [Bleistein, 1987].

The volume integral form of the Kirchhoff approximation requires introduction of the singular function of the reflecting surface. This is a Dirac delta function of normal distance from the reflector. For the inverse problem, it is necessary to isolate all dependence on the reflector in the reflectivity function, which is the singular function multiplied by the plane wave (or geometrical optics) specular reflection coefficient at a distinguished incidence angle. This isolation requires an asymptotic approximation for the obliquity factor appearing in the Kirchhoff representation and also requires that the reflection coefficient be evaluated at specularity. This is the new idea that we bring to the derivation of the basic inversion formula. Once that is done, a direct inversion procedure from the volume Kirchhoff-approximate forward model to an asymptotic inversion that produces the reflectivity function is carried out.

KDM options.

Below, is a list some possible mappings of data sets from an input macro-model and source/receiver configuration to an alternative data set on output. All of these can be carried out in 2D, 2.5D and 3D.

(i) Offset continuation and TZO. KDM is not limited to transforming to zero-offset; the formula lends itself to analysis of the transformation of data from one offset to another, with TZO as a special case. That is, (10) and (16), below, provide a platform for offset continuation along the lines of Fomel [1995a, 1996, 1997] and Fomel et al [1996]. However, as soon as one applies to the platform equations the type of asymptotic analysis that leads to the classical NMO/DMO-type data mapping, the mapping requires "large" (e.g., a few wave lengths) offset to be valid.

(ii) Transformation of common-offset data to common-shot data. In this case, the transformed data represents the response from a single shot at an array of receivers covering the upper surface. Such data has the advantage that it is the solution of the wave equation, whereas common-offset (and zero-offset) data are a collection of single responses to an ensemble of wave equations, one for each shot. This ensemble data is not a solution of the wave equation, although it is treated as such in wave equation migration.

(iii) Mapping of data from variable background propagation parameters to constant background parameters. Time sections in constant background media are easier to interpret. It is not clear to us at this point, however, how multi-paths will map, nor how data from vertical and overhung reflectors will map. We expect singularities of the mapping process for these cases. On the other hand, where the method works, it opens the mapped data set to a much broader suite of applicable migration/inversion programs and related analysis techniques.

(iv) "Unconverting" mode-converted waves. For example, one could map the scalar components of

\[ \text{Kirchhoff data mapping} \] 213
P-SV data to the scalar components of P-P data [Chan and Stewart, 1996]. If the "true" P-P data were available, a comparison of this latter data set with the mapped data set could provide a check on the assumptions made in the macro-model for the converted wave propagation. Furthermore, again, there are many more processing options available for P-P data than for mode converted data. This mapping would provide a means of extending the range of processing options once the data is mapped.

(v) Velocity analysis. When data from a suite of offsets are all mapped to zero offset, events should line up. To the extent that they do not, they provide the same type of information about velocity errors as does a common trace gather of a suite of prestack migrations/inversions.

(vi) Wave equation datuming. The acquisition surface can be changed for both the sources and receivers. Downward continuation of sources and receivers or mapping from irregular acquisition topography to a planar topography, are two potential datuming applications for this platform. See Sheaffer and Bleistein [1998]. For small increments in depth, the implementation of KDM provides an alternative to phase shift migration [Gazdag and Squazzero, 1984], while for larger increments in depth, the implementation of KDM provides an alternative to wave equation datuming [Berryhill, 1979].

(vii) Mapping of swath data to a single line at zero azimuth. Swath shooting is a process whereby multiple lines of receivers are used with a single shot, as might occur if a boat towed more than one sonophone line. The data from the separate lines could be mapped to a single line that could be "straightened" to be along the line of the survey—given sufficient information about the deviation of the swath survey from that line and about the path of the boat [Biondi and Chemingui, 1994].

(viii) Combinations of the above. For example, consider the application of downward continuation of receivers (or sources). The continuation process always yields output: data over a shorter line than that of the input data. Starting from a prescribed shot gather with a given cable length of receivers, the cable length of asymptotically accurate mapped data decreases from the input cable length with increasing depth. On the other hand, consider first: creating a single common shot data set from the full array of common offset gathers. This new data set effectively has a "cable length" equal to the length of the survey, typically, much longer than the cable length for each shot. Now, the range of validity of the downward continuation of receivers "shrinks" from an initial length equal to the survey length. One can expect that the data can be continued much deeper into the subsurface and maintain properly transformed geometrical spreading effects and travel time corrections over a cable length that will be adequate for further processing. As a second example of cascading, consider the process of first downward continuing the receivers and then downward continuing the sources. This should provide a true amplitude simultaneous downward continuation of sources and receivers.

This paper proceeds as follows. In the next section, the fundamental three dimensional KDM platform equation is derived. The restrictions are that this is a platform for scalar data acquisition, no multiple reflections and no multi-pathing of rays from points at depth to either the input or output acquisition surface. Following that, the specialization to 2.5D processing is discussed. The result is a KDM platform for processing a line of data while honoring 3D propagation.

Then, a simple proof of the validity of the 2.5D data acquisition platform formula is presented. Under the restrictions stated above, this proof is carried out in all of the generality of the mapping formulation. The proof is valid for synclinal reflectors that might produce buried foci; their multiple arrivals do not cause a problem in this method. The exclusion of multi-pathing, applies only to structure sufficiently complex that the rays of the Green's function have multiple paths from a point at depth to a point on the upper surface.

Derivation of a 3D Kirchhoff data mapping formula

In this section, the fundamental equation for space-frequency domain KDM in 3D will be derived. This will be done by cascading a Kirchhoff-approximate forward modeling formula with an inversion formula. Crucial to this analysis is the representation of the Kirchhoff modeling formula as a volume integral in which the only unknown from the point of view of the inverse problem will be the reflectivity function described in the introduction. The derivation starts with the Kirchhoff approximation for the upward scattered wave from a single reflector. This representation can be found in many sources; we use equation (5.3.1) in Bleistein, et al [1997].

\[
\begin{align*}
\psi_O(x, \omega) & \sim i\omega F(\omega) \int_{S_R} a_O(x, \xi_O) \\
& \quad \cdot n_R \cdot \nabla x R(x, \xi_O) dS_R \\
& \quad : e^{i\omega x \cdot R(x, \xi_O)} dS_R.
\end{align*}
\]

In this equation, the subscript \( O \) is used to denote output variables. Later, input variables with subscript \( I \) will be introduced. The two component vector, \( \xi_O \), is used
to parametrize the source and receiver locations, \( x_s(\xi_o) \) and \( x_r(\xi_o) \), respectively, and \( \omega_o \) denotes the frequency of the output wave. The upward unit normal to the reflecting surface, \( S_R \), is denoted by \( \hat{n}_R \). The phase and amplitude are given by

\[
\tau_0(x, \xi_o) = \tau(x, x_s(\xi_o)) + \tau(x, x_r(\xi_o));
\]

\[
\cdot a_0(x, \xi_o) = A(x, x_s(\xi_o)) \cdot A(x, x_r(\xi_o)),
\]

with the separate traveltimes and phases being solutions of appropriate eikonal and transport equations, respectively, with initial point, \( x_s \) or \( x_r \), and final point, \( x \). We choose not to be more specific here, allowing for different propagation speeds in the eikonal equations (mode conversion) and transport equations appropriate to the degree of generality of the propagation model under consideration. The amplitudes can also include products of transmission coefficients arising from interfaces above the surface, \( S_R \).

It should be noted that, except for the obliquity factor, \( \hat{n}_R \cdot \nabla \tau_0(x, \xi_o) \), and the reflection coefficient, \( R(x, x_s) \), the elements of the integrand are defined globally in \( x \) and not just on the reflecting surface. That restricted dependence will be isolated further, to \( R \), alone, in order to derive an inversion formula. Thus, what is needed is an approximation of the factor, \( \hat{n}_R \cdot \nabla \tau_0(x, \xi_o) \) that is independent of surface information, that is, independent of \( \hat{n}_R \). The right choice here is the stationary value of this factor, at the stationary point(s) of the original surface integral, (1),

\[
\n_R \cdot \nabla \tau_0(x, \xi_o) = -|\nabla \tau_0(x, \xi_o)|.
\]

The right side, here, is independent of the coordinates of the reflecting surface and, so, this approximation is substituted into (1). This is consistent with the intent of deriving an inversion for the reflectivity function, only. It acknowledges that high frequency modeling and inversion uses reflection data to detect reflecting surfaces. With this approximation, (1) is rewritten as

\[
u_0(\xi_o, \omega_o) = -i\omega_o F(\omega_o) \int_{S_R} a_0(x, \xi_o) \cdot |\nabla \tau_0(x, \xi_o)| R(x, x_s) \cdot e^{i\omega_o \tau_0(x, \xi_o)} dS_R.
\]

The next goal is to rewrite the integral in (4) as a volume integral by introducing the singular function of the reflecting surface, \( \delta(n_R) \), where \( n_R \) measures normal distance from any point on the reflector. The new representation becomes

\[
u_0(\xi_o, \omega_o) \sim -i\omega_o F(\omega_o) \int a_0(x, \xi_o) \cdot |\nabla \tau_0(x, \xi_o)| R(x, x_s) \cdot \delta(n_R) \cdot e^{i\omega_o \tau_0(x, \xi_o)} dV
\]

Here, \( dS_R \cdot n_R = dV \). At worst, a reasonable continuation of the reflection coefficient off the reflecting surface must be introduced in this equation. Of course, that continuation need only correspond to the true \( R \) on the support of the singular function, that is, on the reflector. Even for bandlimited delta functions, this is easy to achieve.

Just as the normal derivative of the travel time has been replaced by its stationary value, the reflection coefficient is replaced by its stationary value, as well. Then, the product, \( R \delta \), appearing in this equation is just the reflectivity function, \( \beta(x) \), of the Bleistein/Cohen inversion theory [Bleistein et al., 1997, eq. 5.1.21]. In this case, (5) can be rewritten as

\[
u_0(\xi_o, \omega_o) \sim -i\omega_o F(\omega_o) \int a_0(x, \xi_o) \cdot |\nabla \tau_0(x, \xi_o)| \beta(x) \cdot e^{i\omega_o \tau_0(x, \xi_o)} dV
\]

**Side remark**

This volume integral representation of Kirchhoff approximate forward modeling could be used to derive an inversion formula for \( \beta \), following the generalized Radon transform method introduced by Belykin [1985] for this problem or by the asymptotic Fourier approach in Bleistein et al. [1997]. Such a derivation would have the advantage of not assuming small perturbations in medium parameters across the reflector, although it would assume primaries-only propagation above the reflector, an accurate background model ("macro-model") above the reflector, and no multi-pathing.

The derivation of an inversion formula would then follow along the lines used for inverting the volume integral forward model of Born-approximate data, but without the added small perturbation constraints of that model. Of course, the inversion formula that would then be derived is exactly the formula for \( \beta \). Rather than deriving that formula, we need only identify the corresponding constituents of the forward modeling formula for \( \alpha \), equation (5.1.7) in Bleistein et al. [1997], and use this correspondence to deduce the inversion formula for \( \beta \) from the inversion formula (5.1.19) for \( \alpha \), in Bleistein et al. [1997].

The correspondence of elements is as follows, with constituents of (5.1.7) appearing on the left and con-
stituents of (6), appearing on the right.

\[ \omega^2 F(\omega) \leftrightarrow -i\omega F(\omega) \]

\[ \frac{a(x, \xi)}{c^2(\xi)} \leftrightarrow a_0(x, \xi_0) |\nabla_x \tau_0(x, \xi_0)| \]

\[ a(x) \leftrightarrow R(x, x_0) \delta(n(x)) \]

Written in a set of variables consistent with this discussion, the reflectivity function is given by

\[ \beta(x) = \frac{1}{8\pi^2} \int d^2 \xi_j \frac{|h(x, \xi_j)|}{a_j(x, \xi_j)|\nabla_{x \xi_j} (x, \xi_j)|} \]

\[ \cdot \int i\omega_l \, dw_l \, d^2 \xi_l \, u_l(\xi_l, \omega_l). \]  

In this equation, the subscript \( I \) is used to denote input variables (i.e., the original variables). The two component vector, \( \xi_i \), is used to parametrize the source and receiver of the input data, \( y_s(\xi_i) \) and \( y_r(\xi_i) \), respectively, and \( \omega_l \) denotes the frequency of the input wave. The phase and amplitude are given by

\[ \tau_l(x, \xi_i) = \tau(x, y_s(\xi_i)) + \tau(x, y_r(\xi_i)); \]

\[ a_l(x, \xi_i) = A(x, y_s(\xi_i)) \cdot A(x, y_r(\xi_i)). \]  

The traveltimes and amplitudes are again solutions of the eikonal and transport equations, except that now the initial points will be chosen from \( y_s(\xi_i) \) and \( y_r(\xi_i) \). Furthermore,

\[ h(x, \xi_i) = \det \begin{bmatrix} \nabla_x \tau(x, \xi_i) \\ \frac{\partial}{\partial \xi_{11}} \nabla_x \tau(x, \xi_i) \\ \frac{\partial}{\partial \xi_{12}} \nabla_x \tau(x, \xi_i) \end{bmatrix} \]  

is the Beylkin [1985] determinant.

Equation (7) is an inversion formula for \( \beta \) derived directly from a Kirchhoff-approximate forward model for a single reflector. Here, however, we have another objective. We wish to map an input data set, with its source/receiver configuration and background parameters (macro-model) to an output data set, with its source/receiver configuration and background parameters. Thus, the different subscripts, \( I \) and \( O \) on the variables. To achieve this, the representation (7) is substituted into (6) to obtain the following representation for the mapping of data from any input source/receiver configuration and background model to any output source/receiver configuration and background model:

\[ u_O(\xi_O, \omega_O) \sim -\frac{i\omega_O}{8\pi} \int i\omega_l \, dw_l \, d^2 \xi_l \, u_l(\xi_l, \omega_l) \]

\[ \cdot \int a_0(x, \xi_0) \left| \frac{\nabla_x \tau_0(x, \xi_O)}{\nabla_{x \xi} (x, \xi_l)} \right| |h(x, \xi_l)| \]

\[ \cdot e^{i\omega_l \tau_0(x, \xi_O) - i\omega_l \tau_1(x, \xi_l))} \, dV \]

Note that only the first line on the right side in these equations depends on the input data. The integration over the interior volume in the second and third lines depends on the modeling and inversion parameters, and not on the data. Hence, for each choice of input and output earth model and each choice of input and output source/receiver configuration, the integrations indicated in the second and third lines could be carried out to obtain an operator kernel that is a function of \( \xi_l, \omega_l, \xi_O, \) and \( \omega_O \). Indeed, we anticipate carrying out those integrations by analytical methods including asymptotic methods such as multi-dimensional stationary phase. Numerical integration is out of the question. There are \( O(n^3) \) coordinates of integration, with \( O(n^3) \) input variables and \( O(n^3) \) output variables. Clearly, the n's are different, but this is still an intractably large set of variables. The processing of this formula would require an integration over \( O(n^3) \) variables to produce a function of \( O(n^3) \) variables. A better choice is the analysis of the volume integral represented by the second and third line to obtain an analytically explicit kernel that depends only on the input and output variables, \( \xi_l, \omega_l, \xi_O, \) and \( \omega_O \).

From the derivation and the specific example of 2.5D DMO, it is expected that the result will not transform the reflection coefficient of the input data configuration to the reflection coefficient of the output configuration. However, we do anticipate that the geometrical spreading effects and curvature effects of the input configuration will be transformed to the correct effects for the output configuration. In fact, this has been proven by Tygel, et al., [1998], using a somewhat different approach to the same problem. While their proof does not include all of the cases listed above, it could easily be extended to them. In that sense, we consider their proof as all encompassing for this operator. However, a simpler, more straightforward proof is offered, below, for the general 2.5D KDM.

**Spatial structure of the operator.**

There is a certain amount of symmetry in the spatial structure of the operator that could have predicted in advance. The geometrical spreading of the input data is "undone" by the division by \( a_I \), while the geometrical spreading of the output is introduced through the multiplication by \( a_O \). Similarly, the obliquity effects in the
input data are undone by the division by a gradient of the input travel time, while the obliquity effects of the output data are reintroduced by the gradient in the numerator. The arrival time of the input data manifests itself through the phase in the frequency domain. That is undone by the multiplication by the negative of the input traveltime multiplied by the input frequency in the phase of the operator. Similarly, the positive product of output frequency and output travel time in the phase of the operator introduces the "correct" arrival time on output. All of these variables are evaluated at stationarity of the integrand. In the simplest of such evaluations, the determinant of the matrix of second derivatives, evaluated at the stationary point will correct for curvature effects from input to output data.

The only unexplained factor, then, is the Beylkin determinant, \( h \), which does not lend itself to the same type of symmetry arguments. As in the inversion integral, (7), this is the factor that cannot be predicted from migration arguments. Here, we refer to Beylkin's [1985] original paper introducing this type of inversion and earlier work by Bojarski [1967, 1968, 1982] and by the first author and associates [Bleistein, 1975, 1976; Mager and Bleistein, 1978; Bleistein and Cohen, 1979] on physical optics far field inverse scattering. Analysis of the forward model, (2), suggests that the model data is proportional to a Fourier transform of the function, \( R_0 \). However, the output variables are not a wave vector, \( k \), but a combination, \( \xi, \omega \).

Similarly, then, in (7), the inversion is asymptotically a Fourier inversion with respect to some wave vector, \( k \), although the integration is over variables, \( \xi \), and \( \omega \). In the earlier work by Bojarski and by Bleistein and associates on far field inverse scattering, the relationship between the phase and a wave vector is explicit. When the far field approximation cannot be used, the choice is less obvious.

Beylkin has shown us that for the case of interest here, the correct relationship is that the wave vector is locally defined for each choice of \( x, \xi \), and \( \omega \) by the relationship,

\[
k = \omega \nabla_x \tau_l(x, \xi).
\]

Thus, in (7) and (10) the factor of \( h \) arises through the identity

\[
dk^3 = \left| \frac{\partial(k)}{\partial(\xi, \omega)} \right| d^2 \xi d\omega = \omega_l^2 |h(x, \xi)| d^2 \xi d\omega_l. \]

In fact, if \( h \) in (10) is replaced by the Jacobian appearing on the right side of this equation, then the remaining frequency dependence of the operator becomes \( \omega_l / \omega \) and has the same quotient symmetry and explanation as the spatial factors discussed above. That is,

\[
u_0(\xi, \omega) \sim -\frac{1}{8\pi^3} \int \frac{\omega_0 u_l(\xi_l, \omega_l) d^3 k}{\omega_l^3} \left( \begin{array}{c} a_0(x, \xi) \nabla_x \tau_l(x, \xi) \\ a_l(x, \xi) \nabla_x \tau_l(x, \xi) \end{array} \right) \cdot e^{i(\omega_0 \tau_l(x, \xi) - \omega_l \tau_l(x, \xi))} d^3 x,
\]

with the change of variables from \( \xi_l, \omega_l \) to \( k \) defined above.

Determination of incidence angle

When the input data is dominated by isolated specular reflection returns, the output will be dominated by such returns, as well. Asymptotically, in this case, the cascade of integrals is dominated by stationary phase contributions where the isochrons of the input and output travel times are tangent and share the same normal direction. This direction is also normal to the reflector at the specular point. When there is no mode conversion, at this specular point,

\[
|\nabla_x \tau_l(x, \xi)| = \frac{2 \cos \theta_l}{c(x)},
\]

\[
|\nabla_x \tau_0(x, \xi)| = \frac{2 \cos \theta_0}{c(x)},
\]

where \( 2 \theta_l \) and \( 2 \theta_0 \) are the opening angles between the incident rays of the input and output source/receiver configurations at the point \( x \). Thus, we introduce two other KDM operators,

\[
\cos_l = -\frac{1}{8\pi^3} \int \left( \nabla_x \tau_l(x, \xi) \right)(\ldots) d^3 x,
\]

\[
\cos_0 = -\frac{1}{8\pi^3} \int \left( \nabla_x \tau_0(x, \xi) \right)(\ldots) d^3 x.
\]

Here, \( \ldots \) denotes the three lines of (10) beyond the integral sign. Then, the ratios of outputs,

\[
\frac{\cos_l}{u_l(\xi, \omega)}, \quad \frac{\cos_0}{u_0(\xi, \omega)}
\]

will provide asymptotic estimates of \( \theta_l \) and \( \theta_0 \), respectively. These estimates of incidence angles provide the necessary additional tool for AVO analysis.

Frequency structure of the operator and asymptotic preliminaries.

We can anticipate an aspect of the asymptotic analysis from the form of result (10), alone. Two powers of frequency appear on the right side of this equation. The amplitude of 3D point source data has no power of frequency in its asymptotic (WKBJ) representation and the same should be true for the output. Each spatial integral carried out by stationary phase will produce an inverse
power of the square root of frequency, if they are approximated by simple\(^5\) stationary point contributions. There are five such integrations to be performed. Thus, not all of the spatial integrals can be estimated by simple stationary point contributions because the resulting power of \(\omega\) on the two sides of the equation will not match. Clearly, then, some other method besides stationary phase must play a role in the analysis of this integral.

It is in this context that Tygel, et al., [1998], employ a different approach. Consider the integral over the interior variables, represented by the differential, \(dV\). Ultimately, this integral is recast as an integral over the isochrons of the input configuration, say, \(\tau_1(x, \xi_0) = \text{constant} = t_1\), followed by integration over \(t_1\). See Figure 1.

