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Phase and Moveout Velocity as Functions of the Ray Parameter for Transversely Isotropic Media

Jack K. Cohen

Center for Wave Phenomena
Colorado School of Mines
Golden, Colorado 80401
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ABSTRACT

In their studies of transversely isotropic media, I. Tsvankin and his co-workers have shown the benefits of expressing the P-wave normal moveout velocity in terms of the zero-offset ray parameter instead of the phase angle, but left for future work the actual task of doing this. Here, explicit formulas are given for both the P-wave phase velocity and normal moveout velocity in terms of the ray parameter. Various results of Tsvankin and others are rederived and, in some cases, extended, in a uniform manner using these explicit results.

The small ray-parameter expansion of the moveout velocity is valid for dips up to approximately 15°. In this regime, an analytic solution for the anisotropy parameter η, important for time-related imaging is given in terms of observations of moveout velocity for two different values of the ray parameter.

INTRODUCTION

In their studies of transversely isotropic media, I. Tsvankin and his co-workers have shown the benefits of expressing the P-wave normal moveout velocity in terms of the zero-offset ray parameter instead of the phase angle. Here, explicit formulas are given for both the P-wave phase velocity and normal moveout velocity in terms of the ray parameter. Various consequences of these results are drawn, including an analytic solution for the anisotropy parameter η, important for time-related imaging is given in terms of observations of moveout velocity for two different values of the ray parameter.

Throughout this study, *Mathematica* was used extensively to derive and check results. In particular, the *Mathematica* package, **Thomsen.m** (Cohen, 1995), was used to convert equation (3) to Thomsen notation, in equation (5), and to obtain various results in the limit of weak transverse isotropy. Similarly, the use of *Mathematica* facilitated computing symbolic derivatives, series expansions, etc.

PHASE VELOCITY AS A FUNCTION OF RAY PARAMETER

Begin with the formula for P-wave phase velocity in terms of dip θ (Tsvankin, 1994), equation(6):

$$\begin{aligned}2\rho V^2(\theta) &= (C_{11}+C_{44})\sin^2\theta + (C_{33}+C_{44})\cos^2\theta \\ &\quad + \Big\{\big[(C_{11}-C_{44})\sin^2\theta - (C_{33}-C_{44})\cos^2\theta\big]^2 \\ &\qquad + 4(C_{13}+C_{44})^2\sin^2\theta\cos^2\theta\Big\}^{1/2}.\end{aligned} \tag{1}$$

Use the substitutions

$$p = \sin\theta/V(\theta), \quad m = \cos\theta/V(\theta), \tag{2}$$

to rewrite equation (1) as an equation for the slowness surface:

$$\begin{aligned}2\rho &= (C_{11}+C_{44})p^2 + (C_{33}+C_{44})m^2 \\ &\quad + \Big\{\big[(C_{11}-C_{44})p^2 - (C_{33}-C_{44})m^2\big]^2 \\ &\qquad + 4(C_{13}+C_{44})^2p^2m^2\Big\}^{1/2}.\end{aligned} \tag{3}$$

To obtain a formula for $V(p)$, follow the recipe given in (Alkhalifah & Tsvankin, 1995), Appendix A. Begin by converting equation (3) to Thomsen notation. After introducing the on-axis P and S velocities, c_P and c_S, and the related quantities,

$$k = \frac{1}{c_P^2}, \qquad f = 1 - \frac{c_S^2}{c_P^2}, \tag{4}$$

obtain the Thomsen-notation form of the slowness surface as

$$2k = (2-f)m^2 + (2+2\epsilon-f)p^2 + \sqrt{4f(2\delta+f)m^2p^2 + (f(p^2-m^2)+2\epsilon p^2)^2}. \tag{5}$$

Next, solve equation (5) for m^2 and, from equation (2), form

$$V(p) = \frac{1}{\sqrt{p^2+m^2(p)}}. \tag{6}$$

After some manipulations, find that $V^2(p)$ can be written as

$$V^2(p) = \frac{A+\sqrt{B}}{2C}, \tag{7}$$

where,

$$\begin{aligned} A &= (2-f)k - 2(\epsilon - f\delta)p^2, && (8)\\ B &= f^2k^2 - 4fk\left[\epsilon-(2-f)\delta\right]p^2 + 4\left[2f(1-f)(\epsilon-\delta)+(\epsilon-f\delta)^2\right]p^4, && (9)\\ C &= k^2 - 2k\epsilon p^2 - 2f(\epsilon-\delta)p^4. && (10)\end{aligned}$$

Before turning to some important special cases, observe the following consequences of the definitions in equations (2):

$$\sin\theta = pV(p), \quad \cos\theta = mV(p) = \sqrt{1-(pV)^2}, \quad p^2 + m^2 = \frac{1}{V^2}. \tag{11}$$

These equations will be of great convenience in translating between representations in phase angle and representations in the ray parameter.

Small ray parameter

First observe that for $p = 0$,

$$A = (2-f)k, \tag{12}$$
$$B = f^2k^2, \tag{13}$$
$$C = k^2, \tag{14}$$

so that,

$$V^2(0) = \frac{(2-f)k + fk}{2k^2} = \frac{2k}{2k^2} = \frac{1}{k} = c_P^2, \tag{15}$$

leading to the expected result,

$$V(0) = c_P. \tag{16}$$

For later purposes, we will need to know the more detailed behavior of $V(p)$ for small p. Introducing the dimensionless parameter,

$$z = (c_P p)^2, \tag{17}$$

obtain

$$\begin{aligned} V(p) &= c_P\Big[1 + \delta z + \Big((\epsilon-\delta)(1+2\delta/f) + 3\delta^2/2\Big)z^2 \\ &\quad + \Big((\epsilon-\delta)(1+2\delta/f)(5\delta + 2(\epsilon-2\delta)/f) + 5\delta^3/2\Big)z^3 \\ &\quad + O(z^4)\Big]. \end{aligned} \tag{18}$$

Remark: This expansion shows that the true meaning of "small p" is that $z = (c_P p)^2 << 1$. Since $V \approx c_P$, the inequality can be written as $\sin^2\theta << 1$, so in dimensionless terms, "small p" represents a small dip angle θ.

Elliptic anisotropy

Next study the elliptic case ($\delta = \epsilon$), where

$$A = (2-f)k - 2(1-f)\delta p^2, \tag{19}$$
$$B = [fk + 2(1-f)\delta p^2]^2, \tag{20}$$
$$C = k^2 - 2k\delta p^2, \tag{21}$$

so that,

$$V^2(p) = \frac{1}{k - 2\delta p^2} = \frac{c_P^2}{1 - 2\delta c_P^2 p^2}, \qquad (\delta = \epsilon). \tag{22}$$

Tsvankin (1995) gives the corresponding elliptic limit value of V^2 in terms of the phase angle, θ, as

$$V^2(\theta) = V_0^2 \cos^2\theta + V_{90}^2 \sin^2\theta, \tag{23}$$

where $V_0 = V(0)$ and $V_{90} = V(\pi/2)$ in the phase angle form of V implied by equation (1). These values are readily found from equation (5). When $p = 0$, then $m = 1/V(0)$, and the equation reduces to $2/c_P^2 = 2/V^2(0)$, so that

$$V_0 = c_P. \tag{24}$$

Similarly, when $m = 0$, then $p = 1/V_{90}$, giving $2/c_P^2 = 2(1 + \delta + \epsilon)/V_{90}$, so that

$$V_{90} = c_P\sqrt{1 + \delta + \epsilon}, \tag{25}$$

which reduces to

$$V_{90} = c_P\sqrt{1 + 2\delta} \qquad (\delta = \epsilon) \tag{26}$$

in the elliptic limit. This also confirms the known result that $V_{\mathrm{nmo}}(0) = V_{90}$ for elliptically anisotropic media.

With these results in hand, equation (23) becomes

$$V^2 = c_P^2 + 2\delta c_P^2 \sin^2\theta = c_P^2 + 2\delta c_P^2 p^2 V^2 \qquad (\delta = \epsilon). \tag{27}$$

On isolating V^2, equation (22) is verified.

Tsvankin (1995) likewise gives the form

$$V(\theta) = c_P\sqrt{1 + 2\delta \sin^2\theta} \qquad (\delta = \epsilon) \tag{28}$$

for the elliptic case. Introducing the ray parameter in that expression gives

$$V^2 = c_P^2(1 + 2\delta p^2 V^2) \tag{29}$$

and, once again, isolating V^2 verifies equation (22).

Weak transverse isotropy

The weak TI limit of equation (7) is

$$V^2(p) \approx c_P^2\left[1 + 2(\delta + (\epsilon - \delta)c_P^2 p^2)c_P^2 p^2\right], \tag{30}$$

or

$$V(p) \approx c_P\left[1 + (\delta + (\epsilon - \delta)c_P^2 p^2)c_P^2 p^2\right]. \tag{31}$$

Thomsen's (1986) expression for this quantity expressed as a function of angle is

$$V(\theta) = c_P \left[1 + (\delta \cos^2\theta + \epsilon \sin^2\theta)\sin^2\theta\right]. \tag{32}$$

Tsvankin (1995) shows that this equation implies

$$\sin^2\theta = \frac{p^2 c_P^2}{1 - 2\delta p^2 c_P^2}[1 + 2(\epsilon - \delta)p^4 c_P^4], \tag{33}$$

which, in turn, in the weak limit is equivalent to

$$p^2 V^2 = p^2 c_P^2 [1 + 2\delta p^2 c_P^2 + 2(\epsilon - \delta)p^4 c_P^4]. \tag{34}$$

On cancelling the common factor of p^2, this verifies equation (30) and hence also equation (31).

NORMAL MOVEOUT VELOCITY AS A FUNCTION OF RAY PARAMETER

The derivation of the normal moveout velocity as a function of ray parameter, follows closely the corresponding derivation of its expression in terms of the ray angle given in (Tsvankin, 1995). For convenience, some explicit references are made to equations in this paper.

First, use equations (11) to find the following relation for $dp/d\theta$:

$$mV = \cos\theta = \frac{d\sin\theta}{d\theta} = (pV)'\,\frac{dp}{d\theta}, \tag{35}$$

where, here and below, the prime notation is used for p-differentiation. Thus, we find that

$$\frac{dV}{d\theta} = V'\,\frac{dp}{d\theta} = \frac{mVV'}{(pV)'}. \tag{36}$$

The derivation of $V_{\text{nmo}}(p)$ begins with Tsvankin's equation (4),

$$V^2_{\text{nmo}}(p) = \frac{2z_0}{t_0}\lim_{h\to 0}\frac{d\tan\psi}{dp}. \tag{37}$$

The second component of this expression is evaluated using Tsvankin's equation (6), equations (11), and equation (36):

$$\begin{aligned}
\tan\psi &= \frac{V\sin\theta + \frac{dV}{d\theta}\cos\theta}{V\cos\theta - \frac{dV}{d\theta}\sin\theta} \\
&= \frac{pV^2 + mV\frac{dV}{d\theta}}{mV^2 - pV\frac{dV}{d\theta}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{pV + m\frac{dV}{d\theta}}{mV - p\frac{dV}{d\theta}} \\
&= \frac{pV(pV)' + m^2VV'}{mV(pV)' - mpVV'} \\
&= \frac{p(pV)' + m^2V'}{m\left[(pV)' - pV'\right]} \\
&= \frac{p^2V' + m^2V' + pV}{mV} \\
&= \frac{V'/V^2 + pV}{mV} \\
&= \frac{V' + pV^3}{mV^3}.
\end{aligned} \tag{38}
$$

The first component of equation (37) is similarly evaluated using Tsvankin's equation (8):

$$
\begin{aligned}
\frac{2z_0}{t_0} &= V\cos\theta\left(1 - \frac{\tan\theta}{V}\frac{dV}{d\theta}\right) \\
&= mV^2\left(1 - \frac{p}{mV}\frac{mVV'}{(pV)'}\right) \\
&= mV^2\left(1 - \frac{pV'}{(pV)'}\right) \\
&= mV^2\frac{V + pV' - pV'}{(pV)'} \\
&= \frac{mV^3}{(pV)'}.
\end{aligned} \tag{39}
$$

Using the previous two results in equation (37), find that

$$
V_{\mathrm{nmo}}^2(p) = \frac{mV^3}{(pV)'}\frac{d}{dp}\left(\frac{V' + pV^3}{mV^3}\right) \tag{40}
$$

or, on eliminating m,

$$
V_{\mathrm{nmo}}^2(p) = \frac{V^2\sqrt{1-(pV)^2}}{(pV)'}\frac{d}{dp}\left(\frac{V' + pV^3}{V^2\sqrt{1-(pV)^2}}\right). \tag{41}
$$

Carrying out the indicated derivative gives an explicit formula for the normal moveout velocity as a function of ray parameter:

$$
V_{\mathrm{nmo}}^2(p) = \frac{2(1-p^2V^2)VV'' + (3p^2V^2 - 2)V'^2 + 2pV^3V' + V^4}{(1-p^2V^2)V(pV)'} \tag{42}
$$

Before turning to the special cases for V, note that when it is necessary to treat the general V, it may be better to avoid a layer of square roots by writing $V_{\text{nmo}}^2(p)$ in terms of $W \equiv V^2$:

$$V_{\text{nmo}}^2(p) = \frac{2(1-p^2W)WW'' + (4p^2W-3)W'^2 + 4pW^2W' + 4W^3}{2(1-p^2W)W(pW'+2W)} \tag{43}$$

Small ray parameter

First, use the two leading terms of equation (18) to obtain the approximations,

$$V = c_P + c_P^3\delta p^2 + O(p^4), \quad V' = 2c_P^3\delta p + O(p^3), \quad V'' = 2c_P^3\delta + O(p^2). \tag{44}$$

On inserting these small-p approximations into equation (42), find that

$$V_{\text{nmo}}^2(p) = c_P^2(1+2\delta) + O(p^2), \tag{45}$$

so

$$V_{\text{nmo}}(0) = c_P\sqrt{1+2\delta}. \tag{46}$$

Now use the next order terms of equation (18) in equation (42) to get the next term in $V_{\text{nmo}}^2(p)$:

$$V_{\text{nmo}}^2(p) = c_P^2\left(1+2\delta+\left((-24\delta^2+24\delta\epsilon+f-8\delta f+4\delta^2 f+12\epsilon f)\frac{(c_Pp)^2}{f}\right)+O(c_Pp)^4\right). \tag{47}$$

Remark: Again, the expansion shows that the true meaning of "small p" is that $(c_Pp)^2 << 1$. Since $V \approx c_P$, and the error is fourth order, this means we can interpret "small p" as meaning $\sin^4\theta << 1$. Numerical tests show that using the small p series for a dip of 15° incurs about a 2% error for typical values of f and δ.

The theory discussed by (Alkhalifah & Tsvankin, 1995) suggests that it is better to express this result in terms of the parameters $V_{\text{nmo}}(0)$ and

$$\eta = \frac{\epsilon-\delta}{1+2\delta}. \tag{48}$$

First introduce η,

$$V_{\text{nmo}}^2(p) \approx c_P^2(1+2\delta) + \left(c_P^4(1+2\delta)(24\delta\eta+f+2\delta f+12\eta f)p^2\right)/f + O(p^4), \tag{49}$$

then use equation (46) to eliminate the explicit appearances of c_P^2 in favor of $V_{\text{nmo}}^2(0)$:

$$V_{\text{nmo}}^2(p) = V_{\text{nmo}}^2(0)\left[\left(1+(1+12\eta\frac{1+2\delta/f}{1+2\delta})V_{\text{nmo}}^2(0)p^2\right)+O(V_{\text{nmo}}^4(0)p^4)\right]. \tag{50}$$

The importance of the anisotropic parameter η, is that Alkhalifah and Tsvankin (1995) have observed, both numerically and empirically, that $V_{\text{nmo}}^2(p)$ depends mainly

on η and $V_{\mathrm{nmo}}^2(0)$. They exploit this reduction to surface-observable parameters to develop time-domain seismic processing algorithms that take account of transverse anisotropy. Equation (50) gives analytic support to the Alkhalifah-Tsvankin theory, since the only deviation to it occurs in the ratio

$$g = \frac{1 + 2\delta/f}{1 + 2\delta} \tag{51}$$

that multiplies η. Figures (1) and (2) show plots of this function over the ranges of f and δ that are relevant in practice. Observe that the function $g(\delta, f)$ varies slowly over these ranges—indeed—ignoring it altogether (i.e., replacing it by the constant 1) is usually justified—at least in the small-angle approximation.

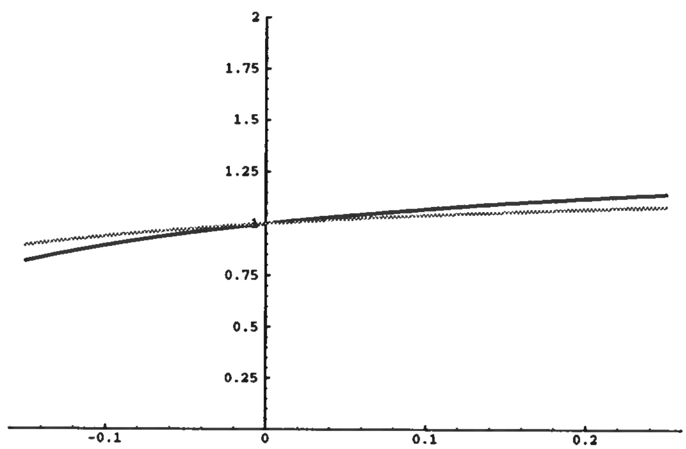

FIG. 1. Plot of the factor $g = (1 + 2\delta/f)/(1 + 2\delta)$ as a function of δ for $f = 0.7$ (dark) and $f = 0.8$ (light).

The small-ray-parameter result in equation (50) gives a means for estimating η from surface observations. For example, suppose one has observations of $V_{\mathrm{nmo}}(p)$ at $p = 0$ and some other (not too large) value $p = p_1$. The solution for η is given by

$$\eta \approx \frac{1}{12g}\left(\frac{V_{\mathrm{nmo}}^2(p_1) - V_{\mathrm{nmo}}^2(0)}{p_1^2 V_{\mathrm{nmo}}^4(0)} - 1\right), \tag{52}$$

where once again, in the absence of information on δ and f, the factor g can be replaced by the constant 1 without much error.

More generally, if one uses two nonzero "small p" values (that is, two separated dips, each less than 15°), then the estimate for $V_{\mathrm{nmo}}^2(0)$ is given by

$$V_{\mathrm{nmo}}^2(0) = \frac{p_2^2 V_{\mathrm{nmo}}^2(p_1) - p_1^2 V_{\mathrm{nmo}}^2(p_2)}{p_2^2 - p_1^2}, \tag{53}$$

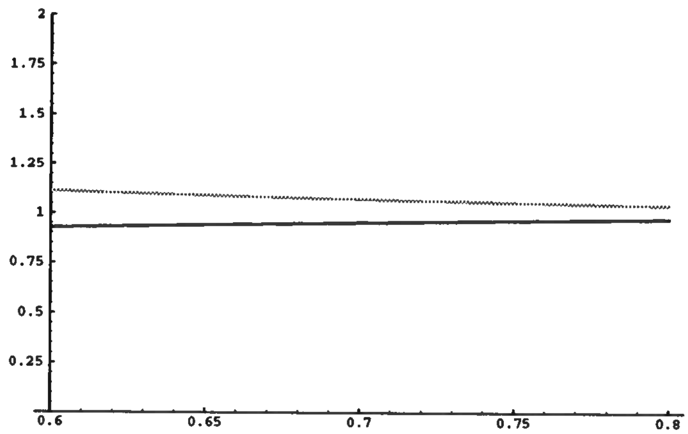

FIG. 2. Plot of the factor $g = (1 + 2\delta/f)/(1 + 2\delta)$ as a function of f for $\delta = -0.05$ (dark) and $\delta = +0.1$ (light).

and the estimate for η is

$$\eta \approx \frac{1}{12g}\left(\frac{(p_1^2 - p_2^2)(V_{\text{nmo}}^2(p_2) - V_{\text{nmo}}^2(p_1))}{(p_1^2 V_{\text{nmo}}^2(p_2) - p_2^2 V_{\text{nmo}}^2(p_1))^2} - 1\right). \tag{54}$$

Finally, note that the full form of the series for $V_{\text{nmo}}^2(p)$ is given by

$$V_{\text{nmo}}^2(p) = V_{\text{nmo}}^2(0)\left[1 + c_2 V_{\text{nmo}}^2(0)p^2 + c_4 V_{\text{nmo}}^4(0)p^4 + \cdots\right], \tag{55}$$

where, as we have seen earlier,

$$c_2 = 1 + 12g\eta \approx 1 + 12\eta. \tag{56}$$

By using the full form of equation (18), we find that

$$c_4 = 1 + 6g(6 - 5g)\eta + \frac{60g}{f}\eta^2 \approx 1 + 6\eta + \frac{60}{f}\eta^2. \tag{57}$$

Here, the approximations result from replacing the factor g defined in equation (51) by the constant 1. Note that the final term in c_4 indicates the first serious divergence from the theory that $V_{\text{nmo}}^2(p)$ depends only on the parameters $V_{\text{nmo}}^2(0)$ and η. However, this term is multiplied by both p^2 and η^2, which ameliorates the effect of replacing the f in this term by, say, 3/4 instead of the true value.

Elliptic anisotropy

Inserting the elliptic P-wave phase velocity as a function of ray parameter given in equation (22) into the general NMO equation (42) gives at once

$$V_{\text{nmo}}^2(p) = \frac{c_P^2(1 + 2\delta)}{(1 - (1 + 2\delta)c_P^2 p^2)}. \tag{58}$$

Recognizing the quantity $V_{\rm nmo}(0)$ from equation (46) gives

$$V_{\rm nmo}(p) = \frac{V_{\rm nmo}(0)}{\sqrt{1 - p^2 V_{\rm nmo}^2(0)}}, \qquad (\delta = \epsilon), \tag{59}$$

in agreement with the result in (Alkhalifah & Tsvankin, 1995). For future use, introduce the notation,

$$V_{\rm ell}(p) \equiv \frac{V_{\rm nmo}(0)}{\sqrt{1 - p^2 V_{\rm nmo}^2(0)}}, \tag{60}$$

for the elliptic result.

Weak transverse isotropy

Using equation (31) in equation (42) gives

$$V_{\rm nmo}^2(p) = \frac{c_P^2}{1-z}\left(1 + \frac{2(\delta + (\epsilon - \delta)\, z(6 - 9z + 4z^2))}{1-z}\right), \tag{61}$$

where we have again used the shorthand notation, $z = (c_P p)^2$. One would have to seek a more sophisticated expansion if p became large enough to approach $1/c_P$. On the other hand, always $p < 1/V$ and in the weak limit, $V \approx c_P$, so this is an unusual circumstance.