Suppose, for the moment, that the first two integrals are carried out by the method of stationary phase, with the last integral in the direction of increasing traveltime analyzed separately. In general, an isochron of \(\tau_1\) is cut by the isochrons of \(\tau_0\). It is fairly straightforward to show that stationarity occurs when the normals of the two travel times line up; that is, the phase is stationary when the isochron of \(\tau_0\) is tangent to the isochron of \(\tau_1\). Indeed, if there were a reflector at such a point with its normal being colinear with the normals to these two isochrons, then, on the data traces of the input and output data sets, there would be a specular return at the travel time evaluated on the input or output isochron and at the trace locations assigned to the input and output values of the parameters, \(\xi_1\) and \(\xi_0\), respectively.

Figure 2. A point of tangency of isochrons from two travel time functions, corresponding to a simple stationary point in the integration over an isochron.

Figure 3. Tangency of isochrons along a curve of revolution.

For this case, consider the mapping from input to output source/receiver configurations along parallel lines. It is easy to show that there are no stationary points unless the out-of-plane input and output variables are the same; \(\xi_{12} = \xi_{02}\). Then, if the isochrons are tangent at a point, they are tangent along the entire curve of revolution through that point. See Figure 3. Thus, asymptotically, the \(\xi_{12}\) integration behaves like a delta function and the integration in the direction of the generating curve of revolution of the isochron is an entire curve of stationarity. The order of the stationary point in this case is infinite and all orders of directional derivative of the phase in this direction vanish. This is a manifestation of a known fact about 3D DMO/TZO in constant background: while its kinematics are straightforward, its dynamics are not, and determination of amplitude dependence of the operator requires great care. This also should serve as a warning about 3D processing for the nonconstant background case. Suppose that the propagation speed(s) of the model are nearly constant. Then, formally, the straightforward multi-dimensional stationary phase described at the beginning of this section will seem to work. However, the second derivative in the direction of the curve of near-revolution will be small and the resulting amplitude will not be accurate for realistic frequencies of the exploration experiment. Thus, uniform

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\(^5\) A stationary point of a single integral is simple if the first derivative is zero, but the second derivative is not. In higher dimensions, all first derivatives are zero, but the matrix of second derivatives must have nonzero determinant.
**asymptotic analysis** is called for, if one is to obtain accurate amplitude information from the mapping process.

More generally, for the problem of offset continuation, a similar pathology occurs. If the offset difference is small, then the two isochrons are quite similar. Again, near total contact at stationarity will imply a small determinant of the matrix of second derivatives arising in the denominator of the stationary phase formula. In this case, the asymptotics breaks down and the predicted amplitude is not valid. For example, true amplitude offset continuation through small offsets cannot be achieved through the same process that produces TZO or its constant background equivalent, NMO/DMO. Here, the alternative method, Fomel [1995a, b, 1996, 1997] and Fomel et al [1996], for offset continuation is the method of choice.

**2.5D KDM**

Two-and-one-half (2.5D) processing is a technique for treating a single line of point source data in three dimensions. In this section, a KDM platform for processing such a single line of data is derived. First we develop the appropriate **thought experiment** from which to derive this 2.5D counterpart of (10). Consider a medium in which the propagation speed and other medium parameters are independent of one transverse direction, say $x_2$. It is further assumed that the input data is gathered on lines of constant $x_2$, say, $x_2 = \xi_{12}$. Finally, consider an output source/receiver configuration that is also confined to lines of constant value of this out-of-plane coordinate. In this case, the input data is independent of the out-of-plane variable, namely, $\xi_{12}$, and the integration in $\xi_{12}$ and in $x_2$ can be carried out by the method of stationary phase. Actually, it is easiest to first apply the stationary phase analysis in $\xi_{12}$ and then in $x_2$ as iterated integrals, rather than to do them together as a two-dimensional stationary phase calculation. The analysis has already been done in Bleistein, et al., [1997] and only the results will be reported here.

For the integration in $\xi_{12}$, the phase to be considered is $\tau_1(x, \xi_{1})$. The stationary point in $\xi_{12}$ is at $\xi_{12} = x_2$ and the second component of the gradient of the phase is zero at this point. Furthermore,

$$\frac{\partial^2 \tau_1}{\partial \xi_{12}^2} \bigg|_{\xi_{12} = x_2} = \frac{1}{\sigma_{1s}} + \frac{1}{\sigma_{1g}}$$

(12)

In this equation, $\sigma_{1s}$ ($\sigma_{1g}$) is a running ray parameter along the ray from the source (receiver) to the scattering point, $x$. It is the natural parameter that arises to describe out-of-plane geometrical spreading in the 2.5D case. It is related to travel time through the equation,

$$\frac{d\sigma}{d\tau} = c^2(x),$$

(13)

with appropriate subscripts on both variables on the left. One can readily verify from this equation that $\sigma$ has the dimensions of length$^2$/time. See Bleistein [1986] for further discussion.

When this value of $\xi_{12}$ is substituted into the phase,

$$\tau_1(x, \xi_{1})$$

is shown to be independent of $x_2$, making the second stationary phase analysis easier, since we only have to consider the phase, $\tau_0(x, \xi_{0})$. The analysis on this phase proceeds exactly as described above. However, stationarity requires that $x_2 = \xi_{02}$, essentially eliminating the out-of-plane coordinate from further consideration. All second component variables take on the same value.

Finally, the Beylkin determinant, (9), also becomes simpler in this case. It is given by

$$h(x, \xi_{1}) = \left[ \frac{1}{\sigma_{1s}} + \frac{1}{\sigma_{1g}} \right] H(x, \xi_{1}),$$

(14)

with

$$H(x, \xi_{1}) = \det \left[ \begin{array}{c} \nabla_x \tau(x, \xi_{1}) \\ \frac{\partial}{\partial \xi_{1}} \nabla_x \tau(x, \xi_{1}) \end{array} \right]$$

(15)

In this last equation, the gradient is a two component operator in $(x_1, x_3)$ and $\xi_{1}$ is a scalar variable.

Applying the method of stationary phase to the integral in (10) in both variables and taking account of the results stated above leads to the following 2.5D analog of that earlier result:

$$u_0(\xi_{0}, \omega_0) \sim \frac{\sqrt{|\omega_0|} e^{-i \frac{\text{sgn} (\omega_0)}{4} \frac{\tau(x, \xi_{1})}{4\pi^2}}}{\int \sqrt{|\omega_1|} e^{-i \frac{\text{sgn} (\omega_1)}{4}} d\omega_1 d\xi_{1}}$$

$$\cdot \int u_1(\xi_{1}, \omega_1) \frac{\omega_0(x, \xi_{0})}{\omega_1(x, \xi_{1})} \frac{\nabla_x \tau_0(x, \xi_{0})}{\nabla_x \tau_1(x, \xi_{1})} \left| H(x, \xi_{1}) \right|$$

$$\cdot \frac{\sqrt{\sigma_{1s} + \sigma_{1g}}}{\sqrt{\sigma_{0s} + \sigma_{0g}}} \frac{\sqrt{\sigma_{0s} \sigma_{1g}}}{\sqrt{\sigma_{1s} \sigma_{0g}}} |H(x, \xi_{1})|$$

$$\cdot e^{i \omega_0 \tau_0(x, \xi_{0}) - i \omega_1 \tau_1(x, \xi_{1})} d^2x.$$  

(16)

In this equation, we have rewritten $\xi_{1}$ for $\xi_{11}$ and $\xi_{0}$ for $\xi_{01}$, since the second component is no longer in the representation and the formerly two component vectors are now scalars.

Equation (16) provides a platform for mapping data from any linear input source-receiver configuration and input medium parameters to data from a linear output source-receiver configuration and output medium parameters. It is a 2.5D transformation, meaning that it accounts for out-of-plane geometrical spreading but assumes modeling parameters that do not depend on the
out-of-plane variable. We view it as a "platform," only, because it still requires an integration of the operator over the interior scattering variables. As above, for each choice of source/ receiver configuration and medium parameters of input and output, this integral should be carried out in advance, preferably analytically, invoking asymptotic methods as appropriate. Then, for a given data set, one reeds only to process the line of data by carrying out the integrals in the first line with the simplified weighting function obtained by the preprocessing analysis of the second and third lines. This is the usual form of TZO (NMO/DMO), for example.

Application of KDM to Kirchhoff data in 2.5D.

In this section, the application of the 2.5D KDM platform equation (16), to 2.5D Kirchhoff data is analyzed. That application leads to 2.5D Kirchhoff data, again, but with the input variables of the test data transformed to output variables. Had the derivation of (16) not used the obliquity approximation and had the reflection coefficient not been replaced by its stationary value, this could be argued to be a meaningless exercise. After all, the derivation started with the Kirchhoff model of the forward scattering problem. These extra approximations that were made along the way, however, make this a useful analysis, showing that these auxiliary approximations are valid within the context of the objective of our KDM, namely, to get the mapping of specular returns on the data trace correct!

To begin this analysis, define the reflector \( S_R \) through the curve,

\[
x = x_R(\ell), \quad x = (x_1, x_3).
\]  

(17)

Here, with no loss of generality, \( \ell \) can be taken to be arclength along this curve which defines the reflector. The 2.5D Kirchhoff approximate data then has the representation [Bleistein, 1986]

\[
u_1(\xi, \omega_1) = -\sqrt{|\omega_1|}e^{-i\omega_1 \text{ sgn}(\omega_1)/4} F(\omega_1)
\]

\[
\int R(x_R(\ell), x_3(\ell))
\]

\[
\nabla_x \tau(\nabla_x R(\ell), \xi) a_1(\xi, \xi_1)
\]

\[
\frac{\sqrt{\sigma_{1s}(x_R, \xi_1)\sigma_{1s}(x_R, \xi_1)}}{\sqrt{\sigma_{1s}(x_R, \xi_1) + \sigma_{1s}(x_R, \xi_1)}}
\]

\[\cdot \nabla_x \tau(\nabla_x R(\ell), \xi_1) dl.
\]

(18)

In this equation, \( F(\omega_1) \) represents the source signature and \( \hat{n}_R \) is the upward unit normal on the reflector. Other expressions are defined as in earlier equations, except that now one of the points is \( x_R(\ell) \) on the reflecting surface.

This upward scattered field is inserted into (16). The result is

\[
u_0(\xi_0, \omega_0) = -\sqrt{|\omega_0|}e^{-i\omega_0 \text{ sgn}(\omega_0)/4}
\]

\[\cdot \int |\omega_1|d\omega_1 d\xi_1 d\ell d^2 x G e^{i\Psi}.\]

(19)

In this equation,

\[
\Psi = \omega_0 \tau(\xi_0) - \omega_1 \tau(\xi_1) + \omega_1 \tau(\xi_1) + \omega_1 \tau(\xi_1)
\]

(20)

and

\[
G = \frac{R(x_R(\ell), x_3(\ell))}{4\pi^2}
\]

\[\cdot \nabla_x \tau(\nabla_x R(\ell), \xi) a_1(\xi, \xi_1)
\]

\[
\cdot \frac{\sqrt{\sigma_{1s}(x_R, \xi_1)\sigma_{1s}(x_R, \xi_1)}}{\sqrt{\sigma_{1s}(x_R, \xi_1) + \sigma_{1s}(x_R, \xi_1)}}
\]

\[
\cdot \frac{\frac{\sigma_{1s}(x, \xi_1)}{\sigma_{1s}(x, \xi_1)}}{\frac{\sigma_{1s}(x, \xi)}{\sigma_{1s}(x, \xi)}} [H(x, \xi)]
\]

is the cascade of the integrands of (18) and (16), exclusive of the explicit frequency dependence appearing here. In the last line, all \( \sigma \)'s are functions of \( x \) rather than \( x_R \).

The method of stationary phase in the variable \( \ell \) will be applied to this representation. The phase of interest is

\[
\Phi_1(\ell) = \tau(\xi, \xi_1)
\]

(22)

To carry out the method of stationary phase, we need the derivatives,

\[
\frac{\partial \Phi_1}{\partial \ell} = \nabla_x \tau(\xi, \xi_1) \cdot \frac{dx_R}{d\ell},
\]

\[
\frac{\partial^2 \Phi_1}{\partial \ell^2} = \frac{\partial^2 \tau_R}{\partial x_R \partial x_{Rj}} \frac{dx_{Ri}}{d\ell} \frac{dx_{Rj}}{d\ell} + \frac{\partial \tau_R}{\partial x_R} \frac{d^2 x_R}{d\ell^2}.
\]

(23)

In these equations and those below, summation over the repeated indices, \( i, j \), from 1 to 3, is to be understood. Setting the first derivative of \( \Phi_1 \) equal to zero picks out the specular reflection point as the stationary point. Denote the stationary value of \( \ell \) by \( \ell(\xi_1) \). Detailed analysis of the second derivative is deferred for the moment. After stationary phase in \( \ell \), (19) becomes
\[ u_0(\xi_0, \omega_0) = -\sqrt{2\pi|\omega_0|e^{-i \text{sgn}(\omega_0)/4}} \]
\[ \int \sqrt{|\omega_1|} F(\omega_1) e^{-i \text{sgn}(\omega_1) \Phi''_{\xi}}/4 \]
\[ -d\omega_1 d\xi_1 d^2 z G - \frac{e^{i \Psi}}{\sqrt{|\Phi'|}}, \]
\[ \ell = \ell(\xi_1). \]

Here, \( \Phi''_{\xi} \) denotes the value of the second derivative at the stationary point,
\[ \Phi''_{\xi} = \frac{\partial^2 \Phi(\ell)}{\partial t^2} \bigg|_{t = \ell}, \] (25)
assumed to be nonzero. That is, the situation in which a caustic of the reflected wavefield is coincident with, or less than a “few” wavelengths from, the observation surface is excluded from consideration. However, the reflected rays may have passed through a caustic arising from the geometry of the surface, allowing \( \Phi''_{\xi} \) to be positive or negative. The case in which point source rays have caustics has been excluded, initially, by not including a KMAH index in our original Green’s functions. Stationary phase in \( \xi_1 \) is now to be applied to the integral in (24). The \( \xi \)-dependent part of the phase is
\[ \Phi_2 = \tau_1(x_R(\ell(\xi_1)), \xi_1) - \tau_1(x, \xi_1). \] (26)

The derivatives of this expression with respect to \( \xi_1 \) are given by
\[ \frac{\partial \Phi_2}{\partial \xi_1} = \frac{\partial \tau_1(x_R(\ell(\xi_1)), \xi_1)}{\partial \xi_1} - \frac{\partial \tau_1(x, \xi_1)}{\partial \xi_1} + \nabla_x \tau_1(x_R(\ell(\xi_1)), \xi_1) \cdot \frac{dx_R}{dt} \frac{d\ell}{dt} \frac{d\xi_1}{dt}, \] (27)
\[ \frac{\partial^2 \Phi_2}{\partial \xi_1^2} = \frac{\partial^2 \tau_1(x_R(\ell(\xi_1)), \xi_1)}{\partial \xi_1^2} \frac{dx_R}{dt} \frac{d\ell}{dt} \frac{d\xi_1}{dt} + \Delta \tau_1, \]
where,
\[ \Delta \tau_1 = \frac{\partial^2 \tau_1(x_R(\ell(\xi_1)), \xi_1)}{\partial \xi_1^2} - \frac{\partial^2 \tau_1(x, \xi_1)}{\partial \xi_1^2}. \] (28)

In the first line in (27), the last term is zero because this is just the stationarity condition imposed on \( \Phi_1 \). In the second line, summation over the repeated index, \( j \), from 1 to 3 is understood.

Consider the condition that the first derivative of \( \Phi_2 \) is equal to zero. The first travel time in \( \Phi_2 \) is just the time for the specular ray path from source to \( S_R \) to the receiver. This travel time remains finite as \( \xi \) varies. On the other hand, the second term represents the travel time from the source to a fixed point at depth to a receiver. If the source/receiver array were of infinite extent, this travel time would increase beyond all bounds as the source/receiver pair moves off towards infinity in either direction. However, this travel time would achieve at least one local extremum (a minimum) at some finite value of \( \xi_1 \). Thus, the travel time difference will approach \(-\infty\) at the extremes and reach some finite maximum for some value(s) \( \xi_1 \). If this \( \xi_1 \) is in the range of integration, that is, in the range of source/receiver pairs for which data was collected, then the integral has a stationary point. If this \( \xi \) is not in the range, then, for that choice of \( \xi \), there is no stationary point and the contribution to the total integral is of lower order. We proceed as if there is an interior stationary point, \( \xi_1 = \ell(\xi_1) \).

Note that if \( x \) is on the reflector, then the \( \xi_1 \) and \( \ell(\xi_1) \) for which this point is the specular reflection point satisfies both stationary phase conditions. An easy way to see this is to note that in this case, the rays from \( x \) and \( x_R \) to the source and receiver are the same and are specular. The fact that they are specular makes \( \Phi_1 \) stationary; the fact that these two points are the same makes their derivatives with respect to \( \xi_1 \) the same and the difference of derivatives appearing in \( \partial \Phi_2 / \partial \xi_1 \) is then equal to zero. It is for this reason that the difference of second derivatives are combined into the expression \( \Delta \tau_1 \). This difference is equal to zero on the reflector and, therefore, near zero for \( x \) near the reflector. This will be important, below.

In order to determine the second derivative at stationarity, the first derivative of \( \ell \) with respect to \( \xi_1 \) is needed. This derivative is determined by first setting \( \partial \Phi_1 / \partial t = 0 \) in (23) and then differentiating implicitly with respect to \( \xi_1 \). This leads to the solution,
\[ \frac{d\ell}{d\xi_1} = -\frac{\partial^2 \tau_1(x_R(\ell(\xi_1)), \xi_1)}{\partial \xi_1 \partial x_R} \frac{dx_R}{dt} \frac{d\ell}{dt} \frac{d\xi_1}{dt}. \]

With this result, (27) is replaced by
\[ \frac{\partial^2 \Phi_2}{\partial \xi_1^2} = \left[ \frac{\partial^2 \tau_1(x_R(\ell(\xi_1)), \xi_1)}{\partial \xi_1 \partial x_R} \frac{dx_R}{dt} \frac{d\ell}{dt} \frac{d\xi_1}{dt} \right]^2 \frac{[\Phi''_{\xi}]}{[\Phi''_{\xi}]}, \] (29)
\[ + \Delta \tau_1. \]

The first factor on the right, here, can be simplified as follows,
\[ \frac{\partial^2 \tau_1(x_R(\ell(\xi_1)), \xi_1)}{\partial \xi_1 \partial x_R} \frac{dx_R}{dt} \frac{d\ell}{dt} \frac{d\xi_1}{dt} = \frac{\partial \nabla_x \tau_1}{\partial \ell} \cdot \frac{dx_R}{dt} \frac{d\ell}{dt} \frac{d\xi_1}{dt}, \] (30)
\[ = \frac{\partial \nabla_x \tau_1}{\partial \ell} \times \hat{n}_x \]
\[ = \frac{\partial \nabla_x \tau_1}{\partial \ell} \times \nabla_x \tau_1 \]

where \( \hat{n}_x \) is the unit normal vector to the reflector.
In the second line, the two dimensional tangent to the reflector has been replaced by the two-dimensional normal to the reflector. In the next line, the colinearity (within a sign) of the surface normal and the traveltime gradient at stationarity is used. The last line, in turn, rewrites this two dimensional cross product as a determinant, the same Belykin determinant as appears in the inversion formula. However, it is now evaluated at the point \( x_R \) on the reflector, subject to the two stationarity conditions, above. Now (30) can be rewritten as

\[
\Phi_2' = -\left( \frac{|H(x_R, \xi_I)|}{|\nabla_x \tau_I|} \right)^2 [\Phi_1''']^{-1} + \Delta \tau_I. \tag{32}
\]

As with \( \Phi_1 \), the notation, \( \Phi_2'' \), is introduced for the evaluation of the second derivative at the stationary point. We remark that for \( x \) near the reflector, this second derivative is dominated by the first term and

\[
\text{sgn} (\Phi_2'') = -\text{sgn} (\Phi_1'''),
\]

while this sign might change "sufficiently far" from the reflector, presumably, more than three wavelengths away, for the sake of asymptotic analysis. The discussion of this possible latter region is postponed until later, and the analysis proceeds in the restricted range where the signs of the second derivative satisfy the stated relationship, above. In this case, application of the method of stationary phase to (24) leads to the result,

\[
u_0(\xi_O, \omega_O) = -2\pi \sqrt{\omega_O} e^{-i\pi \text{sgn}(\omega_O)/4} \cdot \int F(\omega) d\omega_1 d^2 x \frac{e^{i\Phi'}}{\sqrt{|\Phi_1''|}}. \tag{33}
\]

Here, the amplitude and the phase are to be evaluated at the dual stationary points in \( \ell \) and \( \xi_I \).