The approximation, $\eta = \epsilon - \delta$, is valid in the weak limit, so equation (61) may be recast as

$$V_{\rm nmo}^2(p) = \frac{c_P^2}{1-z}\left(1 + \frac{2\delta}{1-z} + 2\eta\, F(z)\right). \tag{62}$$

with

$$F(z) = \frac{z(6 - 9z + 4z^2)}{1-z}. \tag{63}$$

Apparently, we have a disappointing dependence on δ in addition to that on $V_{\rm nmo}(0)$ and η. However, since the equation (59) in the exact elliptic case does not depend on δ, we are encouraged to look deeper. Indeed, on introducing

$$y = (V_{\rm nmo}(0)\, p)^2 = (c_P\, p)^2 (1 + 2\delta) = z(1 + 2\delta), \tag{64}$$

extracting the elliptic result in the notation of equation (60), and again ignoring quadratic terms in the anisotropy parameters, one obtains the expression,

$$V_{\rm nmo}^2(p) = V_{\rm ell}^2(p)\,(1 + 2\eta\, F(y)), \qquad V_{\rm ell}(p) = \frac{V_{\rm nmo}(0)}{\sqrt{1-y}}. \tag{65}$$

in which δ does not appear and which is in agreement with the corresponding equation in (Alkhalifah & Tsvankin). This last equation also implies the weak limit estimate,

$$\eta \approx \frac{1}{2F(y)}\left(\frac{V_{\rm nmo}^2(p)}{V_{\rm ell}^2(p)} - 1\right). \tag{66}$$

At the next order in the anisotropy parameters,

$$V_{\text{nmo}}^2(p) = V_{\text{ell}}^2(p)\,(1 + 2\eta\,F(y) + \frac{4\eta y}{f(1-y)^2}R(\delta,\eta,y)), \tag{67}$$

where,

$$\begin{aligned} R(\delta,\eta,y) \;=\; & 6\delta(1-f)(1-y)^3(1-2y) \\ & +\eta y(15-(69-26f)y+(117-68f)y^2 \\ & \qquad -3(29-21f)y^3+4(6-5f)y^4). \end{aligned} \tag{68}$$

Observe that, as the second order small-ray-parameter expansion, the higher order term here does introduce δ and f in violation of the Alkhalifah-Tsvankin theory. However, observe first the consistency check, that in the elliptic limit, the higher order vanishes entirely because of an overall factor of η that appears in it. Second, notice that in the higher order term, the δ's are always multiplied by $1-f$, which somewhat mitigates their contribution. Indeed, in the common approximation of ignoring shear speed contributions by taking $f=1$, the δ terms drop out completely along with the f contributions. Finally, observe that the function $y(1-2y)(1-y)$ multiplying the δ term has absolute maximum value less than 0.1 on the interval $0 \le y \le 1$, again mitigating the effect of δ on $V_{\text{nmo}}^2(p)$ in this expansion. The overall observation that using, say, $f = 3/4$, instead of the true value has little numerical effect remains true here.

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\begin{aligned}
2\rho V^2(\theta) = {} & (C_{11}+C_{44})\sin^2\theta + (C_{33}+C_{44})\cos^2\theta \\
& + \Big\{\Big[(C_{11}-C_{44})\sin^2\theta - (C_{33}-C_{44})\cos^2\theta\Big]^2 \\
& \qquad + 4(C_{13}+C_{44})^2\sin^2\theta\cos^2\theta\Big\}^{1/2}.
\end{aligned}
\tag{1}
$$

Use the substitutions

$$
p = \sin\theta/V(\theta), \quad m = \cos\theta/V(\theta), \tag{2}
$$

to rewrite equation (1) as an equation for the slowness surface:

$$
\begin{aligned}
2\rho = {} & (C_{11}+C_{44})p^2 + (C_{33}+C_{44})m^2 \\
& + \Big\{\Big[(C_{11}-C_{44})p^2 - (C_{33}-C_{44})m^2\Big]^2 \\
& \qquad + 4(C_{13}+C_{44})^2p^2m^2\Big\}^{1/2}.
\end{aligned}
\tag{3}
$$

To obtain a formula for $V(p)$, follow the recipe given in (Alkhalifah & Tsvankin, 1995), Appendix A. Begin by converting equation (3) to Thomsen notation. After introducing the on-axis P and S velocities, c_P and c_S, and the related quantities,

$$
k = \frac{1}{c_P^2}, \qquad f = 1 - \frac{c_S^2}{c_P^2}, \tag{4}
$$

obtain the Thomsen-notation form of the slowness surface as

$$
2k = (2-f)m^2 + (2+2\epsilon-f)p^2 + \sqrt{4f(2\delta+f)m^2p^2 + (f(p^2-m^2)+2\epsilon p^2)^2}. \tag{5}
$$

Next, solve equation (5) for m^2 and, from equation (2), form

$$
V(p) = \frac{1}{\sqrt{p^2+m^2(p)}}. \tag{6}
$$

After some manipulations, find that $V^2(p)$ can be written as

$$
V^2(p) = \frac{A+\sqrt{B}}{2C}, \tag{7}
$$

where,

$$
\begin{aligned}
A &= (2-f)k - 2(\epsilon - f\delta)p^2, && (8)\\
B &= f^2k^2 - 4fk\left[\epsilon-(2-f)\delta\right]p^2 + 4\left[2f(1-f)(\epsilon-\delta)+(\epsilon-f\delta)^2\right]p^4, && (9)\\
C &= k^2 - 2k\epsilon p^2 - 2f(\epsilon-\delta)p^4. && (10)
\end{aligned}
$$

Before turning to some important special cases, observe the following consequences of the definitions in equations (2):

$$\sin\theta = pV(p), \quad \cos\theta = mV(p) = \sqrt{1-(pV)^2}, \quad p^2+m^2 = \frac{1}{V^2}. \tag{11}$$

These equations will be of great convenience in translating between representations in phase angle and representations in the ray parameter.

Small ray parameter

First observe that for $p = 0$,

$$\begin{aligned} A &= (2-f)k, && (12)\\ B &= f^2k^2, && (13)\\ C &= k^2, && (14) \end{aligned}$$

so that,

$$V^2(0) = \frac{(2-f)k+fk}{2k^2} = \frac{2k}{2k^2} = \frac{1}{k} = c_P^2, \tag{15}$$

leading to the expected result,

$$V(0) = c_P. \tag{16}$$

For later purposes, we will need to know the more detailed behavior of $V(p)$ for small p. Introducing the dimensionless parameter,

$$z = (c_P p)^2, \tag{17}$$

obtain

$$\begin{aligned} V(p) \;=\; c_P\Big[&1+\delta z + \big((\epsilon-\delta)(1+2\delta/f)+3\delta^2/2\big)z^2 \\ &+\big((\epsilon-\delta)(1+2\delta/f)(5\delta+2(\epsilon-2\delta)/f)+5\delta^3/2\big)z^3 \\ &+O(z^4)\Big]. \end{aligned} \tag{18}$$

Remark: This expansion shows that the true meaning of "small p" is that $z = (c_P p)^2 << 1$. Since $V \approx c_P$, the inequality can be written as $\sin^2\theta << 1$, so in dimensionless terms, "small p" represents a small dip angle θ.

Elliptic anisotropy

Next study the elliptic case ($\delta = \epsilon$), where

$$\begin{aligned} A &= (2-f)k - 2(1-f)\delta p^2, && (19)\\ B &= [fk+2(1-f)\delta p^2]^2, && (20)\\ C &= k^2 - 2k\delta p^2, && (21) \end{aligned}$$

so that,

$$V^2(p) = \frac{1}{k - 2\delta p^2} = \frac{c_P^2}{1 - 2\delta c_P^2 p^2}, \qquad (\delta = \epsilon). \tag{22}$$

Tsvankin (1995) gives the corresponding elliptic limit value of V^2 in terms of the phase angle, θ, as

$$V^2(\theta) = V_0^2 \cos^2\theta + V_{90}^2 \sin^2\theta, \tag{23}$$

where $V_0 = V(0)$ and $V_{90} = V(\pi/2)$ in the phase angle form of V implied by equation (1). These values are readily found from equation (5). When $p = 0$, then $m = 1/V(0)$, and the equation reduces to $2/c_P^2 = 2/V^2(0)$, so that

$$V_0 = c_P. \tag{24}$$

Similarly, when $m = 0$, then $p = 1/V_{90}$, giving $2/c_P^2 = 2(1 + \delta + \epsilon)/V_{90}$, so that

$$V_{90} = c_P\sqrt{1 + \delta + \epsilon}, \tag{25}$$

which reduces to

$$V_{90} = c_P\sqrt{1 + 2\delta} \qquad (\delta = \epsilon) \tag{26}$$

in the elliptic limit. This also confirms the known result that $V_{\text{nmo}}(0) = V_{90}$ for elliptically anisotropic media.

With these results in hand, equation (23) becomes

$$V^2 = c_P^2 + 2\delta c_P^2 \sin^2\theta = c_P^2 + 2\delta c_P^2 p^2 V^2 \qquad (\delta = \epsilon). \tag{27}$$

On isolating V^2, equation (22) is verified.

Tsvankin (1995) likewise gives the form

$$V(\theta) = c_P\sqrt{1 + 2\delta \sin^2\theta} \qquad (\delta = \epsilon) \tag{28}$$

for the elliptic case. Introducing the ray parameter in that expression gives

$$V^2 = c_P^2(1 + 2\delta p^2 V^2) \tag{29}$$

and, once again, isolating V^2 verifies equation (22).

Weak transverse isotropy

The weak TI limit of equation (7) is

$$V^2(p) \approx c_P^2 \left[1 + 2(\delta + (\epsilon - \delta)c_P^2 p^2)c_P^2 p^2\right], \tag{30}$$

or

$$V(p) \approx c_P \left[1 + (\delta + (\epsilon - \delta)c_P^2 p^2)c_P^2 p^2\right]. \tag{31}$$

Thomsen's (1986) expression for this quantity expressed as a function of angle is

$$V(\theta) = c_P \left[1 + (\delta \cos^2\theta + \epsilon \sin^2\theta)\sin^2\theta\right]. \tag{32}$$

Tsvankin (1995) shows that this equation implies

$$\sin^2\theta = \frac{p^2 c_P^2}{1 - 2\delta p^2 c_P^2}[1 + 2(\epsilon - \delta)p^4 c_P^4], \tag{33}$$

which, in turn, in the weak limit is equivalent to

$$p^2 V^2 = p^2 c_P^2 [1 + 2\delta p^2 c_P^2 + 2(\epsilon - \delta)p^4 c_P^4]. \tag{34}$$

On cancelling the common factor of p^2, this verifies equation (30) and hence also equation (31).

NORMAL MOVEOUT VELOCITY AS A FUNCTION OF RAY PARAMETER

The derivation of the normal moveout velocity as a function of ray parameter, follows closely the corresponding derivation of its expression in terms of the ray angle given in (Tsvankin, 1995). For convenience, some explicit references are made to equations in this paper.

First, use equations (11) to find the following relation for $dp/d\theta$:

$$mV = \cos\theta = \frac{d\sin\theta}{d\theta} = (pV)' \frac{dp}{d\theta}, \tag{35}$$

where, here and below, the prime notation is used for p-differentiation. Thus, we find that

$$\frac{dV}{d\theta} = V' \frac{dp}{d\theta} = \frac{mVV'}{(pV)'}. \tag{36}$$

The derivation of $V_{\text{nmo}}(p)$ begins with Tsvankin's equation (4),

$$V_{\text{nmo}}^2(p) = \frac{2z_0}{t_0} \lim_{h\to 0} \frac{d\tan\psi}{dp}. \tag{37}$$

The second component of this expression is evaluated using Tsvankin's equation (6), equations (11), and equation (36):

$$\begin{aligned} \tan\psi &= \frac{V\sin\theta + \frac{dV}{d\theta}\cos\theta}{V\cos\theta - \frac{dV}{d\theta}\sin\theta} \\ &= \frac{pV^2 + mV\frac{dV}{d\theta}}{mV^2 - pV\frac{dV}{d\theta}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{pV + m\frac{dV}{d\theta}}{mV - p\frac{dV}{d\theta}} \\
&= \frac{pV(pV)' + m^2VV'}{mV(pV)' - mpVV'} \\
&= \frac{p(pV)' + m^2V'}{m\left[(pV)' - pV'\right]} \\
&= \frac{p^2V' + m^2V' + pV}{mV} \\
&= \frac{V'/V^2 + pV}{mV} \\
&= \frac{V' + pV^3}{mV^3}. \qquad (38)
\end{aligned}
$$

The first component of equation (37) is similarly evaluated using Tsvankin's equation (8):

$$
\begin{aligned}
\frac{2z_0}{t_0} &= V\cos\theta\left(1 - \frac{\tan\theta}{V}\frac{dV}{d\theta}\right) \\
&= mV^2\left(1 - \frac{p}{mV}\frac{mVV'}{(pV)'}\right) \\
&= mV^2\left(1 - \frac{pV'}{(pV)'}\right) \\
&= mV^2\frac{V + pV' - pV'}{(pV)'} \\
&= \frac{mV^3}{(pV)'}. \qquad (39)
\end{aligned}
$$

Using the previous two results in equation (37), find that

$$
V^2_{\text{nmo}}(p) = \frac{mV^3}{(pV)'}\frac{d}{dp}\left(\frac{V' + pV^3}{mV^3}\right) \qquad (40)
$$

or, on eliminating m,

$$
V^2_{\text{nmo}}(p) = \frac{V^2\sqrt{1-(pV)^2}}{(pV)'}\frac{d}{dp}\left(\frac{V' + pV^3}{V^2\sqrt{1-(pV)^2}}\right). \qquad (41)
$$

Carrying out the indicated derivative gives an explicit formula for the normal moveout velocity as a function of ray parameter:

$$
V^2_{\text{nmo}}(p) = \frac{2(1-p^2V^2)VV'' + (3p^2V^2 - 2)V'^2 + 2pV^3V' + V^4}{(1-p^2V^2)V(pV)'} \qquad (42)
$$

Before turning to the special cases for V, note that when it is necessary to treat the general V, it may be better to avoid a layer of square roots by writing $V^2_{\rm nmo}(p)$ in terms of $W \equiv V^2$:

$$V^2_{\rm nmo}(p) = \frac{2(1-p^2W)WW'' + (4p^2W-3)W'^2 + 4pW^2W' + 4W^3}{2(1-p^2W)W(pW'+2W)} \tag{43}$$

Small ray parameter

First, use the two leading terms of equation (18) to obtain the approximations,

$$V = c_P + c_P^3\delta p^2 + O(p^4), \quad V' = 2c_P^3\delta p + O(p^3), \quad V'' = 2c_P^3\delta + O(p^2). \tag{44}$$

On inserting these small-p approximations into equation (42), find that

$$V^2_{\rm nmo}(p) = c_P^2(1+2\delta) + O(p^2), \tag{45}$$

so

$$V_{\rm nmo}(0) = c_P\sqrt{1+2\delta}. \tag{46}$$

Now use the next order terms of equation (18) in equation (42) to get the next term in $V^2_{\rm nmo}(p)$:

$$V^2_{\rm nmo}(p) = c_P^2\left(1+2\delta+\left((-24\delta^2+24\delta\epsilon+f-8\delta f+4\delta^2 f+12\epsilon f)\frac{(c_Pp)^2}{f}\right)+O(c_Pp)^4\right). \tag{47}$$

Remark: Again, the expansion shows that the true meaning of "small p" is that $(c_Pp)^2 << 1$. Since $V \approx c_P$, and the error is fourth order, this means we can interpret "small p" as meaning $\sin^4\theta << 1$. Numerical tests show that using the small p series for a dip of 15° incurs about a 2% error for typical values of f and δ.

The theory discussed by (Alkhalifah & Tsvankin, 1995) suggests that it is better to express this result in terms of the parameters $V_{\rm nmo}(0)$ and

$$\eta = \frac{\epsilon-\delta}{1+2\delta}. \tag{48}$$

First introduce η,

$$V^2_{\rm nmo}(p) \approx c_P^2(1+2\delta) + \left(c_P^4(1+2\delta)(24\delta\eta+f+2\delta f+12\eta f)p^2\right)/f + O(p^4), \tag{49}$$

then use equation (46) to eliminate the explicit appearances of c_P^2 in favor of $V^2_{\rm nmo}(0)$:

$$V^2_{\rm nmo}(p) = V^2_{\rm nmo}(0)\left[\left(1+(1+12\eta\frac{1+2\delta/f}{1+2\delta})V^2_{\rm nmo}(0)p^2\right)+O(V^4_{\rm nmo}(0)p^4)\right]. \tag{50}$$

The importance of the anisotropic parameter η, is that Alkhalifah and Tsvankin (1995) have observed, both numerically and empirically, that $V^2_{\rm nmo}(p)$ depends mainly

on η and $V^2_{\text{nmo}}(0)$. They exploit this reduction to surface-observable parameters to develop time-domain seismic processing algorithms that take account of transverse anisotropy. Equation (50) gives analytic support to the Alkhalifah-Tsvankin theory, since the only deviation to it occurs in the ratio

$$g = \frac{1+2\delta/f}{1+2\delta} \tag{51}$$

that multiplies η. Figures (1) and (2) show plots of this function over the ranges of f and δ that are relevant in practice. Observe that the function $g(\delta, f)$ varies slowly over these ranges—indeed—ignoring it altogether (i.e., replacing it by the constant 1) is usually justified—at least in the small-angle approximation.

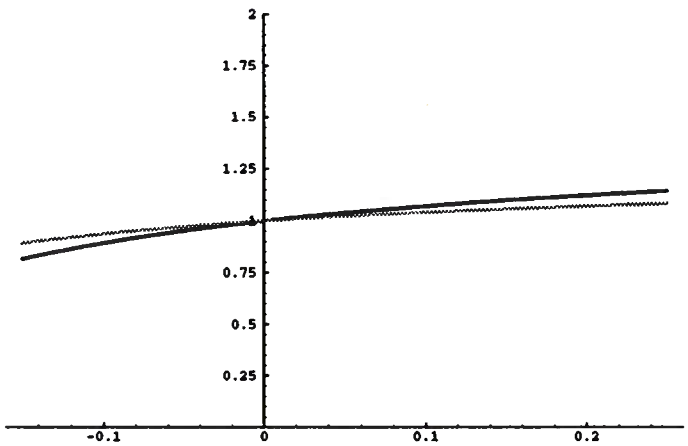

FIG. 1. Plot of the factor $g = (1+2\delta/f)/(1+2\delta)$ as a function of δ for $f = 0.7$ (dark) and $f = 0.8$ (light).

The small-ray-parameter result in equation (50) gives a means for estimating η from surface observations. For example, suppose one has observations of $V_{\text{nmo}}(p)$ at $p = 0$ and some other (not too large) value $p = p_1$. The solution for η is given by

$$\eta \approx \frac{1}{12g}\left(\frac{V^2_{\text{nmo}}(p_1) - V^2_{\text{nmo}}(0)}{p_1^2 V^4_{\text{nmo}}(0)} - 1\right), \tag{52}$$

where once again, in the absence of information on δ and f, the factor g can be replaced by the constant 1 without much error.

More generally, if one uses two nonzero "small p" values (that is, two separated dips, each less than 15°), then the estimate for $V^2_{\text{nmo}}(0)$ is given by

$$V^2_{\text{nmo}}(0) = \frac{p_2^2 V^2_{\text{nmo}}(p_1) - p_1^2 V^2_{\text{nmo}}(p_2)}{p_2^2 - p_1^2}, \tag{53}$$

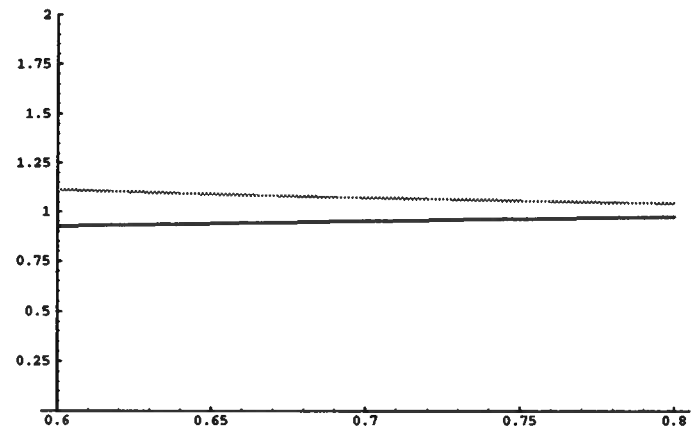

FIG. 2. Plot of the factor $g = (1 + 2\delta/f)/(1 + 2\delta)$ as a function of f for $\delta = -0.05$ (dark) and $\delta = +0.1$ (light).

and the estimate for η is

$$\eta \approx \frac{1}{12g} \left(\frac{(p_1^2 - p_2^2)(V_{\text{nmo}}^2(p_2) - V_{\text{nmo}}^2(p_1))}{(p_1^2 V_{\text{nmo}}^2(p_2) - p_2^2 V_{\text{nmo}}^2(p_1))^2} - 1 \right). \tag{54}$$

Finally, note that the full form of the series for $V_{\text{nmo}}^2(p)$ is given by

$$V_{\text{nmo}}^2(p) = V_{\text{nmo}}^2(0) \left[1 + c_2 V_{\text{nmo}}^2(0) p^2 + c_4 V_{\text{nmo}}^4(0) p^4 + \cdots \right], \tag{55}$$

where, as we have seen earlier,

$$c_2 = 1 + 12g\eta \approx 1 + 12\eta. \tag{56}$$

By using the full form of equation (18), we find that

$$c_4 = 1 + 6g(6 - 5g)\eta + \frac{60g}{f}\eta^2 \approx 1 + 6\eta + \frac{60}{f}\eta^2. \tag{57}$$

Here, the approximations result from replacing the factor g defined in equation (51) by the constant 1. Note that the final term in c_4 indicates the first serious divergence from the theory that $V_{\text{nmo}}^2(p)$ depends only on the parameters $V_{\text{nmo}}^2(0)$ and η. However, this term is multiplied by both p^2 and η^2, which ameliorates the effect of replacing the f in this term by, say, $3/4$ instead of the true value.