The dependence on \( \omega_I \) has not become particularly simple. There is the linear dependence in \( \Psi \), as defined by (20), and also the amplitude factor, \( F(\omega) \). If \( F = 1 \), the \( \omega_I \)-integration yields a delta function. We take the point of view that \( F \) is a filter that leads to a bandlimited version of the delta function that we will denote by \( \delta_B : \)

\[
\delta_B(t) = \frac{1}{2\pi} \int F(\omega) e^{-i\omega t} d\omega. \tag{34}
\]

By using this identity to carry out the \( \omega_I \) integration in (33), we obtain

\[
u_0(\xi_O, \omega_O) = -4\pi^2 \sqrt{\omega_O} e^{-i\pi \text{sgn}(\omega_O)/4} \cdot \int d^2 x \frac{e^{i\Phi'}}{\sqrt{|\Phi_1''|}} \frac{\delta_B(\tau_I(x_R(\ell), \xi_I) - \tau_I(x, \xi_I))}{|\nabla_x \tau_I|}. \tag{35}
\]

The last factor here is a scalar delta function. Its argument is zero when \( x \) is on the reflector where the stationary conditions yield \( x = x_R \) and the value of \( \xi_I \) makes the corresponding source/receiver pair specular. Furthermore, this zero is isolated; the gradient of the argument is just the gradient of the travel time, which is normal to the reflector. Thus, the direction of maximal change of argument of the delta function is initially normal to the reflector. Within a scale factor, then, this delta function is the singular function of the surface, \( S_R \). The scale factor is just the magnitude of the gradient of the travel time, that is,

\[
\delta_B(\tau_I(x_R(\ell), \xi_I) - \tau_I(x, \xi_I)) = \frac{|\delta(\eta_R)\nabla_x \tau_I(x_R(\ell), \xi_I)|}{|\nabla_x \tau_I|}. \tag{36}
\]

Consequently, replacing the bandlimited delta function by the delta function, itself, (35) can be rewritten as

\[
u_0(\xi_O, \omega_O) = -4\pi^2 \sqrt{\omega_O} e^{-i\pi \text{sgn}(\omega_O)/4} \cdot \int d^2 x \frac{e^{i\Phi'}}{|\nabla_x \tau_I|} \frac{\delta_B(\tau_I(x_R(\ell), \xi_I))}{|\nabla_x \tau_I|}. \tag{37}
\]

In this equation, the stationary conditions define \( \xi_I = \xi_I(\ell) \), choosing the value of \( \xi_I \) for which the input source/receiver pair are specular at \( x_R(\ell) \). Now, the amplitude in this equation must be evaluated at stationarity and for \( x = x_R \). In this limit, from (21),

\[
\mathcal{G} = \frac{R(x_R(\ell), x_R(\xi_I))}{4\pi^2} a_O(x_R, \xi_O)
\]

\[
\cdot \nabla_x \tau_I(x_R(\ell), \xi_I) \left| \frac{\nabla_x \tau_I(x_R, \xi_O)}{\nabla_x \tau_I(x_R, \xi_I)} \right| \frac{\sigma_{O_R}(x_R, \xi_O) \sigma_{O_P}(x_R, \xi_O)}{\sigma_{O_R}(x_R, \xi_O) + \sigma_{O_P}(x_R, \xi_O)} |H(x, \xi_I)|. \tag{38}
\]

Furthermore, the term, \( \Delta \tau_I \), defined by (28), is zero and, from (32),

\[
|\Phi_1''| = \frac{|H(x_R, \xi_I)|}{|\nabla_x \tau_I|}. \tag{39}
\]

These results are used in (37) to obtain
\[ u_0(\xi_0, \omega_0) = -\sqrt{\omega_0} e^{-i \pi \text{sgn}(\omega_0)/4} \cdot \int dl R(x_0(\ell), z_0(\ell)) a_0(x_0, \xi_0) \cdot \nabla_T \cdot \nabla_T \left( \sum_{\ell} \frac{V_{T0}(x_0, z_0)}{V_{T1}(x_0, \xi_1)} \right) \]

\[ \cdot \frac{\sqrt{\sigma_0(x_0, \xi_0) \sigma_0(x_0, \xi_0)}}{\sqrt{\sigma_0(x_0, \xi_0) + \rho_0(x_0, \xi_0)}} e^{i \omega_0 \tau_0(x_0, \xi_0)} \]

Since the integrand is evaluated subject to the stationarity relation between \( \ell \) and \( \xi_1 \),

\[ \nabla_T \cdot \nabla_T \left( \sum_{\ell} \frac{V_{T0}(x_0, z_0)}{V_{T1}(x_0, \xi_1)} \right) = -|\nabla_T \cdot \nabla_T \left( \sum_{\ell} V_{T0}(x_0, z_0) \right)|. \]

Just as in (4), we set

\[ |\nabla_T \cdot \nabla_T \left( \sum_{\ell} V_{T0}(x_0, z_0) \right)| = -\nabla_T \cdot \nabla_T \left( \sum_{\ell} V_{T0}(x_0, z_0) \right). \]

With these substitutions,

\[ u_0(\xi_0, \omega_0) = -\sqrt{\omega_0} e^{-i \pi \text{sgn}(\omega_0)/4} \cdot \int R(x_0(\ell), z_0(\ell)) \cdot \nabla_T \cdot \nabla_T \left( \sum_{\ell} V_{T0}(x_0, z_0) \right) a_0(x_0, \xi_0) \cdot \frac{\sqrt{\sigma_0(x_0, \xi_0) \sigma_0(x_0, \xi_0)}}{\sqrt{\sigma_0(x_0, \xi_0) + \rho_0(x_0, \xi_0)}} e^{i \omega_0 \tau_0(x_0, \xi_0)} dl. \]

The Kirchhoff representation in the input source/receiver coordinates has been transformed into the Kirchhoff representation in the output source/receiver coordinates. In obtaining this result, a region of the \( x \)-domain where \( \text{sgn}(\Phi_0'') = \text{sgn}(\Phi_1') \) has been neglected. In this region, the delta function in time is replaced by a principal-value-1/\( t \) function. This function also has its singular support centered around \( t = 0 \). However, the \( x \)-domain where this is the correct value for the signature is bounded away from \( t = 0 \), which corresponded to the neighborhood of the reflector. Thus, any contribution that might be obtained from this combined integration over \( \omega_0 \) and \( x \) will be of lower order asymptotically than the result given here.

The source signature of the input data, \( F(\omega) \) in (18), was used to define the bandlimited delta function that confined the \( x \)-domain integration to the reflecting surface. This, again, is a leading order asymptotic result. In another context, Tygel has suggested that such operators should be viewed as providing a “sinc-like” interpolation of the data in the neighborhood of the peak of the signal. This means an interpolation of the spatial part of the operator, appearing in lines two and three of (16), in the neighborhood of the reflector.

It is well known that the Kirchhoff integral provides the leading order asymptotic expansion of the return from specular reflections. Hence, travel time and all geometrical spreading and curvature effects, including effects of “buried foci,” caustics produced by synclises, will be properly transformed by the KDM process. Where the caustic pierces the upper surface, the arrival time is expected to be accurate, but no claims are made about the accuracy of the amplitude. The factor, \( \Phi_0'' \), is zero in this case and the asymptotic analysis is invalid. However, it produces an integrable singularity in the Kirchhoff integral, with the correct travel times in the phase, hence, our claim that the arrival time is correct, but the amplitude need not be.

For edge-diffracted returns, the Kirchhoff integral produces the correct arrival time, but an inaccurate diffraction coefficient, except at the shadow boundary of the last reflected ray. Thus, the mapped data is expected to contain mapped diffraction arrival times with inaccurate amplitudes.

There is another source of “error” in the amplitude. Note that the reflection coefficient is evaluated at an incidence angle associated with the input source/receiver configuration, through its dependence on the stationarity value of \( \xi_1 \). For correctly mapped data, it would be preferable to have this dependence mapped to \( \xi_0 \). However, this is simply not the case. The input reflection coefficient is preserved, not mapped. This is known from the TZO case and is therefore not surprising in this general result.

In summary, we have shown that the leading order asymptotic input data is mapped to the leading order asymptotic output data, except for the reflection coefficient, which maintains its input value everywhere.

**Conclusions**

We have derived platforms for 3D and 2.5D KDM of scalar wavefields. The formalism assumes knowledge of a physical model for both the input and output data and prescribed input and output source/receiver configurations. By cascading an inversion formula with a modeling formula, we obtain the KDM platform formula. This cascade is a single reflector formalism in the absence of multiple reflections and multi-pathing. In that sense, it is still a somewhat limited result, at the level of generality of standard migration or DMO formalisms.

In the absence of a specific application, the formula includes a multifold integration over the physical model space that must be evaluated asymptotically for each ex-
ample of KDM in order to derive a computationally feasible formalism for implementation. Application of this formalism in constant background 2.5D DMO produces the same formula as was derived in Bleistein, et al., [1998]. This is a straightforward exercise that is not included in this paper.

On the other hand, we show how Kirchhoff approximate model data in a given input configuration is mapped to Kirchhoff data in a different output configuration for the 2.5D case. We have done this in great generality, without specifying any particular configuration transformation. From this result, we conclude that the travel time and geometrical spreading effects of the input model are properly mapped to their counterparts in the output model, while the reflection coefficient is not.

In future papers, we will specialize the mapping platforms to achieve specific KDM formulas. Work is currently in progress on 3D constant background DMO and wave-equation-datuming.

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Approximate dispersion relations for qP-qSV waves in transversely isotropic media

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ABSTRACT

A sequence of rational approximations of the solution of the dispersion relation for a TI medium in the square slownesses is introduced. The zeroth order approximation is merely that of elliptical TI anisotropy. The first order approximation yields the correct values, slopes, and curvatures for the slowness at vertical and grazing incidence; subsequent orders of approximation yield the correct third, fourth, . . . derivatives also. The approximations differ from parabolic approximations in that they are expansions about vertical and grazing incidence combined. Each rational approximation expresses the square of the vertical slowness as an explicit rational function of the square of the horizontal slowness, thus retaining the property of employing a single square root in the calculation of the slowness surface, and conserving branch points. Thus, implementation of various f-k migration schemes for a rational approximation of transverse isotropy is as straightforward as for isotropic media. The rational approximations account for possible off-axis qSV wave triplication and are particularly accurate for qP propagation in TI media with positive anellipticity, such as occurs in most shales. For positive anellipticity and small negative anellipticity, a rational approximation, valid for any direction of propagation, can be generated for any desired accuracy. For practical purposes, the first order approximation for qP propagation and the second order approximation for qSV propagation compare favorably with the exact slowness curve for the range of anisotropy encountered in many shales which occupy much of the volume of the world’s sedimentary basins.

Introduction

For velocity analysis and migration in transversely isotropic (TI) media, it is useful to have a simple (i.e., with the root structure of an isotropic medium) and accurate algorithm to calculate vertical slowness as a function of horizontal slowness. In isotropic and TI media, with no loss of generality, one can confine oneself to propagation in a single vertical plane (assuming the medium’s axis of symmetry is vertical), and that plane will be denoted here as the $(x, z)$-plane, with the $z$-direction taken as vertical. For an isotropic medium, this dispersion, or slowness, relation is of the form,

$$s_z^2 = \frac{1}{\alpha^2} - s_x^2,$$

a straight line in the $(s_x^2, s_z^2)$-plane, where $s_x$ is the horizontal slowness and $s_z$ is the vertical slowness. For elliptical transverse isotropy (a very special case), the slowness relation is also a straight line, albeit with a different slope, because of the difference between horizontal and vertical wave speeds.

In a general TI medium, each slowness curve in the $(s_x^2, s_z^2)$-plane is part of a conic section; for rock-like media, the conic section is usually a hyperbola, one branch for the qP dispersion relation, the other for the qSV. Although closed form expressions for this hyperbola are known, they are not so easy to work with, as they themselves involve a square root. On the other hand, we have developed an interest in understanding rational approximations of slowness surfaces in anisotropic media, in particular for shear waves that exhibit singularities. In this note, a sequence of expressions will be derived giving an approximate relation for $s_z^2$ as a sum of rational functions of $s_x^2$ and a dimensionless anellipticity parameter, $\epsilon_A$. The $n$th order approximation has the form,
\[ s^2 = \frac{1}{\alpha V} \left[ 1 - \alpha_H^2 s^2 + b_1 \frac{N(s^2)\varepsilon_A}{D(s^2; \varepsilon_A)} + b_2 \frac{N^2(s^2)\varepsilon_A^2}{D^2(s^2; \varepsilon_A)} + \ldots + b_n \frac{N^n(s^2)\varepsilon_A^n}{D^{2n-1}(s^2; \varepsilon_A)} \right], \]

where the \( b_j \) are constants, \( N = \alpha_H^2 s^2 (1 - \alpha_H^2 s^2) \) and \( D \) is a linear function of \( s^2 \) in which the coefficient of the linear term is itself a linear function of \( \varepsilon_A \). For weak anisotropy, by definition, only a single rational term is needed. For usual shales, which often have significant positive anisotropy, one, or at most two, rational terms should be sufficient for the qP slowness relation, while two, or at most three, terms should be sufficient for the qSV slowness relation.

The approximations are based on the Taylor series expansion of \( 1 - \sqrt{1 - \zeta} \) in the small quantity \( \zeta \). The derivation of the approximations and their properties in the complex plane will follow a discussion of 'mild' anisotropy and the introduction of a set of dimensionless anisotropy parameters suitable to describing the anisotropic behavior at grazing as well as vertical incidence, and at all angles in between.

The rational approximations will then be compared with the slowness curve from Muir's elegant and quite accurate implicit bi-elliptic approximation (Dellinger et al., 1993) after a short discussion of its properties. Applications to f-k migration are then discussed. Approximate dispersion curves are shown for Greenhorn shale, and for media which are perturbations to it.

**Exact dispersion relations**

For a TI medium, the stress-strain relation, in condensed notation, is

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\sigma_{xy} \\
\sigma_{xz} \\
\sigma_{yz}
\end{bmatrix} = \rho
\begin{bmatrix}
c_{11} & c_{11} - 2c_{66} & c_{13} & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3 \\
\epsilon_4 \\
\epsilon_5 \\
\epsilon_6
\end{bmatrix},
\]

where, according to convention for condensed notation,

\[
[\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 \epsilon_5 \epsilon_6]^t \equiv [\epsilon_{xx} \epsilon_{yy} \epsilon_{zz} 2\epsilon_{xy} 2\epsilon_{xz} 2\epsilon_{yz}]^t,
\]

superscript \( t \) denoting the transpose. The \( c_{ij} \) are the stiffness moduli divided by density \( \rho \) so they have dimension velocity\(^2\), and hence will be called the squared velocity moduli. For positive strain energy to be guaranteed, i.e., for the medium to be stable, the \( 6 \times 6 \) elastic squared velocity matrix must be positive definite. We use \( c_{55} \) (instead of \( c_{66} \)) for the vertical shear velocity squared and the horizontal qSV velocity squared since we are discussing propagation in the \((x, z)\)-plane, and from (1) subscripts \( zz \) \( \rightarrow \) 5.

Assume a plane wave propagating in the \((x, z)\)-plane, with an associated particle motion lying in that plane. The displacement of such a plane wave, of arbitrary signature \( W \), may be written,

\[
\begin{bmatrix}
u_x \\
u_z
\end{bmatrix} W(s_x x + s_z z - t).
\]

Substitution of plane wave form (2) and stress-strain relation (1) into the equations of motion for a homogeneous elastic medium yields the Christoffel equations for waves with their polarization in the plane of propagation:

\[
\begin{bmatrix}
c_{11}s^2 + c_{55}s^2 - 1 & (c_{55} + c_{13})s_z s_x \\
(c_{55} + c_{13})s_z s_x & c_{55}s^2 + c_{33}s^2 - 1
\end{bmatrix}
\begin{bmatrix}
u_x \\
u_z
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \end{bmatrix}.
\]

These equations completely describe qP and qSV wave propagation in the \((x, z)\)-plane which depends only on the four squared velocity moduli, \( c_{11}, c_{33}, c_{55} \) and \( c_{13} \). The slowness curve for these waves is the solution of the dispersion relation which expresses the vanishing of the determinant of the \( 2 \times 2 \) matrix of coefficients acting on the polarization vector \([u_x, u_z]^t\) in (3). This dispersion relation:

\[
c_{11}c_{55}(s^2)^2 + [(c_{11} + c_{33})c_{55} + E^2]s^2s^2 + c_{53}c_{55}(s^2)^2
\]

\[-(c_{11} + c_{55})s^2 - (c_{33} + c_{55})s^2 + 1 = 0,
\]

where,
\[ E^2 \equiv (c_{11} - c_{55})(c_{33} - c_{55}) - (c_{13} + c_{55})^2. \]

Note that \( E^2 \) is already introduced by Gassmann (1964), can be positive, negative or zero, but has the dimension of the square of squared velocity moduli; hence this notation. Note that \( c_{11} = c_{33} \) and \( E^2 = 0 \) are necessary and sufficient conditions for P-SV wave isotropy.

It is important to emphasize at this stage the physical interpretation of \( E^2 \). First note that the intersections of the slowness curves with the coordinate axes are fixed by the values of \( c_{11}, c_{33} \) and \( c_{55} \). These intersections with the coordinate axes may be called the ‘anchor points’, and for the qP slowness curve, the anchor points are \([1/\sqrt{c_{11}}, 0]\) and \([0, 1/\sqrt{c_{33}}]\), while for the qSV slowness curve, they are \([1/\sqrt{c_{55}}, 0]\) and \([0, 1/\sqrt{c_{55}}]\). The value of \( E^2 \), which depends on the somewhat enigmatic modulus \( c_{13} \), determines the shape of the slowness curves between the anchor points. \( E^2 \) increases as \( c_{13} \) decreases, until \( c_{13} + c_{55} = 0 \). The value of \( E^2 \) does not depend on the sign of \( c_{13} + c_{55} \), and for still more negative values of \( c_{13}, E^2 \) starts to decrease again. Thus, the slowness curves and wavefronts are independent of the sign of \( c_{13} + c_{55} \) although the polarization is strongly affected; \( c_{13} + c_{55} < 0 \) is associated with ‘anomalous polarization’, see Helbig and Schoenberg (1986).

The value of \( E^2 \) controls the bulging of the slowness curves, and hence the possible triplication of the qSV wavefront. The special case \( E^2 = 0 \) is the case of elliptical anisotropy. Then the qP slowness curve is an ellipse connecting the qP anchor points while the qSV slowness curve is a circle connecting the qSV anchor points. When \( E^2 > 0 \), the qP slowness curve bulges out from the ellipse connecting the qP anchor points (the plane waves thus are slower) between the anchor points, while \( E^2 < 0 \) implies that this slowness curve is pulled in from the ellipse (the plane waves thus are faster) between the anchor points. On the other hand, for the qSV slowness curve, when \( E^2 > 0 \), the qSV slowness curve is pulled in from the circle that connects the qSV anchor points. If \( E^2 \) is large enough, that qSV curve is pulled in enough to allow it to become concave in an angular region centered on a given oblique direction. This concavity, or change in the sign of the curvature, is manifest in the qSV wavefront by the presence of triplication centered about that oblique direction. Transversely isotropic shales often exhibit such concave regions of the qSV slowness curve. No other wavefront can triplicate.

Transversely isotropic rocks almost always have \( E^2 > 0 \). There are several physical mechanisms that cause positive \( E^2 \). Using equivalent medium theory, it can be shown that any transversely isotropic medium that is equivalent in the long wavelength limit to a stationary finely layered medium made up of isotropic layers, \( E^2 \) must be positive (see, for example, Schoenberg (1994)). In addition, it can be shown that a transversely isotropic medium that is equivalent to an isotropic medium in which are embedded a set of parallel linear slip planes with an axisymmetric compliance matrix gives \( E^2 \) with the same sign as \( Z_T - Z_N \), where \( Z_T \) is the tangential compliance, and \( Z_N \) is the normal compliance, of the slip interfaces per unit distance perpendicular to the interfaces. Thus, \( E^2 = 0 \) when \( Z_T = Z_N \), which implies the traction and displacement discontinuity across the slip interfaces are collinear.

The fit of any approximation will not depend on the actual magnitudes of the four relevant elastic moduli, but on the shape of the solution for the slowness curves, which can depend on at most three parameters. To this end, consider the following notation:

1) \( C \) to denote the arithmetic mean of \( c_{11} \) and \( c_{33} \), i.e.,
\[
C \equiv \frac{1}{2} (c_{33} + c_{11}).
\]  \hspace{1cm} (5)

\( C \), the only dimensional parameter, provides the scaling required, but has no effect on the shape of the solution to the dispersion relation.

2) \( \gamma \) (unrelated to Thomsen’s (1986) parameter used for the ellipticity of qSH waves) to denote the common vertical and horizontal shear velocity squared, normalized by \( C \), i.e.,
\[
\gamma \equiv \frac{c_{55}}{C}.
\]  \hspace{1cm} (6)

\( \gamma \) is the ratio of the square of the shear speed along the coordinate axes to the mean of the square of the compressional speeds along the coordinate axes, analogous to \((u_S/u_P)^2\) for isotropic media.

3) \( \epsilon_P \) to designate the relative difference between \( c_{11} \), the horizontal qP velocity squared, and \( c_{33} \), the vertical qP velocity squared, i.e.,
\[
\epsilon_P \equiv \frac{1}{2} \left( \frac{c_{11} - c_{33}}{C} \right),
\]  \hspace{1cm} (7)

Note that since \( c_{11}, c_{33} > 0 \), \( |\epsilon_P| < 1 \). Positive \( \epsilon_P \) denotes the usual case when the medium’s horizontal P wave speed is greater than its vertical P wave speed. \( \epsilon_P \) is a renormalized version of Thomsen’s (1986) parameter \( \epsilon^T \);
\[
\epsilon^T \equiv \frac{c_{11} - c_{33}}{2c_{33}} = \frac{\epsilon_P}{1 - \epsilon_P}, \quad \text{and} \quad \epsilon_P = \frac{\epsilon^T}{1 + \epsilon^T}.
\]
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Since the rational approximations will be seen to be applicable over the entire range of direction, out to horizontal, the use of \( \varepsilon_P \), which is normalized symmetrically with respect to horizontal and vertical compressional moduli, is preferable here. For 'weak' anisotropy, \( \varepsilon_P \approx \varepsilon_T \ll 1 \).