Elliptic anisotropy

Inserting the elliptic P-wave phase velocity as a function of ray parameter given in equation (22) into the general NMO equation (42) gives at once

$$V_{\text{nmo}}^2(p) = \frac{c_P^2(1 + 2\delta)}{(1 - (1 + 2\delta)c_P^2 p^2)}. \tag{58}$$

Recognizing the quantity $V_{\rm nmo}(0)$ from equation (46) gives

$$V_{\rm nmo}(p) = \frac{V_{\rm nmo}(0)}{\sqrt{1 - p^2 V_{\rm nmo}^2(0)}}, \qquad (\delta = \epsilon), \tag{59}$$

in agreement with the result in (Alkhalifah & Tsvankin, 1995). For future use, introduce the notation,

$$V_{\rm ell}(p) \equiv \frac{V_{\rm nmo}(0)}{\sqrt{1 - p^2 V_{\rm nmo}^2(0)}}, \tag{60}$$

for the elliptic result.

Weak transverse isotropy

Using equation (31) in equation (42) gives

$$V_{\rm nmo}^2(p) = \frac{c_P^2}{1-z}\left(1 + \frac{2(\delta + (\epsilon - \delta)\, z(6 - 9z + 4z^2))}{1-z}\right), \tag{61}$$

where we have again used the shorthand notation, $z = (c_P p)^2$. One would have to seek a more sophisticated expansion if p became large enough to approach $1/c_P$. On the other hand, always $p < 1/V$ and in the weak limit, $V \approx c_P$, so this is an unusual circumstance.

The approximation, $\eta = \epsilon - \delta$, is valid in the weak limit, so equation (61) may be recast as

$$V_{\rm nmo}^2(p) = \frac{c_P^2}{1-z}\left(1 + \frac{2\delta}{1-z} + 2\eta\, F(z)\right). \tag{62}$$

with

$$F(z) = \frac{z(6 - 9z + 4z^2)}{1-z}. \tag{63}$$

Apparently, we have a disappointing dependence on δ in addition to that on $V_{\rm nmo}(0)$ and η. However, since the equation (59) in the exact elliptic case does not depend on δ, we are encouraged to look deeper. Indeed, on introducing

$$y = (V_{\rm nmo}(0)\, p)^2 = (c_P\, p)^2(1 + 2\delta) = z(1 + 2\delta), \tag{64}$$

extracting the elliptic result in the notation of equation (60), and again ignoring quadratic terms in the anisotropy parameters, one obtains the expression,

$$V_{\rm nmo}^2(p) = V_{\rm ell}^2(p)\,(1 + 2\eta\, F(y)), \qquad V_{\rm ell}(p) = \frac{V_{\rm nmo}(0)}{\sqrt{1-y}}. \tag{65}$$

in which δ does not appear and which is in agreement with the corresponding equation in (Alkhalifah & Tsvankin). This last equation also implies the weak limit estimate,

$$\eta \approx \frac{1}{2F(y)}\left(\frac{V_{\rm nmo}^2(p)}{V_{\rm ell}^2(p)} - 1\right). \tag{66}$$

At the next order in the anisotropy parameters,

$$V_{\mathrm{nmo}}^2(p) = V_{\mathrm{ell}}^2(p)\,(1 + 2\eta\, F(y) + \frac{4\eta y}{f(1-y)^2} R(\delta, \eta, y)), \tag{67}$$

where,

$$\begin{aligned} R(\delta,\eta,y) \;=\;& 6\delta(1-f)(1-y)^3(1-2y) \\ & +\eta y(15 - (69 - 26f)y + (117 - 68f)y^2 \\ & \qquad -3(29 - 21f)y^3 + 4(6 - 5f)y^4). \end{aligned} \tag{68}$$

Observe that, as the second order small-ray-parameter expansion, the higher order term here does introduce δ and f in violation of the Alkhalifah-Tsvankin theory. However, observe first the consistency check, that in the elliptic limit, the higher order vanishes entirely because of an overall factor of η that appears in it. Second, notice that in the higher order term, the δ's are always multiplied by $1 - f$, which somewhat mitigates their contribution. Indeed, in the common approximation of ignoring shear speed contributions by taking $f = 1$, the δ terms drop out completely along with the f contributions. Finally, observe that the function $y(1 - 2y)(1 - y)$ multiplying the δ term has absolute maximum value less than 0.1 on the interval $0 \le y \le 1$, again mitigating the effect of δ on $V_{\mathrm{nmo}}^2(p)$ in this expansion. The overall observation that using, say, $f = 3/4$, instead of the true value has little numerical effect remains true here.

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Phase and Moveout Velocity as Functions of the Ray Parameter for Transversely Isotropic Media

Jack K. Cohen

Center for Wave Phenomena
Colorado School of Mines
Golden, Colorado 80401
303/273-3557

Phase and Moveout Velocity as Functions of the Ray Parameter for Transversely Isotropic Media

Jack K. Cohen

Fall, 1995

ABSTRACT

In their studies of transversely isotropic media, I. Tsvankin and his co-workers have shown the benefits of expressing the P-wave normal moveout velocity in terms of the zero-offset ray parameter instead of the phase angle, but left for future work the actual task of doing this. Here, explicit formulas are given for both the P-wave phase velocity and normal moveout velocity in terms of the ray parameter. Various results of Tsvankin and others are rederived and, in some cases, extended, in a uniform manner using these explicit results.

The small ray-parameter expansion of the moveout velocity is valid for dips up to approximately 15°. In this regime, an analytic solution for the anisotropy parameter η, important for time-related imaging is given in terms of observations of moveout velocity for two different values of the ray parameter.

INTRODUCTION

In their studies of transversely isotropic media, I. Tsvankin and his co-workers have shown the benefits of expressing the P-wave normal moveout velocity in terms of the zero-offset ray parameter instead of the phase angle. Here, explicit formulas are given for both the P-wave phase velocity and normal moveout velocity in terms of the ray parameter. Various consequences of these results are drawn, including an analytic solution for the anisotropy parameter η, important for time-related imaging is given in terms of observations of moveout velocity for two different values of the ray parameter.

Throughout this study, *Mathematica* was used extensively to derive and check results. In particular, the *Mathematica* package, **Thomsen.m** (Cohen, 1995), was used to convert equation (3) to Thomsen notation, in equation (5), and to obtain various results in the limit of weak transverse isotropy. Similarly, the use of *Mathematica* facilitated computing symbolic derivatives, series expansions, etc.

PHASE VELOCITY AS A FUNCTION OF RAY PARAMETER

Begin with the formula for P-wave phase velocity in terms of dip θ (Tsvankin, 1994), equation(6):

$$
\begin{aligned}
2\rho V^2(\theta) = (C_{11} + C_{44}) \sin^2\theta + (C_{33} + C_{44}) \cos^2\theta \\
+ \Big\{ \big[(C_{11} - C_{44}) \sin^2\theta - (C_{33} - C_{44}) \cos^2\theta \big]^2 \\
+ 4(C_{13} + C_{44})^2 \sin^2\theta \cos^2\theta \Big\}^{1/2} .
\end{aligned}
\tag{1}
$$

Use the substitutions

$$
p = \sin\theta / V(\theta), \qquad m = \cos\theta / V(\theta), \tag{2}
$$

to rewrite equation (1) as an equation for the slowness surface:

$$
\begin{aligned}
2\rho = (C_{11} + C_{44}) p^2 + (C_{33} + C_{44}) m^2 \\
+ \Big\{ \big[(C_{11} - C_{44}) p^2 - (C_{33} - C_{44}) m^2 \big]^2 \\
+ 4(C_{13} + C_{44})^2 p^2 m^2 \Big\}^{1/2} .
\end{aligned}
\tag{3}
$$

To obtain a formula for $V(p)$, follow the recipe given in (Alkhalifah & Tsvankin, 1995), Appendix A. Begin by converting equation (3) to Thomsen notation. After introducing the on-axis P and S velocities, c_P and c_S, and the related quantities,

$$
k = \frac{1}{c_P^2}, \qquad f = 1 - \frac{c_S^2}{c_P^2}, \tag{4}
$$

obtain the Thomsen-notation form of the slowness surface as

$$
2k = (2 - f)m^2 + (2 + 2\epsilon - f)p^2 + \sqrt{4f(2\delta + f)m^2p^2 + (f(p^2 - m^2) + 2\epsilon p^2)^2}. \tag{5}
$$

Next, solve equation (5) for m^2 and, from equation (2), form

$$
V(p) = \frac{1}{\sqrt{p^2 + m^2(p)}}. \tag{6}
$$

After some manipulations, find that $V^2(p)$ can be written as

$$
V^2(p) = \frac{A + \sqrt{B}}{2C}, \tag{7}
$$

where,

$$
\begin{aligned}
A &= (2 - f)k - 2(\epsilon - f\delta)p^2, && (8)\\
B &= f^2k^2 - 4fk\left[\epsilon - (2 - f)\delta\right]p^2 + 4\left[2f(1 - f)(\epsilon - \delta) + (\epsilon - f\delta)^2\right]p^4, && (9)\\
C &= k^2 - 2k\epsilon p^2 - 2f(\epsilon - \delta)p^4. && (10)
\end{aligned}
$$

Before turning to some important special cases, observe the following consequences of the definitions in equations (2):

$$\sin\theta = pV(p), \quad \cos\theta = mV(p) = \sqrt{1-(pV)^2}, \quad p^2+m^2 = \frac{1}{V^2}. \tag{11}$$

These equations will be of great convenience in translating between representations in phase angle and representations in the ray parameter.

Small ray parameter

First observe that for $p = 0$,

$$\begin{aligned} A &= (2-f)k, && (12)\\ B &= f^2k^2, && (13)\\ C &= k^2, && (14) \end{aligned}$$

so that,

$$V^2(0) = \frac{(2-f)k+fk}{2k^2} = \frac{2k}{2k^2} = \frac{1}{k} = c_P^2, \tag{15}$$

leading to the expected result,

$$V(0) = c_P. \tag{16}$$

For later purposes, we will need to know the more detailed behavior of $V(p)$ for small p. Introducing the dimensionless parameter,

$$z = (c_P p)^2, \tag{17}$$

obtain

$$\begin{aligned} V(p) \;=\; c_P \Big[& 1 + \delta z + \left((\epsilon-\delta)(1+2\delta/f) + 3\delta^2/2\right) z^2 \\ & + \left((\epsilon-\delta)(1+2\delta/f)(5\delta + 2(\epsilon-2\delta)/f) + 5\delta^3/2\right) z^3 \\ & + O(z^4) \Big]. \end{aligned} \tag{18}$$

Remark: This expansion shows that the true meaning of "small p" is that $z = (c_P p)^2 << 1$. Since $V \approx c_P$, the inequality can be written as $\sin^2\theta << 1$, so in dimensionless terms, "small p" represents a small dip angle θ.

Elliptic anisotropy

Next study the elliptic case ($\delta = \epsilon$), where

$$\begin{aligned} A &= (2-f)k - 2(1-f)\delta p^2, && (19)\\ B &= [fk + 2(1-f)\delta p^2]^2, && (20)\\ C &= k^2 - 2k\delta p^2, && (21) \end{aligned}$$

so that,

$$V^2(p) = \frac{1}{k - 2\delta p^2} = \frac{c_P^2}{1 - 2\delta c_P^2 p^2}, \qquad (\delta = \epsilon). \tag{22}$$

Tsvankin (1995) gives the corresponding elliptic limit value of V^2 in terms of the phase angle, θ, as

$$V^2(\theta) = V_0^2 \cos^2\theta + V_{90}^2 \sin^2\theta, \tag{23}$$

where $V_0 = V(0)$ and $V_{90} = V(\pi/2)$ in the phase angle form of V implied by equation (1). These values are readily found from equation (5). When $p = 0$, then $m = 1/V(0)$, and the equation reduces to $2/c_P^2 = 2/V^2(0)$, so that

$$V_0 = c_P. \tag{24}$$

Similarly, when $m = 0$, then $p = 1/V_{90}$, giving $2/c_P^2 = 2(1 + \delta + \epsilon)/V_{90}$, so that

$$V_{90} = c_P\sqrt{1 + \delta + \epsilon}, \tag{25}$$

which reduces to

$$V_{90} = c_P\sqrt{1 + 2\delta} \qquad (\delta = \epsilon) \tag{26}$$

in the elliptic limit. This also confirms the known result that $V_{\mathrm{nmo}}(0) = V_{90}$ for elliptically anisotropic media.

With these results in hand, equation (23) becomes

$$V^2 = c_P^2 + 2\delta c_P^2 \sin^2\theta = c_P^2 + 2\delta c_P^2 p^2 V^2 \qquad (\delta = \epsilon). \tag{27}$$

On isolating V^2, equation (22) is verified.

Tsvankin (1995) likewise gives the form

$$V(\theta) = c_P\sqrt{1 + 2\delta \sin^2\theta} \qquad (\delta = \epsilon) \tag{28}$$

for the elliptic case. Introducing the ray parameter in that expression gives

$$V^2 = c_P^2(1 + 2\delta p^2 V^2) \tag{29}$$

and, once again, isolating V^2 verifies equation (22).

Weak transverse isotropy

The weak TI limit of equation (7) is

$$V^2(p) \approx c_P^2\left[1 + 2(\delta + (\epsilon - \delta)c_P^2 p^2)c_P^2 p^2\right], \tag{30}$$

or

$$V(p) \approx c_P\left[1 + (\delta + (\epsilon - \delta)c_P^2 p^2)c_P^2 p^2\right]. \tag{31}$$

Thomsen's (1986) expression for this quantity expressed as a function of angle is

$$V(\theta) = c_P \left[1 + (\delta \cos^2\theta + \epsilon \sin^2\theta)\sin^2\theta\right]. \tag{32}$$

Tsvankin (1995) shows that this equation implies

$$\sin^2\theta = \frac{p^2 c_P^2}{1 - 2\delta p^2 c_P^2}[1 + 2(\epsilon - \delta)p^4 c_P^4], \tag{33}$$

which, in turn, in the weak limit is equivalent to

$$p^2 V^2 = p^2 c_P^2 [1 + 2\delta p^2 c_P^2 + 2(\epsilon - \delta)p^4 c_P^4]. \tag{34}$$

On cancelling the common factor of p^2, this verifies equation (30) and hence also equation (31).

NORMAL MOVEOUT VELOCITY AS A FUNCTION OF RAY PARAMETER

The derivation of the normal moveout velocity as a function of ray parameter, follows closely the corresponding derivation of its expression in terms of the ray angle given in (Tsvankin, 1995). For convenience, some explicit references are made to equations in this paper.

First, use equations (11) to find the following relation for $dp/d\theta$:

$$mV = \cos\theta = \frac{d\sin\theta}{d\theta} = (pV)' \frac{dp}{d\theta}, \tag{35}$$

where, here and below, the prime notation is used for p-differentiation. Thus, we find that

$$\frac{dV}{d\theta} = V' \frac{dp}{d\theta} = \frac{mVV'}{(pV)'}. \tag{36}$$

The derivation of $V_{\text{nmo}}(p)$ begins with Tsvankin's equation (4),

$$V_{\text{nmo}}^2(p) = \frac{2z_0}{t_0} \lim_{h \to 0} \frac{d\tan\psi}{dp}. \tag{37}$$

The second component of this expression is evaluated using Tsvankin's equation (6), equations (11), and equation (36):

$$\begin{aligned} \tan\psi &= \frac{V\sin\theta + \frac{dV}{d\theta}\cos\theta}{V\cos\theta - \frac{dV}{d\theta}\sin\theta} \\ &= \frac{pV^2 + mV\frac{dV}{d\theta}}{mV^2 - pV\frac{dV}{d\theta}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{pV + m\frac{dV}{d\theta}}{mV - p\frac{dV}{d\theta}} \\
&= \frac{pV(pV)' + m^2VV'}{mV(pV)' - mpVV'} \\
&= \frac{p(pV)' + m^2V'}{m\left[(pV)' - pV'\right]} \\
&= \frac{p^2V' + m^2V' + pV}{mV} \\
&= \frac{V'/V^2 + pV}{mV} \\
&= \frac{V' + pV^3}{mV^3}. \qquad (38)
\end{aligned}
$$

The first component of equation (37) is similarly evaluated using Tsvankin's equation (8):

$$
\begin{aligned}
\frac{2z_0}{t_0} &= V\cos\theta\left(1 - \frac{\tan\theta}{V}\frac{dV}{d\theta}\right) \\
&= mV^2\left(1 - \frac{p}{mV}\frac{mVV'}{(pV)'}\right) \\
&= mV^2\left(1 - \frac{pV'}{(pV)'}\right) \\
&= mV^2\frac{V + pV' - pV'}{(pV)'} \\
&= \frac{mV^3}{(pV)'}. \qquad (39)
\end{aligned}
$$

Using the previous two results in equation (37), find that

$$
V_{\text{nmo}}^2(p) = \frac{mV^3}{(pV)'}\frac{d}{dp}\left(\frac{V' + pV^3}{mV^3}\right) \qquad (40)
$$

or, on eliminating m,

$$
V_{\text{nmo}}^2(p) = \frac{V^2\sqrt{1 - (pV)^2}}{(pV)'}\frac{d}{dp}\left(\frac{V' + pV^3}{V^2\sqrt{1 - (pV)^2}}\right). \qquad (41)
$$

Carrying out the indicated derivative gives an explicit formula for the normal moveout velocity as a function of ray parameter:

$$
V_{\text{nmo}}^2(p) = \frac{2(1 - p^2V^2)VV'' + (3p^2V^2 - 2)V'^2 + 2pV^3V' + V^4}{(1 - p^2V^2)V(pV)'} \qquad (42)
$$

Before turning to the special cases for V, note that when it is necessary to treat the general V, it may be better to avoid a layer of square roots by writing $V_{\mathrm{nmo}}^2(p)$ in terms of $W \equiv V^2$:

$$V_{\mathrm{nmo}}^2(p) = \frac{2(1-p^2W)WW'' + (4p^2W-3)W'^2 + 4pW^2W' + 4W^3}{2(1-p^2W)W(pW'+2W)} \tag{43}$$

Small ray parameter

First, use the two leading terms of equation (18) to obtain the approximations,

$$V = c_P + c_P^3\delta p^2 + O(p^4), \quad V' = 2c_P^3\delta p + O(p^3), \quad V'' = 2c_P^3\delta + O(p^2). \tag{44}$$

On inserting these small-p approximations into equation (42), find that

$$V_{\mathrm{nmo}}^2(p) = c_P^2(1+2\delta) + O(p^2), \tag{45}$$

so

$$V_{\mathrm{nmo}}(0) = c_P\sqrt{1+2\delta}. \tag{46}$$

Now use the next order terms of equation (18) in equation (42) to get the next term in $V_{\mathrm{nmo}}^2(p)$:

$$V_{\mathrm{nmo}}^2(p) = c_P^2\left(1+2\delta+\left((-24\delta^2+24\delta\epsilon+f-8\delta f+4\delta^2 f+12\epsilon f)\frac{(c_Pp)^2}{f}\right)+O(c_Pp)^4\right). \tag{47}$$

Remark: Again, the expansion shows that the true meaning of "small p" is that $(c_Pp)^2 << 1$. Since $V \approx c_P$, and the error is fourth order, this means we can interpret "small p" as meaning $\sin^4\theta << 1$. Numerical tests show that using the small p series for a dip of 15° incurs about a 2% error for typical values of f and δ.

The theory discussed by (Alkhalifah & Tsvankin, 1995) suggests that it is better to express this result in terms of the parameters $V_{\mathrm{nmo}}(0)$ and

$$\eta = \frac{\epsilon-\delta}{1+2\delta}. \tag{48}$$

First introduce η,

$$V_{\mathrm{nmo}}^2(p) \approx c_P^2(1+2\delta) + \left(c_P^4(1+2\delta)(24\delta\eta+f+2\delta f+12\eta f)p^2\right)/f + O(p^4), \tag{49}$$

then use equation (46) to eliminate the explicit appearances of c_P^2 in favor of $V_{\mathrm{nmo}}^2(0)$:

$$V_{\mathrm{nmo}}^2(p) = V_{\mathrm{nmo}}^2(0)\left[\left(1+(1+12\eta\frac{1+2\delta/f}{1+2\delta})V_{\mathrm{nmo}}^2(0)p^2\right)+O(V_{\mathrm{nmo}}^4(0)p^4)\right]. \tag{50}$$

The importance of the anisotropic parameter η, is that Alkhalifah and Tsvankin (1995) have observed, both numerically and empirically, that $V_{\mathrm{nmo}}^2(p)$ depends mainly on η and $V^2_{\mathrm{nmo}}(0)$. They exploit this reduction to surface-observable parameters to develop time-domain seismic processing algorithms that take account of transverse anisotropy. Equation (50) gives analytic support to the Alkhalifah-Tsvankin theory, since the only deviation to it occurs in the ratio

$$g = \frac{1 + 2\delta/f}{1 + 2\delta} \tag{51}$$

that multiplies η. Figures (1) and (2) show plots of this function over the ranges of f and δ that are relevant in practice. Observe that the function $g(\delta, f)$ varies slowly over these ranges—indeed—ignoring it altogether (i.e., replacing it by the constant 1) is usually justified—at least in the small-angle approximation.

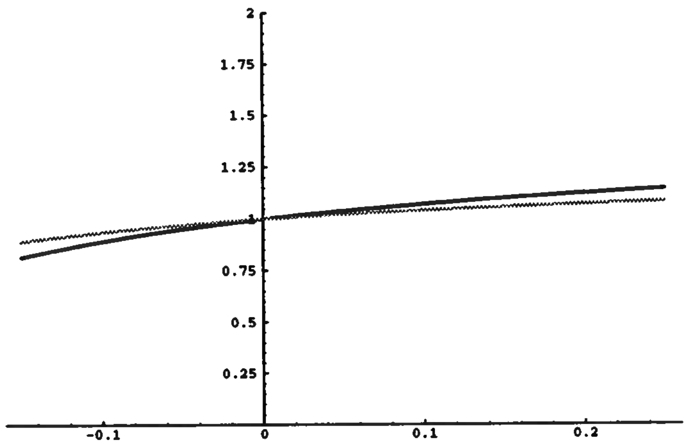

FIG. 1. Plot of the factor $g = (1 + 2\delta/f)/(1 + 2\delta)$ as a function of δ for $f = 0.7$ (dark) and $f = 0.8$ (light).

The small-ray-parameter result in equation (50) gives a means for estimating η from surface observations. For example, suppose one has observations of $V_{\mathrm{nmo}}(p)$ at $p = 0$ and some other (not too large) value $p = p_1$. The solution for η is given by

$$\eta \approx \frac{1}{12g}\left(\frac{V^2_{\mathrm{nmo}}(p_1) - V^2_{\mathrm{nmo}}(0)}{p_1^2 V^4_{\mathrm{nmo}}(0)} - 1\right), \tag{52}$$

where once again, in the absence of information on δ and f, the factor g can be replaced by the constant 1 without much error.