4) \( \varepsilon_A \) to denote a dimensionless version of \( E^2 \), i.e., a normalized version, such that its maximum value (assuming \( c_{11} \) and \( c_{33} \) are both greater than or both less than \( c_{55} \),) over all values of \( c_{12} \), holding \( c_{11}, c_{33} \) and \( c_{55} \) constant, is unity, i.e.,

\[
\varepsilon_A \equiv \frac{E^2}{E_{\text{max}}} = \frac{(c_{11} - c_{55})(c_{33} - c_{55}) - (c_{13} + c_{55})^2}{(c_{11} - c_{55})(c_{33} - c_{55})} \leq 1 .
\]

The minimum value of \( E^2 \) and \( \varepsilon_A \) occurs when \( c_{13} + c_{55} \) is as large as possible. Since stability requires \( c_{13}^2 < (c_{11} - c_{66})c_{33} \), and since it is allowable for \( c_{66} \to 0 \), the absolute minimum value of \( \varepsilon_A \) is,

\[
\varepsilon_{A_{\text{min}}} = \frac{(c_{11} - c_{55})(c_{33} - c_{55}) - (\sqrt{c_{11}c_{33}} + c_{55})^2}{(c_{11} - c_{55})(c_{33} - c_{55})} = \frac{-c_{55} + c_{11} + 2\sqrt{c_{11}c_{55}}}{(c_{11} - c_{55})(c_{33} - c_{55})} = -2\gamma - \frac{1 + \sqrt{1 - \varepsilon_P^2}}{(1 - \gamma)^2 - \varepsilon_P^2} .
\]

However, a reasonable assumption for sedimentary rocks is that \( c_{66} > c_{55} \). Then, letting \( c_{66} \to c_{55} \), stability would require that \( c_{13}^2 < (c_{11} - c_{55})c_{33} \) and the minimum value of \( \varepsilon_A \) becomes,

\[
\varepsilon_{A_{\text{min}}} = \frac{(c_{11} - c_{55})(c_{33} - c_{55}) - (\sqrt{c_{11}c_{33}} + c_{55})^2}{(c_{11} - c_{55})(c_{33} - c_{55})} = \frac{-c_{55} + c_{11} + 2\sqrt{c_{11}c_{55}}}{(c_{11} - c_{55})(c_{33} - c_{55})} = -2\gamma + \frac{1 + \varepsilon_P + 2\sqrt{1 - \varepsilon_P^2} - \gamma(1 - \varepsilon_P)}{(1 - \gamma)^2 - \varepsilon_P^2} .
\]

Anellipticity parameter \( \varepsilon_A \) may be written in terms of Thomsen's parameters,

\[
\varepsilon_A = \frac{2c_{55}}{c_{11} - c_{55}}(\varepsilon_T - \delta_T) .
\]

The key dimensionless parameter for time processing of qP waves in TI media has been found by Alkhalifah and Tsvankin (1995) to be \( \eta \equiv (\varepsilon_T - \delta_T)/(1 - 2\delta_T) \) which is equivalent to \( \delta_T = (\varepsilon_T - \eta)/(1 + 2\eta) \). Thus, anellipticity parameter \( \varepsilon_A \) is related to \( \eta \) by

\[
\varepsilon_A = \frac{2c_{11}}{c_{11} - c_{55}} \eta \left( \frac{1}{1 + 2\eta} \right) ,
\]

and since stability implies \( 1 + 2\eta > 0 \), we see that \( \varepsilon_A, \varepsilon_T - \delta_T \) and \( \eta \) are of the same sign. Note that \( \varepsilon_A \) depends explicitly on shear-wave velocity through \( c_{55} \), whereas \( \eta \) does not.

The parameters \( \varepsilon_P \) and \( \varepsilon_A \) specify the qP-qSV wave anisotropy; the vanishing of both, together with the vanishing of a third anisotropy parameter proportional to \( c_{66} - c_{55} \), associated with the elliptical SH wave, implies isotropy. Each of these parameters control different aspects of the anisotropy, and the effects can be seen clearly by letting these parameters vary independently.

From equations (5), (6), (7) and (8), the squared velocity moduli normalized by \( C \) may be written in terms of the dimensionless parameters \( \gamma, \varepsilon_P \) and \( \varepsilon_A \) as,

\[
\frac{c_{11}}{C} = 1 + \varepsilon_P, \quad \frac{c_{33}}{C} = 1 - \varepsilon_P, \quad \frac{c_{55}}{C} = \gamma, \quad \frac{c_{13}}{C} = \pm \sqrt{(1 - \gamma)^2 - \varepsilon_P^2}(1 - \varepsilon_A - \gamma) .
\]

In the next section, we will constrain the value of \( c_{13} \) so that only the positive root will be allowed.

Mild anisotropy

It is useful to impose a set of conditions to restrict the range of TI media that will be considered. The restrictive conditions will not be of the form that certain parameters are small enough so that squares of the parameters can be neglected, i.e., they will not be conditions of 'weak anisotropy'. The conditions will limit the range of allowable elastic behavior, while including the commonly assumed properties of geological TI media. Media satisfying this set of conditions — which are concerned with 1) the ratio of shear to compressional velocities, 2) polarization (or a stronger condition on apparent Poisson's ratio), and 3) triplication — are called 'mildly anisotropic'; see, for example, Carrion et al (1992). The conditions are:
1) The slowest compressional wave along any coordinate axis is faster than the fastest shear wave along any coordinate axis, which is equivalent to
$$\max[c_{55}, c_{66}] < \min[c_{11}, c_{33}],$$
or, in terms of the dimensionless parameters (ignoring the constraint on $c_{66}$)
$$\gamma < 1 - |\epsilon_P|.$$  \hspace{1cm} (12)

2) In any direction, if a longitudinal wave and a transverse wave polarized in the vertical plane exist, the longitudinal wave is always faster than the transverse wave. This essentially states that anomalous polarization is not allowed, which is equivalent to
$$c_{13} + c_{55} > 0$$ \hspace{1cm} (13)
(Helbig & Schoenberg, 1986). This provides no condition on $\epsilon_A$, since that parameter is a function of $(c_{13} + c_{55})^2$, but it requires the use of the positive square root in the fourth of equations (11). A useful, and somewhat stronger condition one might choose to impose is that, for a rod of the TI medium, with its axis parallel to the symmetry axis, the Poisson’s ratio is positive, which is equivalent to positive $c_{13}$. In terms of the dimensionless parameters, this is equivalent to,
$$\epsilon_A < 1 - \frac{\gamma^2}{(1 - \gamma)^2 - \epsilon_P^2},$$  \hspace{1cm} (14)
which then supercedes inequality (13).

3) The simplest condition concerning triplication would be merely that there is no triplication, and hence no concavity of the qSV slowness curve. However, recent evidence shows that shales often violate the ‘no triplication’ criterion — according to measurements on various shales, both in the laboratory (for example on Greenhorn shale (Jones & Wang, 1981) and in various case studies with in situ measurements (for example Miller et al. (1993)). All the evidence for the presence of a concave region of the qSV slowness curve in certain shales occurs for the case of positive anellipticity, implying that the triplicating region is centered about an oblique direction (near 45°) between the vertical axis of symmetry and the horizontal axis.

This is not too serious for the rational approximations we are proposing, since the horizontal and vertical components of the squared qSV slowness (and thus of the qSV slowness in each quadrant) are still in a one-to-one relationship with each other, as demonstrated by the squared slowness curve in Figure 1 a). This one-to-one relationship of the qSV curve still exists for moderately negative anellipticity, Figure 1 b), but this relationship is violated if there is a triplicating region centered about the horizontal and/or vertical axis which occurs for strongly negative anellipticity. Figure 1 c) shows a case of strong negative anellipticity which triplicates about both the horizontal and vertical axes. This is manifest by the positive slope of the qSV squared slowness curve at both the horizontal and vertical axes. Thus the mild anisotropy condition concerning the absence of qSV triplication will be relaxed to read that there is no triplication centered on either the vertical or the horizontal axis. No triplication about the vertical z-axis is equivalent to $c_{13} + c_{55} < \sqrt{c_{11}(c_{33} - c_{55})}$ or, in terms of $\epsilon_A$,
$$\epsilon_A > -\frac{c_{55}}{c_{11} - c_{55}} = -\frac{\gamma}{1 + \epsilon_P - \gamma} \equiv \epsilon_{A_{no\ tripl}},$$
no triplication about the horizontal z-axis is equivalent to $c_{13} + c_{55} < \sqrt{c_{33}(c_{11} - c_{55})}$ or,
$$\epsilon_A > -\frac{c_{55}}{c_{33} - c_{55}} = -\frac{\gamma}{1 - \epsilon_P - \gamma} \equiv \epsilon_{A_{no\ tripl}},$$
see for example Payton (1983). For our purpose, these inequalities can be combined to yield,
$$\epsilon_A > -\frac{c_{55}}{\max[c_{11}, c_{33}]} - c_{55} = -\frac{\gamma}{1 + |\epsilon_P| - \gamma} \equiv \epsilon_{A_{no\ tripl}}.$$ \hspace{1cm} (15)
It is worthwhile to point out here that if $c_{33} < c_{11}$, i.e., $\epsilon_P > 0$, triplication occurs first (as $B^2$ decreases or equivalently as $c_{13}$ increases) at the vertical axis, while for $c_{33} > c_{11}$, i.e., $\epsilon_P < 0$, triplication occurs first at the horizontal axis.

For mild anisotropy, with the strong condition that $c_{13}$ be positive, and with the weak condition that no triplication occur about either of the coordinate axes, from inequalities (14) and (15), the restriction on anellipticity $\epsilon_A$ becomes,
$$-\frac{\gamma}{1 + |\epsilon_P| - \gamma} < \epsilon_A < 1 - \frac{\gamma^2}{(1 - \gamma)^2 - \epsilon_P^2}.$$ \hspace{1cm} (16)
Normalized dispersion relations

The rational approximation will be taken about the case of zero anellipticity, i.e., the elliptical anisotropy case. When $E^2 = 0$, the two roots of quadratic equation (4) are,

qP: \[ s^2_{qP} = \frac{1 - c_{11}s^2_e}{c_{33}} ; \]

qSV: \[ s^2_{qSV} = \frac{1}{c_{55}} - s^2_e , \]

clearly an ellipse and a circle, respectively, in the slowness domain. Both solutions are straight lines in the squared slowness plane which intersect each other at $s^2_e = (1 - c_{33}/c_{55})/(c_{11} - c_{33})$. Note that for the usual case when $c_{11} > c_{33}$, this intersection occurs at a negative value of $s^2_e$, i.e., at an imaginary value of horizontal slowness. When $c_{11} < c_{33}$, this intersection occurs at a value of $s^2_e$ greater than $1/c_{55}$, i.e., at a real value of horizontal slowness greater than the grazing slowness for qSV waves.

Equation (4) will be non-dimensionalized by stretching the squared slowness axes suitably, i.e., depending on whether we are interested in the qP or qSV slowness curve. In both cases, the curve in the stretched squared slowness plane about which we are perturbing will be a straight line connecting points (0,1) and (1,0). To this end, we set,

qP: \[ X \equiv c_{11}s^2_e , \quad Z \equiv c_{33}s^2_e ; \]

qSV: \[ X \equiv c_{55}s^2_e , \quad Z \equiv c_{55}s^2_e . \]

The dispersion relations become,

qP: \[ \frac{c_{55}}{c_{11}} X^2 + \left[ \frac{c_{55}}{c_{11}c_{33}} + \frac{E^2}{c_{11}c_{33}} \right] XZ + \frac{c_{55}}{c_{33}} Z^2 \]

\[ - \left( 1 + \frac{c_{55}}{c_{11}} \right) X - \left( 1 + \frac{c_{55}}{c_{33}} \right) Z + 1 = 0 ; \]

qSV: \[ \frac{c_{11}}{c_{55}} X^2 + \left[ \frac{c_{11} + c_{33}}{c_{55}} + \frac{E^2}{c_{55}} \right] XZ + \frac{c_{33}}{c_{55}} Z^2 \]

\[ - \left( 1 + \frac{c_{11}}{c_{55}} \right) X - \left( 1 + \frac{c_{33}}{c_{55}} \right) Z + 1 = 0 . \]

Now, let

\[ Z = 1 - X + f(X; E^2) , \]

and note that for $E^2 = 0$ we indeed obtain the straight line solutions $Z = 1 - X$. Substituting equation (20) into equation (19) yields a quadratic equation in the perturbation $f$ about the elliptically anisotropic case,

\[ f^2 - B(X; \delta) f + \delta X(1 - X) = 0 , \]

in which

qP: \[ \delta \equiv \frac{E^2}{c_{11}c_{55}} = \frac{(1 - \gamma)^2 - \epsilon^2_F}{\gamma(1 + \epsilon_p) - \epsilon_A} , \]

\[ B(X; \delta) \equiv \frac{c_{33}}{c_{55}} - 1 + \left( 1 - \frac{c_{33}}{c_{11}} - \delta \right) X = \frac{1 - \gamma - \epsilon_p}{1 + \epsilon_p + \delta} \]

qSV: \[ \delta \equiv \frac{E^2}{c_{33}c_{55}} = \frac{(1 - \gamma)^2 - \epsilon^2_F}{\gamma(1 + \epsilon_p) - \epsilon_A} , \]

\[ B(X; \delta) \equiv - \left( 1 - \frac{c_{55}}{c_{33}} \right) + \left( 1 - \frac{c_{11}}{c_{33}} - \delta \right) X = - \frac{1 - \gamma - \epsilon_p}{1 - \epsilon_p} - \left( \frac{2\epsilon_p}{1 - \epsilon_p} + \delta \right) X \]

Regarding the expressions for $B$, first note that $B(0; \delta)$ is independent of $\delta$, and hence may be written as $B(0)$. In view of mild anisotropy condition (12), simple substitution shows that for qP, $B(0)$ is positive and $B(1; \delta)$ is positive for all $\delta$. Similarly, for qSV, $B(0)$ is negative and $B(1; \delta)$ is negative for $\epsilon_A > \epsilon_{A_{\text{no trip}}}$, i.e., in the range of $\epsilon_A$ such that there is no triplication centered
Rational approximations

on the horizontal axis, this condition being subsumed in (15). Thus, subject to mild anisotropy,
\[ B(X; \delta) \neq 0, \quad 0 \leq X \leq 1, \] (22)
i.e., over the entire pre-critical range of $X$. For qP, $B$ is positive over this range; for qSV, $B$ is negative.

Of the two roots of quadratic equation (21), the desired one is the one which vanishes when $\delta X(1 - X) = 0$, i.e.,
\[ f = \frac{1}{2} \left[ B(X; \delta) - \chi(B(X; 0)) \sqrt{B^2(X; \delta) - 4X(1 - X)\delta} \right], \] (23)
where
\[ \chi(\xi) \equiv \begin{cases} 
+1 & \text{if } \text{Re} \{\xi\} > 0, \\
-1 & \text{if } \text{Re} \{\xi\} < 0.
\end{cases} \]

For the purpose of a Taylor expansion of the square root, we rewrite the root as follows:
\[ f = \frac{B(X; \delta)}{2} \left[ 1 - \chi(B(X; 0)) \chi(B(X; \delta)) \sqrt{1 - \frac{4X(1 - X)\delta}{B^2(X; \delta)}} \right]. \] (24)

This latter expression shows more explicitly how branch cuts form.

Note that quadratic equations (19) for the qP and qSV waves can be written conveniently in terms of the appropriate $\delta$ and $B(X; \delta)$ by the substitution of $f \equiv X + Z - 1$ into equation (21). This substitution yields,
\[ F(X, Z; \delta) \equiv (X + Z - 1)^2 - B(X; \delta)(X + Z - 1) + \delta X(1 - X) = 0, \] (25)
as the normalized form of the exact dispersion relation. This form is most suitable for expressing derivatives
\[ \frac{d^n Z}{dX^n} \bigg|_{X=0, Z=1} \quad \text{and} \quad \frac{d^n X}{dZ^n} \bigg|_{X=1, Z=0} \]
in terms of $\delta$ and $B$, since $X + Z - 1$ vanishes both at $X = 0, Z = 1$ and at $X = 1, Z = 0$. In particular, from setting the total derivative of $F$ with respect to $X$, and with respect to $Z$, to zero
\[ \frac{dZ}{dX} \bigg|_{X=0, Z=1} = -1 + \frac{\delta}{B(0)}, \quad \frac{dX}{dZ} \bigg|_{X=1, Z=0} = -1 + \frac{\delta}{B(1)} \] (26)
Similarly, from differentiating $F$ once again, and setting the second order total derivatives to zero, we find,
\[ \frac{d^2 Z}{dX^2} \bigg|_{X=0, Z=1} = \frac{2\delta[B - B(0)B(1; \delta)]}{B^3(0)}, \quad \frac{d^2 X}{dZ^2} \bigg|_{X=1, Z=0} = \frac{2\delta[B - B(0)B(1; \delta)]}{B^3(1; 0)} \] (27)
These expressions will be compared with analogous expressions based on the approximation to be developed below.

Rational approximations for squared slowness curves

Expanding the square root in equation (24) in a Taylor series and limiting the number of terms yield a sequence of approximations which apply for small $X(1 - X)$ (i.e., simultaneously about near vertical and near critical horizontal slowness) or for small $\delta$. If the series converges globally over the full real slowness surface, these approximations differ essentially from a ‘parabolic’ approximation, as the appropriate conditions are satisfied at grazing incidence as well as at vertical incidence; if not, and the series diverges in some range between vertical and grazing incidence, the approximation would be bi-‘parabolic’ like about vertical and grazing incidence.

Now, Taylor expanding the square-root expression of equation (24) yields, for $f$, the following series of rational functions of $X$:
\[ f = \frac{B(X; \delta)}{2} \left[ 1 - \chi(B(X; \delta)) \chi(B(X; 0)) \left( 1 - \sum_{n=1}^{\infty} \frac{[2X(X - X)\delta]n}{n!B^{2n}(X; \delta)} \prod_{m=1}^{n} |2m - 3| \right) \right] \] (28)
\[ = \frac{B(X; \delta)}{2} \left[ 1 - \chi(B(X; \delta)) \chi(B(X; 0)) \left( 1 - \frac{2X(1 - X)\delta}{B^2(X; \delta)} - \frac{2X(1 - X)\delta^2}{B^4(X; \delta)} - \ldots \right) \right], \]
and hence, from equations (20) and (18), the series for the squared vertical slownesses. This series converges *globally* only so long as, 
\[
\frac{4X(1-X)\delta}{B^2(X;\delta)} < 1.
\]
(29) 

The issue of convergence is relevant for pre-critical directions of propagation, i.e., on the interval \(0 < X < 1\). Further, a pole in \(Z\), hence an algebraic branch point in \(s_z\), is introduced by the approximation at \(X = X_p\); \(X_p\) satisfies \(B(X_p,\delta) = 0\). From equation (22) and the discussion preceding it, \(X_p\) cannot lie on the pre-critical interval between 0 and 1. The branch point will lie on the real axis (beyond the critical horizontal slowness) or on the imaginary axis in the complex horizontal slowness plane.

The branches of the elliptic, zeroth-order approximation (the straight lines in the squared slowness domain) cross at \(X = X_x\) such that \(B(X_x;0) = 0\) for both qP and qSV. This crossing point is, 
\[
s_x^2 = -\frac{c_{33} - c_{55}}{c_{33}(c_{11} - c_{33})}, \quad s_x^2 = -\frac{c_{11} - c_{55}}{c_{55}(c_{11} - c_{33})}.
\]
(30) 

At this point, the function \(\chi(B(X;0))\) in equation (24) changes sign and the rational approximation jumps from one branch or root to the other. For \(c_{11} > c_{33}\), the jump occurs on the imaginary horizontal slowness axis, for \(c_{11} < c_{33}\), on the real horizontal slowness axis beyond critical qSV slowness. These jumps are necessary for the asymptotic behavior of the approximations at infinity in the complex horizontal slowness plane to be consistent with, although not exactly equal to, the asymptotic behavior of the exact solution. In particular, along the imaginary axis in the complex horizontal slowness plane, i.e., \(X \to -\infty\), the vertical slowness \(Z \to \infty\). This is of practical importance if these rational approximations are to be applied over a range of pre- and post-critical horizontal slowness. The right asymptotic behavior guarantees the convergence of the propagator in the space-time domain based on our spectral-domain approximation, see for example, De Hoop and De Hoop (1994).

As far as inequality (29) is concerned, first note that there are two cases to consider, \(\delta > 0\) (positive anellipticity) and \(\delta < 0\) (negative anellipticity). Since the quadratic equation for \(f\) has two distinct real roots for all \(X, 0 < X < 1\), the expression in the square root of equation (24) is positive and, for \(\delta > 0\), of the form one *minus* the positive quantity in equation (29). Thus that positive quantity must be less than unity. Hence, only the case \(\delta < 0\) requires further analysis to find conditions for which (29) is satisfied. This analysis is carried out in Appendix A, and the results are as follows:

**For qP waves when \(\delta < 0\):**

The conditions of mild anisotropy are sufficient to ensure that equation (29) is satisfied under any plausible geological setting. A more stringent condition limiting the maximum value of \(c_{13}\) (cf. equation (A5)) or equivalently, the minimum value of \(\delta\), is necessary only when
\[
c_{33} \max[c_{11}, c_{33}] < 2c_{55},
\]
which means that for \(c_{11} < c_{33}\), the ratio of vertical shear velocity to vertical compressional velocity is greater than \(\sqrt{2}/2 \approx 0.84\), while for \(c_{11} > c_{33}\), the ratio of vertical shear velocity to the geometric mean of vertical and horizontal compressional velocities is greater than \(\sqrt{2}/2\).

**For qSV waves when \(\delta < 0\):**

The conditions of mild anisotropy are not sufficient to ensure that inequality (29) is satisfied, except for the slightly extreme case when \(c_{11} > 2c_{33}\), see Appendix A. In all other situations, for series of equation (28) to converge for all \(X\) when \(\delta < 0\), the additional inequality (A8) must be satisfied, which, in terms of dimensionless parameters, may be written,
\[
\delta > -\frac{(c_{11} - c_{33})(c_{33} - c_{55})}{c_{33}(2c_{33} - c_{55})} = -\frac{c_{55}}{2c_{33} - c_{55}} = -\frac{\gamma}{2(1 - \epsilon_p) - \gamma}
\]

Consider the pre-critical directions of propagation. The first order approximation to either the qP or qSV slowness curve is returned by retaining just the first term in equation (28),
\[
Z = 1 - X + f = 1 - X + \frac{X(1-X)\delta}{B(X;\delta)} = (1-X)\frac{B(X;0)}{B(X;\delta)} \equiv (1-X) R_1(X);
\]
(31)
the second order approximation is returned by retaining the first two terms,

\[ Z = 1 - X + \frac{X(1 - X)\delta}{B(X; \delta)} + \frac{[X(1 - X)\delta]^2}{B^2(X; \delta)} \]

\[ = (1 - X) \frac{B(X; 0)B^2(X; \delta) + X^2(1 - X)\delta^2}{B^3(X; \delta)} \]

\[ \equiv (1 - X) R_2(X). \quad \text{(32)} \]

These expressions are the key results of this paper. The first order expression satisfies the correct curvature at both normal and grazing incidence; the second order expression satisfies, in addition, the correct third derivative of the slowness at normal and grazing incidence. Note that the curvatures of the slowness surface and the wave front are reciprocal, while the curvature of the wave front determines the short-spread moveout velocity; hence matching the curvatures implies matching the zero offset moveout velocities for both a horizontal and a vertical array of receivers. Higher order approximations satisfy higher order derivatives at normal and grazing incidence. The \( n \)th order rational function \( R_n(X) \) has \( 2n - 1 \) degree polynomials in \( X \) for both numerator and denominator, with the denominator given by \( B^{2n-1}(X; \delta) \).

It is important to point out that the approximation to a given order for the qSV curve is not as accurate as the one for the qP curve. So to achieve the same degree of accuracy for qSV, more terms must be taken into account. Each additional term matches the next higher derivative at \( X = 0, Z = 1 \) and \( X = 1, Z = 0 \). If, for example, one wished only to match the curvature (second derivative) of the slowness surface, one could add to the first order rational approximation the second order rational approximation term times a scaling factor chosen so that either a) a given point between \( X = 0 \) and \( X = 1 \) is matched exactly, or b) a set of points between \( X = 0 \) and \( X = 1 \) is matched in an optimum way by a least squares criterion. Thus the third derivative at the endpoints will be slightly off in return for a much closer fit in the intermediate region between the axes. This is equivalent to an interpolation approach using a higher-order correction term.

**Other expansions**

Note that the square root in equation (24) can expanded in several different ways. One could consider a power series in \( X \), which would be fine near vertical, but such an expansion and others like it will not be considered here because we are looking for approximations that are valid over the entire range, \( 0 \leq X \leq 1 \). A naive application of a power series in \( X \) never even gives \( Z|_{X=1} = 0 \).

Another possibility is to expand the series in equation (28) in a power series in \( \delta \), yielding,

\[ f = \frac{1 - X}{B(X; 0)} X\delta + \frac{1 - X}{B^3(X; 0)} [1 - X + B(X; 0)] X^2 \delta^2 + \ldots. \quad \text{(33)} \]

Retaining only the first term yields,

\[ Z = (1 - X) \left[ 1 + \frac{X}{B(0; \delta)} \delta \right], \]

which gives the exact value for \( dZ/dX|_{X=0} \) but not for \( dX/dZ|_{X=1} \). The power of the rational approximation proposed above is that even a single term gives so much of the character of the exact slowness curve.

**A simplification of the qP slowness relation**

The qP slowness relation will depend strongly on \( c_{13} + 2c_{55} \) but only weakly on \( c_{13} \) or \( c_{55} \) individually. Hence, we are led to introduce the ratio

\[ \lambda = \frac{c_{13}}{c_{13} + 2c_{55}}. \quad \text{(34)} \]

We express \( c_{13} \) and \( c_{55} \) in terms of the combination \( c_{13} + 2c_{55} = a \), say, and \( \lambda \):

\[ c_{55} = \frac{1}{2}(1 - \lambda) a, \quad c_{13} = \lambda a. \quad \text{(35)} \]

Then

\[ E^2 = c_{11} c_{33} - \frac{1}{2}(1 - \lambda) a (c_{11} + c_{33}) - \lambda a^2. \quad \text{(36)} \]
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In terms of \( \lambda \) and \( a \) the dispersion relation (4) becomes

\[
[c_{11}s_z^2 + \frac{1}{2}(1 - \lambda)\alpha s_z^2 - 1] \left[ \frac{1}{2}(1 - \lambda)\alpha s_z^2 + c_{33}s_z^2 - 1 \right] - \left[ \frac{1}{2}(1 + \lambda)\alpha s_z^2 s_z^2 \right] = 0 .
\]  
(37)

The qP slowness relation can be shown to be insensitive to variations in \( \lambda \). The mild anisotropy conditions imply that \( \lambda > -1 \) while \( a > 0 \). Also, mildly anisotropic media satisfy \( c_{55} = \frac{1}{2}(1 - \lambda) \left( c_{13} + 2c_{33} \right) < \min[c_{11}, c_{33}] \). If \( c_{55} \) and \( c_{66} \) are allowed to approach zero, then \( c_{13} > 0 \) and \( \lambda \uparrow 1 \), while \( a \) is kept equal to its original value. In this limit, the exact dispersion relation for qP waves simplifies (cf. equation (19)):

\[
\left[ 1 - \left( 1 - \frac{a^2}{c_{11}c_{33}} \right) \frac{\alpha}{c_{33}} \right] Z = 1 - X,
\]

(38)

directly leading to a rational approximation. In terms of our parameters \( \lambda \) and \( a \), our first-order rational approximation (31) becomes

\[
Z = (1 - X) \frac{1 - \frac{1}{2}(1 - \lambda) \frac{a}{c_{33}} \left[ 1 - \left( 1 - \frac{c_{33}}{c_{11}} \right) \frac{\alpha}{c_{33}} \right] Z}{1 - \frac{1}{2}(1 - \lambda) \frac{a}{c_{33}} + \left( 1 - \lambda \right) \frac{a}{c_{33}} + \left( 1 - \frac{\lambda a^2}{c_{11}c_{33}} \right)} X,
\]

(39)

which in the limit \( \lambda \uparrow 1 \) reduces to the solution of the exact dispersion relation (38).

Comparison with Muir's bi-elliptic approximation for slowness curves

For either qP or qSV slowness surfaces, equation (20) with \( f = 0 \) allows us to express the case of elliptical anisotropy in dimensionless form as \( X + Z = 1 \), or, equivalently,

\[
\frac{X^3 + 3X^2Z + 3XZ^2 + Z^3}{(X + Z)^2} = 1 .
\]

The bi-elliptic approximation to the slowness curve is a variation on this expression which ensures that the slope of the squared slowness surface along the coordinate axes, i.e., at the anchor points \( X = 0, Z = 1 \) and \( Z = 0, X = 1 \) (equal to the curvature of the slowness surface at the anchor points) of the bi-elliptic approximate curve and of the exact curve are the same ( Dellinger et al., 1993). To accomplish this, let

\[
G(X, Z) \equiv \frac{X^3 + (2 - b_X)X^2Z + (2 - b_Z)XZ^2 + Z^3}{(X + Z)^2} - 1 = 0 .
\]

(40)

The fact that the total derivative of \( G(X, Z) \) in equation (40) with respect to \( X \) and with respect to \( Z \) must vanish (since \( G \) is equal to a constant) enables us to find that,

\[
\frac{dZ}{dX} \bigg|_{X=0,Z=1} = b_Z, \quad \frac{dX}{dZ} \bigg|_{Z=0,X=1} = b_X.
\]

(41)

The exact values of \( dZ/dX \big|_{X=0,Z=1} \) and \( dX/dZ \big|_{X=1,Z=0} \) are given in equation (26). Thus in order to preserve these slopes at normal and grazing incidence, the bi-elliptic approximation requires that,

\[
b_Z = -1 + \frac{\delta}{B(0)}, \quad b_X = -1 + \frac{\delta}{B(1;0)}.
\]

(42)

For the qP curve, \( b_X \) and \( b_Z \) are negative; for the qSV curve, \( b_X \) and \( b_Z \) can be negative or positive. Positive \( b_Z \) corresponds to a triplicating region around the vertical \( z \)-axis; positive \( b_X \) to a triplicating region around the horizontal \( x \)-axis.

These values of \( b_X \) and \( b_Z \) substituted into equation (40) yield Muir's bi-elliptic approximate slowness curves, i.e.,

\[
G(X, Z) = \frac{X^3 + \left[ 3 - \frac{\delta}{B(1;0)} \right] X^2Z + \left[ 3 - \frac{\delta}{B(0)} \right] XZ^2 + Z^3}{(X + Z)^2} - 1
\]

\[= X + Z - 1 - \frac{\delta XZ}{(X + Z)^2} \left[ \frac{X}{B(1;0)} + \frac{Z}{B(0)} \right] = 0 .
\]

(43)

Because curvature at the anchor points is preserved, this approximation returns the correct horizontal and vertical zero offset moveout velocity, as does our first-order rational approximation.

However, the problem with the bi-elliptic approximation is that it is implicit in \( X \) and \( Z \) and thus is as difficult to apply as the exact dispersion relation when \( Z \) as a function of \( X \) is required. The power of the bi-elliptic approximation comes from the fact
Rational approximations

that it is equally well suited for approximating the group velocity as a function of direction or the vertical component of the group velocity as a function of the horizontal component, without the need to evaluate the associated slowness vector. To demonstrate, note that all group velocity vectors which are associated with real slowness vectors lie on a curve, the wave surface, which is polar reciprocal to the slowness curve. Further note that the polar reciprocal of an ellipse is also an ellipse. For elliptical anisotropy, the wave surface then is given by \( U + W = 1 \), or, equivalently,

\[
\frac{U^3 + 3U^2W + 3UW^2 + W^3}{(U + W)^2} = 1,
\]

where,

\[ q_P: \quad U \equiv \frac{v_{2x}}{c_{11}}, \quad W \equiv \frac{v_{2x}}{c_{33}}, \]

\[ q_{SV}: \quad U \equiv \frac{v_{2x}}{c_{55}}, \quad W \equiv \frac{v_{2x}}{c_{55}}. \]

In similar fashion to the procedure for the slowness approximation, one approximates the wave surface by,

\[
V(U, W) \equiv \frac{U^3 + (2 - b_U)U^2W + (2 - b_W)UW^2 + W^3}{(U + W)^2} - 1 = 0, \tag{44}
\]

and from the fact that the total derivative of \( V \) must vanish,

\[
\frac{dW}{dU} \bigg|_{U=0, W=1} = b_W, \quad \frac{dU}{dW} \bigg|_{W=0, U=1} = b_U.
\]

Thus it remains only to evaluate \( dW/dU \big|_{U=0, W=1} \) and \( dU/dW \big|_{W=0, U=1} \) from the exact expression for group velocity (which is parameterized in terms of slowness). These derivatives are evaluated in Appendix B; the results, from equations (B9) and (B12), are that,

\[
b_W = -1 - \frac{\delta}{B(0) - \delta}, \quad b_U = -1 - \frac{\delta}{B(1; \delta)}. \tag{45}
\]

Substitution of these values into equation (44) gives Muir's bi-elliptic approximation for the wave surface,

\[
V(U, W) = \frac{U^3 + \left[3 + \frac{\delta}{B(1; \delta)}\right]U^2W + \left[3 + \frac{\delta}{B(0) - \delta}\right]UW^2 + W^3}{(U + W)^2} - 1 = 0,
\]

or, in a more compact form,

\[
U + W + \frac{\delta UW}{(U + W)^2} \left[\frac{U}{B(1; \delta)} + \frac{W}{B(0) - \delta}\right] = 1. \tag{46}
\]

This is a very good approximation for both qP and qSV wave surfaces except that it fails, by design, around triplications.

For both qP and qSV waves, to evaluate group velocity magnitude \( v_g \) as a function of group direction \( \theta_g \) from equation (46), merely let \( v_{2x} = v_g(\theta_g) \sin \theta_g \) and \( v_{2y} = v_g(\theta_g) \cos \theta_g \) and solve the resulting linear equation on \( v_g^2(\theta_g) \). Having \( v_g(\theta_g) \) is particularly useful to evaluate travel time along a fixed (even if not precisely correct) ray path.

Application to f-k migration schemes

For phase shift migration, one must use an expression for vertical wavenumber \( k_z \). For isotropic (or elliptically transversely isotropic) media, the vertical wavenumber is given by,

\[
k_z = \omega s_z = \frac{\omega}{\alpha_V} \sqrt{Z} = \frac{\omega}{\alpha_V} \sqrt{1 - X},
\]

where \( \alpha_V \) is the vertical wavespeed. The power of these rational approximations is that, for transversely isotropic media, these expressions are replaced simply by,

\[
k_z = \frac{\omega}{\alpha_V} \sqrt{(1 - X)R_n(X)}. \tag{47}
\]

In a homogeneous medium, only propagating modes, i.e., pre-critical waves, are required.
From zero-offset (post-stack) data, Stolt single mode migration requires an expression for $\omega$ in terms of $k_x$ and $k_z$. The needed approximation can be obtained in a very similar manner as that carried out above, taking as a starting point the exact dispersion relation as given in equation (25) multiplied by

$$\Omega \equiv \omega^2 .$$

Then defining,

$$K_X \equiv \omega^2 X , \quad K_Z \equiv \omega^2 Z ,$$

equation (25) assumes the form,

$$(K_X + K_Z - \Omega)^2 - [B(0) \Omega + [B(1; \delta) - B(0)] K_X] (K_X + K_Z - \Omega) + \delta K_X (\Omega - K_X) = 0 .$$

Introducing into this form an expansion about the elliptically anisotropic case, viz.,

$$\Omega = K_X + K_Z + f_{1n} ,$$
yields, after expansion in powers of $f_{1n},$

$$[1 - B(0)] f_{1n}^2 - [[B(1; \delta) - \delta] K_X + B(0) K_Z] f_{1n} + \delta K_X K_Z = 0 .$$

As above, there are a sequence of rational approximations to the solution of this quadratic equation that vanishes when $\delta = 0$ based on the Taylor series expansion of $1 - \sqrt{1 - \xi}$. The first order rational approximation, after noting that $B(1; \delta) - \delta = B(1; 2\delta)$, is given by,

$$f_{1n} \approx \frac{\delta K_X K_Z}{B(1; 2\delta) K_X + B(0) K_Z} ,$$

or, equivalently,

$$\Omega \approx \frac{[B(1; 2\delta) K_X + B(0) K_Z] (K_X + K_Z) + \delta K_X K_Z}{B(1; 2\delta) K_X + B(0) K_Z}$$

$$= \frac{B(1; 2\delta) K_{1n}^2 + [B(0) + B(1; \delta)] K_Z K_X + B(0) K_{1n}^2}{B(1; 2\delta) K_X + B(0) K_Z} .$$

Higher order approximations are easily derived as well.

Examples and discussion

The approximations will be illustrated using the measured moduli of Greenhorn shale (Jones & Wang, 1981) as a starting model. The relevant squared velocity moduli,

\[
c_{11} = 14.47 , \quad c_{33} = 9.57 , \quad c_{55} = 2.28 , \quad c_{13} = 4.51 ,
\]
give dimensionless parameters:

$$\gamma = 0.190 , \quad \varepsilon_p = 0.204 , \quad \varepsilon_A = 0.482 .$$

Other examples considered will have these same parameters except for anellipticity $\varepsilon_A$, which will be varied by changing the value of $c_{13}$. Thus all examples will have the same anchor points, but different anellipticity. We compare the first and second order rational approximations with the exact slowness surfaces, with Muir’s bi-elliptic approximation, and with the elliptical TI medium.

Figure 2 a) shows the exact qP and qSV slowness curves, their first order rational approximations and their bi-elliptic approximations for Greenhorn shale. As a reference, for this figure and all the remaining figures, the associated elliptically anisotropic medium curves will be shown in grey. Figure 2 b) shows, instead of the rational first order approximation, the second order approximation. This and subsequent figures are normalized so the qSV curve traverses points (0,1) and (1,0), i.e., the curves are scaled by normalizing all moduli by $c_{55}$.

Next, we perturb the Greenhorn shale by increasing $\varepsilon_A$ to the value 0.910 (by decreasing $c_{13}$ to 0.547) and Figure 3 shows the the same curves for this very anelliptic medium as were plotted in Figure 2. Figure 3 a) contains the first order rational approximation and 3 b) the second order approximation. In Figure 3 c), the approximation (38) is compared with the first-order rational approximation, for qP waves only. Finally, we perturb the Greenhorn shale by decreasing $\varepsilon_A$ to the negative value −0.126 (by increasing $c_{13}$ to 7.72). The slowness curves are shown in Figure 4.
Rational approximations

Note that for all three cases, the first order rational approximation is very accurate for the qP curves while the second order approximation is acceptable for the qSV curves, even for the very high anellipticity case shown in Figure 3. For all three cases, the first order rational approximation is closer to the exact dispersion relation than the bi-elliptic for qP but further for qSV. Even the second order rational approximation only approaches (but not quite attains) the accuracy of the bi-elliptic approximation for qSV. The bi-elliptic approximation too can be generalized to satisfy higher order derivatives at the origin, although this is somewhat tedious. For positive anellipticity, the higher order rational approximations approach the exact curves monotonically from inside for qP and from outside for qSV. For negative anellipticity, if a given order approximation is inside the exact curve, the next order approximation will be outside, and vice versa, to be expected from the fact that the series expansion of equation (28) is an alternating series for \( \delta < 0 \). This can be seen in Figure 4 for the qSV curve.

In Figure 5 we show slowness curves of a TI medium with large \( c_{55} \) (relative to \( c_{11} \) and \( c_{33} \)), specified by \( \gamma \), and large negative anellipticity, specified by \( \epsilon_A \). The dimensionless parameters used are,

\[
\gamma = 0.5, \quad \epsilon_P = 0.2, \quad \epsilon_A = -0.7143.
\]

For these values of \( \gamma \) and \( \epsilon_P \), incipient triplication centered on the vertical occurs at just this value of \( \epsilon_A \) as can be seen by the zero curvature of the exact and approximate curves at the vertical axis. Divergence of series expansion (28) for the qSV surface for some \( X \) occurs for the less negative value of \( \epsilon_A < -0.4545 \). At this much greater negative value of \( \epsilon_A \), qSV divergence occurs for \( 0.338 < s_e < 0.902 \). The figure shows the first, fifth and ninth order rational approximations. The qP curve converges everywhere, and even the first order approximation is quite accurate. The approximate qSV curves in the divergence interval move away from the exact curve as the order increases.

In Figures 6 and 7 segments of the exact dispersion curve and the first order rational approximation are shown which correspond to \( X \) lying out of the range \([0,1] \). The behavior is illustrated for the case of Greenhorn shale. In Figure 6, the vertical slowness is imaginary (corresponding to negative \( Z \)) along the real horizontal slowness axis beyond the critical angles (corresponding to \( X > 1 \)). Note the singularity for the qP curve at \( s_e = 0.725 \) which is associated with \( B_{qP}(X; \delta) = 0 \). In Figure 7, the horizontal slowness is imaginary (corresponding to negative \( X \)). Note the singularity for the qSV curve at \( s_e = 0.555 i \) corresponding to \( B_{qSV}(X; \delta) = 0 \). Further note the equal and opposite discontinuities in the qP and qSV curves at \( s_e = 1.22 i \) corresponding to \( X \), where \( B(X; \delta) = 0 \) for both qP and qSV. This is necessarily at the same horizontal slowness where the elliptical qP and qSV curves (shown in grey) cross, the coordinates of the crossing point being given in (30).

In summary, it is emphasized that our rational approximations are explicit, i.e., of the form \( Z = Z(X) \), whereas Muir’s bi-elliptic approximation to the dispersion relation is implicit. Our purpose was to obtain a sequence of such explicit approximations. From (31), the first order explicit formula for the approximate qP dispersion curve is,

\[
s_e^2 = \left[ \frac{1}{c_{33}} - \frac{c_{11} c_{55} s_e^2}{c_{33} - c_{55} + (c_{11} - c_{33}) c_{55} s_e^2} \right] \left[ c_{33} - c_{55} + (c_{11} - c_{33}) c_{55} - E^2 \right] s_e^2, \tag{51}
\]

where,

\[E^2 = (c_{11} - c_{55})(c_{33} - c_{55}) - (c_{13} + c_{55})^2.\]

For \( E^2 = 0 \) the slowness curve is easily seen to be an ellipse. This elliptical approximation is a poor approximation for the usual transverse isotropy encountered in sedimentary basins for which the dimensionless anellipticity \( \epsilon_A \neq 1 \). Only the first power of the anellipticity appears in this expression, but the approximation is much better than first order in that anellipticity since this approximation matches the curvature of the exact dispersion curve at the vertical and horizontal axes.