More generally, if one uses two nonzero "small p" values (that is, two separated dips, each less than 15°), then the estimate for $V^2_{\mathrm{nmo}}(0)$ is given by

$$V^2_{\mathrm{nmo}}(0) = \frac{p_2^2 V^2_{\mathrm{nmo}}(p_1) - p_1^2 V^2_{\mathrm{nmo}}(p_2)}{p_2^2 - p_1^2}, \tag{53}$$

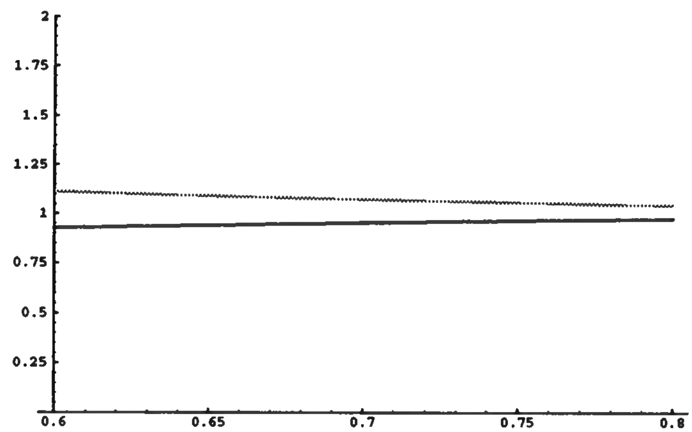

FIG. 2. Plot of the factor $g = (1 + 2\delta/f)/(1 + 2\delta)$ as a function of f for $\delta = -0.05$ (dark) and $\delta = +0.1$ (light).

and the estimate for η is

$$\eta \approx \frac{1}{12g}\left(\frac{(p_1^2 - p_2^2)(V_{\rm nmo}^2(p_2) - V_{\rm nmo}^2(p_1))}{(p_1^2 V_{\rm nmo}^2(p_2) - p_2^2 V_{\rm nmo}^2(p_1))^2} - 1\right). \tag{54}$$

Finally, note that the full form of the series for $V_{\rm nmo}^2(p)$ is given by

$$V_{\rm nmo}^2(p) = V_{\rm nmo}^2(0)\left[1 + c_2 V_{\rm nmo}^2(0)p^2 + c_4 V_{\rm nmo}^4(0)p^4 + \cdots\right], \tag{55}$$

where, as we have seen earlier,

$$c_2 = 1 + 12g\eta \approx 1 + 12\eta. \tag{56}$$

By using the full form of equation (18), we find that

$$c_4 = 1 + 6g(6 - 5g)\eta + \frac{60g}{f}\eta^2 \approx 1 + 6\eta + \frac{60}{f}\eta^2. \tag{57}$$

Here, the approximations result from replacing the factor g defined in equation (51) by the constant 1. Note that the final term in c_4 indicates the first serious divergence from the theory that $V_{\rm nmo}^2(p)$ depends only on the parameters $V_{\rm nmo}^2(0)$ and η. However, this term is multiplied by both p^2 and η^2, which ameliorates the effect of replacing the f in this term by, say, 3/4 instead of the true value.

Elliptic anisotropy

Inserting the elliptic P-wave phase velocity as a function of ray parameter given in equation (22) into the general NMO equation (42) gives at once

$$V_{\rm nmo}^2(p) = \frac{c_P^2(1 + 2\delta)}{(1 - (1 + 2\delta)c_P^2 p^2)}. \tag{58}$$

Recognizing the quantity $V_{\rm nmo}(0)$ from equation (46) gives

$$V_{\rm nmo}(p) = \frac{V_{\rm nmo}(0)}{\sqrt{1 - p^2 V_{\rm nmo}^2(0)}}, \qquad (\delta = \epsilon), \tag{59}$$

in agreement with the result in (Alkhalifah & Tsvankin, 1995). For future use, introduce the notation,

$$V_{\rm ell}(p) \equiv \frac{V_{\rm nmo}(0)}{\sqrt{1 - p^2 V_{\rm nmo}^2(0)}}, \tag{60}$$

for the elliptic result.

Weak transverse isotropy

Using equation (31) in equation (42) gives

$$V_{\rm nmo}^2(p) = \frac{c_P^2}{1-z}\left(1 + \frac{2(\delta + (\epsilon - \delta)\, z(6 - 9z + 4z^2))}{1-z}\right), \tag{61}$$

where we have again used the shorthand notation, $z = (c_P p)^2$. One would have to seek a more sophisticated expansion if p became large enough to approach $1/c_P$. On the other hand, always $p < 1/V$ and in the weak limit, $V \approx c_P$, so this is an unusual circumstance.

The approximation, $\eta = \epsilon - \delta$, is valid in the weak limit, so equation (61) may be recast as

$$V_{\rm nmo}^2(p) = \frac{c_P^2}{1-z}\left(1 + \frac{2\delta}{1-z} + 2\eta\, F(z)\right). \tag{62}$$

with

$$F(z) = \frac{z(6 - 9z + 4z^2)}{1-z}. \tag{63}$$

Apparently, we have a disappointing dependence on δ in addition to that on $V_{\rm nmo}(0)$ and η. However, since the equation (59) in the exact elliptic case does not depend on δ, we are encouraged to look deeper. Indeed, on introducing

$$y = (V_{\rm nmo}(0)\, p)^2 = (c_P\, p)^2(1 + 2\delta) = z(1 + 2\delta), \tag{64}$$

extracting the elliptic result in the notation of equation (60), and again ignoring quadratic terms in the anisotropy parameters, one obtains the expression,

$$V_{\rm nmo}^2(p) = V_{\rm ell}^2(p)\,(1 + 2\eta\, F(y)), \qquad V_{\rm ell}(p) = \frac{V_{\rm nmo}(0)}{\sqrt{1-y}}. \tag{65}$$

in which δ does not appear and which is in agreement with the corresponding equation in (Alkhalifah & Tsvankin). This last equation also implies the weak limit estimate,

$$\eta \approx \frac{1}{2F(y)}\left(\frac{V_{\rm nmo}^2(p)}{V_{\rm ell}^2(p)} - 1\right). \tag{66}$$

At the next order in the anisotropy parameters,

$$V_{\text{nmo}}^2(p) = V_{\text{ell}}^2(p)\,(1 + 2\eta\, F(y) + \frac{4\eta y}{f(1-y)^2} R(\delta,\eta,y)), \tag{67}$$

where,

$$\begin{aligned} R(\delta,\eta,y) &= 6\delta(1-f)(1-y)^3(1-2y) \\ &\quad + \eta y(15 - (69-26f)y + (117-68f)y^2 \\ &\qquad\quad -3(29-21f)y^3 + 4(6-5f)y^4). \end{aligned} \tag{68}$$

Observe that, as the second order small-ray-parameter expansion, the higher order term here does introduce δ and f in violation of the Alkhalifah-Tsvankin theory. However, observe first the consistency check, that in the elliptic limit, the higher order vanishes entirely because of an overall factor of η that appears in it. Second, notice that in the higher order term, the δ's are always multiplied by $1-f$, which somewhat mitigates their contribution. Indeed, in the common approximation of ignoring shear speed contributions by taking $f = 1$, the δ terms drop out completely along with the f contributions. Finally, observe that the function $y(1-2y)(1-y)$ multiplying the δ term has absolute maximum value less than 0.1 on the interval $0 \le y \le 1$, again mitigating the effect of δ on $V_{\text{nmo}}^2(p)$ in this expansion. The overall observation that using, say, $f = 3/4$, instead of the true value has little numerical effect remains true here.

ACKNOWLEDGEMENT

Thanks to Tariq Alkhalifah for performing numerical tests that established the range of validity of the small dip η estimate, Ken Larner for his encouragement and for a careful critique of the text, and especially to Ilya Tsvankin for sharing his deep understanding of anisotropic issues at several crucial junctures in this research.

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Center for Wave Phenomena

303/273-3557
FAX: 303/273-3478
Telex: 910/934-0190
e-mail: cwpcsm@dix.mines.edu

COLORADO SCHOOL OF MINES
GOLDEN, COLORADO 80401-1887

October 9, 1995

To: Consortium Sponsors, Seismic Inverse Methods for Complex Structures

From: Jo Ann Fink, Program Assistant JAF

Subject: Research Report, CWP-191, Phase and Moveout Velocity as Functions of the Ray Parameter for Transversely Isotropic Media *by* Jack K. Cohen.

Enclosed are three copies of a research paper recently completed by Professor Jack K. Cohen. A short description of the paper is below:

> Following some recent work of Ilya Tsvankin, this paper gives a systematic derivation of phase and NMO velocity results in terms of ray parameter. Together with Tariq Alkhalifah, Ilya has established a complete theory of time-domain seismic processing taking account of transverse anisotropy based on a new parameter they call η. This paper gives some analytic justification backing up the numerical and empirical success of their η-theory.

This paper will be submitted to GEOPHYSICS following the Consortium proprietary period.

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Center for Wave Phenomena

303/273-3557
FAX: 303/273-3478
Telex: 910/934-0190
e-mail: cwpcsm@dix.mines.edu

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Center for Wave Phenomena

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FAX: 303/273-3478
Telex: 910/934-0190
e-mail: cwpcsm@dix.mines.edu

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Date: Mon, 13 Nov 1995 13:41:49 -0700
From: "Jack K. Cohen" <jkc@dix.Mines.EDU>
To: klarner@dix.Mines.EDU, ilya@dix.Mines.EDU, talkhali@dix.Mines.EDU
CC: norm@dix.Mines.EDU, barbara@dix.Mines.EDU
Subject: What else?

I think I've addressed all the recent suggestions and once again, the report is improved because of them. Thank you all! I hesitate to inflict another version on anyone, but I would be glad to make xeroxes for those who ask. This version has a real abstract and introduction and readable references (thanks to Barbara & Ken) and contains pictures illustrating the fit of the weak/moderate anisotropy expansions and small-p expansions to the respective EXACT phase velocity and moveout functions (this material is new--formerly I had nothing on the exact functions).

I didn't feel good about the DMO in the title, but my current effort is a long way from the short, catchy title that Tariq liked. Here it is:

Analytic Study of the Effective Parameters for
Determination of the NMO Velocity Function in
Transversely Isotropic Media

Aside from this cumbersome title, I feel good about this report. Yesterday, it occurred to me that it is one of the few things I done that has no co-authors! Which brought to my mind Ken's recent note denigrating "teams." While it is true that one needs some space to create anything, I have not personally experienced great success when I felt myself isolated. I know that this goes up against the American icon of the lonely hero (e.g., Ayn Rand), but just think:

I would never have been interested in this problem if Ilya were not here. I would have been lost down unrewarding pathways if Ilya and Tariq hadn't been patient with a non-geophysicist's repeated dumb questions and wrong assertions. And I couldn't have written it up so coherently (I hope it's coherent!) without the lessons that Ken taught me both in connection with this report and previously. Not to mention the kindness of my fourth grade arithmetic teacher, etc. So it isn't really true that I have no co-authors, you see.

And I think we have a great team.

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Jack K. Cohen, Colorado School of Mines,
Center for Wave Phenomena, Golden CO, 80401
Email: jkc@dix.mines.edu Phone: (303)273-3512 Fax: (303) 273-3478

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Phase and Moveout Velocity as Functions of the Ray Parameter for Transversely Isotropic Media

Jack K. Cohen

Center for Wave Phenomena
Colorado School of Mines
Golden, Colorado 80401
303/273-3557

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Jack K. Cohen

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ABSTRACT

In their studies of transversely isotropic media, I. Tsvankin and his co-workers have shown the benefits of expressing the P-wave normal moveout velocity in terms of the zero-offset ray parameter instead of the phase angle, but left for future work the actual task of doing this. Here, explicit formulas are given for both the P-wave phase velocity and normal moveout velocity in terms of the ray parameter. Various results of Tsvankin and others are rederived and, in some cases, extended, in a uniform manner using these explicit results.

The small ray-parameter expansion of the moveout velocity is valid for dips up to approximately 15°. In this regime, an analytic solution for the anisotropy parameter η, important for time-related imaging is given in terms of observations of moveout velocity for two different values of the ray parameter.

INTRODUCTION

In their studies of transversely isotropic media, I. Tsvankin and his co-workers have shown the benefits of expressing the P-wave normal moveout velocity in terms of the zero-offset ray parameter instead of the phase angle. Here, explicit formulas are given for both the P-wave phase velocity and normal moveout velocity in terms of the ray parameter. Various consequences of these results are drawn, including an analytic solution for the anisotropy parameter η, important for time-related imaging is given in terms of observations of moveout velocity for two different values of the ray parameter.

Throughout this study, *Mathematica* was used extensively to derive and check results. In particular, the *Mathematica* package, **Thomsen.m** (Cohen, 1995), was used to convert equation (3) to Thomsen notation, in equation (5), and to obtain various results in the limit of weak transverse isotropy. Similarly, the use of *Mathematica* facilitated computing symbolic derivatives, series expansions, etc.

PHASE VELOCITY AS A FUNCTION OF RAY PARAMETER

Begin with the formula for P-wave phase velocity in terms of dip θ (Tsvankin, 1994), equation(6):

$$2\rho V^2(\theta) = (C_{11} + C_{44})\sin^2\theta + (C_{33} + C_{44})\cos^2\theta \\ + \left\{\left[(C_{11} - C_{44})\sin^2\theta - (C_{33} - C_{44})\cos^2\theta\right]^2 \right. \\ \left. + 4(C_{13} + C_{44})^2 \sin^2\theta\cos^2\theta\right\}^{1/2}. \tag{1}$$

Use the substitutions

$$p = \sin\theta/V(\theta), \quad m = \cos\theta/V(\theta), \tag{2}$$

to rewrite equation (1) as an equation for the slowness surface:

$$2\rho = (C_{11} + C_{44})p^2 + (C_{33} + C_{44})m^2 \\ + \left\{\left[(C_{11} - C_{44})p^2 - (C_{33} - C_{44})m^2\right]^2 \right. \\ \left. + 4(C_{13} + C_{44})^2 p^2 m^2\right\}^{1/2}. \tag{3}$$

To obtain a formula for $V(p)$, follow the recipe given in (Alkhalifah & Tsvankin, 1995), Appendix A. Begin by converting equation (3) to Thomsen notation. After introducing the on-axis P and S velocities, c_P and c_S, and the related quantities,

$$k = \frac{1}{c_P^2}, \qquad f = 1 - \frac{c_S^2}{c_P^2}, \tag{4}$$

obtain the Thomsen-notation form of the slowness surface as

$$2k = (2 - f)m^2 + (2 + 2\epsilon - f)p^2 + \sqrt{4f(2\delta + f)m^2p^2 + (f(p^2 - m^2) + 2\epsilon p^2)^2}. \tag{5}$$

Next, solve equation (5) for m^2 and, from equation (2), form

$$V(p) = \frac{1}{\sqrt{p^2 + m^2(p)}}. \tag{6}$$

After some manipulations, find that $V^2(p)$ can be written as

$$V^2(p) = \frac{A + \sqrt{B}}{2C}, \tag{7}$$

where,

$$\begin{aligned} A &= (2 - f)k - 2(\epsilon - f\delta)p^2, & (8)\\ B &= f^2k^2 - 4fk\left[\epsilon - (2 - f)\delta\right]p^2 + 4\left[2f(1 - f)(\epsilon - \delta) + (\epsilon - f\delta)^2\right]p^4, & (9)\\ C &= k^2 - 2k\epsilon p^2 - 2f(\epsilon - \delta)p^4. & (10) \end{aligned}$$

Before turning to some important special cases, observe the following consequences of the definitions in equations (2):

$$\sin\theta = pV(p), \quad \cos\theta = mV(p) = \sqrt{1-(pV)^2}, \quad p^2+m^2 = \frac{1}{V^2}. \tag{11}$$

These equations will be of great convenience in translating between representations in phase angle and representations in the ray parameter.

Small ray parameter

First observe that for $p = 0$,

$$\begin{aligned} A &= (2-f)k, && (12)\\ B &= f^2k^2, && (13)\\ C &= k^2, && (14) \end{aligned}$$

so that,

$$V^2(0) = \frac{(2-f)k+fk}{2k^2} = \frac{2k}{2k^2} = \frac{1}{k} = c_P^2, \tag{15}$$

leading to the expected result,

$$V(0) = c_P. \tag{16}$$

For later purposes, we will need to know the more detailed behavior of $V(p)$ for small p. Introducing the dimensionless parameter,

$$z = (c_Pp)^2, \tag{17}$$

obtain

$$\begin{aligned} V(p) \quad = \quad & c_P\Big[1+\delta z+\Big((\epsilon-\delta)(1+2\delta/f)+3\delta^2/2\Big)z^2\\ & +\Big((\epsilon-\delta)(1+2\delta/f)(5\delta+2(\epsilon-2\delta)/f)+5\delta^3/2\Big)z^3 \qquad (18)\\ & +O(z^4)\Big]. \end{aligned}$$

Remark: This expansion shows that the true meaning of "small p" is that $z = (c_Pp)^2 << 1$. Since $V \approx c_P$, the inequality can be written as $\sin^2\theta << 1$, so in dimensionless terms, "small p" represents a small dip angle θ.

Elliptic anisotropy

Next study the elliptic case ($\delta = \epsilon$), where

$$\begin{aligned} A &= (2-f)k-2(1-f)\delta p^2, && (19)\\ B &= [fk+2(1-f)\delta p^2]^2, && (20)\\ C &= k^2-2k\delta p^2, && (21) \end{aligned}$$

so that,

$$V^2(p) = \frac{1}{k - 2\delta p^2} = \frac{c_P^2}{1 - 2\delta c_P^2 p^2}, \qquad (\delta = \epsilon). \tag{22}$$

Tsvankin (1995) gives the corresponding elliptic limit value of V^2 in terms of the phase angle, θ, as

$$V^2(\theta) = V_0^2 \cos^2\theta + V_{90}^2 \sin^2\theta, \tag{23}$$

where $V_0 = V(0)$ and $V_{90} = V(\pi/2)$ in the phase angle form of V implied by equation (1). These values are readily found from equation (5). When $p = 0$, then $m = 1/V(0)$, and the equation reduces to $2/c_P^2 = 2/V^2(0)$, so that

$$V_0 = c_P. \tag{24}$$

Similarly, when $m = 0$, then $p = 1/V_{90}$, giving $2/c_P^2 = 2(1 + \delta + \epsilon)/V_{90}$, so that

$$V_{90} = c_P\sqrt{1 + \delta + \epsilon}, \tag{25}$$

which reduces to

$$V_{90} = c_P\sqrt{1 + 2\delta} \qquad (\delta = \epsilon) \tag{26}$$

in the elliptic limit. This also confirms the known result that $V_{\text{nmo}}(0) = V_{90}$ for elliptically anisotropic media.

With these results in hand, equation (23) becomes

$$V^2 = c_P^2 + 2\delta c_P^2 \sin^2\theta = c_P^2 + 2\delta c_P^2 p^2 V^2 \qquad (\delta = \epsilon). \tag{27}$$

On isolating V^2, equation (22) is verified.

Tsvankin (1995) likewise gives the form

$$V(\theta) = c_P\sqrt{1 + 2\delta \sin^2\theta} \qquad (\delta = \epsilon) \tag{28}$$

for the elliptic case. Introducing the ray parameter in that expression gives

$$V^2 = c_P^2(1 + 2\delta p^2 V^2) \tag{29}$$

and, once again, isolating V^2 verifies equation (22).

Weak transverse isotropy

The weak TI limit of equation (7) is

$$V^2(p) \approx c_P^2 \left[1 + 2(\delta + (\epsilon - \delta)c_P^2 p^2)c_P^2 p^2\right], \tag{30}$$

or

$$V(p) \approx c_P \left[1 + (\delta + (\epsilon - \delta)c_P^2 p^2)c_P^2 p^2\right]. \tag{31}$$

Thomsen's (1986) expression for this quantity expressed as a function of angle is

$$V(\theta) = c_P \left[1 + (\delta \cos^2\theta + \epsilon \sin^2\theta)\sin^2\theta\right]. \tag{32}$$

Tsvankin (1995) shows that this equation implies

$$\sin^2\theta = \frac{p^2 c_P^2}{1 - 2\delta p^2 c_P^2}[1 + 2(\epsilon - \delta)p^4 c_P^4], \tag{33}$$

which, in turn, in the weak limit is equivalent to

$$p^2 V^2 = p^2 c_P^2 [1 + 2\delta p^2 c_P^2 + 2(\epsilon - \delta)p^4 c_P^4]. \tag{34}$$

On cancelling the common factor of p^2, this verifies equation (30) and hence also equation (31).

NORMAL MOVEOUT VELOCITY AS A FUNCTION OF RAY PARAMETER

The derivation of the normal moveout velocity as a function of ray parameter, follows closely the corresponding derivation of its expression in terms of the ray angle given in (Tsvankin, 1995). For convenience, some explicit references are made to equations in this paper.