Note that the accuracy of the low order approximations differ from one another; the first order rational approximation to the qP slowness curve is better than the bi-elliptic approximation but the first order qSV approximation is poorer. Muir’s bi-elliptic approximation is still meaningful away from mild anisotropy, whereas for certain large enough negative values of anellipticity, the rational qSV approximation can diverge.

Acknowledgment

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References


Figure 1. Squared slowness curves denoted by the solid black lines for three different TI media. All three media have $\gamma = 0.19$ and $\epsilon_P = 0.20$. For reference, the elliptical TI medium curves $\epsilon_A = 0$, (straight lines in the squared slowness domain), for the same $\gamma$ and $\epsilon_P$ are the grey lines. a) Positive anisotropy medium, with $c_{13} = 0.24$ equivalent to $\epsilon_A = 0.91$. b) Moderate negative anisotropy medium, with $c_{13} = 3.39$ equivalent to $\epsilon_A = -0.13$. c) Strong negative anisotropy medium, with $c_{13} = 4.50$ equivalent to $\epsilon_A = -0.77$, exhibits shear wave triplication about both the vertical and horizontal axes. In this and all other figures, all slownesses are normalized by $\sqrt{c_{33}}$, so that the anchor points of the qSV curves are always (0,1) and (1,0).
Figure 2. Slowness curves for Greenhorn shale. The elastic moduli are normalized by $c_{05}$. The dimensionless parameters: $\gamma = 0.190$, $\epsilon_P = 0.204$, $\epsilon_A = 0.482$. The exact slowness curves are the black solid curves; the rational approximations are the long dashes; Muir's bi-elliptic approximations are the short dashes; for reference the associated elliptical TI curves are shown in grey.

a) First order rational approximation.

b) Second order rational approximation.

Figure 3. Exactly as Figure 2 except the medium is a perturbed Greenhorn shale with anellipticity parameter $\epsilon_A$ increased to 0.910.
Figure 5. The exact slowness curves and the first, fifth and ninth order rational approximations for a medium with $\gamma = 0.5$, $\epsilon_P = 0.2$, $\epsilon_A = -0.7143$. Because of the large value of $\gamma$, this large negative value of $\epsilon_A$ is negative enough for the qSV rational approximation to diverge over a significant range, but is not negative enough for triplcation to occur, as this value of $\epsilon_A$ is exactly the value for incipient triplcation. In the diverging interval, the closest dashed curve to the exact solid curve is the first order rational approximation, the next furthest is the fifth order and the furthest is the ninth order. Since the anellipticity is negative, the even orders would lie inside the qSV curve.

Figure 4. Exactly as Figure 2 except the medium is a perturbed Greenhorn shale with negative anellipticity. $\epsilon_A$ is decreased to $-0.126$.

Figure 6. Exact imaginary vertical slownesses and the first order rational approximation for post-critical horizontal slowness (corresponding to $X > 1$ and $Z < 0$). Curves shown are for Greenhorn shale.
Figure 7. Exact vertical slownesses and the first order rational approximation along the imaginary horizontal slowness axis (corresponding to $X < 0$). Curves shown are for Greenhorn shale.
APPENDIX A: The convergence of series (28) for negative anellipticity

The condition for convergence of the series in equation (28) is inequality (29). As discussed in the paragraph leading to equation (22), under the conditions of mild anisotropy, for qP, $B > 0$ and for qSV, $B < 0$, for $0 \leq X \leq 1$, while the numerator is positive except at the endpoints where it vanishes. Thus the expression in equation (29) has a maximum value $M$ at some value of $X$, say at $X = X_{\text{max}}$, between 0 and 1. Differentiating this expression yields, after setting the result to zero,

$$X_{\text{max}} = \frac{B(0)}{B(0) + B(1; \delta)}, \quad M = \frac{4X(1 - X)|\delta|}{B^2(X; \delta)} \bigg|_{X = X_{\text{max}}} = \frac{|\delta|}{B(0)B(1; \delta)}.$$  \hspace{1cm} (A1)

Thus

$M < 1$

becomes the condition for convergence of the series in equation (28) for all pre-critical horizontal slownesses. It is necessarily satisfied for positive anellipticity; the condition must be analysed in the context of mild anisotropy for negative anellipticity, $\delta < 0$, when, from equation (21),

$$B(1; \delta) = \begin{cases} \frac{(1 - \varepsilon_P)(1 + \varepsilon_P - \gamma)}{\gamma(1 + \varepsilon_P)} + |\delta| & \text{for qP,} \\ \frac{1 + \varepsilon_P - \gamma}{1 - \varepsilon_P} + |\delta| & \text{for qSV.} \end{cases}$$ \hspace{1cm} (A2)

FOR QP WAVES ($B(0) > 0, B(1; \delta) > 0$):

From equations (A1) and (A2),

$$M = \frac{|\delta|}{B(0) \left[ \frac{(1 - \varepsilon_P)(1 + \varepsilon_P - \gamma)}{\gamma(1 + \varepsilon_P)} + |\delta| \right]} < 1,$$  \hspace{1cm} (A3)

which is satisfied by inspection for $B(0) > 1$, since

$$\frac{(1 - \varepsilon_P)(1 + \varepsilon_P - \gamma)}{\gamma(1 + \varepsilon_P)} > 0$$

from mild anisotropy. Thus we only have to consider the case $B(0) < 1$, i.e., $2c_{55} > c_{33}$. In the unlikely event that this is the case (since it means that the ratio of vertical shear velocity to vertical compressional velocity is greater than $\sqrt{2}/2$), $M < 1$ is equivalent to,

$$|\delta| < \left[ \frac{(1 - \varepsilon_P)(1 + \varepsilon_P - \gamma)}{\gamma(1 + \varepsilon_P)} \right] \frac{B(0)}{1 - B(0)}.$$  \hspace{1cm} (A4)

Equivalently, from the definition of $B(0)$ and $\delta$ in equation (21),

$$|\varepsilon_A| < \frac{1 - \varepsilon_P}{2\gamma - (1 - \varepsilon_P)} = \gamma + \frac{(1 - \varepsilon_P)(1 + |\varepsilon_P|) - 2\gamma}{[2\gamma - (1 - \varepsilon_P)(1 + |\varepsilon_P| - \gamma)]}$$

$$= |\varepsilon_{A_{\text{no trip}}}| + \frac{c_{33} \max[c_{11}, c_{33}] - 2c_{55}^2}{(2c_{55} - c_{33})(\max[c_{11}, c_{33}] - c_{55})}.$$ \hspace{1cm} (A5)

Thus, the sign of the numerator of the second term on the right hand side of inequality (A5) — since the sign of the denominator is positive — determines whether or not the conditions of mild anisotropy automatically imply $M < 1$ for the case $2c_{55} > c_{33}$. If the numerator is non-negative, $\varepsilon_A$ satisfying the no triplication condition of mild anisotropy implies that the inequality is satisfied, and thus that $M < 1$. If the sign of the numerator is negative, i.e., $\sqrt{2} c_{55} > \sqrt{c_{33} \max[c_{11}, c_{33}]}$, then $M < 1$ is a more stringent condition than mild anisotropy; the more stringent condition is given by inequality (A5).
For qSV waves \( E(0) < 0, B(1; \delta) < 0 \): 

From equations (A1) and (A2),

\[
M = \frac{\left| \epsilon \right|}{B(0) \left[ \frac{1 + \epsilon_p - \gamma}{1 - \epsilon_p} + \left| \epsilon \right| \right]} < 1 ,
\]

(A6)

which is satisfied if and only if,

\[
\left| \epsilon \right| < \frac{1 + \epsilon_p - \gamma}{1 - \epsilon_p} \frac{-B(0)}{1 - B(0)} .
\]

(A7)

Equivalently, from the definition of \( B(0) \) and \( \delta \) in equation (21), when,

\[
\left| \epsilon_A \right| < \frac{\gamma}{2(1 - \epsilon_p) - \gamma} = \frac{\gamma}{1 + |\epsilon_p| - \gamma} + \frac{\gamma}{2(1 - \epsilon_p) - \gamma(1 + |\epsilon_p| - \gamma)} = \frac{c_{55}}{(2c_{33} - c_{55})(\max[c_{11}, c_{22}] - c_{55})} .
\]

(A8)

As for the qP case, if the numerator of the second term on the right hand side of inequality (A8) is non-negative — which is most unlikely since the non-negativity requires \( c_{11} \geq 2c_{33} \) — the conditions of mild anisotropy automatically imply \( M < 1 \). If the sign of the numerator is negative, the expected situation, \( M < 1 \) is a more stringent condition than mild anisotropy; the more stringent condition is given by inequality (A8). Observe that often for \( \epsilon_A \) substantially negative, the series will diverge for qSV and converge for qP. However, it is possible for it to converge for qSV and still diverge for qP, or to diverge for both qP and qSV.

APPENDIX B: The evaluation of \( dW/dU |_{U=0, W=1} \) and \( dW/dU |_{W=0, U=1} \) from the exact dispersion relation for use in Muir’s bi-elliptic approximation for group velocity

The form of the dispersion relation suitable for either qP and qSV waves equation (25) is reproduced here,

\[
F(X, Z; \delta) = (X + Z - 1)^2 - B(X; \delta)(X + Z - 1) + \delta X(1 - X) = 0 ,
\]

(B1)

Polar reciprocity of the wave and slowness surfaces is equivalent to the group velocity vector associated with a given slowness vector being given by \( v_g = \nabla_s / s \nabla_s \) so that the components of group velocity are given by,

\[
v_{g_x} = \frac{\partial F}{\partial s_x} \left[ s_x \frac{\partial F}{\partial s_z} + s_z \frac{\partial F}{\partial s_x} \right]^{-1} , \quad v_{g_z} = \frac{\partial F}{\partial s_z} \left[ s_x \frac{\partial F}{\partial s_z} + s_z \frac{\partial F}{\partial s_x} \right]^{-1} .
\]

(B2)

Changing variables from \( s_x, s_z \) to \( X, Z \) yields,

\[
v_{g_x} = \frac{\delta X \frac{\partial F}{\partial s_x}}{s_x \frac{\partial F}{\partial s_z} + s_z \frac{\partial F}{\partial s_x}} , \quad v_{g_z} = \frac{\delta Z \frac{\partial F}{\partial s_z}}{s_x \frac{\partial F}{\partial s_z} + s_z \frac{\partial F}{\partial s_x}} ,
\]

and, after noting that for both qP and qSV, \( s_x \partial X / \partial s_x = 2X \) and \( s_z \partial Z / \partial s_z = 2Z \), the group velocity components become,

\[
v_{g_x} = \frac{1}{s_x} \frac{X \frac{\partial F}{\partial X}}{X \frac{\partial F}{\partial X} + Z \frac{\partial F}{\partial Z}} , \quad v_{g_z} = \frac{1}{s_z} \frac{Z \frac{\partial F}{\partial Z}}{X \frac{\partial F}{\partial X} + Z \frac{\partial F}{\partial Z}} .
\]

Squaring and dividing by the appropriate elastic moduli (for qP or qSV) yields,

\[
U = \frac{X \left( \frac{\partial F}{\partial X} \right)^2}{\left[ X \frac{\partial F}{\partial X} + Z \frac{\partial F}{\partial Z} \right]^2} , \quad W = \frac{Z \left( \frac{\partial F}{\partial Z} \right)^2}{\left[ X \frac{\partial F}{\partial X} + Z \frac{\partial F}{\partial Z} \right]^2} .
\]

(B3)

The needed expressions for \( \partial F / \partial X \) and \( \partial F / \partial Z \) are,

\[
\frac{\partial F}{\partial X} = (2 - B') (X + Z - 1) - B(X; \delta) + \delta (1 - 2X) , \quad \frac{\partial F}{\partial Z} = 2(X + Z - 1) - B(X; \delta) ,
\]

(B4)

where,
\[ B' = \frac{dB}{dX} = B(1; \delta) - B(0) \, . \]

Then, using the vanishing of \( F \) for any possible wave, we find,

\[ X \frac{\partial F}{\partial X} + Z \frac{\partial F}{\partial Z} = (2 + B(0)) (X + Z - 1) - B(X; 0) \, , \tag{B5} \]

and substitution of equations (B4) and (B5) into equation (B3) yields,

\[ U = \frac{X [(2 - B')(X + Z - 1) - B(X; \delta) + \delta (1 - 2X)]^2}{(2 + B(0)) (X + Z - 1) - B(X; 0)} \, , \tag{B6} \]

\[ W = \frac{Z [2(X + Z - 1) - B(X; \delta)]^2}{(2 + B(0)) (X + Z - 1) - B(X; 0)} \, , \]

explicit expressions for \( U \) and \( W \) in terms of the corresponding \( X \) and \( Z \) (which satisfy the exact dispersion relation).

The derivative,

\[ \frac{dW}{dU} \bigg|_{U=0, W=1} = \lim_{\Delta X, \Delta Z \to 0} \frac{W(\Delta X, 1 + \Delta Z) - W(0, 1)}{U(\Delta X, 1 + \Delta Z) - U(0, 1)} \]

\[ = \lim_{\Delta X \to 0} \frac{W(\Delta X, 1 + \Delta Z(\Delta X)) - 1}{U(\Delta X, 1 + \Delta Z(\Delta X))} \, , \tag{B7} \]

may be evaluated easily if \( \Delta Z(\Delta X) \), i.e. \( \Delta Z \) in terms of \( \Delta X \) is known. From equation (26),

\[ \Delta Z = \frac{dZ}{dX} \bigg|_{X=0, Z=1} \Delta X + \mathcal{O}(\Delta X)^2 = \left[ -1 + \frac{\delta}{B(0)} \right] \Delta X + \mathcal{O}(\Delta X)^2 \, , \]

and thus,

\[ X + Z - 1 \bigg|_{X=\Delta X, Z=1+\Delta Z} = \Delta X + \Delta Z = \frac{\delta}{B(0)} \Delta X + \mathcal{O}(\Delta X)^2 \, . \]

Substitution into equation (B6) yields,

\[ U(\Delta X) = \frac{\Delta X [-B(0) + \delta]^2 + \mathcal{O}(\Delta X)^2}{B^2(0) + [2B(0)B' - 4\delta] \Delta X + \mathcal{O}(\Delta X)^2} \, , \tag{B8} \]

\[ W(\Delta X) = \frac{B'^2(0) + [2B(0)B' - 4\delta - B^2(0) + B(0)\delta] \Delta X + \mathcal{O}(\Delta X)^2}{B^2(0) + [2B(0)B' - 4\delta] \Delta X + \mathcal{O}(\Delta X)^2} \, , \]

and further substitution into equation (B7) then gives,

\[ \frac{dW}{dU} \bigg|_{U=0, W=1} = -\frac{B(0)}{B(0) - \delta} = -1 - \frac{\delta}{B(0) - \delta} \, , \tag{B9} \]

the needed slope of the \( U, W \) curve at the vertical \( W \)-axis. Note that from equation (26),

\[ \frac{dW}{dU} \bigg|_{U=0, W=1} \times \frac{dZ}{dX} \bigg|_{X=0, Z=1} = 1 \, . \]

This then constitutes a proof that the slopes at the vertical axis of the squared slowness and squared group velocity curves are reciprocal. A simple change of variables yields the result that the curvatures at the vertical axis of the slowness and group velocity curves are also reciprocal.

The derivative,

\[ \frac{dU}{dW} \bigg|_{W=0, U=1} = \lim_{\Delta X, \Delta Z \to 0} \frac{U(1 + \Delta X, \Delta Z) - U(1, 0)}{W(1 + \Delta X, \Delta Z) - W(1, 0)} \]

\[ = \lim_{\Delta Z \to 0} \frac{U(1 + \Delta X(\Delta Z), \Delta Z) - 1}{W(1 + \Delta X(\Delta Z), \Delta Z)} \, , \tag{B10} \]

may be evaluated in the same way as the previous slope on the vertical axis was evaluated. This derivative is one over the slope of the \( U, V \) curve on the horizontal \( U \) axis. Here we need know \( \Delta X(\Delta Z) \), i.e. \( \Delta X \) in terms of \( \Delta Z \). From equation (26),
\[ \Delta X = \left. \frac{dX}{dZ} \right|_{x=1, z=0} \Delta Z + O(\Delta Z)^2 = \left[ -1 + \frac{\delta}{B(1; 0)} \right] \Delta Z + O(\Delta Z)^2 , \]

and thus,

\[ X + Z - 1 \big|_{x=1+\Delta x, z=\Delta z} = \Delta X + \Delta Z = \frac{\delta}{B(1; 0)} \Delta Z + O(\Delta Z)^2 . \]

The procedure is exactly the same as that carried out above. Substitution into equation (B6) yields,

\[ U(\Delta Z) = \frac{B^2(1; 0) - \delta B(1; 0) B(1; \delta) + 2 B(1; \delta) (B' + \delta) + 2 (2 + B(0)) \delta \Delta Z + O(\Delta Z)^2}{B^2(1; 0) - 2 [B(1; \delta) (B' + \delta) + (2 + B(0)) \delta] \Delta Z + O(\Delta Z)^2} , \tag{B11} \]

\[ W(\Delta Z) = \frac{\Delta Z B^2(1; \delta) + O(\Delta Z)^2}{B^2(1; 0) - 2 [B(1; \delta) (B' + \delta) + (2 + B(0)) \delta] \Delta Z + O(\Delta Z)^2} , \]

and further substitution into equation (B10) then gives,

\[ \left. \frac{dU}{dW} \right|_{U=1, W=0} = - \frac{B(1; 0)}{B(1; \delta)} = -1 - \frac{\delta}{B(1; \delta)} . \tag{B12} \]

the needed slope of the \( U, W \) curve at the horizontal \( U \)-axis. As was the case on the vertical axis, note that from equation (26),

\[ \left. \frac{dU}{dW} \right|_{U=1, W=0} \times \left. \frac{dX}{dZ} \right|_{x=1, z=0} \Delta Z = 1 , \]

a proof that at the horizontal axis as well, the slopes of the squared slowness and squared group velocity curves are reciprocal, as are the curvatures of the slowness and group velocity curves.
What is noise?

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Lest men suspect your tale untrue,
Keep probability in view
— John Gay

The concept of "noise" plays a crucial role in the statistical analysis of data. As an example of a noisy record consider Figure 1 which shows the ground motion of the seismological station NES1 in St. Petersburg after an earthquake in Egypt. (In earthquake seismology periods may be orders of magnitude larger than in exploration seismology, but the principles are the same.) This time series shows no distinct arrivals or other apparent signatures of an organized nature. Given the proximity of the recording station to a major population center and to the coast such a noisy record does not seem to be very surprising.

But what is noise exactly? In the context of seismic prospecting Dobrin and Savit (1988) define noise as "spurious seismic signals from ground motion not associated with reflections." They have in mind such things as surface waves, near-surface reverberations and so on; coherent but uninteresting signal in other words. Fair enough. One might dispute the use of the term noise here, but these authors are certainly within their rights to identify certain signal as being uninteresting. But they go on to speak of "incoherent noise, sometimes referred to as random noise ... usually associated with scattering from near-surface irregularities." (Emphasis in the original.) By identifying random noise with incoherency they have sailed into rough waters. For although the signal associated with scattering from near-surface irregularities may well be incoherent (though that is debatable), it is clearly reproducible, so does it make sense to call it random? And further, there is no law that says that random processes must be uncorrelated. (Just take an uncorrelated ("white") process and apply a smoothing operator to it.)

It turns out to be extraordinarily difficult to give a precise mathematical definition of randomness, so we won't try. (A brief perusal of randomness in Volume 2 of Knuth's great The Art of Computer Programming is edifying and frustrating in equal measures.) In any case it is undoubtedly more satisfying to think in terms of observations of physical experiments. Here is Parzen's (1960) definition, which is as good as any:

A random (or chance) phenomenon is an empirical phenomenon characterized by the property that its observation under a given set of circumstances does not always lead to the same observed outcomes (so that there is no deterministic regularity) but rather to different outcomes in such a way that there is statistical regularity. By this is meant that numbers exist between 0 and 1 that represent the relative frequency with which the different possible outcomes may be observed in a series of observations of independent occurrences of the phenomenon. . . . A random event is one whose relative frequency of occurrence, in a very long sequence of observations
of randomly selected situations in which the event may occur, approaches a stable limit value as the number of observations is increased to infinity; the limit value of the relative frequency is called the probability of the random event.

It is precisely this lack of deterministic reproducibility that allows us to reduce random noise by averaging over many repetitions of the experiment. Using this definition, the "incoherent noise" of Dobrin and Savit is not random.\(^*\)

But why should we care about the definition of noise? As geophysicists, the data at our disposal will always contain some features which we will not bother to explain. If we accepted our data as being absolutely precise and reproducible, then no model whose response disagreed with the observations even to the slightest degree could be correct. But we don’t believe that our data are exact and exactly reproducible. And further, because we cannot calculate the exact response of our Earth models (because we cannot afford to put all the physics on the computer) and because we have only approximate models anyway (we cannot use an infinite number of parameters), there are likely to be deterministic aspects of the data which we cannot or do not want to explain. Keep in mind, however, that with enough degrees of freedom one can fit any data, even if it’s not worth fitting. And the resulting model might be excessively complicated or physically unreasonable.

In fact, in many situations “noise” is highly reproducible between different experiments and corresponds therefore to a deterministic process. For example, let us return to the seismogram of Figure 1. In Figure 2 the same seismogram is shown (on the same scale) but now the signal before the first arriving P-wave around 400 s is shown as well. The signal before the P-wave consists purely of ambient noise. It can be seen that this noise level is negligible compared to the later parts of the signal. This means that the signal shown in Figure 1 is Earth response that corresponds to a multitude of different arrivals rather than random noise. Some of these arrivals can be explained by a simple 1D earth model, but as shown by Neely and Snieder (1991) this part of the signal also contains exotic arrivals such as body waves that have been converted to surface waves. It may be that one chooses not to explain this part of the signal, discarding it as noise.

* The term “coherency spectrum” was coined by Wiener to denote the absolute value of cross covariance of two signals divided by the square root of the product of the respective autocovariance functions. Cf. Priestley (1981), page 661. If two stationary processes are uncorrelated, for example if they are independent, then the coherency spectrum is zero at all frequencies.

\(^{†}\) We are purposely using the term "explain" rather than "fit" since the latter seems to be burdened with certain psychological baggage. But note that fitting is nothing more than a quantitative attempt at explanation.
mately it doesn’t really matter whether nature admits truly random processes or not, unless you’re doing quantum mechanics.

As a more concrete example, suppose we have a well log which we have modeled as a Markovian process. In other words we have estimated the $n$-dimensional joint distribution function of a Markovian process, one realization of which we take to be the well log. Operationally we could do this by making histograms to approximate the 1 and 2-dimensional marginals, which are sufficient to describe a Markovian process. Then we generate pseudo-random samples of this model Markovian process. This works exactly like the pseudo-random number generator described above, but the samples come out in accordance with this Markovian distribution that we have estimated. We pass the original well log and one of the pseudo-randomly simulated logs to a fancy statistical hypothesis test and it says that at the 99% confidence level, for instance, the two are drawn from the same distribution. Then, whether or not we believe the real log is a realization of a random process, we can usefully model it as such.

So what does all this have to do with geophysics? There are three main implications:

- **Stacking of data:** The notion that averaging over repeated realizations of an experiment (stacking) reduces noise (compared to signal) presupposes that the noise in the different experiments is uncorrelated because only then do we get the desired noise-suppression. The criterion here is that the correlation in the noise is zero for different experiments.

- **Prescription of a-priori errors in Bayesian inversion:** In such an inversion we need to prescribe the joint distribution function of the data errors (e.g., the data covariance matrix and mean if the distribution is assumed to be Gaussian). It is fine to include signal-generated “noise” in this (although a purist might argue against this). However, this type of noise will in general be highly correlated (between different samples, between different shots, and between different receivers). The correlations are crucial here, but in practice they may be difficult to quantify (Gouveia and Scales (1997)).

- **Making the decision how well to fit the data:** In a least-squares fitting procedure, one must use use chi-square or some other measure of the misfit to determine how well the model explains the data. Here, one needs to know which parts of the data are real (or interesting), and which part should be considered noise. (The latter should not be fitted.) This is not only an issue of noise levels or correlations (e.g., knowing the mean and variance of Gaussian noise), but in some applications it is important to identify data and noise in a more subtle way. For example, ground roll may be considered noise that should not be fitted by weird reflectors.

This means that the issue “what is noise” is of more than academic interest. If we define noise as being that data we choose not to fit, then we must have a model that explains the rest of the data. If not, this could be a sign that the “noise” is carrying important information. The association of noise with non-deterministic processes may be misleading since the concept of noise is also used as the garbage-can of unexplained deterministic phenomena. A treatment of this type of noise on purely statistical grounds may lead to conceptual as well as practical problems.

**References**


**Note:**

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Time reversed imaging as a diagnostic of chaotic behavior of waves and particles

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ABSTRACT
In the presence of multiple scattering, waves and particles behave differently. Particles are associated with well-defined trajectories. If the position of a particle is perturbed the trajectory may change dramatically; this is one of the hallmarks of chaos. Waves are not associated with particular trajectories and multiply-scattered waves interfere. Little is known about the stability properties of waves. We introduce the concept of time reversed imaging as a measure of the stability of propagation for both particles and waves. Analytic expressions are derived for the conditions under which waves and particles propagate stably. These are confirmed by numerical experiments.

Introduction
Time reversed imaging (TRI) of acoustic and elastic waves has recently been demonstrated experimentally ([1], [2], [3]). TRI, which relies on the invariance of the wave equation under time reversal, is achieved by driving one or more piezoelectric transducers with a time-reversed version of the recorded wave field. The process has proven to be surprisingly stable, even for an experiment involving a medium with 2000 strong scatterers [1]. In [3] the ergodicity of stadium boundaries has been exploited to achieve TRI of elastic waves with only a single receiver.

The degree of focusing of waves under TRI depends on the aperture of the receiver array, the initial errors in the reversed wave field and on the stability properties of wave propagation. For closed systems of particles with perfectly reflecting walls (i.e., billiards), certain boundary geometries are known to be ergodic (e.g., [4]). This ergodicity, combined with a sensitive dependence on initial conditions is the definition of chaos [5]. For open systems of particles and for waves the situation is less clear. TRI provides a diagnostic of the stability of propagation that can be used for both particles and waves in open or closed systems. The idea is that for a system that is time-reversal invariant, both particles and waves should return to their source when at a certain time the waves and particles are reversed. A complete focusing on the source will only take place when the velocity and position are known exactly and when the scattering medium is exactly the same before and after the time reversal. The degree to which errors in these quantities destroy the imaging on the source is a diagnostic of the stability of the wave or particle propagation, and hence of chaotic behavior of particles and waves.

In [6] we show that the delicate interference required to achieve TRI of waves can be destroyed by relatively small perturbations in the position of the scatterers. Here we address systematically the stability of TRI under various kinds of perturbations for open systems of both particles and waves in the presence of multiple scattering. The system used here is similar to the one used in [1] and is shown in Figure 1. Particles or waves are emitted from a source and propagate through a system of 200 strong isotropic point scatterers. For the waves, 96 receivers are located on the line indicated in Figure 1. A particle is recorded for the time-reversed imaging when it traverses the receiver line.

When comparing the stability of wave or particle propagation, one can either specify the medium (e.g. a quantum mechanical potential) or the scattering properties of the waves and particles. In this work the latter approach is taken by using isotropic point scatterers for both waves and particles. This choice ensures that the only difference between the waves and particles lies in the dynamics of propagation, rather than in a different interaction with the scatterers.
Scattering of particles and waves

Isotropic scattering of particles that is invariant under time reversal is ensured by requiring that both the velocity $v (=1500 \text{ m/s})$ and the impact parameter $b$ of the particles are conserved during scattering and that (in two dimensions) the scattering angle $\Theta$ is linear in the impact parameter:

$$\Theta = \pi \left( \frac{\sigma}{\sigma} - \frac{2b}{\sigma} \right) \quad \text{for} \quad |b| \leq \sigma/2,$$

(1)

where $\sigma$ is the scattering cross-section. For larger values of the impact parameter the particle is not scattered (i.e. $\Theta = 0$). See Table 1 for the values of parameters in the numerical experiment. Figure 2 shows the mean number of encounters $n$ with scatterers for the particles that cross the receiver line as a function of time $t$. A least-squares fit of the line in Figure 2 gives the mean time between consecutive encounters with scatterers, this can be used to infer that the mean free path $l$ is given by $l = 15.56 \text{ mm}$. This quantity is much less than the size of the scattering region (80mm), which implies that the particles are strongly scattered. For the TRI of the particles, the velocity of the particles is reversed when they cross the receiver line and $t$ is replaced by $-t$; in ideal circumstances the particle should then return to the source at $t = 0$.

For the scattering of waves the formalism of Groenenboom and Snieder [7] for isotropic point scatterers is used. This approach gives the exact response of a system of isotropic point scatterers and allows us to use the same scattering cross-section for the waves and the particles. The wave-field can be written as a sum over all possible scattering paths $P$:

$$u(r) = \sum_P e^{ikLP} \left( \prod C \right) u^0.$$

(2)

In this expression $L_P$ is the path length of the path ending at location $r$ and $u^0$ is the source signal emitted from the sources and $\prod C$ gives the product of geometrical spreading and scattering coefficients for the paths between scatterers. In case multiple sources are present a summation over these sources is implied. The significance of this expression is that the total wave-field is written as a sum over all possible paths joining the scatterers, in this way it constitutes a discrete version of the Feynman path integral.

Time reversed imaging of the waves is carried out by recording the wave-field at 96 equidistant receivers on the receiver line, and by using the complex conjugate of the wave-field in the frequency domain as source signals that are emitted from the receivers.

Stability analysis for scattered particles

Consider a particle that is scattered once with impact parameter $b$ and with a perturbed impact parameter $b + \Delta$. Using (1) the divergence of the trajectories is given by $|r_{b+\Delta}(t) - r_b(t)| \approx vt(\Theta(b + \Delta) - \Theta(b)) = 2\pi vt \Delta/\sigma$.
This implies that the error $\Delta_{out}$ at time $t$ since the scattering is related to the initial error $\Delta_{in}$ by $\Delta_{out} = 2\pi (vt/l) \Delta_{in}$. On average, $vt$ is the mean-free path $l$, hence

$$\Delta_{out} = 2\pi (l/s) \Delta_{in} .$$

(3)

When a wave is scattered $n$ times, the error $\Delta_{n}$ follows by recursion:

$$\Delta_{n} = (2\pi l/s)^n \Delta_{0} .$$

(4)

The number of scatterer encounters is on average given by $n = vt/l$, hence the Lyapunov exponent $\mu$ associated with the exponential divergence of trajectories is given by

$$\mu = \ln (2\pi l/s) v/l .$$

(5)

Equation (4) gives the error in the trajectory after $n$ scattering encounters. The error $\delta$ in the TRI is given by $\delta = D(\Delta t) = D(\Delta t) n \Delta_{n}$, hence

$$\delta = \frac{2nD}{\sigma} \left( \frac{2\pi l}{\sigma} \right)^n \Delta_{0} .$$

(6)

When the error in the trajectory is of the order of $\sigma/2$ the trajectory will be completely different because the particle then encounters different scatterers. The associated critical perturbation $\delta_c$ follows from (4):

$$\delta_{c} = \left( \frac{\sigma}{2\pi l} \right)^n \frac{\sigma}{2} .$$

(7)

For the numerical experiments the critical length scale is shown in Table 2. Also indicated is the precision with which the numerical simulations have been carried out. (All calculations were done in 64 bit arithmetic on an SGI Power Challenge.) Since the mean free path $l$ is much larger than the scattering cross-section $\sigma$ (Table 1) the critical length scale decreases dramatically with the number of scattering encounters.

The previous analysis applies for a perturbation of the starting point of a particle. When the scatterer locations are perturbed over a distance $\delta$, a term $\delta$ should be added to the right-hand side of (3). The error after $n$ scattering encounters is then given by

$$\Delta_{n} = \left( \frac{\left( \frac{2\pi l}{\sigma} \right)^n - 1}{\left( \frac{2\pi l}{\sigma} \right)^n - 1} \right) \Delta_{0} .$$

(8)

However, given the high numerical value of $2\pi l/\sigma$ ($\approx 61$) in the numerical experiments this result is similar to (4) for the perturbation of initial conditions. The associated critical length scale is shown in Table 3.

It follows that for the particles the critical length scale depends on the scattering cross section and the mean free path, and that this quantity depends exponentially on the number of scattering encounters (and thus on time). Due to this dependence the critical length scale $\delta_{c}$ is dramatically smaller than the scattering cross section $\sigma$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\delta_{c}(\text{nm})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0129</td>
</tr>
<tr>
<td>2</td>
<td>$2.11 \times 10^{-4}$</td>
</tr>
<tr>
<td>3</td>
<td>$3.43 \times 10^{-6}$</td>
</tr>
<tr>
<td>4</td>
<td>$5.60 \times 10^{-8}$</td>
</tr>
<tr>
<td>5</td>
<td>$9.11 \times 10^{-10}$</td>
</tr>
<tr>
<td>6</td>
<td>$1.48 \times 10^{-11}$</td>
</tr>
<tr>
<td>7</td>
<td>$2.41 \times 10^{-13}$</td>
</tr>
<tr>
<td>8</td>
<td>$3.33 \times 10^{-15}$</td>
</tr>
<tr>
<td>9</td>
<td>$6.41 \times 10^{-17}$</td>
</tr>
</tbody>
</table>

Table 2. Critical error defined in equation (7) for different number of scattering encounters. Also indicated is the employed machine precision.

<table>
<thead>
<tr>
<th>Scattering location</th>
<th>Source location</th>
</tr>
</thead>
<tbody>
<tr>
<td>Particles</td>
<td>$\frac{1}{2} \left( \frac{2\pi l}{\sigma} \right)^n \frac{\sigma}{2} \left( \frac{2\pi l}{\sigma} - 1 \right)$</td>
</tr>
<tr>
<td>Ballistic wave</td>
<td>$\frac{\sqrt{3} \sqrt{\pi L}}{4 \sqrt{n+1}}$</td>
</tr>
<tr>
<td>Coda wave</td>
<td>$\frac{1}{\sqrt{2n \pi}}$</td>
</tr>
</tbody>
</table>

Table 3. Critical length scales $\delta_{c}$ for different perturbations.

Stability analysis for scattered waves

A fundamental difference between time reversed imaging of waves and particles is that TRI of particles occurs because a trajectory returns to the source at $t = 0$ whereas for waves TRI is achieved because the waves interfere at $t = 0$ constructively only near the source. When either the sources or the scatterers are perturbed for the waves, the dominant effect on the wave-field is the perturbation of the path lengths $L_{P}$ in (2). When the variance $\sigma_{L}$ of the path length is of the order of a quarter wavelength (denoted by $\lambda$) the resulting interference pattern is destroyed. Hence TRI of waves will break down when $\sigma_{L} \approx \lambda/4$.

For the coda (the later part of the wave field consisting of multiply-scattered waves), one can show that the variance in the path length joining $n$ scatterers is given by

$$\sigma_{L,\text{coda}}^2 = \sqrt{2n} \delta .$$

(9)
TRI of the coda breaks down when this quantity equals \( \lambda/4 \). The critical length scale for perturbations of the scatterer locations is thus given by \( \delta_{c\text{coda}} = \lambda / (4\sqrt{2n}) \); see Table 3. Note that in contrast to the situation for particles this critical length scale does not depend exponentially on \( n \).

For the ballistic wave (the wave that propagates along the line of sight from source to receiver), perturbations of the scattering locations have an effect that is of the order \( \lambda/L \) smaller. A detailed analysis shows that

\[
\sigma_L^{\text{ball}} \approx \sqrt{\frac{n+1}{3}} \frac{\lambda}{L} \delta.
\]

(10)

When \( \sigma_L^{\text{ball}} \) is about a quarter wavelength the interference is upset, hence the critical length scale is given by \( \delta^{\text{ball}}_c \approx \sqrt{\lambda L / \sqrt{n+1} \times \sqrt{3}/4} \) (see Table 3). Note that this length scale is proportional to the width \( \sqrt{\lambda L} \) of the first Fresnel zone; when a scatterer is moved over the width of the Fresnel zone it contributes in a fundamentally different way to the ballistic wave.

When the source locations are perturbed over a distance \( \delta \) but the scatterers remain fixed, only the length of the trajectory to the first scatterer is perturbed. This means that for this perturbation for both the coda and the ballistic wave \( \sigma_L = \delta \). Thus, the critical length scale for perturbation of the source locations for both the coda and the ballistic wave is given by \( \delta_{\text{c,source}} = \lambda/4 \), see Table 3.

**Numerical simulations**

In the TRI of particles 20,000 particles are propagated from the source to the receiver line and after time-reversal back-propagated to the source. For the case when the receivers and the scatterers are not perturbed, the only relevant error is the error in the numerical calculations. It follows from Table 2 that particles with more than 8 scattering encounters will not be focused on the source during TRI. The numerical experiments confirm this conclusion. Figure 3 shows the natural logarithm of the number of particles at \( t = 0 \) as a function of position. In the ideal case, all particles are imaged on the source at \( x = z = 0 \). For the particles with 6 or fewer scattering encounters (shown in blue) this is indeed the case, whereas the particles with 10 or more encounters (shown in green) are imaged quasi-randomly over the whole region. The particles with 7-9 encounters (shown in red) are at \( t = 0 \) localized near the source region, but the imaging is degraded.

The quality of the time-reversed-image is quantified by \( \exp(-\text{error}/D) \), where \( \text{error} \) denotes the mean distance of the particles to the source at \( t = 0 \). This imaging quality is shown in Figure 4 as a function of the error in the source position for various values of the scattering encounter \( n \). The critical length scale shown in Table 2 is indicated with the vertical arrows. The horizontal scale ends at the left with the machine precision. When the TRI degrades, the imaging quality decays from unity to zero, it follows from Figure 4 that the analytical estimates of section agree well with the numerical results. When the scatterer locations are perturbed rather than the source locations, the results are virtually the same. This is due to the fact that for large values of \( 2n \lambda / \sigma \) the expressions (4) and (8) are almost identical.

For the waves, TRI has been carried for several time windows, see Table 4. The imaged section along the line \( x = 0 \) of Figure 1 is shown in Figure 5 by the thick solid line. In this example, a short time window of the coda from 0.25 s to 0.30 s has been used. The energy is focused on the source location at \( x = 0 \), the nonvanishing energy at other locations is due to the finite aperture of the receiver array used in the TRI. This section compares favorably with the experimental results in [1]. The thin lines in Figure 5 give the imaged section for various values of the perturbation in source position; the number indicates the variance in the perturbation of the source location measured in wavelengths. It can be seen that TRI indeed breaks down when the source locations are perturbed over about a quarter wavelength. The quality of the TRI can be quantified by computing the ratio of the amplitudes of the imaging peak of TRI with perturbation to the imaging peaks without perturbation.
Figure 4. Imaging quality defined as $\exp(-\text{error}/D)$ as a function of the perturbation in the initial position of the time reversal of particles. Estimates of the critical perturbation defined in equation (7) are indicated by vertical arrows.

<table>
<thead>
<tr>
<th>Wave</th>
<th>Time window (s)</th>
<th>Number of encounters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ballistic</td>
<td>0.11-0.13</td>
<td>2</td>
</tr>
<tr>
<td>Coda 1</td>
<td>0.20-0.25</td>
<td>13</td>
</tr>
<tr>
<td>Coda 2</td>
<td>0.30-0.35</td>
<td>22</td>
</tr>
<tr>
<td>Coda 3</td>
<td>0.40-0.45</td>
<td>32</td>
</tr>
</tbody>
</table>

Table 4. Time windows used in the different numerical experiments with waves and number of scattering encounters.

The resulting relative peak heights are shown in Figure 6 as a function of the error in source or scatterer locations for TRI experiments with the time windows shown in Table 4. The critical length scales shown in Table 3 are for each case indicated by vertical arrows. The curves for the perturbation of the source position are the four solid lines in the middle. These curves are identical for the four employed time windows and show a decay when the perturbation is of the order $\lambda/4$ (which has the numerical value 0.625 mm). For the perturbation of scatterers for TRI of the ballistic waves, the critical length scale is significantly larger, and agrees well with the critical length scale shown in Table 3. For TRI of the three coda intervals the critical length scale is appreciably less than a wavelength. The reason is that the number of scattering encounters is large for these waves (see Table 4). The agreement between numerical simulations and the estimates shown in Table 3 is very good. This confirms the assumption that the dominant effect of the perturbation of the time reversed imaging of waves is the perturbation of the path length.

Discussion

It follows from the TRI of particles and waves that the stability of particle and wave motion for the perturbation of initial conditions or scatterer locations is fundamentally different. In the numerical experiments particles that have encountered 8 or more scatterers do not return to the source after TRI, whereas waves that have encountered up to at least 30 scatterers (Table 4) focus well on the source after TRI. The physical reason for this difference is that particles follow a single trajectory. When the initial conditions or a scatterer along the trajectory are perturbed, the whole trajectory is perturbed, often in a dramatic fashion. Because of the chaotic nature of trajectories, the critical length scale is significantly less than the scattering cross-section by a factor that depends exponentially on the number of encountered scatterers and hence exponentially on time. This pertains both to the perturbation of the source position as well as to the perturbation of the scatterer position.

For the waves when the source or scatterer location is perturbed, the different wave-paths are not perturbed fundamentally; only the length of the wave-paths is changed. However, this perturbation only leads to appreciable effects when this perturbation is around a
quarter-wavelength, because it is the interference of the waves along all possible wave-paths that determines the total wave-field. For both the coda and the ballistic wave the critical perturbation of the source location is a quarter-wavelength. For the perturbation of the scatterers the critical wavelength for the coda is proportional to the wavelength, but much smaller with a factor $1/\sqrt{2\pi}$. In contrast, the ballistic wave is only sensitive to perturbations of the scatterer position that are of the order of the width of the Fresnel zone.

This implies that waves and particles react in fundamentally different ways to perturbations of the initial conditions or the medium. The reason for this is that particles "select" a certain trajectory whereas waves travel along all possible trajectories visiting all the scatterers in all possible combinations. It is the "selection process" of a particle trajectory that creates the fundamentally larger instability of particle propagation than of wave propagation.