First, use equations (11) to find the following relation for $dp/d\theta$:

$$mV = \cos\theta = \frac{d\sin\theta}{d\theta} = (pV)'\frac{dp}{d\theta}, \tag{35}$$

where, here and below, the prime notation is used for p-differentiation. Thus, we find that

$$\frac{dV}{d\theta} = V'\frac{dp}{d\theta} = \frac{mVV'}{(pV)'}. \tag{36}$$

The derivation of $V_{\mathrm{nmo}}(p)$ begins with Tsvankin's equation (4),

$$V_{\mathrm{nmo}}^2(p) = \frac{2z_0}{t_0}\lim_{h\to 0}\frac{d\tan\psi}{dp}. \tag{37}$$

The second component of this expression is evaluated using Tsvankin's equation (6), equations (11), and equation (36):

$$\begin{aligned}\tan\psi &= \frac{V\sin\theta + \frac{dV}{d\theta}\cos\theta}{V\cos\theta - \frac{dV}{d\theta}\sin\theta} \\ &= \frac{pV^2 + mV\frac{dV}{d\theta}}{mV^2 - pV\frac{dV}{d\theta}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{pV + m\frac{dV}{d\theta}}{mV - p\frac{dV}{d\theta}} \\
&= \frac{pV(pV)' + m^2VV'}{mV(pV)' - mpVV'} \\
&= \frac{p(pV)' + m^2V'}{m\left[(pV)' - pV'\right]} \\
&= \frac{p^2V' + m^2V' + pV}{mV} \\
&= \frac{V'/V^2 + pV}{mV} \\
&= \frac{V' + pV^3}{mV^3}. \qquad (38)
\end{aligned}
$$

The first component of equation (37) is similarly evaluated using Tsvankin's equation (8):

$$
\begin{aligned}
\frac{2z_0}{t_0} &= V\cos\theta\left(1 - \frac{\tan\theta}{V}\frac{dV}{d\theta}\right) \\
&= mV^2\left(1 - \frac{p}{mV}\frac{mVV'}{(pV)'}\right) \\
&= mV^2\left(1 - \frac{pV'}{(pV)'}\right) \\
&= mV^2\frac{V + pV' - pV'}{(pV)'} \\
&= \frac{mV^3}{(pV)'}. \qquad (39)
\end{aligned}
$$

Using the previous two results in equation (37), find that

$$
V_{\text{nmo}}^2(p) = \frac{mV^3}{(pV)'}\frac{d}{dp}\left(\frac{V' + pV^3}{mV^3}\right) \qquad (40)
$$

or, on eliminating m,

$$
V_{\text{nmo}}^2(p) = \frac{V^2\sqrt{1-(pV)^2}}{(pV)'}\frac{d}{dp}\left(\frac{V' + pV^3}{V^2\sqrt{1-(pV)^2}}\right). \qquad (41)
$$

Carrying out the indicated derivative gives an explicit formula for the normal moveout velocity as a function of ray parameter:

$$
V_{\text{nmo}}^2(p) = \frac{2(1-p^2V^2)VV'' + (3p^2V^2-2)V'^2 + 2pV^3V' + V^4}{(1-p^2V^2)V(pV)'} \qquad (42)
$$

Before turning to the special cases for V, note that when it is necessary to treat the general V, it may be better to avoid a layer of square roots by writing $V^2_{\rm nmo}(p)$ in terms of $W \equiv V^2$:

$$V^2_{\rm nmo}(p) = \frac{2(1-p^2W)WW'' + (4p^2W-3)W'^2 + 4pW^2W' + 4W^3}{2(1-p^2W)W(pW'+2W)} \tag{43}$$

Small ray parameter

First, use the two leading terms of equation (18) to obtain the approximations,

$$V = c_P + c_P^3\delta p^2 + O(p^4), \quad V' = 2c_P^3\delta p + O(p^3), \quad V'' = 2c_P^3\delta + O(p^2). \tag{44}$$

On inserting these small-p approximations into equation (42), find that

$$V^2_{\rm nmo}(p) = c_P^2(1+2\delta) + O(p^2), \tag{45}$$

so

$$V_{\rm nmo}(0) = c_P\sqrt{1+2\delta}. \tag{46}$$

Now use the next order terms of equation (18) in equation (42) to get the next term in $V^2_{\rm nmo}(p)$:

$$V^2_{\rm nmo}(p) = c_P^2\left(1+2\delta+\left((-24\delta^2+24\delta\epsilon+f-8\delta f+4\delta^2 f+12\epsilon f)\frac{(c_Pp)^2}{f}\right)+O(c_Pp)^4\right). \tag{47}$$

Remark: Again, the expansion shows that the true meaning of "small p" is that $(c_Pp)^2 << 1$. Since $V \approx c_P$, and the error is fourth order, this means we can interpret "small p" as meaning $\sin^4\theta << 1$. Numerical tests show that using the small p series for a dip of 15° incurs about a 2% error for typical values of f and δ.

The theory discussed by (Alkhalifah & Tsvankin, 1995) suggests that it is better to express this result in terms of the parameters $V_{\rm nmo}(0)$ and

$$\eta = \frac{\epsilon-\delta}{1+2\delta}. \tag{48}$$

First introduce η,

$$V^2_{\rm nmo}(p) \approx c_P^2(1+2\delta) + \left(c_P^4(1+2\delta)(24\delta\eta+f+2\delta f+12\eta f)p^2\right)/f + O(p^4), \tag{49}$$

then use equation (46) to eliminate the explicit appearances of c_P^2 in favor of $V^2_{\rm nmo}(0)$:

$$V^2_{\rm nmo}(p) = V^2_{\rm nmo}(0)\left[\left(1+(1+12\eta\frac{1+2\delta/f}{1+2\delta})V^2_{\rm nmo}(0)p^2\right)+O(V^4_{\rm nmo}(0)p^4)\right]. \tag{50}$$

The importance of the anisotropic parameter η, is that Alkhalifah and Tsvankin (1995) have observed, both numerically and empirically, that $V^2_{\rm nmo}(p)$ depends mainly

on η and $V^2_{\mathrm{nmo}}(0)$. They exploit this reduction to surface-observable parameters to develop time-domain seismic processing algorithms that take account of transverse anisotropy. Equation (50) gives analytic support to the Alkhalifah-Tsvankin theory, since the only deviation to it occurs in the ratio

$$g = \frac{1+2\delta/f}{1+2\delta} \tag{51}$$

that multiplies η. Figures (1) and (2) show plots of this function over the ranges of f and δ that are relevant in practice. Observe that the function $g(\delta, f)$ varies slowly over these ranges—indeed—ignoring it altogether (i.e., replacing it by the constant 1) is usually justified—at least in the small-angle approximation.

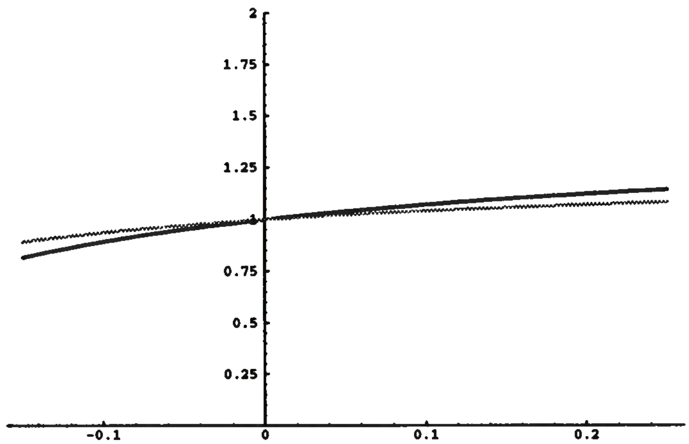

FIG. 1. Plot of the factor $g = (1+2\delta/f)/(1+2\delta)$ as a function of δ for $f = 0.7$ (dark) and $f = 0.8$ (light).

The small-ray-parameter result in equation (50) gives a means for estimating η from surface observations. For example, suppose one has observations of $V_{\mathrm{nmo}}(p)$ at $p = 0$ and some other (not too large) value $p = p_1$. The solution for η is given by

$$\eta \approx \frac{1}{12g}\left(\frac{V^2_{\mathrm{nmo}}(p_1) - V^2_{\mathrm{nmo}}(0)}{p_1^2 V^4_{\mathrm{nmo}}(0)} - 1\right), \tag{52}$$

where once again, in the absence of information on δ and f, the factor g can be replaced by the constant 1 without much error.

More generally, if one uses two nonzero "small p" values (that is, two separated dips, each less than 15°), then the estimate for $V^2_{\mathrm{nmo}}(0)$ is given by

$$V^2_{\mathrm{nmo}}(0) = \frac{p_2^2 V^2_{\mathrm{nmo}}(p_1) - p_1^2 V^2_{\mathrm{nmo}}(p_2)}{p_2^2 - p_1^2}, \tag{53}$$

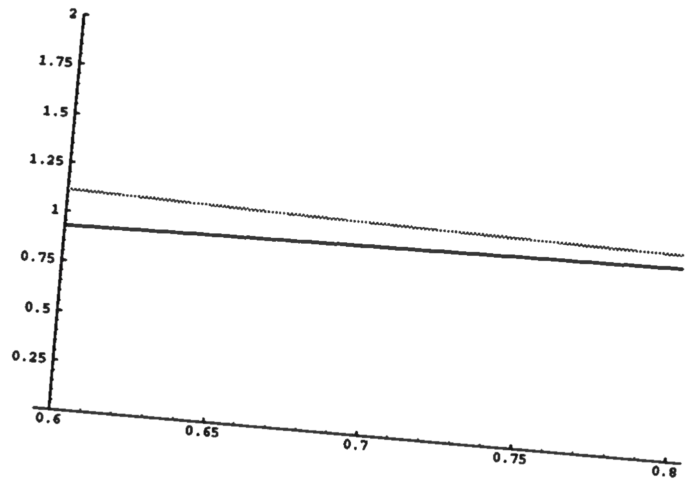

FIG. 2. Plot of the factor $g = (1 + 2\delta/f)/(1 + 2\delta)$ as a function of f for $\delta = -0.05$ (dark) and $\delta = +0.1$ (light).

and the estimate for η is

$$\eta \approx \frac{1}{12g}\left(\frac{(p_1^2 - p_2^2)(V_{\rm nmo}^2(p_2) - V_{\rm nmo}^2(p_1))}{(p_1^2 V_{\rm nmo}^2(p_2) - p_2^2 V_{\rm nmo}^2(p_1))^2} - 1\right). \tag{54}$$

Finally, note that the full form of the series for $V_{\rm nmo}^2(p)$ is given by

$$V_{\rm nmo}^2(p) = V_{\rm nmo}^2(0)\left[1 + c_2 V_{\rm nmo}^2(0) p^2 + c_4 V_{\rm nmo}^4(0) p^4 + \cdots\right], \tag{55}$$

where, as we have seen earlier,

$$c_2 = 1 + 12g\eta \approx 1 + 12\eta. \tag{56}$$

By using the full form of equation (18), we find that

$$c_4 = 1 + 6g(6 - 5g)\eta + \frac{60g}{f}\eta^2 \approx 1 + 6\eta + \frac{60}{f}\eta^2. \tag{57}$$

Here, the approximations result from replacing the factor g defined in equation (51) by the constant 1. Note that the final term in c_4 indicates the first serious divergence from the theory that $V_{\rm nmo}^2(p)$ depends only on the parameters $V_{\rm nmo}^2(0)$ and η. However, this term is multiplied by both p^2 and η^2, which ameliorates the effect of replacing the f in this term by, say, 3/4 instead of the true value.

Elliptic anisotropy

Inserting the elliptic P-wave phase velocity as a function of ray parameter given in equation (22) into the general NMO equation (42) gives at once

$$V_{\rm nmo}^2(p) = \frac{c_P^2(1 + 2\delta)}{(1 - (1 + 2\delta)c_P^2 p^2)}. \tag{58}$$

Recognizing the quantity $V_{\rm nmo}(0)$ from equation (46) gives

$$V_{\rm nmo}(p) = \frac{V_{\rm nmo}(0)}{\sqrt{1 - p^2 V_{\rm nmo}^2(0)}}, \qquad (\delta = \epsilon), \tag{59}$$

in agreement with the result in (Alkhalifah & Tsvankin, 1995). For future use, introduce the notation,

$$V_{\rm ell}(p) \equiv \frac{V_{\rm nmo}(0)}{\sqrt{1 - p^2 V_{\rm nmo}^2(0)}}, \tag{60}$$

for the elliptic result.

Weak transverse isotropy

Using equation (31) in equation (42) gives

$$V_{\rm nmo}^2(p) = \frac{c_P^2}{1-z}\left(1 + \frac{2(\delta + (\epsilon - \delta)\, z(6 - 9z + 4z^2))}{1-z}\right), \tag{61}$$

where we have again used the shorthand notation, $z = (c_P p)^2$. One would have to seek a more sophisticated expansion if p became large enough to approach $1/c_P$. On the other hand, always $p < 1/V$ and in the weak limit, $V \approx c_P$, so this is an unusual circumstance.

The approximation, $\eta = \epsilon - \delta$, is valid in the weak limit, so equation (61) may be recast as

$$V_{\rm nmo}^2(p) = \frac{c_P^2}{1-z}\left(1 + \frac{2\delta}{1-z} + 2\eta\, F(z)\right). \tag{62}$$

with

$$F(z) = \frac{z(6 - 9z + 4z^2)}{1-z}. \tag{63}$$

Apparently, we have a disappointing dependence on δ in addition to that on $V_{\rm nmo}(0)$ and η. However, since the equation (59) in the exact elliptic case does not depend on δ, we are encouraged to look deeper. Indeed, on introducing

$$y = (V_{\rm nmo}(0)\, p)^2 = (c_P\, p)^2(1 + 2\delta) = z(1 + 2\delta), \tag{64}$$

extracting the elliptic result in the notation of equation (60), and again ignoring quadratic terms in the anisotropy parameters, one obtains the expression,

$$V_{\rm nmo}^2(p) = V_{\rm ell}^2(p)\,(1 + 2\eta\, F(y)), \qquad V_{\rm ell}(p) = \frac{V_{\rm nmo}(0)}{\sqrt{1-y}}. \tag{65}$$

in which δ does not appear and which is in agreement with the corresponding equation in (Alkhalifah & Tsvankin). This last equation also implies the weak limit estimate,

$$\eta \approx \frac{1}{2F(y)}\left(\frac{V_{\rm nmo}^2(p)}{V_{\rm ell}^2(p)} - 1\right). \tag{66}$$

At the next order in the anisotropy parameters,

$$V^2_{\rm nmo}(p) = V^2_{\rm ell}(p)\,(1 + 2\eta\, F(y) + \frac{4\eta y}{f(1-y)^2} R(\delta,\eta,y)), \tag{67}$$

where,

$$\begin{aligned} R(\delta,\eta,y) &= 6\delta(1-f)(1-y)^3(1-2y) \\ &\quad +\eta y(15-(69-26f)y+(117-68f)y^2 \\ &\qquad -3(29-21f)y^3+4(6-5f)y^4). \end{aligned} \tag{68}$$

Observe that, as the second order small-ray-parameter expansion, the higher order term here does introduce δ and f in violation of the Alkhalifah-Tsvankin theory. However, observe first the consistency check, that in the elliptic limit, the higher order vanishes entirely because of an overall factor of η that appears in it. Second, notice that in the higher order term, the δ's are always multiplied by $1-f$, which somewhat mitigates their contribution. Indeed, in the common approximation of ignoring shear speed contributions by taking $f=1$, the δ terms drop out completely along with the f contributions. Finally, observe that the function $y(1-2y)(1-y)$ multiplying the δ term has absolute maximum value less than 0.1 on the interval $0 \le y \le 1$, again mitigating the effect of δ on $V^2_{\rm nmo}(p)$ in this expansion. The overall observation that using, say, $f = 3/4$, instead of the true value has little numerical effect remains true here.

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Jack K. Cohen

Center for Wave Phenomena
Colorado School of Mines
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The small ray-parameter expansion of the moveout velocity is valid for dips up to approximately 15°. In this regime, an analytic solution for the anisotropy parameter η, important for time-related imaging is given in terms of observations of moveout velocity for two different values of the ray parameter.

INTRODUCTION

In their studies of transversely isotropic media, I. Tsvankin and his co-workers have shown the benefits of expressing the P-wave normal moveout velocity in terms of the zero-offset ray parameter instead of the phase angle. Here, explicit formulas are given for both the P-wave phase velocity and normal moveout velocity in terms of the ray parameter. Various consequences of these results are drawn, including an analytic solution for the anisotropy parameter η, important for time-related imaging is given in terms of observations of moveout velocity for two different values of the ray parameter.

Throughout this study, *Mathematica* was used extensively to derive and check results. In particular, the *Mathematica* package, **Thomsen.m** (Cohen, 1995), was used to convert equation (3) to Thomsen notation, in equation (5), and to obtain various results in the limit of weak transverse isotropy. Similarly, the use of *Mathematica* facilitated computing symbolic derivatives, series expansions, etc.

PHASE VELOCITY AS A FUNCTION OF RAY PARAMETER

Begin with the formula for P-wave phase velocity in terms of dip θ (Tsvankin, 1994), equation(6):

$$\begin{aligned} 2\rho V^2(\theta) = (C_{11}+C_{44})\sin^2\theta + (C_{33}+C_{44})\cos^2\theta \\ + \Big\{\left[(C_{11}-C_{44})\sin^2\theta - (C_{33}-C_{44})\cos^2\theta\right]^2 \\ + 4(C_{13}+C_{44})^2\sin^2\theta\cos^2\theta\Big\}^{1/2}. \end{aligned} \tag{1}$$

Use the substitutions

$$p = \sin\theta/V(\theta), \quad m = \cos\theta/V(\theta), \tag{2}$$

to rewrite equation (1) as an equation for the slowness surface:

$$\begin{aligned} 2\rho = (C_{11}+C_{44})p^2 + (C_{33}+C_{44})m^2 \\ + \Big\{\left[(C_{11}-C_{44})p^2 - (C_{33}-C_{44})m^2\right]^2 \\ + 4(C_{13}+C_{44})^2p^2m^2\Big\}^{1/2}. \end{aligned} \tag{3}$$

To obtain a formula for $V(p)$, follow the recipe given in (Alkhalifah & Tsvankin, 1995), Appendix A. Begin by converting equation (3) to Thomsen notation. After introducing the on-axis P and S velocities, c_P and c_S, and the related quantities,

$$k = \frac{1}{c_P^2}, \qquad f = 1 - \frac{c_S^2}{c_P^2}, \tag{4}$$

obtain the Thomsen-notation form of the slowness surface as

$$2k = (2-f)m^2 + (2+2\epsilon-f)p^2 + \sqrt{4f(2\delta+f)m^2p^2 + (f(p^2-m^2)+2\epsilon p^2)^2}. \tag{5}$$

Next, solve equation (5) for m^2 and, from equation (2), form

$$V(p) = \frac{1}{\sqrt{p^2+m^2(p)}}. \tag{6}$$

After some manipulations, find that $V^2(p)$ can be written as

$$V^2(p) = \frac{A+\sqrt{B}}{2C}, \tag{7}$$

where,

$$\begin{aligned} A &= (2-f)k - 2(\epsilon - f\delta)p^2, && (8) \\ B &= f^2k^2 - 4fk\left[\epsilon - (2-f)\delta\right]p^2 + 4\left[2f(1-f)(\epsilon-\delta) + (\epsilon - f\delta)^2\right]p^4, && (9) \\ C &= k^2 - 2k\epsilon p^2 - 2f(\epsilon-\delta)p^4. && (10) \end{aligned}$$

Before turning to some important special cases, observe the following consequences of the definitions in equations (2):

$$\sin\theta = pV(p), \quad \cos\theta = mV(p) = \sqrt{1-(pV)^2}, \quad p^2+m^2 = \frac{1}{V^2}. \tag{11}$$

These equations will be of great convenience in translating between representations in phase angle and representations in the ray parameter.

Small ray parameter

First observe that for $p=0$,

$$\begin{aligned} A &= (2-f)k, & (12)\\ B &= f^2k^2, & (13)\\ C &= k^2, & (14) \end{aligned}$$

so that,

$$V^2(0) = \frac{(2-f)k+fk}{2k^2} = \frac{2k}{2k^2} = \frac{1}{k} = c_P^2, \tag{15}$$

leading to the expected result,

$$V(0) = c_P. \tag{16}$$

For later purposes, we will need to know the more detailed behavior of $V(p)$ for small p. Introducing the dimensionless parameter,

$$z = (c_P p)^2, \tag{17}$$

obtain

$$\begin{aligned} V(p) = c_P \Big[& 1 + \delta z + \big((\epsilon-\delta)(1+2\delta/f) + 3\delta^2/2\big) z^2 \\ & + \big((\epsilon-\delta)(1+2\delta/f)(5\delta+2(\epsilon-2\delta)/f) + 5\delta^3/2\big) z^3 \\ & + O(z^4)\Big]. \end{aligned} \tag{18}$$

Remark: This expansion shows that the true meaning of "small p" is that $z = (c_P p)^2 << 1$. Since $V \approx c_P$, the inequality can be written as $\sin^2\theta << 1$, so in dimensionless terms, "small p" represents a small dip angle θ.

Elliptic anisotropy

Next study the elliptic case ($\delta = \epsilon$), where

$$\begin{aligned} A &= (2-f)k - 2(1-f)\delta p^2, & (19)\\ B &= [fk + 2(1-f)\delta p^2]^2, & (20)\\ C &= k^2 - 2k\delta p^2, & (21) \end{aligned}$$

so that,

$$V^2(p) = \frac{1}{k - 2\delta p^2} = \frac{c_P^2}{1 - 2\delta c_P^2 p^2}, \qquad (\delta = \epsilon). \tag{22}$$

Tsvankin (1995) gives the corresponding elliptic limit value of V^2 in terms of the phase angle, θ, as

$$V^2(\theta) = V_0^2 \cos^2\theta + V_{90}^2 \sin^2\theta, \tag{23}$$

where $V_0 = V(0)$ and $V_{90} = V(\pi/2)$ in the phase angle form of V implied by equation (1). These values are readily found from equation (5). When $p = 0$, then $m = 1/V(0)$, and the equation reduces to $2/c_P^2 = 2/V^2(0)$, so that

$$V_0 = c_P. \tag{24}$$

Similarly, when $m = 0$, then $p = 1/V_{90}$, giving $2/c_P^2 = 2(1 + \delta + \epsilon)/V_{90}$, so that

$$V_{90} = c_P\sqrt{1 + \delta + \epsilon}, \tag{25}$$

which reduces to

$$V_{90} = c_P\sqrt{1 + 2\delta} \qquad (\delta = \epsilon) \tag{26}$$

in the elliptic limit. This also confirms the known result that $V_{\rm nmo}(0) = V_{90}$ for elliptically anisotropic media.

With these results in hand, equation (23) becomes

$$V^2 = c_P^2 + 2\delta c_P^2 \sin^2\theta = c_P^2 + 2\delta c_P^2 p^2 V^2 \qquad (\delta = \epsilon). \tag{27}$$

On isolating V^2, equation (22) is verified.

Tsvankin (1995) likewise gives the form

$$V(\theta) = c_P\sqrt{1 + 2\delta \sin^2\theta} \qquad (\delta = \epsilon) \tag{28}$$

for the elliptic case. Introducing the ray parameter in that expression gives

$$V^2 = c_P^2(1 + 2\delta p^2 V^2) \tag{29}$$

and, once again, isolating V^2 verifies equation (22).

Weak transverse isotropy

The weak TI limit of equation (7) is

$$V^2(p) \approx c_P^2\left[1 + 2(\delta + (\epsilon - \delta)c_P^2 p^2)c_P^2 p^2\right], \tag{30}$$

or

$$V(p) \approx c_P\left[1 + (\delta + (\epsilon - \delta)c_P^2 p^2)c_P^2 p^2\right]. \tag{31}$$

Thomsen's (1986) expression for this quantity expressed as a function of angle is

$$V(\theta) = c_P \left[1 + (\delta \cos^2\theta + \epsilon \sin^2\theta)\sin^2\theta\right]. \tag{32}$$

Tsvankin (1995) shows that this equation implies

$$\sin^2\theta = \frac{p^2 c_P^2}{1 - 2\delta p^2 c_P^2}[1 + 2(\epsilon - \delta)p^4 c_P^4], \tag{33}$$

which, in turn, in the weak limit is equivalent to

$$p^2 V^2 = p^2 c_P^2 [1 + 2\delta p^2 c_P^2 + 2(\epsilon - \delta)p^4 c_P^4]. \tag{34}$$

On cancelling the common factor of p^2, this verifies equation (30) and hence also equation (31).

NORMAL MOVEOUT VELOCITY AS A FUNCTION OF RAY PARAMETER

The derivation of the normal moveout velocity as a function of ray parameter, follows closely the corresponding derivation of its expression in terms of the ray angle given in (Tsvankin, 1995). For convenience, some explicit references are made to equations in this paper.

First, use equations (11) to find the following relation for $dp/d\theta$:

$$mV = \cos\theta = \frac{d\sin\theta}{d\theta} = (pV)' \frac{dp}{d\theta}, \tag{35}$$

where, here and below, the prime notation is used for p-differentiation. Thus, we find that

$$\frac{dV}{d\theta} = V' \frac{dp}{d\theta} = \frac{mVV'}{(pV)'}. \tag{36}$$

The derivation of $V_{\text{nmo}}(p)$ begins with Tsvankin's equation (4),

$$V_{\text{nmo}}^2(p) = \frac{2z_0}{t_0} \lim_{h \to 0} \frac{d\tan\psi}{dp}. \tag{37}$$

The second component of this expression is evaluated using Tsvankin's equation (6), equations (11), and equation (36):

$$\begin{aligned} \tan\psi &= \frac{V\sin\theta + \frac{dV}{d\theta}\cos\theta}{V\cos\theta - \frac{dV}{d\theta}\sin\theta} \\ &= \frac{pV^2 + mV\frac{dV}{d\theta}}{mV^2 - pV\frac{dV}{d\theta}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{pV + m\frac{dV}{d\theta}}{mV - p\frac{dV}{d\theta}} \\
&= \frac{pV(pV)' + m^2VV'}{mV(pV)' - mpVV'} \\
&= \frac{p(pV)' + m^2V'}{m\left[(pV)' - pV'\right]} \\
&= \frac{p^2V' + m^2V' + pV}{mV} \\
&= \frac{V'/V^2 + pV}{mV} \\
&= \frac{V' + pV^3}{mV^3}. \qquad (38)
\end{aligned}
$$

The first component of equation (37) is similarly evaluated using Tsvankin's equation (8):

$$
\begin{aligned}
\frac{2z_0}{t_0} &= V\cos\theta\left(1 - \frac{\tan\theta}{V}\frac{dV}{d\theta}\right) \\
&= mV^2\left(1 - \frac{p}{mV}\frac{mVV'}{(pV)'}\right) \\
&= mV^2\left(1 - \frac{pV'}{(pV)'}\right) \\
&= mV^2\frac{V + pV' - pV'}{(pV)'} \\
&= \frac{mV^3}{(pV)'}. \qquad (39)
\end{aligned}
$$

Using the previous two results in equation (37), find that

$$
V^2_{\text{nmo}}(p) = \frac{mV^3}{(pV)'}\frac{d}{dp}\left(\frac{V' + pV^3}{mV^3}\right) \qquad (40)
$$

or, on eliminating m,

$$
V^2_{\text{nmo}}(p) = \frac{V^2\sqrt{1-(pV)^2}}{(pV)'}\frac{d}{dp}\left(\frac{V' + pV^3}{V^2\sqrt{1-(pV)^2}}\right). \qquad (41)
$$

Carrying out the indicated derivative gives an explicit formula for the normal moveout velocity as a function of ray parameter:

$$
V^2_{\text{nmo}}(p) = \frac{2(1-p^2V^2)VV'' + (3p^2V^2 - 2)V'^2 + 2pV^3V' + V^4}{(1-p^2V^2)V(pV)'} \qquad (42)
$$

Before turning to the special cases for V, note that when it is necessary to treat the general V, it may be better to avoid a layer of square roots by writing $V^2_{\rm nmo}(p)$ in terms of $W \equiv V^2$:

$$V^2_{\rm nmo}(p) = \frac{2(1-p^2W)WW'' + (4p^2W-3)W'^2 + 4pW^2W' + 4W^3}{2(1-p^2W)W(pW'+2W)} \tag{43}$$

Small ray parameter

First, use the two leading terms of equation (18) to obtain the approximations,

$$V = c_P + c_P^3\delta p^2 + O(p^4), \quad V' = 2c_P^3\delta p + O(p^3), \quad V'' = 2c_P^3\delta + O(p^2). \tag{44}$$

On inserting these small-p approximations into equation (42), find that

$$V^2_{\rm nmo}(p) = c_P^2(1+2\delta) + O(p^2), \tag{45}$$

so

$$V_{\rm nmo}(0) = c_P\sqrt{1+2\delta}. \tag{46}$$

Now use the next order terms of equation (18) in equation (42) to get the next term in $V^2_{\rm nmo}(p)$:

$$V^2_{\rm nmo}(p) = c_P^2\left(1 + 2\delta + \left((-24\delta^2 + 24\delta\epsilon + f - 8\delta f + 4\delta^2 f + 12\epsilon f)\frac{(c_Pp)^2}{f}\right) + O(c_Pp)^4\right). \tag{47}$$

Remark: Again, the expansion shows that the true meaning of "small p" is that $(c_Pp)^2 << 1$. Since $V \approx c_P$, and the error is fourth order, this means we can interpret "small p" as meaning $\sin^4\theta << 1$. Numerical tests show that using the small p series for a dip of 15° incurs about a 2% error for typical values of f and δ.

The theory discussed by (Alkhalifah & Tsvankin, 1995) suggests that it is better to express this result in terms of the parameters $V_{\rm nmo}(0)$ and

$$\eta = \frac{\epsilon - \delta}{1+2\delta}. \tag{48}$$

First introduce η,

$$V^2_{\rm nmo}(p) \approx c_P^2(1+2\delta) + \left(c_P^4(1+2\delta)(24\delta\eta + f + 2\delta f + 12\eta f)p^2\right)/f + O(p^4), \tag{49}$$

then use equation (46) to eliminate the explicit appearances of c_P^2 in favor of $V^2_{\rm nmo}(0)$:

$$V^2_{\rm nmo}(p) = V^2_{\rm nmo}(0)\left[\left(1 + (1 + 12\eta\frac{1+2\delta/f}{1+2\delta})V^2_{\rm nmo}(0)p^2\right) + O(V^4_{\rm nmo}(0)p^4)\right]. \tag{50}$$

The importance of the anisotropic parameter η, is that Alkhalifah and Tsvankin (1995) have observed, both numerically and empirically, that $V^2_{\rm nmo}(p)$ depends mainly

on η and $V^2_{\text{nmo}}(0)$. They exploit this reduction to surface-observable parameters to develop time-domain seismic processing algorithms that take account of transverse anisotropy. Equation (50) gives analytic support to the Alkhalifah-Tsvankin theory, since the only deviation to it occurs in the ratio

$$g = \frac{1 + 2\delta/f}{1 + 2\delta} \tag{51}$$

that multiplies η. Figures (1) and (2) show plots of this function over the ranges of f and δ that are relevant in practice. Observe that the function $g(\delta, f)$ varies slowly over these ranges—indeed—ignoring it altogether (i.e., replacing it by the constant 1) is usually justified—at least in the small-angle approximation.

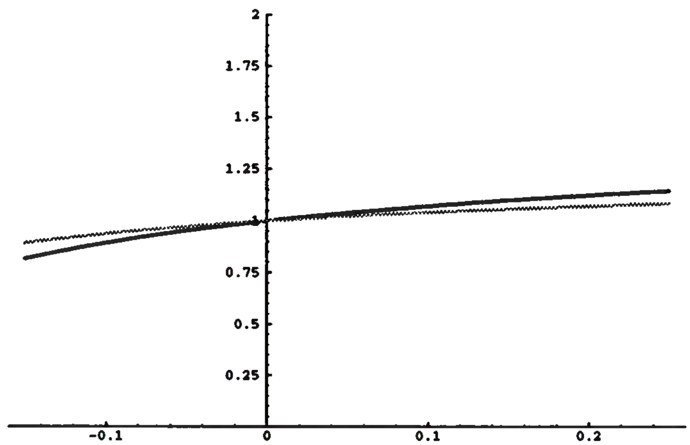

FIG. 1. Plot of the factor $g = (1 + 2\delta/f)/(1 + 2\delta)$ as a function of δ for $f = 0.7$ (dark) and $f = 0.8$ (light).

The small-ray-parameter result in equation (50) gives a means for estimating η from surface observations. For example, suppose one has observations of $V_{\text{nmo}}(p)$ at $p = 0$ and some other (not too large) value $p = p_1$. The solution for η is given by

$$\eta \approx \frac{1}{12g}\left(\frac{V^2_{\text{nmo}}(p_1) - V^2_{\text{nmo}}(0)}{p_1^2 V^4_{\text{nmo}}(0)} - 1\right), \tag{52}$$

where once again, in the absence of information on δ and f, the factor g can be replaced by the constant 1 without much error.

More generally, if one uses two nonzero "small p" values (that is, two separated dips, each less than 15°), then the estimate for $V^2_{\text{nmo}}(0)$ is given by

$$V^2_{\text{nmo}}(0) = \frac{p_2^2 V^2_{\text{nmo}}(p_1) - p_1^2 V^2_{\text{nmo}}(p_2)}{p_2^2 - p_1^2}, \tag{53}$$

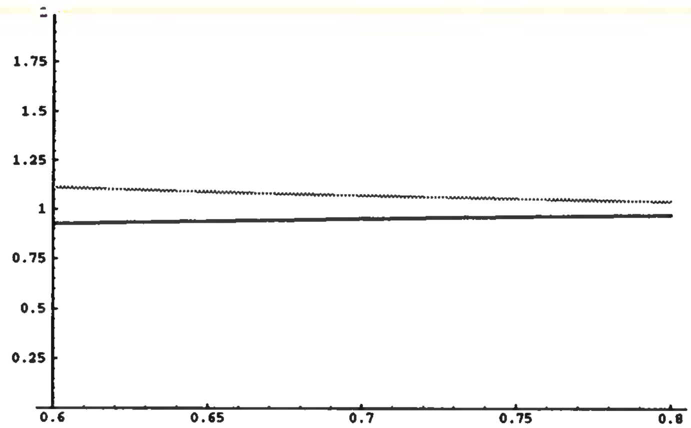

FIG. 2. Plot of the factor $g = (1 + 2\delta/f)/(1 + 2\delta)$ as a function of f for $\delta = -0.05$ (dark) and $\delta = +0.1$ (light).

and the estimate for η is

$$\eta \approx \frac{1}{12g}\left(\frac{(p_1^2 - p_2^2)(V_{\text{nmo}}^2(p_2) - V_{\text{nmo}}^2(p_1))}{(p_1^2 V_{\text{nmo}}^2(p_2) - p_2^2 V_{\text{nmo}}^2(p_1))^2} - 1\right). \tag{54}$$

Finally, note that the full form of the series for $V_{\text{nmo}}^2(p)$ is given by

$$V_{\text{nmo}}^2(p) = V_{\text{nmo}}^2(0)\left[1 + c_2 V_{\text{nmo}}^2(0)p^2 + c_4 V_{\text{nmo}}^4(0)p^4 + \cdots\right], \tag{55}$$

where, as we have seen earlier,

$$c_2 = 1 + 12g\eta \approx 1 + 12\eta. \tag{56}$$

By using the full form of equation (18), we find that

$$c_4 = 1 + 6g(6 - 5g)\eta + \frac{60g}{f}\eta^2 \approx 1 + 6\eta + \frac{60}{f}\eta^2. \tag{57}$$

Here, the approximations result from replacing the factor g defined in equation (51) by the constant 1. Note that the final term in c_4 indicates the first serious divergence from the theory that $V_{\text{nmo}}^2(p)$ depends only on the parameters $V_{\text{nmo}}^2(0)$ and η. However, this term is multiplied by both p^2 and η^2, which ameliorates the effect of replacing the f in this term by, say, 3/4 instead of the true value.

Elliptic anisotropy

Inserting the elliptic P-wave phase velocity as a function of ray parameter given in equation (22) into the general NMO equation (42) gives at once

$$V_{\text{nmo}}^2(p) = \frac{c_P^2(1 + 2\delta)}{(1 - (1 + 2\delta)c_P^2 p^2)}. \tag{58}$$

Recognizing the quantity $V_{\text{nmo}}(0)$ from equation (46) gives

$$V_{\text{nmo}}(p) = \frac{V_{\text{nmo}}(0)}{\sqrt{1 - p^2 V_{\text{nmo}}^2(0)}}, \qquad (\delta = \epsilon), \tag{59}$$

in agreement with the result in (Alkhalifah & Tsvankin, 1995). For future use, introduce the notation,

$$V_{\text{ell}}(p) \equiv \frac{V_{\text{nmo}}(0)}{\sqrt{1 - p^2 V_{\text{nmo}}^2(0)}}, \tag{60}$$

for the elliptic result.

Weak transverse isotropy

Using equation (31) in equation (42) gives

$$V_{\text{nmo}}^2(p) = \frac{c_P^2}{1 - z}\left(1 + \frac{2(\delta + (\epsilon - \delta)\, z(6 - 9z + 4z^2))}{1 - z}\right), \tag{61}$$

where we have again used the shorthand notation, $z = (c_P p)^2$. One would have to seek a more sophisticated expansion if p became large enough to approach $1/c_P$. On the other hand, always $p < 1/V$ and in the weak limit, $V \approx c_P$, so this is an unusual circumstance.

The approximation, $\eta = \epsilon - \delta$, is valid in the weak limit, so equation (61) may be recast as

$$V_{\text{nmo}}^2(p) = \frac{c_P^2}{1 - z}\left(1 + \frac{2\delta}{1 - z} + 2\eta\, F(z)\right). \tag{62}$$

with

$$F(z) = \frac{z(6 - 9z + 4z^2)}{1 - z}. \tag{63}$$

Apparently, we have a disappointing dependence on δ in addition to that on $V_{\text{nmo}}(0)$ and η. However, since the equation (59) in the exact elliptic case does not depend on δ, we are encouraged to look deeper. Indeed, on introducing

$$y = (V_{\text{nmo}}(0)\, p)^2 = (c_P\, p)^2 (1 + 2\delta) = z(1 + 2\delta), \tag{64}$$

extracting the elliptic result in the notation of equation (60), and again ignoring quadratic terms in the anisotropy parameters, one obtains the expression,

$$V_{\text{nmo}}^2(p) = V_{\text{ell}}^2(p)\,(1 + 2\eta\, F(y)), \qquad V_{\text{ell}}(p) = \frac{V_{\text{nmo}}(0)}{\sqrt{1 - y}}. \tag{65}$$

in which δ does not appear and which is in agreement with the corresponding equation in (Alkhalifah & Tsvankin). This last equation also implies the weak limit estimate,

$$\eta \approx \frac{1}{2F(y)}\left(\frac{V_{\text{nmo}}^2(p)}{V_{\text{ell}}^2(p)} - 1\right). \tag{66}$$

At the next order in the anisotropy parameters,

$$V^2_{\rm nmo}(p) = V^2_{\rm ell}(p)\,(1 + 2\eta\,F(y) + \frac{4\eta y}{f(1-y)^2}R(\delta,\eta,y)), \tag{67}$$

where,

$$\begin{aligned} R(\delta,\eta,y) \;=\; & 6\delta(1-f)(1-y)^3(1-2y) \\ & +\eta y(15-(69-26f)y+(117-68f)y^2 \\ & \qquad -3(29-21f)y^3+4(6-5f)y^4). \end{aligned} \tag{68}$$

Observe that, as the second order small-ray-parameter expansion, the higher order term here does introduce δ and f in violation of the Alkhalifah-Tsvankin theory. However, observe first the consistency check, that in the elliptic limit, the higher order vanishes entirely because of an overall factor of η that appears in it. Second, notice that in the higher order term, the δ's are always multiplied by $1-f$, which somewhat mitigates their contribution. Indeed, in the common approximation of ignoring shear speed contributions by taking $f=1$, the δ terms drop out completely along with the f contributions. Finally, observe that the function $y(1-2y)(1-y)$ multiplying the δ term has absolute maximum value less than 0.1 on the interval $0 \le y \le 1$, again mitigating the effect of δ on $V^2_{\rm nmo}(p)$ in this expansion. The overall observation that using, say, $f = 3/4$, instead of the true value has little numerical effect remains true here.

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Phase and Moveout Velocity as Functions of the Ray Parameter for Transversely Isotropic Media

Jack K. Cohen

Center for Wave Phenomena
Colorado School of Mines
Golden, Colorado 80401
303/273-3557

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Fall, 1995

ABSTRACT

In their studies of transversely isotropic media, I. Tsvankin and his co-workers have shown the benefits of expressing the P-wave normal moveout velocity in terms of the zero-offset ray parameter instead of the phase angle, but left for future work the actual task of doing this. Here, explicit formulas are given for both the P-wave phase velocity and normal moveout velocity in terms of the ray parameter. Various results of Tsvankin and others are rederived and, in some cases, extended, in a uniform manner using these explicit results.

The small ray-parameter expansion of the moveout velocity is valid for dips up to approximately 15°. In this regime, an analytic solution for the anisotropy parameter η, important for time-related imaging is given in terms of observations of moveout velocity for two different values of the ray parameter.

INTRODUCTION

In their studies of transversely isotropic media, I. Tsvankin and his co-workers have shown the benefits of expressing the P-wave normal moveout velocity in terms of the zero-offset ray parameter instead of the phase angle. Here, explicit formulas are given for both the P-wave phase velocity and normal moveout velocity in terms of the ray parameter. Various consequences of these results are drawn, including an analytic solution for the anisotropy parameter η, important for time-related imaging is given in terms of observations of moveout velocity for two different values of the ray parameter.

Throughout this study, *Mathematica* was used extensively to derive and check results. In particular, the *Mathematica* package, **Thomsen.m** (Cohen, 1995), was used to convert equation (3) to Thomsen notation, in equation (5), and to obtain various results in the limit of weak transverse isotropy. Similarly, the use of *Mathematica* facilitated computing symbolic derivatives, series expansions, etc.

PHASE VELOCITY AS A FUNCTION OF RAY PARAMETER

Begin with the formula for P-wave phase velocity in terms of dip θ (Tsvankin, 1994), equation(6):

$$
\begin{aligned}
2\rho V^2(\theta) = {} & (C_{11} + C_{44}) \sin^2\theta + (C_{33} + C_{44}) \cos^2\theta \\
& + \Big\{ \big[(C_{11} - C_{44}) \sin^2\theta - (C_{33} - C_{44}) \cos^2\theta \big]^2 \\
& \qquad + 4(C_{13} + C_{44})^2 \sin^2\theta \cos^2\theta \Big\}^{1/2} .
\end{aligned}
\tag{1}
$$

Use the substitutions

$$
p = \sin\theta / V(\theta), \quad m = \cos\theta / V(\theta), \tag{2}
$$

to rewrite equation (1) as an equation for the slowness surface:

$$
\begin{aligned}
2\rho = {} & (C_{11} + C_{44}) p^2 + (C_{33} + C_{44}) m^2 \\
& + \Big\{ \big[(C_{11} - C_{44}) p^2 - (C_{33} - C_{44}) m^2 \big]^2 \\
& \qquad + 4(C_{13} + C_{44})^2 p^2 m^2 \Big\}^{1/2} .
\end{aligned}
\tag{3}
$$

To obtain a formula for $V(p)$, follow the recipe given in (Alkhalifah & Tsvankin, 1995), Appendix A. Begin by converting equation (3) to Thomsen notation. After introducing the on-axis P and S velocities, c_P and c_S, and the related quantities,

$$
k = \frac{1}{c_P^2}, \qquad f = 1 - \frac{c_S^2}{c_P^2}, \tag{4}
$$

obtain the Thomsen-notation form of the slowness surface as

$$
2k = (2 - f) m^2 + (2 + 2\epsilon - f) p^2 + \sqrt{4f(2\delta + f) m^2 p^2 + (f(p^2 - m^2) + 2\epsilon p^2)^2}. \tag{5}
$$

Next, solve equation (5) for m^2 and, from equation (2), form

$$
V(p) = \frac{1}{\sqrt{p^2 + m^2(p)}}. \tag{6}
$$

After some manipulations, find that $V^2(p)$ can be written as

$$
V^2(p) = \frac{A + \sqrt{B}}{2C}, \tag{7}
$$

where,

$$
\begin{aligned}
A &= (2 - f)k - 2(\epsilon - f\delta) p^2, && (8) \\
B &= f^2 k^2 - 4fk \left[\epsilon - (2 - f)\delta \right] p^2 + 4 \left[2f(1 - f)(\epsilon - \delta) + (\epsilon - f\delta)^2 \right] p^4, && (9) \\
C &= k^2 - 2k\epsilon p^2 - 2f(\epsilon - \delta) p^4. && (10)
\end{aligned}
$$

Before turning to some important special cases, observe the following consequences of the definitions in equations (2):

$$\sin\theta = pV(p), \quad \cos\theta = mV(p) = \sqrt{1-(pV)^2}, \quad p^2+m^2 = \frac{1}{V^2}. \tag{11}$$

These equations will be of great convenience in translating between representations in phase angle and representations in the ray parameter.

Small ray parameter

First observe that for $p = 0$,

$$\begin{aligned} A &= (2-f)k, && (12)\\ B &= f^2k^2, && (13)\\ C &= k^2, && (14) \end{aligned}$$

so that,

$$V^2(0) = \frac{(2-f)k + fk}{2k^2} = \frac{2k}{2k^2} = \frac{1}{k} = c_P^2, \tag{15}$$

leading to the expected result,

$$V(0) = c_P. \tag{16}$$

For later purposes, we will need to know the more detailed behavior of $V(p)$ for small p. Introducing the dimensionless parameter,

$$z = (c_P p)^2, \tag{17}$$

obtain

$$\begin{aligned} V(p) \;=\; c_P\Big[1 &+ \delta z + \Big((\epsilon-\delta)(1+2\delta/f) + 3\delta^2/2\Big)z^2 \\ &+ \Big((\epsilon-\delta)(1+2\delta/f)(5\delta + 2(\epsilon-2\delta)/f) + 5\delta^3/2\Big)z^3 \qquad (18)\\ &+ O(z^4)\Big]. \end{aligned}$$

Remark: This expansion shows that the true meaning of "small p" is that $z = (c_P p)^2 << 1$. Since $V \approx c_P$, the inequality can be written as $\sin^2\theta << 1$, so in dimensionless terms, "small p" represents a small dip angle θ.