References
Estimating data uncertainties for least squares optimization

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ABSTRACT
When fitting model parameters to observed data, as in inverse calculations, it is essential to have some estimate of the data uncertainties. If these estimates are too small, then the fitting procedure will likely put features into the model that are not required by the data. If the estimated data uncertainties are too large, then the fitting will not extract all the available information from the data. In practice, these uncertainties are often asserted a priori or estimated globally for the entire data set. We develop several simple and robust optimization procedures that estimate data uncertainty automatically from the data. The simplest approach is to obtain a global estimate of the errors from ordinary least-squares. We then show how binning the data can be used to compute a more detailed, estimate of the error distribution. Finally, we propose and iterative extension of this approach. This uncertainty information can then be incorporated into the least squares optimization procedure by weighting the observations in each bin with the reciprocal of their locally computed variance. This leads to a robust least squares solution since bins with outliers in them will be down-weighted. We apply these methods to a synthetic vertical seismic profile (VSP), and to a real data set involving bathymetry measurements from the Sea of Galilee. These examples illustrate the fact that better estimates of data errors lead to a better distinction between data uncertainty and the effects of local model features.

Introduction

Geophysical inversion problems are fundamentally problems of statistical inference. In this paper we study a key component of such calculations, namely the calculation (by optimization) of finite dimensional vectors of model parameters whose response agrees with the observed data to some tolerance. To solve the complete inverse problem we would need to address many other issues such as the accuracy of the forward modeling, the level of discretization (since earth models are really functions) and a priori information. Here, instead, we focus on the problem of determining the data uncertainties directly from the data, so that we can measure the extent to which models fit the data. Thus our problem can be stated mathematically as follows.

Let $d \in \mathbb{R}^n$ be a vector of observations, $z \in \mathbb{R}^m$ be a vector of model parameters, and $A \in \mathbb{R}^{n \times m}$ a linear operator (the forward modeling operator) mapping model vectors into data vectors. We assume that the observed data $d$ are related to the true data $d_T$ (the data associated with a perfect noise-free experiment) by $d = d_T + \epsilon$, where $\epsilon$ is random. Further, we assume that the components of $d$ are independent and can be adequately described by a Gaussian distribution. Finding models that fit the data can then be cast as a weighted least squares problem. For the classical problem of estimating $m$ pa-
rameters from \( n \) observations we use the standard \( \chi^2 \) measure:

\[
\chi^2 = \frac{1}{n} \sum_{i=0}^{n} \left( \frac{\sum_{j=1}^{m} A_{ij} x_j - d_i}{\sigma_i} \right)^2
\]

with \( \sigma_i \) being the standard deviation of the \( i \)-th datum.

The difficulty in practice is that we do not know \( \sigma_i \). Various strategies for assigning or computing the data uncertainties have been proposed. The simplest of course is to fix all the \( \sigma_i \) to some \( \sigma \) a priori, based on the assumed precision of the experiment. For example, Oldenburg et al. (1997) invert a variety of exploration geophysical data sets, but all are assumed to have a constant error of 5%. Another strategy is to try to estimate a global \( \sigma \) from the residuals of a "good" model (e.g., Rogers & Wahr (1993)). Strictly speaking this is not possible since the data errors must be independent of the model, but it is a reasonable strategy in practice, and we will show below how to extend this idea to estimate a spatially varying data variance. In Bayesian inversion it is necessary to estimate the joint distribution function of the data uncertainties. For example, Gouveia & Scales (1998) use subsets of the data, which are assumed to contain only ambient noise, to estimate the mean and covariance of an \( N \)-dimensional normal error distribution. But this is a complicated procedure which may introduce sampling artifacts unless some model of the covariance structure is assumed.

In this paper we will describe three simple methods for estimating the data uncertainty automatically from the data. There are certainly more sophisticated approaches available, but our goal is to develop algorithms that are robust and easy to use.

Let us first state the basic linear estimation problem for known i.i.d. (independent, identically distributed) data uncertainties:

**\( \chi^2 \) estimation** Let \( x \) be the \( m \)-dimensional model vector, and \( d \) the vector with \( n \) observations. Let the vector \( \sigma \) contain an estimate of the data standard deviation.

(i) Let \( \chi^2 = \frac{1}{n} \sum_{i=0}^{N} \left( \frac{\sum_{j=1}^{m} A_{ij} x_j - d_i}{\sigma_i} \right)^2 \)

(ii) Solve \( \hat{x} = \min_{x} \left| \chi^2 - 1 \right| \)

By minimizing \( \chi^2 - 1 \), algorithms will stop when the model predicts the observations on average within one standard deviation. We can replace 1 by some other number, but whatever we choose we must acknowledge the fact that the least squares model (obtained by replacing the 1 with a 0) will usually over-fit the data. We could also use \( \chi^2 = 1 \) as a constraint in a more general optimization problem; the point is simply that we fit the data up to some tolerance. In practice we achieve this by adding features to the model until we just manage to fit the data. This sort of algorithm can be implemented with any least squares code; we refer to the appendix for the weighted conjugate gradient algorithm used in this paper. In any case, \( \chi^2 \) estimation assumes that one knows the data errors \( \sigma_i \).

### Global estimate of data uncertainty from OLS

We will first show that one can obtain an average of the data uncertainty directly from the data. Recall that we have

\[
A x = d = d_T + \epsilon.
\]

\( A \) is the forward operator with \( n \) rows and \( m \) columns, \( d \) is the \( n \) dimensional observed data, and \( x \) is the vector of model parameters of dimension \( m \). The noise vector \( \epsilon \) also has dimension \( n \). To get a global estimate of data uncertainty each component of \( \epsilon \) is assumed to be random with zero mean and variance \( \sigma^2 \).

The ordinary least squares (OLS) estimate of \( x \) is

\[
\hat{x} = A^T d
\]

where \( A^T \) is the pseudo inverse of the \( A \). When \( A^T A \) is nonsingular, the OLS estimate is unique:

\[
\hat{x} = (A^T A)^{-1} A^T d.
\]

In practice, \( A^T A \) is usually not invertible. In our numerical calculations we use the conjugate gradient least squares algorithm (Scales, 1987), which converges to the pseudo inverse solution.

The normal equations are a set of \( m \) linear equations in \( m \) unknowns. The residual vector \( d - A \hat{x} \) is the projection of the error vector \( \epsilon \) into the \( n - p \) dimensional space orthogonal to the column space of \( A \), where \( p \) is the rank of \( A \). We assume that \( p < n \). For this reason, the squared length (i.e., \( L_2 \) norm) of the residual vector has the expected value \( \frac{n-n}{n} \) times the expected squared length of the error vector. Since the expected squared length of \( \epsilon \) is \( n \sigma^2 \), the expected squared length of the residual vector is \( (n-p)\sigma^2 \) (Stuart & Ord, 1987). Therefore, the estimate of the global variance \( \sigma^2 \) is
\[ \hat{\sigma}^2 = \frac{\|d - A\hat{x}\|^2}{n - p} \]

We call the resulting algorithm Global $\chi^2$ estimation:

**Algorithm 1. Global $\chi^2$ estimation** Let $A^T$ be the pseudo inverse of $A$, let $p$ be the rank of $A$ and let $n$ be the number of data, $d$.

(i) Compute the OLS estimate $\hat{x} = A^T d$.
(ii) Define the average data variance $\hat{\sigma}^2 = \frac{\|d - A\hat{x}\|^2}{n-p}$.
(iii) Perform $\chi^2$ estimation with $\sigma_i^2 = \hat{\sigma}^2$.

Assuming locally constant data uncertainty

The Global $\chi^2$ approach leads to a global value for the data uncertainty. This is simple but has the disadvantage that we weight all data equally, regardless of quality. We will now introduce a method to recover an estimate of the data uncertainty that is only locally constant. This will allow us to differentiate between areas of good and bad data quality.

We will assign sub-regions (or bins) where we assume the data uncertainty is constant. For example, bins may represent regions in space or time. Within each bin we compute the variance. The resulting regional data variances are then used as the weights in the basic $\chi^2$ estimation procedure. We call this algorithm Local $\chi^2$ estimation.

**Algorithm 2. Local $\chi^2$ estimation**

(i) Group n observations in b bins.
(ii) Compute the data variance in each bin $i$: \( \{\sigma^2_j\}_{j=1}^b \).
(iii) Perform $\chi^2$ estimation with $\sigma_i^2 = \sigma_j^2$, where $j_i$ is the bin containing the observation $d_i$.

Local $\chi^2$ estimation down-weights regions of the data which exhibit large local fluctuations. This leads to a more robust procedure, since the influence of large outbursts of noise ("spikes") is reduced. On the other hand, bins in which large local fluctuations are systematic will be down-weighted as well. The models generated by this procedure will tend to smooth out local features in the data.

In applying this method, we must make decisions regarding the number and size of the bins. Our goal here is to discuss the effects of binning the data rather than to recommend optimal methods. For some discussion on bin selection methods, see (Freedman & Diaconis, 1981a) and (Freedman & Diaconis, 1981b).

**Extension to an iterative scheme**

In our third algorithm we will iteratively update the weights $\sigma_i$ to achieve higher resolution in the model. The initial weights are determined by computing the variance of the binned residuals resulting from Local $\chi^2$ estimation (algorithm 2). With these weights we perform $\chi^2$ estimation. The resulting model gives rise to a new set of residuals from which updated variances are computed. The process repeats until the weights converge. The advantage of this method is that the component of the data uncertainty as computed in algorithm 2 that was due to structure in the model has been reduced. We refer to this algorithm as Iterative Local $\chi^2$ estimation:

**Algorithm 3. Iterative Local $\chi^2$ estimation** Let the model $\tilde{x}$ be the result of optimization with b bins each with constant data variance (algorithm 2).

(i) Group the n residuals $A\tilde{x} - d$ in b bins.
(ii) Let $\sigma_j^2$ be the variance of the residuals in the j-th bin.
(iii) Perform $\chi^2$ estimation with $\sigma_i^2 = \sigma_j^2$, where $j_i$ is the bin containing the observation $d_i$.

The data variances can be iteratively updated by using the resulting model $\tilde{x}$ in item 1.

Summarizing, we have an optimization procedure ($\chi^2$ estimation) that requires an estimate of the data uncertainty. We have proposed three methods to obtain such an estimate. The first method involves a global data variance estimate, while the other two involve local estimates based on binning the observations. In the next sections we will apply each method to a synthetic example.

**A synthetic VSP example with Gaussian noise**

This synthetic VSP experiment contains one nearly zero-offset shot recorded at 78 geophones in a bore-hole (figure 1). The velocity model of the subsurface is represented by 41 constant velocity layers. Using a ray tracing forward operator, we have computed a synthetic data set from a model with increasing velocity with depth, except
that layers 14 through 20 are low velocity layers (LVL) as shown in figure 2.

We added uncorrelated Gaussian noise with zero mean and a variance of 1 millisecond to the noise free data. All results in this section are the average of 100 synthetic experiments with different realizations of this noise. For simplicity, the rays are treated as straight lines. Also, the first two layers contain no receivers. These conditions make it impossible to resolve the slowness of the first two layers, individually. Further, note that a rapid increase in velocity with depth in the shallow layers makes it possible for the arrival time at the second receiver to be shorter than at the first.

Our first experiment on the synthetic data set is an attempt to retrieve the (correct) 1 ms value of the data uncertainty with Global \( \chi^2 \) estimation. From equation 3 we know that if we have the ordinary least squares estimate of \( z \), we can calculate an estimate of the variance. Using an undamped conjugate gradient algorithm (described in the appendix), the estimate for the variance converges within 40 iterations to 1.01 ms, as shown in figure 3. Figure 4 shows the results of \( \chi^2 \) estimation with this global data variance estimate. Because the sowness of the first two layers cannot be resolved, the estimate is poor near the surface.

Next, we applied Local \( \chi^2 \) estimation (algorithm 2). We grouped the data in bins of six receivers (13 bins). In each bin the data variance is computed. The variances act as weights in the \( \chi^2 \) estimation scheme, the results of which are shown in figure 5. Although the computed model still seems to represent the features of the true model, we see a slight loss of resolution in the flanks of the LVL. Finally, we performed five iterations of Iterative Local \( \chi^2 \) estimation (algorithm 3). These results are shown in 6. We see very similar results to those obtained with Local \( \chi^2 \) estimation, except for an improvement in the flanks of the LVL.

Inversion with a global estimate of the data uncertainty performed best since the observations do in fact have constant data variance. The Local \( \chi^2 \) estimation leads to the over-estimate of the data variance (see figure 7) causing loss of resolution in the LVL. The Iterative Local \( \chi^2 \) estimation manages to diminish the over-estimate. Table 1 shows the true and computed slownesses in depths sur-
rounding the LVL. It is clear that iterating the variance estimates improves the resolution.

The VSP example with Gaussian noise and outliers

In geophysical data, noise spikes are relatively common. In this example we will use the same VSP data as in the last section, but in addition to the Gaussian noise two noise spikes of 10 ms amplitude have been added. These spikes are located at depths of 5 m and 19 m. Rather than apply a rejection criterion we will show how locally updating the data variances down-weights these large deviations automatically.

First, we computed the the global estimate of the data variance (equation 3). The value was 3.73 ms, which, due to the noise spikes, is significantly more than the 1 ms Gaussian noise added. (Without the Gaussian noise, the two spikes lead to a global data variance of 2.7 ms.) Using a global estimate of the data variance of 3.73 ms we have computed the Global $\chi^2$ estimate (figure 8). With the global estimate of the data uncertainty, every data point is of equal weight. Therefore, this algorithm resulted in a model that predicts not only the LVL, but also part of the outliers.
Figure 6. Same as figure 3, but for Iterative Local $\chi^2$ estimation.

<table>
<thead>
<tr>
<th>parameter</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>19</th>
<th>20</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>true model</td>
<td>0.71</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>0.45</td>
</tr>
<tr>
<td>Local $\chi^2$</td>
<td>1.32</td>
<td>1.55</td>
<td>1.68</td>
<td>1.72</td>
<td>1.48</td>
<td>1.16</td>
</tr>
<tr>
<td>Iterative Local $\chi^2$</td>
<td>1.22</td>
<td>1.66</td>
<td>1.90</td>
<td>1.84</td>
<td>1.48</td>
<td>1.01</td>
</tr>
</tbody>
</table>

Table 1. Table shows the true and computed slownesses for the VSP problem with Gaussian noise. The low velocity layer (LVL) starts in the true model at model parameter 14 and ends at parameter 20. The results are the average over 1000 experiments.

Figure 7. The variance distribution from the three different different algorithms assuming 1 ms uncorrelated Gaussian noise.

Second, we applied Local $\chi^2$ estimation (algorithm 2) to the data with outliers. We grouped the receivers in 13 bins again. With the variance per bin as an estimate of the data uncertainty, the resulting model does not follow the spikes (figure 9). The relatively high variance in the bins with outliers (see figure 10) causes down-weighting of these observations.

After grouping the data in 13 bins, algorithm 2 produced the initial model to apply the iterative method of algorithm 3. After 5 iterations, the results also show robust behavior in the presence of outliers (figure 11). Again, we see an improvement in resolution in the flanks of the LVL with respect to Local $\chi^2$ estimation, because of a locally more accurate estimate of the data uncertainty (see figure 10).

Estimating the data variances locally provided robustness in the optimization, while the physical features as the low velocity zone were still well resolved. Although the bins were large due to the limited number of data, inversion with the data variance estimate from binning the data still showed encouraging results. With a more dense data set we could choose a smaller bin size. This could improve resolution. The next section provides an example with a more dense distribution of observations.
Automatic data variance estimation

Example: Sea of Galilee bathymetry

Ben-Avraham et al. (1990) have published depth measurements of the Sea of Galilee. The system they used combined estimates of the boat’s position with echo-sounding data. Tidal and other corrections were applied in real-time. The echo-soundings themselves were calibrated every few hours using a bar that was lowered to the bottom. The depth values were transformed to values below mean sea level using measured values of the water level on the surface. The errors associated with each of these steps combine. A lower bound on the errors in the depth values is .01%, which is the quoted accuracy of the echo-sounder.

The soundings were made on an irregular grid. If we define linear interpolation as extracting values between the known values on a regular grid, then inferring the depths on a regular grid from the irregularly spaced data can be considered a linear inverse problem (Claerbout, 1997). We will now solve this problem by the $\chi^2$ estimation procedure using estimates of the data uncertainties obtained from our three algorithms. We use a four-point linear interpolator as the forward operator. To speed up the calculations, we worked on a subset of the data, highlighted in figure 12. To view the data, we have plotted the mean depth in square bins of 50 meter sides (figure 13).

Computing the least squares estimate of the depths on a
regular grid of 125 by 125 meters, resulted in an estimate of the data variance of 0.41 m$^2$ (figure 14). This means the standard deviation is 0.64 m or 0.3 percent of the average depth. Inversion with the global estimate of a data variance of 0.41 m$^2$ resulted in an overall smooth model with some local structures at (205 km, 243 km) and (207 km, 241 km) (figure 15).

Next we performed the Local $\chi^2$ estimation procedure assuming the data variance to be constant over square areas of 125 by 125 meter. We defined our interpolation grid to match the bin distribution, i.e., the grid points are 125 meters apart. With this estimate of the data variance, we have obtained a model of the subsurface (shown in figure 16), that achieves a normalized $\chi^2$ of slightly less than 1. As can be seen in this figure, typical variances are on the order of 0.075 m$^2$, with isolated bins of higher variance. (A variance of 0.075 m$^2$ corresponds to depth errors on the order of .1%.) This means that in most of the model we will be able to extract more information from the data (put more features into the model) than we could with the relatively large global variance esti-
Figure 13. Contour plot of the mean of the depth measurements in bins of 50 by 50 m.

Figure 14. Global data variance estimate as a function of the number of iterations. The estimate converges to 0.41 m².

Figure 15. Contour plot of the sea floor after $\chi^2$ estimation with a global data variance estimate of 0.41 m². The optimization is stopped when $\chi^2 = 1$. The grey levels refer to variance of the bins (in this case a constant), while the contours are labeled according to depths. Notice that with such a large global data variance the model is quite smooth, but with a few isolated topographic features.

quite smooth, so that the weights do not change much when updated.

Conclusion

The character of least squares estimated models depend on the data variance estimate. If these estimates are too small, then the fitting procedure will likely put features into the model that are not required by the data. If the estimated data uncertainties are too large, then the fitting will not extract all the available information from the data. We have shown three simple, efficient algorithms for automatically estimating the data uncertainty for least squares optimization. One of the algorithms estimates a global value for the data uncertainty (Global $\chi^2$ estimation). The other two algorithms allow regionally varying estimates of the data uncertainty. Local $\chi^2$ estimation computes the variance of the observations in bins. In case of model structure on a scale smaller than
the bin size, such a procedure would lead to an overestimate of the data variance. An iterative up-dating of the weights (Iterative Local $\chi^2$ estimation) can improve the accuracy of the data variance estimate. We have shown that in cases where the data uncertainty varies regionally, these local estimates of the data uncertainty introduce robustness into the optimization scheme by down-weighting regions of large variance. In regions of small data variance we can extract more information about the model parameters. We have illustrated the application of these algorithms on both synthetic examples as well as a bathymetry study of the Sea of Galilee.

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Figure 16. Contoured image of the sea floor after local $\chi^2$ estimation. The shades of grey reflect the data variance in each bin. The prevailing variances are of the order of 0.075 m$^2$. White bins have a data variance larger than 1 m$^2$. Thus most of the data is considerably lower variance than we estimated with Global $\chi^2$, but there a number of bins of higher variance. These get down-weighted by the local algorithms.

Figure 17. Contoured image of the sea floor after five iterations of Iterative Local $\chi^2$ estimation. The shades of grey reflect the data variance in each bin. The prevailing variances are of the order of 0.075 m$^2$. White bins have a data variance larger than 1 m$^2$. This result Iterative Local $\chi^2$ result is very similar that obtained using Local $\chi^2$.

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\[ q_i = B p_i = R^{1/2} A p_i \]
\[ \alpha_i+1 = \frac{(r_i, x_i)}{(x_i, x_i)} \]
\[ x_{i+1} = x_i + \alpha_{i+1} p_i \]
\[ s_{i+1} = s_i - \alpha_{i+1} q_i \]
\[ r_{i+1} = B^T s_{i+1} = A^T \frac{1}{R^{1/2}} s_{i+1} \]
\[ \beta_{i+1} = \frac{(r_{i+1}, x_{i+1})}{(r_i, x_i)} \]
\[ p_{i+1} = r_{i+1} + \beta_{i+1} p_i \]

By moving the calculation of \( q_i \) to the top of the loop, it is not necessary to define a starting value for \( q \).

**APPENDIX A: Conjugate Gradient Method for Weighted Least-Squares**

Conjugate gradient can be extended to the least squares solution of arbitrary linear systems. When you weight the data the normal equations are:

\[ A^T R A x = A^T R h \]  \hspace{1cm} (A1)

Now, the algorithm for weighted least squares will be based on the Conjugate Gradient Least Squares algorithm (CGLS) of Hestenes & Stiefel (1952). Note that \( R \) is a diagonal matrix, with non-negative values. Therefore, \( R = R^{1/2} R^{1/2} \) and \( R^T = R \). Now we decompose \( R \):

\[ A^T R^{1/2} R^{1/2} A x = A^T R^{1/2} R^{1/2} h \iff \]
\[ (R^{1/2} A)^T R^{1/2} A x = (R^{1/2} A)^T R^{1/2} h \iff \]
\[ B^T B x = B^T y \]

where \( B = R^{1/2} A \), and \( y = R^{1/2} h \).

The algorithm for weighted least squares will be: Choose \( x_0 \). Put

- \( s_0 = y - B x_0 = R^{1/2} (h - A x_0) \)
- \( r_0 = p_0 = B^T s_0 = A^T R^{1/2} s_0 \)
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