Elliptic anisotropy

Next study the elliptic case ($\delta = \epsilon$), where

$$\begin{aligned} A &= (2-f)k - 2(1-f)\delta p^2, && (19)\\ B &= [fk + 2(1-f)\delta p^2]^2, && (20)\\ C &= k^2 - 2k\delta p^2, && (21) \end{aligned}$$

so that,

$$V^2(p) = \frac{1}{k - 2\delta p^2} = \frac{c_P^2}{1 - 2\delta c_P^2 p^2}, \qquad (\delta = \epsilon). \tag{22}$$

Tsvankin (1995) gives the corresponding elliptic limit value of V^2 in terms of the phase angle, θ, as

$$V^2(\theta) = V_0^2 \cos^2\theta + V_{90}^2 \sin^2\theta, \tag{23}$$

where $V_0 = V(0)$ and $V_{90} = V(\pi/2)$ in the phase angle form of V implied by equation (1). These values are readily found from equation (5). When $p = 0$, then $m = 1/V(0)$, and the equation reduces to $2/c_P^2 = 2/V^2(0)$, so that

$$V_0 = c_P. \tag{24}$$

Similarly, when $m = 0$, then $p = 1/V_{90}$, giving $2/c_P^2 = 2(1 + \delta + \epsilon)/V_{90}$, so that

$$V_{90} = c_P\sqrt{1 + \delta + \epsilon}, \tag{25}$$

which reduces to

$$V_{90} = c_P\sqrt{1 + 2\delta} \qquad (\delta = \epsilon) \tag{26}$$

in the elliptic limit. This also confirms the known result that $V_{\mathrm{nmo}}(0) = V_{90}$ for elliptically anisotropic media.

With these results in hand, equation (23) becomes

$$V^2 = c_P^2 + 2\delta c_P^2 \sin^2\theta = c_P^2 + 2\delta c_P^2 p^2 V^2 \qquad (\delta = \epsilon). \tag{27}$$

On isolating V^2, equation (22) is verified.

Tsvankin (1995) likewise gives the form

$$V(\theta) = c_P\sqrt{1 + 2\delta \sin^2\theta} \qquad (\delta = \epsilon) \tag{28}$$

for the elliptic case. Introducing the ray parameter in that expression gives

$$V^2 = c_P^2(1 + 2\delta p^2 V^2) \tag{29}$$

and, once again, isolating V^2 verifies equation (22).

Weak transverse isotropy

The weak TI limit of equation (7) is

$$V^2(p) \approx c_P^2\left[1 + 2(\delta + (\epsilon - \delta)c_P^2 p^2)c_P^2 p^2\right], \tag{30}$$

or

$$V(p) \approx c_P\left[1 + (\delta + (\epsilon - \delta)c_P^2 p^2)c_P^2 p^2\right]. \tag{31}$$

Thomsen's (1986) expression for this quantity expressed as a function of angle is

$$V(\theta) = c_P \left[1 + (\delta \cos^2\theta + \epsilon \sin^2\theta)\sin^2\theta\right]. \tag{32}$$

Tsvankin (1995) shows that this equation implies

$$\sin^2\theta = \frac{p^2 c_P^2}{1 - 2\delta p^2 c_P^2}[1 + 2(\epsilon - \delta)p^4 c_P^4], \tag{33}$$

which, in turn, in the weak limit is equivalent to

$$p^2 V^2 = p^2 c_P^2 [1 + 2\delta p^2 c_P^2 + 2(\epsilon - \delta)p^4 c_P^4]. \tag{34}$$

On cancelling the common factor of p^2, this verifies equation (30) and hence also equation (31).

NORMAL MOVEOUT VELOCITY AS A FUNCTION OF RAY PARAMETER

The derivation of the normal moveout velocity as a function of ray parameter, follows closely the corresponding derivation of its expression in terms of the ray angle given in (Tsvankin, 1995). For convenience, some explicit references are made to equations in this paper.

First, use equations (11) to find the following relation for $dp/d\theta$:

$$mV = \cos\theta = \frac{d\sin\theta}{d\theta} = (pV)' \frac{dp}{d\theta}, \tag{35}$$

where, here and below, the prime notation is used for p-differentiation. Thus, we find that

$$\frac{dV}{d\theta} = V' \frac{dp}{d\theta} = \frac{mVV'}{(pV)'}. \tag{36}$$

The derivation of $V_{\mathsf{nmo}}(p)$ begins with Tsvankin's equation (4),

$$V_{\mathsf{nmo}}^2(p) = \frac{2z_0}{t_0} \lim_{h\to 0} \frac{d\tan\psi}{dp}. \tag{37}$$

The second component of this expression is evaluated using Tsvankin's equation (6), equations (11), and equation (36):

$$\begin{aligned} \tan\psi &= \frac{V\sin\theta + \frac{dV}{d\theta}\cos\theta}{V\cos\theta - \frac{dV}{d\theta}\sin\theta} \\ &= \frac{pV^2 + mV\frac{dV}{d\theta}}{mV^2 - pV\frac{dV}{d\theta}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{pV + m\frac{dV}{d\theta}}{mV - p\frac{dV}{d\theta}} \\
&= \frac{pV(pV)' + m^2VV'}{mV(pV)' - mpVV'} \\
&= \frac{p(pV)' + m^2V'}{m\left[(pV)' - pV'\right]} \\
&= \frac{p^2V' + m^2V' + pV}{mV} \\
&= \frac{V'/V^2 + pV}{mV} \\
&= \frac{V' + pV^3}{mV^3}.
\end{aligned} \tag{38}
$$

The first component of equation (37) is similarly evaluated using Tsvankin's equation (8):

$$
\begin{aligned}
\frac{2z_0}{t_0} &= V\cos\theta\left(1 - \frac{\tan\theta}{V}\frac{dV}{d\theta}\right) \\
&= mV^2\left(1 - \frac{p}{mV}\frac{mVV'}{(pV)'}\right) \\
&= mV^2\left(1 - \frac{pV'}{(pV)'}\right) \\
&= mV^2\frac{V + pV' - pV'}{(pV)'} \\
&= \frac{mV^3}{(pV)'}.
\end{aligned} \tag{39}
$$

Using the previous two results in equation (37), find that

$$
V_{\text{nmo}}^2(p) = \frac{mV^3}{(pV)'}\frac{d}{dp}\left(\frac{V' + pV^3}{mV^3}\right) \tag{40}
$$

or, on eliminating m,

$$
V_{\text{nmo}}^2(p) = \frac{V^2\sqrt{1-(pV)^2}}{(pV)'}\frac{d}{dp}\left(\frac{V' + pV^3}{V^2\sqrt{1-(pV)^2}}\right). \tag{41}
$$

Carrying out the indicated derivative gives an explicit formula for the normal moveout velocity as a function of ray parameter:

$$
V_{\text{nmo}}^2(p) = \frac{2(1 - p^2V^2)VV'' + (3p^2V^2 - 2)V'^2 + 2pV^3V' + V^4}{(1 - p^2V^2)V(pV)'} \tag{42}
$$

Before turning to the special cases for V, note that when it is necessary to treat the general V, it may be better to avoid a layer of square roots by writing $V^2_{\mathrm{nmo}}(p)$ in terms of $W \equiv V^2$:

$$V^2_{\mathrm{nmo}}(p) = \frac{2(1-p^2W)WW'' + (4p^2W-3)W'^2 + 4pW^2W' + 4W^3}{2(1-p^2W)W(pW'+2W)} \tag{43}$$

Small ray parameter

First, use the two leading terms of equation (18) to obtain the approximations,

$$V = c_P + c_P^3\delta p^2 + O(p^4), \quad V' = 2c_P^3\delta p + O(p^3), \quad V'' = 2c_P^3\delta + O(p^2). \tag{44}$$

On inserting these small-p approximations into equation (42), find that

$$V^2_{\mathrm{nmo}}(p) = c_P^2(1+2\delta) + O(p^2), \tag{45}$$

so

$$V_{\mathrm{nmo}}(0) = c_P\sqrt{1+2\delta}. \tag{46}$$

Now use the next order terms of equation (18) in equation (42) to get the next term in $V^2_{\mathrm{nmo}}(p)$:

$$V^2_{\mathrm{nmo}}(p) = c_P^2\left(1+2\delta + \left((-24\delta^2+24\delta\epsilon+f-8\delta f+4\delta^2 f+12\epsilon f)\frac{(c_Pp)^2}{f}\right) + O(c_Pp)^4\right). \tag{47}$$

Remark: Again, the expansion shows that the true meaning of "small p" is that $(c_Pp)^2 << 1$. Since $V \approx c_P$, and the error is fourth order, this means we can interpret "small p" as meaning $\sin^4\theta << 1$. Numerical tests show that using the small p series for a dip of 15° incurs about a 2% error for typical values of f and δ.

The theory discussed by (Alkhalifah & Tsvankin, 1995) suggests that it is better to express this result in terms of the parameters $V_{\mathrm{nmo}}(0)$ and

$$\eta = \frac{\epsilon-\delta}{1+2\delta}. \tag{48}$$

First introduce η,

$$V^2_{\mathrm{nmo}}(p) \approx c_P^2(1+2\delta) + \left(c_P^4(1+2\delta)(24\delta\eta+f+2\delta f+12\eta f)p^2\right)/f + O(p^4), \tag{49}$$

then use equation (46) to eliminate the explicit appearances of c_P^2 in favor of $V^2_{\mathrm{nmo}}(0)$:

$$V^2_{\mathrm{nmo}}(p) = V^2_{\mathrm{nmo}}(0)\left[\left(1+(1+12\eta\frac{1+2\delta/f}{1+2\delta})V^2_{\mathrm{nmo}}(0)p^2\right) + O(V^4_{\mathrm{nmo}}(0)p^4)\right]. \tag{50}$$

The importance of the anisotropic parameter η, is that Alkhalifah and Tsvankin (1995) have observed, both numerically and empirically, that $V^2_{\mathrm{nmo}}(p)$ depends mainly

on η and $V^2_{\text{nmo}}(0)$. They exploit this reduction to surface-observable parameters to develop time-domain seismic processing algorithms that take account of transverse anisotropy. Equation (50) gives analytic support to the Alkhalifah-Tsvankin theory, since the only deviation to it occurs in the ratio

$$g = \frac{1 + 2\delta/f}{1 + 2\delta} \tag{51}$$

that multiplies η. Figures (1) and (2) show plots of this function over the ranges of f and δ that are relevant in practice. Observe that the function $g(\delta, f)$ varies slowly over these ranges—indeed—ignoring it altogether (i.e., replacing it by the constant 1) is usually justified—at least in the small-angle approximation.

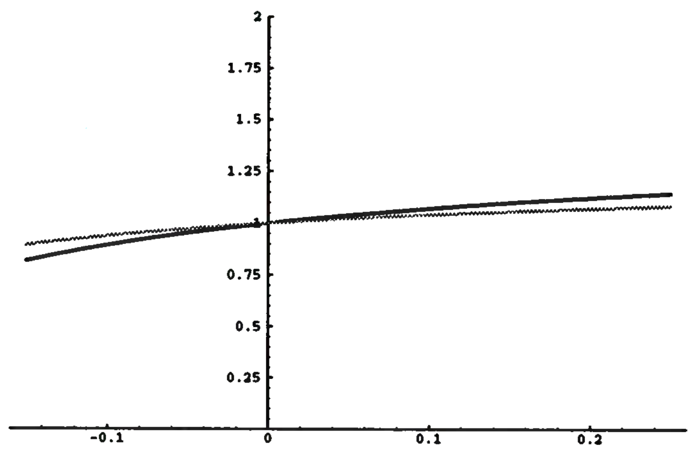

FIG. 1. Plot of the factor $g = (1 + 2\delta/f)/(1 + 2\delta)$ as a function of δ for $f = 0.7$ (dark) and $f = 0.8$ (light).

The small-ray-parameter result in equation (50) gives a means for estimating η from surface observations. For example, suppose one has observations of $V_{\text{nmo}}(p)$ at $p = 0$ and some other (not too large) value $p = p_1$. The solution for η is given by

$$\eta \approx \frac{1}{12g}\left(\frac{V^2_{\text{nmo}}(p_1) - V^2_{\text{nmo}}(0)}{p_1^2 V^4_{\text{nmo}}(0)} - 1\right), \tag{52}$$

where once again, in the absence of information on δ and f, the factor g can be replaced by the constant 1 without much error.

More generally, if one uses two nonzero "small p" values (that is, two separated dips, each less than 15°), then the estimate for $V^2_{\text{nmo}}(0)$ is given by

$$V^2_{\text{nmo}}(0) = \frac{p_2^2 V^2_{\text{nmo}}(p_1) - p_1^2 V^2_{\text{nmo}}(p_2)}{p_2^2 - p_1^2}, \tag{53}$$

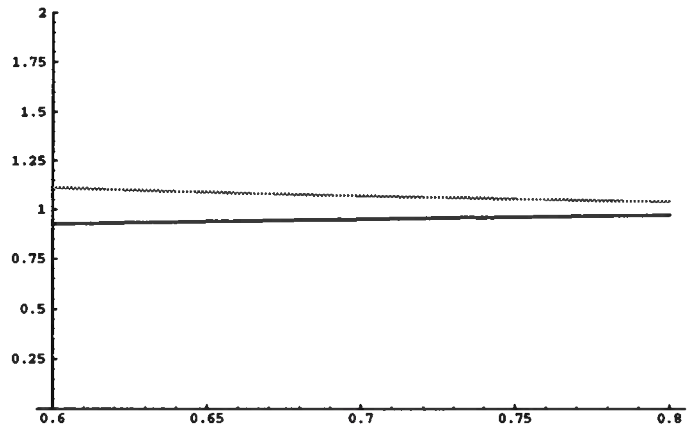

FIG. 2. Plot of the factor $g = (1 + 2\delta/f)/(1 + 2\delta)$ as a function of f for $\delta = -0.05$ (dark) and $\delta = +0.1$ (light).

and the estimate for η is

$$\eta \approx \frac{1}{12g}\left(\frac{(p_1^2 - p_2^2)(V_{\mathrm{nmo}}^2(p_2) - V_{\mathrm{nmo}}^2(p_1))}{(p_1^2 V_{\mathrm{nmo}}^2(p_2) - p_2^2 V_{\mathrm{nmo}}^2(p_1))^2} - 1\right). \tag{54}$$

Finally, note that the full form of the series for $V_{\mathrm{nmo}}^2(p)$ is given by

$$V_{\mathrm{nmo}}^2(p) = V_{\mathrm{nmo}}^2(0)\left[1 + c_2 V_{\mathrm{nmo}}^2(0)p^2 + c_4 V_{\mathrm{nmo}}^4(0)p^4 + \cdots\right], \tag{55}$$

where, as we have seen earlier,

$$c_2 = 1 + 12g\eta \approx 1 + 12\eta. \tag{56}$$

By using the full form of equation (18), we find that

$$c_4 = 1 + 6g(6 - 5g)\eta + \frac{60g}{f}\eta^2 \approx 1 + 6\eta + \frac{60}{f}\eta^2. \tag{57}$$

Here, the approximations result from replacing the factor g defined in equation (51) by the constant 1. Note that the final term in c_4 indicates the first serious divergence from the theory that $V_{\mathrm{nmo}}^2(p)$ depends only on the parameters $V_{\mathrm{nmo}}^2(0)$ and η. However, this term is multiplied by both p^2 and η^2, which ameliorates the effect of replacing the f in this term by, say, $3/4$ instead of the true value.

Elliptic anisotropy

Inserting the elliptic P-wave phase velocity as a function of ray parameter given in equation (22) into the general NMO equation (42) gives at once

$$V_{\mathrm{nmo}}^2(p) = \frac{c_P^2(1 + 2\delta)}{(1 - (1 + 2\delta)c_P^2 p^2)}. \tag{58}$$

Recognizing the quantity $V_{\rm nmo}(0)$ from equation (46) gives

$$V_{\rm nmo}(p) = \frac{V_{\rm nmo}(0)}{\sqrt{1 - p^2 V_{\rm nmo}^2(0)}}, \qquad (\delta = \epsilon), \tag{59}$$

in agreement with the result in (Alkhalifah & Tsvankin, 1995). For future use, introduce the notation,

$$V_{\rm ell}(p) \equiv \frac{V_{\rm nmo}(0)}{\sqrt{1 - p^2 V_{\rm nmo}^2(0)}}, \tag{60}$$

for the elliptic result.

Weak transverse isotropy

Using equation (31) in equation (42) gives

$$V_{\rm nmo}^2(p) = \frac{c_P^2}{1-z}\left(1 + \frac{2(\delta + (\epsilon - \delta)\, z(6 - 9z + 4z^2))}{1-z}\right), \tag{61}$$

where we have again used the shorthand notation, $z = (c_P p)^2$. One would have to seek a more sophisticated expansion if p became large enough to approach $1/c_P$. On the other hand, always $p < 1/V$ and in the weak limit, $V \approx c_P$, so this is an unusual circumstance.

The approximation, $\eta = \epsilon - \delta$, is valid in the weak limit, so equation (61) may be recast as

$$V_{\rm nmo}^2(p) = \frac{c_P^2}{1-z}\left(1 + \frac{2\delta}{1-z} + 2\eta\, F(z)\right). \tag{62}$$

with

$$F(z) = \frac{z(6 - 9z + 4z^2)}{1-z}. \tag{63}$$

Apparently, we have a disappointing dependence on δ in addition to that on $V_{\rm nmo}(0)$ and η. However, since the equation (59) in the exact elliptic case does not depend on δ, we are encouraged to look deeper. Indeed, on introducing

$$y = (V_{\rm nmo}(0)\, p)^2 = (c_P\, p)^2 (1 + 2\delta) = z(1 + 2\delta), \tag{64}$$

extracting the elliptic result in the notation of equation (60), and again ignoring quadratic terms in the anisotropy parameters, one obtains the expression,

$$V_{\rm nmo}^2(p) = V_{\rm ell}^2(p)\,(1 + 2\eta\, F(y)), \qquad V_{\rm ell}(p) = \frac{V_{\rm nmo}(0)}{\sqrt{1-y}}. \tag{65}$$

in which δ does not appear and which is in agreement with the corresponding equation in (Alkhalifah & Tsvankin). This last equation also implies the weak limit estimate,

$$\eta \approx \frac{1}{2F(y)}\left(\frac{V_{\rm nmo}^2(p)}{V_{\rm ell}^2(p)} - 1\right). \tag{66}$$

At the next order in the anisotropy parameters,

$$V^2_{\mathrm{nmo}}(p) = V^2_{\mathrm{ell}}(p)\,(1 + 2\eta\, F(y) + \frac{4\eta y}{f(1-y)^2} R(\delta,\eta,y)), \tag{67}$$

where,

$$\begin{aligned} R(\delta,\eta,y) \;=\;& 6\delta(1-f)(1-y)^3(1-2y) \\ & +\eta y(15-(69-26f)y+(117-68f)y^2 \\ & \qquad -3(29-21f)y^3+4(6-5f)y^4). \end{aligned} \tag{68}$$

Observe that, as the second order small-ray-parameter expansion, the higher order term here does introduce δ and f in violation of the Alkhalifah-Tsvankin theory. However, observe first the consistency check, that in the elliptic limit, the higher order vanishes entirely because of an overall factor of η that appears in it. Second, notice that in the higher order term, the δ's are always multiplied by $1-f$, which somewhat mitigates their contribution. Indeed, in the common approximation of ignoring shear speed contributions by taking $f=1$, the δ terms drop out completely along with the f contributions. Finally, observe that the function $y(1-2y)(1-y)$ multiplying the δ term has absolute maximum value less than 0.1 on the interval $0 \le y \le 1$, again mitigating the effect of δ on $V^2_{\mathrm{nmo}}(p)$ in this expansion. The overall observation that using, say, $f = 3/4$, instead of the true value has little numerical effect remains true here.

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Phase and Moveout Velocity as Functions of the Ray Parameter for Transversely Isotropic Media

Jack K. Cohen

Center for Wave Phenomena
Colorado School of Mines
Golden, Colorado 80401
303/273-3557

Phase and Moveout Velocity as Functions of the Ray Parameter for Transversely Isotropic Media

Jack K. Cohen

Fall, 1995

ABSTRACT

In their studies of transversely isotropic media, I. Tsvankin and his co-workers have shown the benefits of expressing the P-wave normal moveout velocity in terms of the zero-offset ray parameter instead of the phase angle, but left for future work the actual task of doing this. Here, explicit formulas are given for both the P-wave phase velocity and normal moveout velocity in terms of the ray parameter. Various results of Tsvankin and others are rederived and, in some cases, extended, in a uniform manner using these explicit results.

The small ray-parameter expansion of the moveout velocity is valid for dips up to approximately 15°. In this regime, an analytic solution for the anisotropy parameter η, important for time-related imaging is given in terms of observations of moveout velocity for two different values of the ray parameter.

INTRODUCTION

In their studies of transversely isotropic media, I. Tsvankin and his co-workers have shown the benefits of expressing the P-wave normal moveout velocity in terms of the zero-offset ray parameter instead of the phase angle. Here, explicit formulas are given for both the P-wave phase velocity and normal moveout velocity in terms of the ray parameter. Various consequences of these results are drawn, including an analytic solution for the anisotropy parameter η, important for time-related imaging is given in terms of observations of moveout velocity for two different values of the ray parameter.

Throughout this study, *Mathematica* was used extensively to derive and check results. In particular, the *Mathematica* package, **Thomsen.m** (Cohen, 1995), was used to convert equation (3) to Thomsen notation, in equation (5), and to obtain various results in the limit of weak transverse isotropy. Similarly, the use of *Mathematica* facilitated computing symbolic derivatives, series expansions, etc.

PHASE VELOCITY AS A FUNCTION OF RAY PARAMETER

Begin with the formula for P-wave phase velocity in terms of dip θ (Tsvankin, 1994), equation(6):

$$\begin{aligned} 2\rho V^2(\theta) = {} & (C_{11}+C_{44})\sin^2\theta + (C_{33}+C_{44})\cos^2\theta \\ & + \Big\{ \big[(C_{11}-C_{44})\sin^2\theta - (C_{33}-C_{44})\cos^2\theta\big]^2 \\ & \qquad + 4(C_{13}+C_{44})^2 \sin^2\theta\cos^2\theta \Big\}^{1/2}. \end{aligned} \tag{1}$$

Use the substitutions

$$p = \sin\theta / V(\theta), \qquad m = \cos\theta / V(\theta), \tag{2}$$

to rewrite equation (1) as an equation for the slowness surface:

$$\begin{aligned} 2\rho = {} & (C_{11}+C_{44})p^2 + (C_{33}+C_{44})m^2 \\ & + \Big\{ \big[(C_{11}-C_{44})p^2 - (C_{33}-C_{44})m^2\big]^2 \\ & \qquad + 4(C_{13}+C_{44})^2 p^2 m^2 \Big\}^{1/2}. \end{aligned} \tag{3}$$

To obtain a formula for $V(p)$, follow the recipe given in (Alkhalifah & Tsvankin, 1995), Appendix A. Begin by converting equation (3) to Thomsen notation. After introducing the on-axis P and S velocities, c_P and c_S, and the related quantities,

$$k = \frac{1}{c_P^2}, \qquad f = 1 - \frac{c_S^2}{c_P^2}, \tag{4}$$

obtain the Thomsen-notation form of the slowness surface as

$$2k = (2-f)m^2 + (2+2\epsilon-f)p^2 + \sqrt{4f(2\delta+f)m^2p^2 + (f(p^2-m^2)+2\epsilon p^2)^2}. \tag{5}$$

Next, solve equation (5) for m^2 and, from equation (2), form

$$V(p) = \frac{1}{\sqrt{p^2+m^2(p)}}. \tag{6}$$

After some manipulations, find that $V^2(p)$ can be written as

$$V^2(p) = \frac{A+\sqrt{B}}{2C}, \tag{7}$$

where,

$$\begin{aligned} A &= (2-f)k - 2(\epsilon - f\delta)p^2, & (8)\\ B &= f^2k^2 - 4fk\left[\epsilon - (2-f)\delta\right]p^2 + 4\left[2f(1-f)(\epsilon-\delta) + (\epsilon - f\delta)^2\right]p^4, & (9)\\ C &= k^2 - 2k\epsilon p^2 - 2f(\epsilon-\delta)p^4. & (10) \end{aligned}$$

Before turning to some important special cases, observe the following consequences of the definitions in equations (2):

$$\sin\theta = pV(p), \quad \cos\theta = mV(p) = \sqrt{1-(pV)^2}, \quad p^2+m^2 = \frac{1}{V^2}. \tag{11}$$

These equations will be of great convenience in translating between representations in phase angle and representations in the ray parameter.

Small ray parameter

First observe that for $p = 0$,

$$\begin{aligned} A &= (2-f)k, & (12)\\ B &= f^2k^2, & (13)\\ C &= k^2, & (14) \end{aligned}$$

so that,

$$V^2(0) = \frac{(2-f)k + fk}{2k^2} = \frac{2k}{2k^2} = \frac{1}{k} = c_P^2, \tag{15}$$

leading to the expected result,

$$V(0) = c_P. \tag{16}$$

For later purposes, we will need to know the more detailed behavior of $V(p)$ for small p. Introducing the dimensionless parameter,

$$z = (c_P p)^2, \tag{17}$$

obtain

$$\begin{aligned} V(p) \;=\; c_P\Big[1 &+ \delta z + \big((\epsilon-\delta)(1+2\delta/f) + 3\delta^2/2\big)\, z^2 \\ &+ \big((\epsilon-\delta)(1+2\delta/f)(5\delta + 2(\epsilon - 2\delta)/f) + 5\delta^3/2\big)\, z^3 \\ &+ O(z^4)\Big]. \end{aligned} \tag{18}$$

Remark: This expansion shows that the true meaning of "small p" is that $z = (c_P p)^2 << 1$. Since $V \approx c_P$, the inequality can be written as $\sin^2\theta << 1$, so in dimensionless terms, "small p" represents a small dip angle θ.

Elliptic anisotropy

Next study the elliptic case ($\delta = \epsilon$), where

$$\begin{aligned} A &= (2-f)k - 2(1-f)\delta p^2, & (19)\\ B &= [fk + 2(1-f)\delta p^2]^2, & (20)\\ C &= k^2 - 2k\delta p^2, & (21) \end{aligned}$$

so that,

$$V^2(p) = \frac{1}{k - 2\delta p^2} = \frac{c_P^2}{1 - 2\delta c_P^2 p^2}, \qquad (\delta = \epsilon). \tag{22}$$

Tsvankin (1995) gives the corresponding elliptic limit value of V^2 in terms of the phase angle, θ, as

$$V^2(\theta) = V_0^2 \cos^2\theta + V_{90}^2 \sin^2\theta, \tag{23}$$

where $V_0 = V(0)$ and $V_{90} = V(\pi/2)$ in the phase angle form of V implied by equation (1). These values are readily found from equation (5). When $p = 0$, then $m = 1/V(0)$, and the equation reduces to $2/c_P^2 = 2/V^2(0)$, so that

$$V_0 = c_P. \tag{24}$$

Similarly, when $m = 0$, then $p = 1/V_{90}$, giving $2/c_P^2 = 2(1 + \delta + \epsilon)/V_{90}$, so that

$$V_{90} = c_P\sqrt{1 + \delta + \epsilon}, \tag{25}$$

which reduces to

$$V_{90} = c_P\sqrt{1 + 2\delta} \qquad (\delta = \epsilon) \tag{26}$$

in the elliptic limit. This also confirms the known result that $V_{\mathrm{nmo}}(0) = V_{90}$ for elliptically anisotropic media.

With these results in hand, equation (23) becomes

$$V^2 = c_P^2 + 2\delta c_P^2 \sin^2\theta = c_P^2 + 2\delta c_P^2 p^2 V^2 \qquad (\delta = \epsilon). \tag{27}$$

On isolating V^2, equation (22) is verified.

Tsvankin (1995) likewise gives the form

$$V(\theta) = c_P\sqrt{1 + 2\delta \sin^2\theta} \qquad (\delta = \epsilon) \tag{28}$$

for the elliptic case. Introducing the ray parameter in that expression gives

$$V^2 = c_P^2(1 + 2\delta p^2 V^2) \tag{29}$$

and, once again, isolating V^2 verifies equation (22).

Weak transverse isotropy

The weak TI limit of equation (7) is

$$V^2(p) \approx c_P^2 \left[1 + 2(\delta + (\epsilon - \delta)c_P^2 p^2)c_P^2 p^2\right], \tag{30}$$

or

$$V(p) \approx c_P \left[1 + (\delta + (\epsilon - \delta)c_P^2 p^2)c_P^2 p^2\right]. \tag{31}$$

Thomsen's (1986) expression for this quantity expressed as a function of angle is

$$V(\theta) = c_P\left[1 + (\delta\cos^2\theta + \epsilon\sin^2\theta)\sin^2\theta\right]. \tag{32}$$

Tsvankin (1995) shows that this equation implies

$$\sin^2\theta = \frac{p^2c_P^2}{1 - 2\delta p^2c_P^2}[1 + 2(\epsilon - \delta)p^4c_P^4], \tag{33}$$

which, in turn, in the weak limit is equivalent to

$$p^2V^2 = p^2c_P^2[1 + 2\delta p^2c_P^2 + 2(\epsilon - \delta)p^4c_P^4]. \tag{34}$$

On cancelling the common factor of p^2, this verifies equation (30) and hence also equation (31).

NORMAL MOVEOUT VELOCITY AS A FUNCTION OF RAY PARAMETER

The derivation of the normal moveout velocity as a function of ray parameter, follows closely the corresponding derivation of its expression in terms of the ray angle given in (Tsvankin, 1995). For convenience, some explicit references are made to equations in this paper.

First, use equations (11) to find the following relation for $dp/d\theta$:

$$mV = \cos\theta = \frac{d\sin\theta}{d\theta} = (pV)'\frac{dp}{d\theta}, \tag{35}$$

where, here and below, the prime notation is used for p-differentiation. Thus, we find that

$$\frac{dV}{d\theta} = V'\frac{dp}{d\theta} = \frac{mVV'}{(pV)'}. \tag{36}$$

The derivation of $V_{\text{nmo}}(p)$ begins with Tsvankin's equation (4),

$$V_{\text{nmo}}^2(p) = \frac{2z_0}{t_0}\lim_{h\to 0}\frac{d\tan\psi}{dp}. \tag{37}$$

The second component of this expression is evaluated using Tsvankin's equation (6), equations (11), and equation (36):

$$\begin{aligned}\tan\psi &= \frac{V\sin\theta + \frac{dV}{d\theta}\cos\theta}{V\cos\theta - \frac{dV}{d\theta}\sin\theta}\\ &= \frac{pV^2 + mV\frac{dV}{d\theta}}{mV^2 - pV\frac{dV}{d\theta}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{pV + m\frac{dV}{d\theta}}{mV - p\frac{dV}{d\theta}} \\
&= \frac{pV(pV)' + m^2VV'}{mV(pV)' - mpVV'} \\
&= \frac{p(pV)' + m^2V'}{m\left[(pV)' - pV'\right]} \\
&= \frac{p^2V' + m^2V' + pV}{mV} \\
&= \frac{V'/V^2 + pV}{mV} \\
&= \frac{V' + pV^3}{mV^3}.
\end{aligned} \tag{38}
$$

The first component of equation (37) is similarly evaluated using Tsvankin's equation (8):

$$
\begin{aligned}
\frac{2z_0}{t_0} &= V\cos\theta\left(1 - \frac{\tan\theta}{V}\frac{dV}{d\theta}\right) \\
&= mV^2\left(1 - \frac{p}{mV}\frac{mVV'}{(pV)'}\right) \\
&= mV^2\left(1 - \frac{pV'}{(pV)'}\right) \\
&= mV^2\frac{V + pV' - pV'}{(pV)'} \\
&= \frac{mV^3}{(pV)'}.
\end{aligned} \tag{39}
$$

Using the previous two results in equation (37), find that

$$
V_{\text{nmo}}^2(p) = \frac{mV^3}{(pV)'}\frac{d}{dp}\left(\frac{V' + pV^3}{mV^3}\right) \tag{40}
$$

or, on eliminating m,

$$
V_{\text{nmo}}^2(p) = \frac{V^2\sqrt{1-(pV)^2}}{(pV)'}\frac{d}{dp}\left(\frac{V' + pV^3}{V^2\sqrt{1-(pV)^2}}\right). \tag{41}
$$

Carrying out the indicated derivative gives an explicit formula for the normal moveout velocity as a function of ray parameter:

$$
V_{\text{nmo}}^2(p) = \frac{2(1-p^2V^2)VV'' + (3p^2V^2 - 2)V'^2 + 2pV^3V' + V^4}{(1-p^2V^2)V(pV)'} \tag{42}
$$

Before turning to the special cases for V, note that when it is necessary to treat the general V, it may be better to avoid a layer of square roots by writing $V^2_{\rm nmo}(p)$ in terms of $W \equiv V^2$:

$$V^2_{\rm nmo}(p) = \frac{2(1-p^2W)WW'' + (4p^2W-3)W'^2 + 4pW^2W' + 4W^3}{2(1-p^2W)W(pW'+2W)} \tag{43}$$

Small ray parameter

First, use the two leading terms of equation (18) to obtain the approximations,

$$V = c_P + c_P^3\delta p^2 + O(p^4), \quad V' = 2c_P^3\delta p + O(p^3), \quad V'' = 2c_P^3\delta + O(p^2). \tag{44}$$

On inserting these small-p approximations into equation (42), find that

$$V^2_{\rm nmo}(p) = c_P^2(1+2\delta) + O(p^2), \tag{45}$$

so

$$V_{\rm nmo}(0) = c_P\sqrt{1+2\delta}. \tag{46}$$

Now use the next order terms of equation (18) in equation (42) to get the next term in $V^2_{\rm nmo}(p)$:

$$V^2_{\rm nmo}(p) = c_P^2\left(1 + 2\delta + \left((-24\delta^2 + 24\delta\epsilon + f - 8\delta f + 4\delta^2 f + 12\epsilon f)\frac{(c_Pp)^2}{f}\right) + O(c_Pp)^4\right). \tag{47}$$

Remark: Again, the expansion shows that the true meaning of "small p" is that $(c_Pp)^2 << 1$. Since $V \approx c_P$, and the error is fourth order, this means we can interpret "small p" as meaning $\sin^4\theta << 1$. Numerical tests show that using the small p series for a dip of 15° incurs about a 2% error for typical values of f and δ.

The theory discussed by (Alkhalifah & Tsvankin, 1995) suggests that it is better to express this result in terms of the parameters $V_{\rm nmo}(0)$ and

$$\eta = \frac{\epsilon - \delta}{1+2\delta}. \tag{48}$$

First introduce η,

$$V^2_{\rm nmo}(p) \approx c_P^2(1+2\delta) + \left(c_P^4(1+2\delta)(24\delta\eta + f + 2\delta f + 12\eta f)p^2\right)/f + O(p^4), \tag{49}$$

then use equation (46) to eliminate the explicit appearances of c_P^2 in favor of $V^2_{\rm nmo}(0)$:

$$V^2_{\rm nmo}(p) = V^2_{\rm nmo}(0)\left[\left(1 + (1 + 12\eta\frac{1+2\delta/f}{1+2\delta})V^2_{\rm nmo}(0)p^2\right) + O(V^4_{\rm nmo}(0)p^4)\right]. \tag{50}$$

The importance of the anisotropic parameter η, is that Alkhalifah and Tsvankin (1995) have observed, both numerically and empirically, that $V^2_{\rm nmo}(p)$ depends mainly

on η and $V^2_{\rm nmo}(0)$. They exploit this reduction to surface-observable parameters to develop time-domain seismic processing algorithms that take account of transverse anisotropy. Equation (50) gives analytic support to the Alkhalifah-Tsvankin theory, since the only deviation to it occurs in the ratio

$$g = \frac{1 + 2\delta/f}{1 + 2\delta} \tag{51}$$

that multiplies η. Figures (1) and (2) show plots of this function over the ranges of f and δ that are relevant in practice. Observe that the function $g(\delta, f)$ varies slowly over these ranges—indeed—ignoring it altogether (i.e., replacing it by the constant 1) is usually justified—at least in the small-angle approximation.

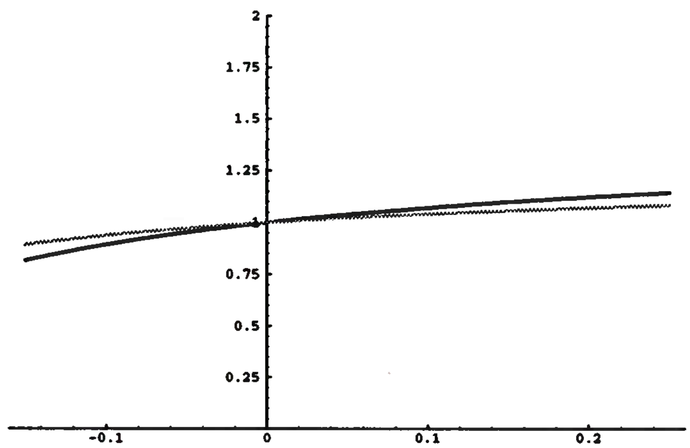

FIG. 1. Plot of the factor $g = (1 + 2\delta/f)/(1 + 2\delta)$ as a function of δ for $f = 0.7$ (dark) and $f = 0.8$ (light).

The small-ray-parameter result in equation (50) gives a means for estimating η from surface observations. For example, suppose one has observations of $V_{\rm nmo}(p)$ at $p = 0$ and some other (not too large) value $p = p_1$. The solution for η is given by

$$\eta \approx \frac{1}{12g}\left(\frac{V^2_{\rm nmo}(p_1) - V^2_{\rm nmo}(0)}{p_1^2 V^4_{\rm nmo}(0)} - 1\right), \tag{52}$$

where once again, in the absence of information on δ and f, the factor g can be replaced by the constant 1 without much error.

More generally, if one uses two nonzero "small p" values (that is, two separated dips, each less than 15°), then the estimate for $V^2_{\rm nmo}(0)$ is given by

$$V^2_{\rm nmo}(0) = \frac{p_2^2 V^2_{\rm nmo}(p_1) - p_1^2 V^2_{\rm nmo}(p_2)}{p_2^2 - p_1^2}, \tag{53}$$

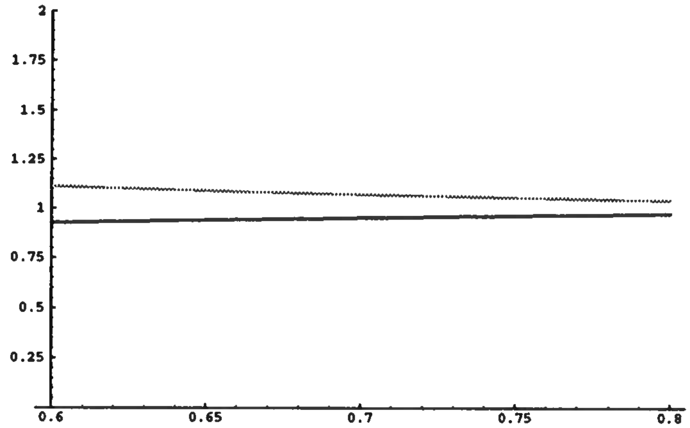

FIG. 2. Plot of the factor $g = (1 + 2\delta/f)/(1 + 2\delta)$ as a function of f for $\delta = -0.05$ (dark) and $\delta = +0.1$ (light).

and the estimate for η is

$$\eta \approx \frac{1}{12g}\left(\frac{(p_1^2 - p_2^2)(V_{\text{nmo}}^2(p_2) - V_{\text{nmo}}^2(p_1))}{(p_1^2 V_{\text{nmo}}^2(p_2) - p_2^2 V_{\text{nmo}}^2(p_1))^2} - 1\right). \tag{54}$$

Finally, note that the full form of the series for $V_{\text{nmo}}^2(p)$ is given by

$$V_{\text{nmo}}^2(p) = V_{\text{nmo}}^2(0)\left[1 + c_2 V_{\text{nmo}}^2(0)p^2 + c_4 V_{\text{nmo}}^4(0)p^4 + \cdots\right], \tag{55}$$

where, as we have seen earlier,

$$c_2 = 1 + 12g\eta \approx 1 + 12\eta. \tag{56}$$

By using the full form of equation (18), we find that

$$c_4 = 1 + 6g(6 - 5g)\eta + \frac{60g}{f}\eta^2 \approx 1 + 6\eta + \frac{60}{f}\eta^2. \tag{57}$$

Here, the approximations result from replacing the factor g defined in equation (51) by the constant 1. Note that the final term in c_4 indicates the first serious divergence from the theory that $V_{\text{nmo}}^2(p)$ depends only on the parameters $V_{\text{nmo}}^2(0)$ and η. However, this term is multiplied by both p^2 and η^2, which ameliorates the effect of replacing the f in this term by, say, 3/4 instead of the true value.

Elliptic anisotropy

Inserting the elliptic P-wave phase velocity as a function of ray parameter given in equation (22) into the general NMO equation (42) gives at once

$$V_{\text{nmo}}^2(p) = \frac{c_P^2(1 + 2\delta)}{(1 - (1 + 2\delta)c_P^2 p^2)}. \tag{58}$$

Recognizing the quantity $V_{\text{nmo}}(0)$ from equation (46) gives

$$V_{\text{nmo}}(p) = \frac{V_{\text{nmo}}(0)}{\sqrt{1 - p^2 V_{\text{nmo}}^2(0)}}, \qquad (\delta = \epsilon), \tag{59}$$

in agreement with the result in (Alkhalifah & Tsvankin, 1995). For future use, introduce the notation,

$$V_{\text{ell}}(p) \equiv \frac{V_{\text{nmo}}(0)}{\sqrt{1 - p^2 V_{\text{nmo}}^2(0)}}, \tag{60}$$

for the elliptic result.

Weak transverse isotropy

Using equation (31) in equation (42) gives

$$V_{\text{nmo}}^2(p) = \frac{c_P^2}{1-z}\left(1 + \frac{2(\delta + (\epsilon - \delta)\, z(6 - 9z + 4z^2))}{1-z}\right), \tag{61}$$

where we have again used the shorthand notation, $z = (c_P p)^2$. One would have to seek a more sophisticated expansion if p became large enough to approach $1/c_P$. On the other hand, always $p < 1/V$ and in the weak limit, $V \approx c_P$, so this is an unusual circumstance.

The approximation, $\eta = \epsilon - \delta$, is valid in the weak limit, so equation (61) may be recast as

$$V_{\text{nmo}}^2(p) = \frac{c_P^2}{1-z}\left(1 + \frac{2\delta}{1-z} + 2\eta\, F(z)\right). \tag{62}$$

with

$$F(z) = \frac{z(6 - 9z + 4z^2)}{1-z}. \tag{63}$$

Apparently, we have a disappointing dependence on δ in addition to that on $V_{\text{nmo}}(0)$ and η. However, since the equation (59) in the exact elliptic case does not depend on δ, we are encouraged to look deeper. Indeed, on introducing

$$y = (V_{\text{nmo}}(0)\, p)^2 = (c_P\, p)^2(1 + 2\delta) = z(1 + 2\delta), \tag{64}$$

extracting the elliptic result in the notation of equation (60), and again ignoring quadratic terms in the anisotropy parameters, one obtains the expression,

$$V_{\text{nmo}}^2(p) = V_{\text{ell}}^2(p)\,(1 + 2\eta\, F(y)), \qquad V_{\text{ell}}(p) = \frac{V_{\text{nmo}}(0)}{\sqrt{1-y}}. \tag{65}$$

in which δ does not appear and which is in agreement with the corresponding equation in (Alkhalifah & Tsvankin). This last equation also implies the weak limit estimate,

$$\eta \approx \frac{1}{2F(y)}\left(\frac{V_{\text{nmo}}^2(p)}{V_{\text{ell}}^2(p)} - 1\right). \tag{66}$$

At the next order in the anisotropy parameters,

$$V_{\rm nmo}^2(p) = V_{\rm ell}^2(p)\,(1 + 2\eta\,F(y) + \frac{4\eta y}{f(1-y)^2}R(\delta,\eta,y)), \tag{67}$$

where,

$$\begin{aligned} R(\delta,\eta,y) &= 6\delta(1-f)(1-y)^3(1-2y) \\ &\quad +\eta y(15-(69-26f)y+(117-68f)y^2 \\ &\qquad -3(29-21f)y^3+4(6-5f)y^4). \end{aligned} \tag{68}$$

Observe that, as the second order small-ray-parameter expansion, the higher order term here does introduce δ and f in violation of the Alkhalifah-Tsvankin theory. However, observe first the consistency check, that in the elliptic limit, the higher order vanishes entirely because of an overall factor of η that appears in it. Second, notice that in the higher order term, the δ's are always multiplied by $1-f$, which somewhat mitigates their contribution. Indeed, in the common approximation of ignoring shear speed contributions by taking $f=1$, the δ terms drop out completely along with the f contributions. Finally, observe that the function $y(1-2y)(1-y)$ multiplying the δ term has absolute maximum value less than 0.1 on the interval $0 \le y \le 1$, again mitigating the effect of δ on $V_{\rm nmo}^2(p)$ in this expansion. The overall observation that using, say, $f = 3/4$, instead of the true value has little numerical effect remains true here.

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Phase and Moveout Velocity as Functions of the Ray Parameter for Transversely Isotropic Media

Jack K. Cohen

Center for Wave Phenomena
Colorado School of Mines
Golden, Colorado 80401
303/273-3557

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